Paris Lectures on Topological Insulators

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Abstract

This is a written record corresponding to three lectures given in Paris in July 2022 on the topic of mathematical aspects of non-interacting topological insulators. The audience was largely Vojkan Jakšić's graduate students from McGill.

Contents

1 Introduction

1.1 Single-particle solid-state physics

We are interested in understanding the motion of electrons in a solid. For that we use quantum mechanics in the single-particle approximation, and hence, in $d \in \mathbb{N}$ space dimensions the Hilbert space is $\ell^2\left(\mathbb{Z}^d\right)$ if we are using the tight-binding approximation or $L^2\left(\mathbb{R}^d\right)$ if not.

The approach in solid state physics is to choose the simplest model so that non-trivial physics may be described. Furthermore, since electrons in solids rarely reach relativistic speeds, it seems unnecessary to speak about operators in the continuum. Hence we will mostly (unless otherwise stated) use the tight-binding approximation. Furthermore, sometimes it is necessary to describe internal degrees of freedom on each lattice site, and so, we will actually use the Hilbert space

$$\mathcal{H} := \ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N \cong \ell^2\left(\mathbb{Z}^d \times \{ 1, \dots, N \}\right) \cong \ell^2\left(\mathbb{Z}^d \to \mathbb{C}^N\right)$$

where $N \in \mathbb{N}$ is the number of degrees of freedom on each lattice site (sub-lattice, iso-spin, spin, etc).

To specify a solid, we must specify a Hamiltonian–a self-adjoint operator on $\mathcal{H}$. It will not be a big restriction to assume these operators are bounded: the bounded linear operators on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. This space of operators forms a *C-star algebra*. Furthermore, Hamiltonians should obey certain locality properties.

1.2 Locality properties

The natural basis on $\mathcal{H}$ is described as follows: we denote by $\{ \delta_x \}_{x\in\mathbb{Z}^d}$ the natural basis for $\ell^2\left(\mathbb{Z}^d\right)$ and by $\{ e_j \}_{j=1}^N$ the natural basis for $\mathbb{C}^N$. In particular we mean that, for any $x \in \mathbb{Z}^d$, the vector with entries

$$(\delta_x)_y \quad\equiv\quad \delta_{x,y} \equiv \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \qquad \left(y \in \mathbb{Z}^d\right) .$$

Then an ONB for $\mathcal{H}$ is given by

$$\{ \delta_x \otimes e_j \}_{x\in\mathbb{Z}^d, j=1,\dots,N}$$

and any operator $A \in \mathcal{B}(\mathcal{H})$ will have matrix elements

$$A_{xy;ij} \quad\equiv\quad \langle \delta_x \otimes e_i, A\delta_y \otimes e_j \rangle .$$

We can also consider partial matrix elements, i.e., define the $N \times N$ matrix A_{xy} whose i, j matrix element is given by

$$(A_{xy})_{ij} \quad:=\quad A_{xy;ij} \equiv \langle \delta_x \otimes e_i, A\delta_y \otimes e_j \rangle .$$

Definition 1.1. An operator $A \in \mathcal{B}(\mathcal{H})$ is called *local* iff its matrix elements in space exhibit exponential off-diagonal decay. I.e., iff there are constants $C_A, \mu_A \in (0, \infty)$ such that

$$\|A_{xy}\| \leq C_A \mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right) .$$

Here the norm on the LHS is the operator norm on $\mathrm{Mat}_N(\mathbb{C})$ and on the RHS the Euclidean norm on $\mathbb{R}^d$.

One may also consider locality with respect to other rates of decay (e.g. any-rate polynomial) or ask what is the slowest rate of decay so that such and such theorem holds. We shall mostly avoid such questions unless otherwise specified.

Definition 1.2. A single-particle *solid* is an operator $H \in \mathcal{B}(\mathcal{H})$ (the Hamiltonian) which is (1) self-adjoint, i.e., $H^* = H$ and (2) local as in Definition 1.1.

The significance of locality is clear from the Lieb-Robinson theorem. It is usually discussed in the context of many-body quantum mechanics [LR72], but here in the single-particle setting, obtains a particularly simple guise, which we take from [AW15, Exercises 2.2 (a)]:

Theorem 1.3. *(Single-particle Lieb-Robinson) If H is a solid as in Definition 1.2 then there is some velocity v_H and some $D < \infty$ such that for any $v > v_H$,*

$$\mathbb{P}[\{ \text{a particle starting at the origin is outside } B_{vt}(0) \}] \leq D\mathrm{e}^{-\frac{\mu_H}{2}(v-v_H)t} \qquad (t \geq 0) . \tag{1.1}$$

Here we mean $B_{vt}(0) \equiv \left\{ x \in \mathbb{Z}^d \,\middle|\, \|x\| < vt \right\}$.

Proof. First we interpret the LHS. We know from quantum mechanics that the state of a particle starting in the origin is δ_0. Since we have internal degrees of freedom we allow for an arbitrary state φ in $\mathbb{C}^N$ so we take the initial state of the particle as $\delta_0 \otimes \varphi$. We know that after time t, its state, according to quantum mechanics, is

$$\mathrm{e}^{-\mathrm{i}tH}\delta_0 \otimes \varphi$$

and finally, the probability to measure its position at some $y \in \mathbb{Z}^d$ (in some internal state $\psi \in \mathbb{C}^N$) is

$$\left|\left\langle \delta_y \otimes \psi, \mathrm{e}^{-\mathrm{i}tH}\delta_0 \otimes \varphi \right\rangle\right|^2 .$$

We thus bound the LHS of (1.1) as

$$\sup_{\varphi,\psi\in\mathbb{C}^N:\|\varphi\|=\|\psi\|=1} \sum_{y\in B_{vt}(0)^c} \left|\left\langle \delta_y \otimes \psi, \mathrm{e}^{-\mathrm{i}tH}\delta_0 \otimes \varphi \right\rangle\right|^2 = \sum_{y\in B_{vt}(0)^c} \left\|\left\langle \delta_y, \mathrm{e}^{-\mathrm{i}tH}\delta_0 \right\rangle\right\|^2 . \tag{1.2}$$

Now, we begin with a few preliminary estimates: For any $n \in \mathbb{N}$,

$$\begin{aligned}
\left\|(H^n)_{xy}\right\| &\equiv \left\| \sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} H_{x,z_1} \dots H_{z_{n-1},y} \right\| \\
&\leq \sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} \|H_{x,z_1}\| \dots \left\|H_{z_{n-1},y}\right\| \\
&\leq \sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} C_H^n \mathrm{e}^{-\mu_H(\|x-z_1\|+\dots+\|z_{n-1}-y\|)} && \text{(Locality of } H\text{)} \\
&\leq C_H^n \mathrm{e}^{-\frac{\mu_H}{2}\|x-y\|} \sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} \mathrm{e}^{-\frac{\mu_H}{2}(\|z_2-z_1\|+\dots+\|z_{n-1}-y\|)} .
\end{aligned}$$

In this last step, we have used the triangle inequality:

$$\|x - z_1\| + \dots + \|z_{n-1} - y\| \quad \geq \quad \|x - y\|$$

as well as dropping the first term since it is clearly positive. Since $\mathbb{Z}^d$ is invariant under translations, we find

$$\sum_{z_1,\dots,z_{n-1}\in\mathbb{Z}^d} \mathrm{e}^{-\nu(\|z_2-z_1\|+\dots+\|z_{n-1}-y\|)} \quad = \quad \left(\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\nu\|z\|} \right)^{n-1}$$

but the inner sum is clearly finite. E.g., $\|z\| \geq \frac{1}{\sqrt{d}}\|z\|_1$ with $\|z\|_1 \equiv \sum_{j=1}^d |z_j|$ which then factorizes:

$$\begin{aligned}
\sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\nu\|z\|} &\leq \left(\sum_{z\in\mathbb{Z}} \mathrm{e}^{-\frac{\nu}{\sqrt{d}}|z|} \right)^d \\
&= \left[\coth\left(\frac{\nu}{2\sqrt{d}} \right) \right]^d \\
&=: D_{\nu,d}
\end{aligned} \tag{1.3}$$

Combining everything together we have the estimate

$$\begin{aligned}
\left\|(H^n)_{xy}\right\| &\leq C_H^n \mathrm{e}^{-\frac{\mu_H}{2}\|x-y\|} \left(\left[\coth\left(\frac{\mu_H}{4\sqrt{d}} \right) \right]^d \right)^{n-1} \\
&= \frac{1}{D_{\frac{\mu_H}{2}}} \left(C_H D_{\frac{\mu_H}{2}} \right)^n \mathrm{e}^{-\frac{\mu_H}{2}\|x-y\|} .
\end{aligned}$$

Next, we have

$$\begin{aligned}\left\|\left\langle\delta_y, \mathrm{e}^{-\mathrm{i}tH}\delta_0\right\rangle\right\| &= \left\|\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}\left\langle\delta_y, H^n\delta_0\right\rangle\right\| \\ &\leq \sum_{n=0}^{\infty}\frac{t^n}{n!}\left\|(H^n)_{y,0}\right\| \\ &\leq \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{1}{D_{\frac{\mu_H}{2}}}\left(C_H D_{\frac{\mu_H}{2}}\right)^n \mathrm{e}^{-\frac{\mu_H}{2}\|y\|} && \text{(Previous estimate)} \\ &= \frac{1}{D_{\frac{\mu_H}{2}}}\mathrm{e}^{tC_H D_{\frac{\mu_H}{2}}-\frac{\mu_H}{2}\|y\|}\end{aligned}$$

and hence the RHS of (1.2) is bounded by

$$\begin{aligned}\sum_{y\in B_{vt}(0)^c}\left(D_{\frac{\mu_H}{2}}\right)^{-2}\mathrm{e}^{2tC_H D_{\frac{\mu_H}{2}}-\mu_H\|y\|} &\leq \left(D_{\frac{\mu_H}{2}}\right)^{-2}\mathrm{e}^{2tC_H D_{\frac{\mu_H}{2}}-\frac{\mu_H}{2}vt}\sum_{y\in\mathbb{Z}^d}\mathrm{e}^{-\frac{\mu_H}{2}\|y\|} \\ &\leq \left(D_{\frac{\mu_H}{2}}\right)^{-2}\mathrm{e}^{2tC_H D_{\frac{\mu_H}{2}}-\frac{\mu_H}{2}vt}D_{\frac{\mu_H}{2},d} \\ &= \frac{1}{D_{\frac{\mu_H}{2}}}\mathrm{e}^{-\frac{\mu_H}{2}\left(v-4\frac{C_H D_{\frac{\mu_H}{2}}}{\mu_H}\right)t}\end{aligned}$$

and so we identify $v_H := 4\frac{C_H D_{\frac{\mu_H}{2}}}{\mu_H}$ and $D := \frac{1}{D_{\frac{\mu_H}{2}}}$. $\square$

While the Lieb-Robinson bound gives an intuitive sense for what locality implies for quantum dynamics, we will find more for the Combes-Thomas estimate. Again, originally presented in the context of many-body quantum mechanics [CT73], the single-particle version ([AW15, Chapter 10.3]), presented here roughly speaking says that the analytic functional calculus of Hamiltonians preserves locality:

Theorem 1.4. *(The Combes-Thomas estimate) If H is a solid as in Definition 1.2, $z\in\mathbb{C}$ with*

$$\delta := \operatorname{dist}(z, \sigma(H)) > 0 \tag{1.4}$$

then there is some $\tilde{\mu}_H > 0$ such that

$$\left\|R(z)_{xy}\right\| \leq \frac{2}{\delta}\mathrm{e}^{-\tilde{\mu}_H\delta\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^d\right)$$

with $R(z)\equiv(H-z\mathbb{1})^{-1}$ being the resolvent operator and $\sigma(H)\subseteq\mathbb{R}$ the spectrum of H.

Corollary 1.5. *(The analytic functional calculus of a local self-adjoint operator is local) Assume that $f:\mathbb{R}\to\mathbb{C}$ is analytic, i.e., that,*

$$f(\lambda) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{1}{z-\lambda}f(z)\,\mathrm{d}z \qquad (\lambda\in\mathbb{R})$$

for some closed CCW contour Γ which encloses λ. Then if H is a solid as in Definition 1.2 then $f(H)$ is local as in Definition 1.1.

Proof. Write

$$f(H) \quad = \quad \frac{\mathrm{i}}{2\pi}\oint_\Gamma R(z)\,f(z)\,\mathrm{d}z$$

where Γ is a closed CCW contour which encloses $\sigma(H)$. Since H is bounded, $\sigma(H)$ has a finite diameter. Let Γ be a contour which always stays distance 1 away from $\sigma(H)$ and not ever more than distance 2, so that, say,

$\oint_\Gamma |\mathrm{d}z| \leq 2\|H\|$. Then

$$\begin{aligned}\left\|f(H)_{xy}\right\| &\leq \frac{1}{2\pi}\sup_{z\in\Gamma}\left\|R(z)_{xy}\right\|\sup_{z\in\Gamma}|f(z)|\oint_\Gamma |\mathrm{d}z| \\ &\leq \frac{1}{\pi}\|H\|\sup_{z\in\Gamma}|f(z)|\sup_{z\in\Gamma}\left\|R(z)_{xy}\right\| \\ &\leq \frac{2}{\pi}\|H\|\sup_{z\in\Gamma}|f(z)|\,\mathrm{e}^{-\nu\|x-y\|}\,. \qquad \text{(Use the Combes-Thomas estimate)}\end{aligned}$$

□

Proof. of Theorem 1.4: Let $f:\mathbb{Z}^d\to\mathbb{C}$ be a *bounded measurable* function such that there is some $\nu\in(0,\infty)$ with

$$|f(x)-f(y)| \quad\leq\quad \nu\|x-y\| \qquad \left(x,y\in\mathbb{Z}^d\right)\,.$$

Define then

$$H_f \quad:=\quad \mathrm{e}^{f(X)}H\mathrm{e}^{-f(X)}$$

which is clearly also bounded. A short calculation yields

$$\begin{aligned}\left[\left(H_f-z\mathbb{1}\right)^{-1}\right]_{xy} &= \left[\left(\mathrm{e}^{f(X)}H\mathrm{e}^{-f(X)}-z\mathbb{1}\right)^{-1}\right]_{xy} \\ &= \left[\left(\mathrm{e}^{f(X)}H\mathrm{e}^{-f(X)}-z\mathrm{e}^{f(X)}\mathrm{e}^{-f(X)}\right)^{-1}\right]_{xy} \\ &= \left[\mathrm{e}^{-f(X)}R(z)\,\mathrm{e}^{f(X)}\right]_{xy} \\ &= \mathrm{e}^{-f(x)}R(z)_{xy}\,\mathrm{e}^{f(y)}\end{aligned}$$

Hence,

$$\begin{aligned}\left\|R(z)_{xy}\right\| &= \mathrm{e}^{f(x)-f(y)}\left\|R_f(z)_{xy}\right\| \\ &\leq \mathrm{e}^{f(x)-f(y)}\|R_f(z)\|\,.\end{aligned}$$

But for any $\varphi\in\mathcal{H}$,

$$\begin{aligned}\|(H_f-z\mathbb{1})\varphi\| &= \|(H-z\mathbb{1})\varphi\|-\|(H_f-H)\varphi\| \\ &\geq \delta\|\varphi\|-\|H_f-H\|\|\varphi\|\end{aligned}$$

where we have used (1.4) in the last step. Using Holmgren's bound (see Lemma 1.6 just below) we have then

$$\begin{aligned}\|H_f-H\| &\leq \max_{x\leftrightarrow y}\sup_x\sum_y\left\|(H_f-H)_{xy}\right\| \\ &= \max_{x\leftrightarrow y}\sup_x\sum_y\left|\mathrm{e}^{f(x)-f(y)}-1\right|\|H_{xy}\| \\ &\leq \max_{x\leftrightarrow y}\sup_x\sum_y\left(\mathrm{e}^{\nu\|x-y\|}-1\right)C_H\mathrm{e}^{-\mu_H\|x-y\|} \\ &= \sum_y\left(\mathrm{e}^{\nu\|y\|}-1\right)C_H\mathrm{e}^{-\mu_H\|y\|} \\ &\leq 2C_H\nu\sum_y\mathrm{e}^{-(\mu_H-2\nu)\|y\|} \qquad \left(\text{Use } \mathrm{e}^{\nu\|y\|}-1\leq 2\nu\mathrm{e}^{2\nu\|y\|}\right) \\ &\leq 2C_H\nu D_{\mu_H-2\nu} \\ &\leq 2C_H\nu D_{\frac{\mu_H}{2}}\end{aligned}$$

where D_α is defined as in (1.3). Hence if we pick ν such that

$$2C_H \nu D_{\frac{\mu_H}{2}} = \frac{\delta}{2}$$

we find $\|R_f(z)\| \leq \frac{\delta}{2}$. I.e.,

$$\nu := \frac{\delta}{4C_H D_{\frac{\mu_H}{2}}}.$$

We thus recognize $\tilde{\mu}_H := \frac{1}{4C_H D_{\frac{\mu_H}{2}}}$ and conclude using the fact that we have the freedom to replace $f \mapsto -f$, and hence, we have

$$\left\| R(z)_{xy} \right\| \leq \frac{\delta}{2} \min\left(\left\{ \mathrm{e}^{f(x)-f(y)}, \mathrm{e}^{-f(x)+f(y)} \right\}\right)$$

and together with $|f(x) - f(y)| \leq \tilde{\mu}_H \delta \|x - y\|$ this implies the result. $\square$

Above we have used the following basic

Lemma 1.6. *(Holmgren's bound) For any operator A on a Hilbert space with an ONB $\{\psi_j\}_j$ we have*

$$\|A\| \leq \sqrt{\sup_j \sum_k |\langle \psi_j, A\psi_k \rangle|} \sqrt{\sup_k \sum_j |\langle \psi_j, A\psi_k \rangle|}$$

Proof. Start by the characterization $\|A\| = \sup(\{ |\langle \varphi, A\psi \rangle| \mid \|\varphi\| = \|\psi\| = 1 \})$, and use

$$\begin{aligned}
|\langle \varphi, A\psi \rangle| &\leq \sum_{i,j} |\varphi_i| \, |A_{ij}| \, |\psi_j| \\
&= \sum_{i,j} \left(|\varphi_i| \sqrt{|A_{ij}|} \right) \left(\sqrt{|A_{ij}|} \, |\psi_j| \right) \\
&\leq \sqrt{\sum_{i,j} |\varphi_i|^2 \, |A_{ij}|} \sqrt{\sum_{i,j} |A_{ij}| \, |\psi_j|^2} && \text{(Cauchy-Schwarz)} \\
&\leq \sqrt{\left(\sup_i \sum_j |A_{ij}| \right) \left(\sum_i |\varphi_i|^2 \right)} \sqrt{\left(\sup_j \sum_i |A_{ij}| \right) \left(\sum_j |\psi_j|^2 \right)} \\
&= \sqrt{\sup_i \sum_j |A_{ij}|} \sqrt{\sup_j \sum_i |A_{ij}|}\,.
\end{aligned}$$

$\square$

1.3 Periodic systems

While periodic (or translation-invariant) materials are not at all realistic from the point of view of solid-state physics, their analysis offers some intuitive pictures which are instructive for the further study of disordered systems–out ultimate goal.

Definition 1.7. An operator $A \in \mathcal{B}(\mathcal{H})$ is called *periodic* iff

$$A_{x+z,y+z} = A_{x,y} \qquad \left(x, y, z \in \mathbb{Z}^d\right).$$

Thus, periodic operators are constant along their diagonal (if we think about operators as infinite matrices). Our main tool to analyze them will be the Fourier series (math) or Bloch decomposition. It sends wave-functions in real space to wave-functions in momentum space, or Fourier space, so to speak.

Definition 1.8. The Fourier series is the map

$$\mathcal{F}:\ell^2\left(\mathbb{Z}^d\right)\quad\to\quad L^2\left(\mathbb{T}^d\right)$$

where $\mathbb{T}^d\equiv[0,2\pi)^d$ is the d-dimensional torus, given by

$$\ell^2\left(\mathbb{Z}\right)^d\ni\psi\quad\overset{\mathcal{F}}{\mapsto}\quad\left(\mathbb{T}^d\ni k\mapsto(2\pi)^{-\frac{d}{2}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\psi_x\right)=:\hat{\psi}$$

and inverse

$$L^2\left(\mathbb{T}^d\right)\ni\hat{\psi}\quad\overset{\mathcal{F}^{-1}}{\mapsto}\quad\left(\mathbb{Z}^d\ni x\mapsto(2\pi)^{-\frac{d}{2}}\int_{k\in\mathbb{T}^d}\mathrm{e}^{\mathrm{i}\langle k,x\rangle}\hat{\psi}\left(k\right)\mathrm{d}k\right).$$

Lemma 1.9. *(1) $\mathcal{F}$ is unitary, (2) If A is periodic as in Definition 1.7 then*

$$\mathcal{F}A\mathcal{F}^*\quad=\quad M_a$$

where M_a is the multiplication operator by the function a: $\left(M_a\hat{\psi}\right)(k)\equiv a\left(k\right)\hat{\psi}\left(k\right)$ and

$$a\left(k\right)\quad=\quad(2\pi)^{-\frac{d}{s}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}A_{0,x}\qquad\left(k\in\mathbb{T}^d\right).$$

Proof. For the first item, we calculate

$$\begin{aligned}
\left\langle\hat{\varphi},\hat{\psi}\right\rangle_{L^2(\mathbb{T}^d)} &\equiv \int_{k\in\mathbb{T}^d}\overline{\left((2\pi)^{-\frac{d}{2}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\varphi_x\right)}\left((2\pi)^{-\frac{d}{2}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\psi_x\right)\mathrm{d}k\\
&= \sum_{x,y\in\mathbb{Z}^d}\overline{\varphi_x}\psi_y\underbrace{(2\pi)^{-d}\int_{k\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\langle k,x-y\rangle}\mathrm{d}k}_{=\delta_{x,y}}\\
&= \sum_{x\in\mathbb{Z}^d}\overline{\varphi_x}\psi_x\\
&\equiv \langle\varphi,\psi\rangle_{\ell^2(\mathbb{Z}^d)}\,.
\end{aligned}$$

For the second item, let us calculate

$$\begin{aligned}
\left(\mathcal{F}A\mathcal{F}^*\hat{\psi}\right)(k) &\equiv (2\pi)^{-\frac{d}{2}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\sum_{y\in\mathbb{Z}^d}A_{xy}\,(2\pi)^{-\frac{d}{2}}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,y\rangle}\hat{\psi}\left(p\right)\mathrm{d}p\\
&= (2\pi)^{-\frac{d}{2}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\sum_{y\in\mathbb{Z}^d}A_{0,y-x}\,(2\pi)^{-\frac{d}{2}}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,y\rangle}\hat{\psi}\left(p\right)\mathrm{d}p\\
&= (2\pi)^{-\frac{d}{2}}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\sum_{z\in\mathbb{Z}^d}A_{0,z}\,(2\pi)^{-\frac{d}{2}}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,z+x\rangle}\hat{\psi}\left(p\right)\mathrm{d}p\\
&= \sum_{z\in\mathbb{Z}^d}A_{0,z}\,(2\pi)^{-d}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,z\rangle}\delta\left(k+p\right)\hat{\psi}\left(p\right)\mathrm{d}p\\
&= \sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,z\rangle}A_{0,z}\hat{\psi}\left(k\right)\,.
\end{aligned}$$

□

Example 1.10. The right-shift in direction $j = 1, \ldots, d$ operator R_j is defined as

$$(R_j\psi)_y \quad \equiv \quad \psi_{y-e_j} \qquad \left(y \in \mathbb{Z}^d, \psi \in \ell^2\left(\mathbb{Z}^d\right)\right)$$

It is a periodic operator and and hence its Fourier representation is a diagonal multiplication operator:

$$\mathcal{F} R_j \mathcal{F}^* \quad = \quad M_{r_j}$$

with $r_j(k) = \mathrm{e}^{-\mathrm{i}k_j}$ for all $k \in \mathbb{T}^d$.

Example 1.11. The discrete Laplacian $-\Delta$ defined via

$$(-\Delta\psi)_x \quad := \quad \sum_{y\sim x} \psi_x - \psi_y \qquad \left(\psi \in \ell^2\left(\mathbb{Z}^d\right), x \in \mathbb{Z}^d\right)$$

is periodic. It may be re-written using the right-shift operator as

$$-\Delta \quad = \quad 2\sum_{j=1}^{d} \mathbb{1} - \mathbb{Re}\left\{R_j\right\}$$

with $\mathbb{Re}\left\{A\right\} \equiv \frac{1}{2}\left(A + A^*\right)$. In momentum (i.e., Fourier) space, it is given as multiplication by the function

$$\mathcal{E}(k) \quad := \quad 2\sum_{j=1}^{d} 1 - \cos\left(k_j\right) \qquad \left(k \in \mathbb{T}^d\right).$$

Example 1.12. The position operator in the jth direction $(j = 1, \ldots, d)$

$$(X_j\psi)_y \equiv y_j\psi_y \qquad \left(y \in \mathbb{Z}^d, \psi \in \ell^2\left(\mathbb{Z}^d\right)\right) \tag{1.5}$$

gets mapped to to derivative with respect to momentum, i.e.,

$$\mathcal{F} X_j \mathcal{F}^* = \mathrm{i}\partial_j \qquad (j = 1, \ldots, d).$$

What does $[X_j, A]$ map to for a periodic operator A?

Example 1.13. A multiplication operator in real space M_v by the function $v : \mathbb{Z}^d \to \mathbb{R}$ is mapped onto the convolution operator C_v in momentum space.

Proof. Use the convolution theorem for Fourier series. □

The following theorem from classical harmonic analysis [Kat04, pp. 27] associates locality in real space to regularity in momentum space:

Theorem 1.14. *(Riemann-Lebesgue) If A is local as in Definition 1.1 and periodic as in Definition 1.7, so that $\mathcal{F} A \mathcal{F}^* = M_a$, then $a : \mathbb{T}^d \to \mathbb{C}$ is analytic.*

Proof. We have from Lemma 1.9 that

$$a(z) \quad = \quad (2\pi)^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}^d} \prod_{j=1}^{d} z_j^{x_j} A_{0,x} \qquad \left(z \in \left(\mathbb{S}^1\right)^d\right).$$

Now deforming z, we write it instead as of $\tilde{z} = rz$ where $r \in (0,\infty)^d$ so that

$$\begin{aligned}|a(\tilde{z})| &\leq (2\pi)^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}^d} \prod_{j=1}^{d} r_j^{x_j} \|A_{0,x}\| \\ &\leq C_A (2\pi)^{-\frac{d}{2}} \prod_{j=1}^{d} \sum_{x_j\in\mathbb{Z}} r_j^{x_j} \mathrm{e}^{-\frac{\mu_A}{\sqrt{d}}|x_j|} . \qquad \text{(Using Definition 1.1)}\end{aligned}$$

Clearly this is finite if $\mathrm{e}^{-\frac{\mu_A}{\sqrt{d}}} < \max\left(\left\{ r_j, \frac{1}{r_j} \right\}\right)$. Hence we get a convergent power series in an annulus around the torus which is equivalent to analyticity on that annulus. □

More generally, any-rate polynomial decay will be mapped to smooth "symbols", and ℓ^p locality will be mapped to C^p reguliarty of the symbol.

2 Insulators

2.1 Non-interacting electrons in a solid

So far we have discussed the various consequences of representing the dynamics of electrons moving in solids via the single-particle, tight-binding approximations as bounded self-adjoint, local operators on Hilbert space. We have studied the relationship between constraints on these operators and *the quantum dynamics*. We now want to take this a step further and discuss what kind of constraints on these operators will lead to consequences on *electronic transport properties*.

We are studying the single-particle approximation, but our particles are still electrons, hence Fermions. There is an extent to which we still take interactions into account: we still put in the Fermi exclusion principle for Fermions into our description. The way this happens is that while in single-particle quantum mechanics, when we compute the expectation value of an observable $B = B^* \in \mathcal{B}(\mathcal{H})$ in a state $\psi \in \mathcal{H}$ as

$$\langle \psi, B\psi \rangle \ .$$

To speak about the expectation value in the state of many non-interacting electrons of the single-particle observable B, it turns out that the appropriate expression to calculate is

$$\mathrm{tr}_{\mathcal{H}}(PB)$$

where $P \equiv \chi_{(-\infty,\mu]}(H)$ is the *Fermi projection* and $\mu \in \mathbb{R}$ is the *Fermi energy*, which characterizes "how many electrons" there are in the solid (the number is mathematically infinite in infinite volume so it rather characterizes the density). The Fermi projection is an example of a *density matrix*, and we now explain why it is the appropriate object to use for the many-body non-interacting ground state of the electrons in a solid.

2.2 Zero-temperature ground state expectation values

In quantum mechanics, the state of a particle is described by a vector in a Hilbert space $\mathcal{H}$. To talk about the state of M *distinguishable* particles simultaneously, we need to consider a vector in the M-fold tensor product Hilbert space $\bigotimes_{j=1}^M \mathcal{H}$. However, if we have M *indistinguishable* particles which are *Fermions,* which is the situation for electrons in a solid, then the state of these particles is actually a vector in the M-fold exterior product Hilbert space $\bigwedge_{j=1}^M \mathcal{H}$, since the state must be anti-symmetric with respect to exchange of any two particles.

If the single-particle Hamiltonian H is acting on each particle separately, then the many-body Hamiltonian is given by

$$\mathrm{d}\Gamma(H) := \sum_{j=1}^{M} \mathbb{1}^{\wedge(j-1)} \wedge H \wedge \mathbb{1}^{\wedge(M-j)} ,$$

i.e., the single particle Hamiltonian acts on the jth particle and doesn't do anything on all other particles.

Then, if we are interested in the *many-body expectation value of a non-interacting single-particle observable*, say, B, we would first raise it to the many-body Hilbert space just as above:

$$B \quad \mapsto \quad \sum_{j=1}^{M} \mathbb{1}^{\wedge(j-1)} \wedge B \wedge \mathbb{1}^{\wedge(M-j)} =: \mathrm{d}\Gamma(B)$$

and then if our system was in the state $\Psi \in \bigwedge_{j=1}^{M} \mathcal{H}$, we would calculate

$$\langle \Psi, \mathrm{d}\Gamma(B)\Psi \rangle\ .$$

Now, if Ψ itself is a product state, i.e., $\Psi = \psi_1 \wedge \cdots \wedge \psi_M$, where $\{\ \psi_j\ \}_{j=1}^{M}$ is an orthonormal state, then this simplifies to

$$\begin{aligned} \langle \Psi, \mathrm{d}\Gamma(B)\Psi \rangle &= \sum_{j=1}^{M} \langle \psi_j, B\psi_j \rangle \\ &= \operatorname{tr}\left(\sum_{i=1}^{M} \psi_i \otimes \psi_i^* B \right) \end{aligned}$$

where we recognize $\sum_{i=1}^{M} \psi_i \otimes \psi_i^*$ as the projection operator onto the space spanned by the orthonormal set $\{\ \psi_j\ \}_{j=1}^{M}$.

Now, say our Hamiltonian of the solid we wish to describe is $H \in \mathcal{B}(\mathcal{H})$, and say its eigenstates are $\{\ \varphi_j\ \}_{j\in\mathbb{N}}$ (ordered so that φ_1 has the lowest energy, etc). Since no two Fermions can occupy the same quantum mechanical state (this is the Pauli exclusion principle), if we fill the solid with M electrons, the ground state (i.e., the state of least energy, *at zero temperature*) is the one where the M electrons occupy the M first levels of H, i.e., $\varphi_1, \dots, \varphi_M$. The corresponding state on the many-body Hilbert space $\bigwedge \mathcal{H}$ (the exterior algebra generated by $\mathcal{H}$) is thus

$$\varphi_1 \wedge \cdots \wedge \varphi_M$$

(which is called *the Slater determinant*). If we calculate the corresponding *one-particle reduced density matrix,* $\rho \in \mathcal{B}(\mathcal{H})$*, whose matrix elements are given by*

$$\rho_{xy} := \sum_{x_2,\dots,x_M \in \mathbb{Z}^d} \overline{(\varphi_1 \wedge \cdots \wedge \varphi_M)(x, x_2, \dots, x_M)}\,(\varphi_1 \wedge \cdots \wedge \varphi_M)(y, x_2, \dots, x_M) \qquad \left(x, y \in \mathbb{Z}^d\right) \tag{2.1}$$

we obtain precisely the projection onto these M lowest states

$$\rho = \chi_{(-\infty,\lambda_M]}(H)\ .$$

where λ_M is the energy corresponding to φ_M. Hence, we use the density matrix ρ in our single-particle calculations, and this corresponds to a zero temperature non-interacting many-body expectation value. Since it makes more sense to speak about the energy up to which electrons are filled rather than how many electrons there are (in infinite volume there are infinitely many particles and it is more appropriate to speak about the density anyway), we characterize the filling of the system by an energy parameter $\mu \in \mathbb{R}$ called the Fermi energy and define the Fermi projection corresponding to H

$$P_\mu := \chi_{(-\infty,\mu]}(H)$$

which is manifestly the reduced one-particle density matrix associated with the non-interacting ground state associated with H filled up to energy level μ. In infinite volume the range of this operator is infinite dimensional. In order to define this operator rigorously one has to apply the measurable functional calculus of bounded self-adjoint operators, see [RS80, Vol. 1, Ch. VII]. It will turn out that the Fermi projection P_μ contains most of the properties we care about in regards to topological insulators. At non-zero temperatures the Fermi-Dirac distribution should be used–we won't make use of this.

2.3 Electric conductivity

We wish to study insulators, for which we would like to calculate their electric conductance, which is phenomenologically defined via Ohms law:

$$\sigma = \frac{I}{V}$$

with I being the current and V the voltage. More generally, the conductivity σ is defined as the matrix relating the current density j with the electric field as follows:

$$j = \sigma E\,.$$

In principle each of these calculations of σ depends on the Fermi energy μ to which we fill the system.

Definition 2.1. An electric insulator at Fermi energy μ is a material filled to μ whose conductivity matrix at that energy is zero on the diagonal:

$$\sigma_{ii}(\mu) \quad = \quad 0\,.$$

Why do we only talk about the diagonal conductivity will become clear later when we consider the Hall conductivity.

In the physics literature, for historical and possibly physical reasons, one usually separates the objects of study in an experimental setup where there is a material (a solid) which is described by a Hamiltonian H and the external driving electric field. Hence, if we calculate the conductivity associated with H alone, it should be zero (since it would typically have no spontaneous currents) and only once we *perturb* with an external electric field it does it actually make to calculate σ. Thus, we are at the task of perturbation theory, by, say a constant electric field. As we know from undergraduate quantum mechanics, this means adding a term of the form

$$E_0 X_i$$

if the field is of strength E_0 in direction $i = 1, \dots, d$.

Typically, however, the type of perturbation theory taught in undergraduate quantum mechanics (Rayleigh-Schrödinger perturbation theory) is inappropriate for most systems we want to deal with, since it only deals with systems with discrete spectrum (finitely degenerate isolated eigenvalues). Also, generally one likes to do perturbation theory of the more general density matrices. The general theory under which this is done is called *linear response theory* [Kub91].

2.4 Linear response theory–the Kubo formula

As we have said the perturbation we are mostly concerned with is something proportional to the position operator and the observable should be the current density, i.e.,

$$j_i \quad = \quad n\mathrm{i}\,[H, X_i] \qquad (i = 1, \dots, d)$$

where n is the density of particles. Indeed, H being the generator of time-translations, $\mathrm{i}\,[H, X_i]$ is associated with $\frac{\mathrm{d}}{\mathrm{d}t} X_i$, i.e., the velocity.

Furthermore, the perturbations we shall consider are *not* constant in time. Instead, they will be turned on very slowly from being zero at the beginning of time.

Theorem 2.2. *(The Kubo formula) Assume a system governed by H is in state described by density matrix ρ_0. Assume further that it is perturbed by the time-dependent operator $\varepsilon f(t) A$, where $f : \mathbb{R} \to [0, 1]$ is some smooth time-modulation function which obeys $f(-\infty) = 0$ and $f(0) = 1$, $\varepsilon > 0$ is some small order parameter, and A is a time-independent self-adjoint operator. Then the first order (in ε) coefficient of the expectation value of an observable $B = B^*$ for which $\mathrm{tr}(\rho_0 B) = 0$ to the perturbation at time zero is given by*

$$\chi_{BA} := -\mathrm{i} \int_{-\infty}^{0} \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH} B \mathrm{e}^{\mathrm{i}tH} [A, \rho_0]\right) f(t)\,\mathrm{d}t\,. \tag{2.2}$$

Proof. The state of the system at time t is governed by the Schrödinger equation for the density matrix, which is

$$\mathrm{i}\dot{\rho}(t) \quad = \quad [H + \varepsilon f(t) A, \rho(t)]\,.$$

We assume that

$$\rho(t) \quad = \quad \rho_0 + \varepsilon \rho_1(t)$$

where ρ_0 is independent of time since the zero order in ε has no time dependence in the Hamiltonian. A short calculation then shows that

$$\mathrm{i}\dot{\rho}_1(t) \quad = \quad f(t)[A, \rho_0] + [H, \rho_1(t)] + \mathcal{O}(\varepsilon)$$

which is solved by

$$\rho_1(t) = -\mathrm{i}\varepsilon \int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}(t-s)H} [A, \rho_0]\, \mathrm{e}^{+\mathrm{i}(t-s)H} f(s)\,\mathrm{d}s \tag{2.3}$$

Hence, we find

$$\begin{aligned}\langle B\rangle_{\rho(t)} &\equiv \operatorname{tr}\left(\rho\left(t\right)B\right)\\ &= \operatorname{tr}\left(\rho_0 B\right)+\varepsilon\operatorname{tr}\left(\rho_1\left(t\right)B\right)\end{aligned}$$

and plugging in (2.3) into this first order expression yields the claim. □

2.5 The conductivity formula from Kubo

We now want to apply this formalism in order to calculate the conductivity of a system. As explained above, the appropriate initial density matrix ρ_0 to use is the Fermi projection, i.e.,

$$\rho_0 \quad = \quad P\equiv\chi_{(-\infty,\mu]}\left(H\right).$$

At non-zero temperature one replaces $\chi_{(-\infty,\mu]}$ with the Fermi-Dirac distribution.

The observable B should be the current density, which is related to the velocity operator in direction i, so we shall take

$$B \quad = \quad \mathrm{i}\left[H,X_i\right].$$

The perturbation shall be the electric field in direction j, i.e., X_j, so that all together we find that to first order in the electric field,

$$\sigma_{ij}\left(\mu\right)=\lim_{f\to1}\operatorname{tr}\int_{-\infty}^{0}\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\left[H,X_i\right]\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[X_j,P\right]f\left(t\right)\mathrm{d}t\,. \tag{2.4}$$

The reason why we take the limit is that eventually we are interested in the static case, where the perturbation is *not* time dependent (or alternatively in the adiabatic limit where the perturbation is turned on infinitely slowly).

Theorem 2.3. *If H is time-reversal invariant (in the sense that (2.4) is symmetric in $i\leftrightarrow j$) then*

$$\sigma_{ij}\left(\mu\right) \quad = \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}x_ix_j\left\|R\left(\mu+\mathrm{i}\varepsilon\right)_{0,x}\right\|^2. \tag{2.5}$$

The proof of this claim is beyond the scope of these notes, see [AG98, Appendix A] for more details. An immediate corollary is that systems which have a spectral gap (so that Theorem 1.4 is useful) have $\sigma_{ij}\left(\mu\right)=0$ if μ is within the gap.

This corollary motivates the main definition of this chapter, which is its main takeaway message:

Definition 2.4. A solid specified by H (as in Definition 1.2), is said to be *an insulator at Fermi energy* $\mu\in\mathbb{R}$ iff $\mu\notin\sigma\left(H\right)$.

Note that since $\sigma\left(H\right)^c\in\operatorname{Open}\left(\mathbb{C}\right)$, there is a whole interval about μ which is *not* in the spectrum.

This is not the most general form of an insulator: it completely missed the phenomenon of being an insulator due to Anderson localization. But we shall not say more about this at the moment, see e.g. [EGS05, Sha20] for more details on the so-called mobility gap regime.

Theorem 2.5. *For systems that do* not *have time-reversal-invariance, such as integer quantum Hall systems, if μ is within a spectral gap of H, one can bring (2.4) to the form*

$$\sigma_{ij}\left(\mu\right) \quad = \quad \mathrm{i}\operatorname{tr}\left(P\left[\left[\Lambda_1,P\right],\left[\Lambda_2,P\right]\right]\right). \tag{2.6}$$

Here, Λ_j is a projection operator onto the positive half-space defined by the jth axis:

$$\left(\Lambda_j\psi\right)_x \quad \equiv \quad \begin{cases}\psi_x & x_j\geq1\\ 0 & x_j\leq0\,.\end{cases}$$

Part of the statement of the theorem is that the above expression is indeed

The proof of this statement is again beyond the scope of these notes but may be found, e.g., in [Sha16].

3 Topology

In this brief chapter, we explain some notions from point-set topology and set up a program for the study of "topological insulators".

Topology is the study of properties of mathematical spaces which are preserved by smooth of continuous deformations. The mathematical field of topology seems, a-posteriori, to be the appropriate way to describe physical properties of materials which are macroscopically stable under experimental noise, impurities and variation. It will turn out that the topology of spaces of quantum mechanical Hamiltonians obeying certain constraints (like locality, the insulator condition and possibly symmetry constraints) will yield interesting and experimentally-relevant descriptions.

3.1 Some words on point-set topology

This section here is only included really mainly to set notation, but is expected to be basic undergraduate material to be found, e.g., in [Mun00].

Definition 3.1. For any set S, a topology $\operatorname{Open}(S)$ on S is a set of subsets of S such that:

1. $S \in \operatorname{Open}(S)$ and $\varnothing \in \operatorname{Open}(S)$.
2. An arbitrary union of elements of $\operatorname{Open}(S)$ again belongs to $\operatorname{Open}(S)$.
3. A *finite* intersection of elements of $\operatorname{Open}(S)$ again belongs to $\operatorname{Open}(S)$.

Naturally, a subset $U \subseteq S$ is called *open* iff $U \in \operatorname{Open}(S)$. A set $F \subseteq S$ is called *closed* iff $S \setminus F \in \operatorname{Open}(S)$. The set of all closed sets is denoted $\operatorname{Closed}(S)$. Note that these two properties are *not* mutually exclusive and one speaks of *clopen* sets as well: $\operatorname{Open}(S) \cap \operatorname{Closed}(S)$.

Unless one is studying a designated class on point-set topology, most likely one will be dealing with metric spaces, in which a pre-given distance $d(x,y)$ between any two points $x, y \in S$ is given. In this setting, there is a much more concrete way to describe $\operatorname{Open}(S)$:

Definition 3.2. (*metric topology*) For any metric space (S,d), a natural topology $\operatorname{Open}(S)$ is specified by

$$\operatorname{Open}(S) \quad := \quad \{\, U \subseteq S \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq U \,\}$$

where we use the notation for an *open ball* of radius ε:

$$B_\varepsilon(x) \quad := \quad \{\, y \in S \mid d(x,y) < \varepsilon \,\} \ .$$

Another important example is the

Definition 3.3. (*discrete topology*) For any set S the *discrete topology* is defined by

$$\operatorname{Open}(S) \quad = \quad \mathscr{P}(S)$$

where $\mathscr{P}(S)$ is the power set of S, i.e., the set of all subsets of S. That is, the discrete topology is a topology where *all* subsets are open. Note that this is also an example of a metric topology if we define the *discrete metric*

$$\rho(x,y) \quad := \quad \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \qquad (x, y \in S) \ .$$

An important notion that will come up frequently is that of subspace topology

Definition 3.4. Let $(S, \operatorname{Open}(S))$ be a topological space and $T \subseteq S$. Then a natural topology on T is given by the *subspace topology descending from S:*

$$\operatorname{Open}(T) \quad := \quad \{\, U \subseteq T \mid U = T \cap V \exists V \in \operatorname{Open}(S) \,\} \ .$$

Definition 3.5. For any point $s \in S$, a *neighborhood* of s is some $U \subseteq S$ such that $s \in U$ and $\exists V \in \text{Open}(S)$ with $s \in V \subseteq U$. The set of all neighborhoods of a given point s is denoted by $\text{Nbhd}(s)$.

Like all mathematical structures on sets (e.g. groups, vector spaces, etc), we are interested in maps $f : S \to T$ which respects a given structure. This leads us to the notion of a *continuous* map:

Definition 3.6. A map $f : S \to T$ between two topological spaces is said to be *continuous* iff

$$f^{-1}(\text{Open}(T)) \subseteq \text{Open}(S) . \tag{3.1}$$

Here, $f^{-1}(\text{Open}(T)) \equiv \left\{ f^{-1}(U) \,\middle|\, U \in \text{Open}(T) \right\}$ and $f^{-1}(B) \equiv \{ s \in S \mid f(s) \in B \}$.

One also uses the slightly more fine-grained definition that $f : S \to T$ *is continuous at* $s \in S$ iff

$$f^{-1}(\text{Nbhd}(f(s))) \subseteq \text{Nbhd}(s) . \tag{3.2}$$

Then f is continuous iff it is continuous at s for all $s \in S$.

Remark 3.7. This notion is more useful than that of an *open* map: $f : S \to T$ is open iff

$$f(\text{Open}(S)) \subseteq \text{Open}(T) . \tag{3.3}$$

In what way does (3.1) respect the topological structure rather than (3.3)? The basic operations on open sets are arbitrary unions and finite intersections. I.e., we want an operation that "commutes" with the operations which preserve openness: unions and intersections of sets. It turns out that the preimage operation, f^{-1}, respects these operations whereas the image operation f does not:

$$\begin{aligned} f^{-1}\left(\bigcup_\alpha U_\alpha\right) &= \bigcup_\alpha f^{-1}(U_\alpha) \\ f^{-1}\left(\bigcap_\alpha U_\alpha\right) &= \bigcap_\alpha f^{-1}(U_\alpha) \end{aligned}$$

and for the image,

$$\begin{aligned} f\left(\bigcup_\alpha U_\alpha\right) &= \bigcup_\alpha f(U_\alpha) \\ f\left(\bigcap_\alpha U_\alpha\right) &\subseteq \bigcap_\alpha f(U_\alpha) . \end{aligned}$$

An example where this last inclusion is proper is the following: $f : \mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto n^2$ has $\mathbb{N}_{\geq 0} \cap \mathbb{Z}_{\leq 0} = \{ 0 \}$ so that

$$f(\mathbb{N}_{\geq 0} \cap \mathbb{Z}_{\leq 0}) = f(\{ 0 \}) = \{ 0 \}$$

and yet $f(\mathbb{N}_{\geq 0}) = f(\mathbb{Z}_{\geq 0}) = (\mathbb{N}_{\geq 0})^2 = \{ 0, 1, 4, 9, \dots \}$.

It is of course comforting to know that these new abstract notions match those studied in real analysis:

Lemma 3.8. *For a map $f : S \to T$ between two metric spaces, f is continuous at $s \in S$ iff for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that*

$$B_{\delta_\varepsilon}(s) \subseteq f^{-1}(B_\varepsilon(f(s))) . \tag{3.4}$$

or stated in the more familiar notation, f is continuous at $s \in S$ iff for every $\varepsilon > 0$ $\exists \delta_\varepsilon > 0$ such that for all $\tilde{s} \in S$ such that $d_S(s, \tilde{s}) < \delta_\varepsilon$ one has $d_T(f(s), f(\tilde{s})) < \varepsilon$.

Proof. First assume f is continuous at $s \in S$, i.e., (3.2). Now let $\varepsilon > 0$ be given. Since f is continuous and $B_\varepsilon(f(s)) \in \text{Open}(T)$,

$$f^{-1}(B_\varepsilon(f(s))) \quad \in \quad \text{Open}(S) \,.$$

Since $s \in f^{-1}(B_\varepsilon(f(s)))$, then using Definition 3.2, there exists some $\delta_\varepsilon > 0$ such that $B_{\delta_\varepsilon}(s) \subseteq f^{-1}(B_\varepsilon(f(s)))$ which is (3.4).

We now tend to showing that (3.4) implies (3.2). Let $U \in \text{Nbhd}(f(s))$. Because U is open, there is some $\varepsilon > 0$ such that $B_\varepsilon(f(s)) \subseteq U$. Then by assumption, there is some $\delta_\varepsilon > 0$ such that $B_{\delta_\varepsilon}(s) \subseteq f^{-1}(B_\varepsilon(f(s)))$. We find $s \in B_{\delta_\varepsilon}(s) \subseteq f^{-1}(U)$ so that $f^{-1}(U) \in \text{Nbhd}(s)$, i.e., (3.2). □

3.2 Connectedness and path-connectedness

For the topological spaces coming from quantum mechanics, we are mainly interested in *connectivity properties*

Definition 3.9. A topological space $(S, \text{Open}(S))$ is said to be *disconnected* iff

$$S \quad = \quad A \cup B$$

where $A, B \in \text{Open}(S) \setminus \{ \varnothing \}$, $A \cap B = \varnothing$. Naturally, $(S, \text{Open}(S))$is said to be *connected* iff it is not disconnected.

Lemma 3.10. *A topological space* $(S, \text{Open}(S))$ *is connected iff*

$$\text{Open}(S) \cap \text{Closed}(S) = \{ S, \varnothing \} \tag{3.5}$$

iff any continuous map $f : S \to \{ 0, 1 \}$ *(where* $\{ 0, 1 \}$ *is taken with the discrete topology) is constant.*

Proof. Let us assume that S is connected, and show (3.5). Say then that there is some

$$A \quad \in \quad \text{Open}(S) \cap \text{Closed}(S) \setminus \{ S, \varnothing \}$$

and define $B := S \setminus A$. Then $B \in \text{Open}(S) \cap \text{Closed}(S)$ too (by definition of closed as complement of open), neither are empty, and their union is S, which implies S is disconnected, a contradiction. Conversely, if (3.5) yet S is disconnected, then A, B are both clopen, which is a contradiction.

Next, assume that $f : S \to \{ 0, 1 \}$ is a continuous map which is *not* constant. Since $\{ 0 \}$ and $\{ 1 \}$ are both clopen, and f is continuous, $f^{-1}(\{ 0 \})$ and $f^{-1}(\{ 1 \})$ are both clopen, not empty (because f is not constant) and together form the entirety of S. So that S is disconnected. Now if S is disconnected, define the map

$$f(s) \quad := \quad \begin{cases} 0 & s \in A \\ 1 & s \in B \end{cases}$$

which is continuous and not constant. □

Definition 3.11. The connected component of a space S is the collection of maximal connected subsets.

We will be particularly interested in the following

Lemma 3.12. *Let* D *be a discrete space and* $f : S \to D$ *be continuous. Then* f *is constant on connected components of* S*. In particular, if* $s, s' \in S$ *have* $f(s) \neq f(s')$ *then necessarily* s, s' *belong to different connected components of* S*, and we have*

$$|f(S)| \quad \leq \quad |\textit{Set of connected components of } S| \,.$$

In this sense, it turns out that such functions f are "experimentally stable" and are called topological invariants of the material. They (hopefully) correspond to physically measurable quantities which do not change when making slight variations in the experimental setup. In lucky situations, we find such an invariant $f : S \to D$ such that

$$D \quad \cong \quad \text{Set of connected components of } S$$

in which case we say we have a *complete* classification of the phases of matter, and D is the set of indices or labels of such phases.

Another related notion is that of *path-connectedness*: For any $s \in S$, define an equivalence class $[s] \subseteq S$ via $s \sim s'$ iff there is a continuous function $\gamma : [0,1] \to S$ with $\gamma(0) = s$ and $\gamma(1) = s'$ (with $[0,1]$ taken with the metric topology associated with the absolute value distance).

Definition 3.13. S is *path-connected* iff $s \sim s'$ for any two $s, s' \in S$. The path-connected components are the set of all equivalence classes and is denoted as $\pi_0(S)$:

$$\pi_0(S) \quad \equiv \quad \{ [s] \mid s \in S \} \ .$$

Similarly to before, also here any continuous map $f : S \to D$ where D is discrete will be constant on path-connected components.

Lemma 3.14. *Path-connectedness implies connectedness but the converse is in general false.*

Proof. Assume $S = A \sqcup B$ and set up a path $\gamma : [0,1] \to S$ with $\gamma(0) \in A$ and $\gamma(1) \in B$. This would violate the connectedness of $[0,1]$, but it is connected, and hence, we reach a contradiction. For a counter-example of the converse, look up the topologist's sine-curve. $\square$

Since the converse is false, the two notions are in fact distinct. There is a notion of "locally path-connected" space in which open and connected implies path-connected. Presumably, all interesting spaces of quantum mechanical Hamiltonians we care about are locally path-connected (although this is far from obvious). If this is false, then it is an interesting question whether connectedness or path-connectedness is the relevant notion which selects experimental stability of invariants (it seems highly dubious that this question should be relevant to real-world experiments, but perhaps?).

To keep matters practical, in the remainder of these notes we pretend the two notions *are* equivalent, and furthermore, since path-connectedness is the simpler notion to study, we shall focus entirely on that.

Problem 3.15. Thus, the program now for the mathematical field of "topological insulators" is to define an "appropriate" topology $\operatorname{Open}(\mathcal{S})$ on the space of relevant quantum mechanical Hamiltonians $\mathcal{S}$, and calculate its set of path-connected components $\pi_0(\mathcal{S})$. In interesting cases $|\pi_0(\mathcal{S})| > 1$. An additional goal of the analysis will be to exhibit an isomorphism of $\pi_0(\mathcal{S})$ with some non-trivial discrete set via an explicit formula which relates the discrete label to a physically measurable quantity (such as a conductivity).

There is also the less ambitious goal, which is not to create a full classification but rather just define an index:

Problem 3.16. Define an "appropriate" topology $\operatorname{Open}(\mathcal{S})$ on the space of relevant quantum mechanical Hamiltonians $\mathcal{S}$, and a *continuous* surjective map $f : \mathcal{S} \to D$ for some discrete space D. Associate f with an experimentally measurable observable of the system.

While this last problem does not yield a *complete* classification it would say that there are at least as many "topological phases" as $|D|$.

3.3 A first non-trivial example of algebraic topology: the winding number

Consider the set $\mathcal{S}$ of continuous functions $\mathbb{S}^1 \to \mathbb{C} \setminus \{ 0 \}$. Here, $\mathbb{S}^1 \equiv [0, 2\pi]$ is the unit circle (with 0 and 2π identified). On this space, we define a topology which is the metric topology (see Definition 3.2) induced by the uniform norm (which itself induces a metric):

$$\|f\|_\infty \quad := \quad \sup_{z \in \mathbb{S}^1} |f(z)| \ .$$

Definition 3.17. (*The winding number*) Define a map $W : \mathcal{S} \to \mathbb{Z}$ via the following prescription. We write any map $f \in \mathcal{S}$ in polar coordinates as

$$f(\theta) \quad := \quad r(\theta)\, \mathrm{e}^{\mathrm{i}a(\theta)} \qquad (\theta \in [0, 2\pi))$$

where

$$r:[0,2\pi]\quad\to\quad(0,\infty)$$

is periodic, and zero is excluded from its codomain as the codomain of f is the *punctured* complex plane, and

$$a:[0,2\pi]\quad\to\quad\mathbb{R}$$

is 2π-periodic. The reason that a is chosen as merely 2π-periodic rather than periodic is so that it can be chosen as continuous. In fact, since f is continuous, clearly $r:=|f|$ is continuous as the composition of two continuous maps, and so is a (it is a basic fact in complex analysis that a continuous choice of the Argument function exists). We then define

$$W(f)\quad:=\quad\frac{a(2\pi)-a(0)}{2\pi}\in\mathbb{Z}\,.$$

Even though a has an ambiguity in its definition, the difference above does *not* due to the continuous requirement. Hence W is well-defined.

Theorem 3.18. *W lifts to a bijection $\pi_0(\mathcal{S})\cong\mathbb{Z}$.*

Proof. First we must show that W is well-defined with its domain $\pi_0(\mathcal{S})$, i.e., that $W([f])=W([g])$ if $[f]=[g]$, i.e., if $f\sim g$. By definition then, there is a continuous map (called a homotopy) $h:[0,1]\to\mathcal{S}$ such that $h(0)=f$ and $h(1)=g$. For any $t\in[0,1]$, $h(t):\mathbb{S}^1\to\mathbb{C}\setminus\{0\}$ and we use the notation

$$h(t,\theta)\quad=\quad r(t,\theta)\,\mathrm{e}^{\mathrm{i}a(t,\theta)}\qquad(t\in[0,1]\,,\theta\in[0,2\pi])\,.$$

Then

$$[0,1]\ni t\quad\mapsto\quad\frac{a(t,2\pi)-a(t,0)}{2\pi}\equiv W(h(t,\cdot))\in\mathbb{Z}$$

is a continuous function from a connected space $[0,1]$ into a discrete space $\mathbb{Z}$ and is hence constant. In particular, its two end-points agree and so W is well-defined with $\pi_0(\mathcal{S})$ as its domain.

Clearly W is surjective with the set of functions $\{\,\theta\mapsto\mathrm{e}^{im\theta}\,\}_{m\in\mathbb{Z}}$, so we need only demonstrate injectivity, i.e., that $W(f)=W(g)$ implies $f\sim g$.

Let Ff be the flipped map associated to f, i.e.,

$$(Ff)(\theta)\quad:=\quad f(2\pi-\theta)\qquad(\theta\in[0,2\pi])\,.$$

Clearly, $F:\mathcal{S}\to\mathcal{S}$ and $W(Rf)=-W(f)$ for all $f\in\mathcal{S}$. Also, for any $z\in\mathbb{C}\setminus\{0\}$ let R_zf be the z-scaled map associated to f, i.e.,

$$(R_zf)(\theta)\quad:=\quad zf(\theta)\qquad(\theta\in[0,2\pi])\,.$$

Clearly, for any z, $R_z:\mathcal{S}\to\mathcal{S}$ and $W\circ R_z=W$.

Now, given any two $f,g\in\mathcal{S}$, it is not clear that both may be concatenated, since their "base-points" $f(2\pi)$, $g(0)$ need not agree. However, the map $R_{\frac{f(2\pi)}{g(0)}}g\in\mathcal{S}$ *may* be concatenated with f. We denote the concatenation operation

$$(f\sharp g)(\theta)\quad:=\quad\begin{cases}f(2\theta) & 0\leq\theta\leq\pi\\ \frac{f(2\pi)}{g(0)}g(2(\theta-\pi)) & \pi\leq\theta\leq 2\pi\end{cases}\qquad(\theta\in[0,2\pi])\,.$$

It holds that $\sharp:\mathcal{S}^2\to\mathcal{S}$ and it is clear from the definition that $W(f\sharp g)=W(f)+W(g)$. Hence we have

$$\begin{aligned}0&=W(f)-W(g)\\&=W(f)+W(\overleftarrow{g})\\&=W(f\sharp\overleftarrow{g})\,.\end{aligned}$$

So that we have reduced the problem to showing that if $W(f) = 0$ then $f \sim 1$ where 1 is the constant map. Since $W(f) = 0$, we have $a(2\pi) = a(0)$. Now, consider the homotopy

$$a_t(\theta) \quad := \quad (1-t)\,a(\theta) + t a(0)\ .$$

Since $a(2\pi) = a(0)$, we have $a_t(2\pi) = a(0)$ so that throughout the homotopy a_t remains a well-defined map $[0, 2\pi] \to \mathbb{R}$ which is periodic (cf. if $a(2\pi) - a(0) \in 2\pi\mathbb{Z}$ whence $a_t(2\pi) = a(2\pi) + t(a(0) - a(2\pi)) \notin 2\pi\mathbb{Z}$ for all $t \in (0, 1)$). This defines a continuous function $h : [0, 1] \to \mathcal{S}$ with $h(0) = f$ and $h(1)$ being a map with a constant angular part. Since we know global rotations do not affect the winding, we make a further rotation from the angle of $f(0)$ to the horizontal axis. A further homotopy is given by

$$r_t(\theta) \quad = \quad (1-t)\,r(\theta) + t$$

and we have $r_t(2\pi) = (1-t)\,r(2\pi) + t = (1-t)\,r(0) + t = r_t(0)$ so that it is indeed periodic. Note that the order of these homotopies was important: first deform the angular part, then rotate to the horizontal axis, then deform the radial part.

Finally, we claim that if $f \sharp \overleftarrow{g}$ is homotopic to the constant 1 map then $f \sim g$. $\square$

We will see below that this example corresponds to an actual class of topological insulators, the one dimensional chiral model.

The winding number is a special case of the following theorem

Theorem 3.19. *Let $\mathcal{M}, \mathcal{N}$ be two smooth compact manifolds* of the same dimension *and $C(\mathcal{M} \to \mathcal{N})$ the space of continuous functions endowed with the topology associated with the norm $\|\cdot\|_\infty$. Then the degree map*

$$D : C(\mathcal{M} \to \mathcal{N}) \quad \to \quad \mathbb{Z}$$

is continuous and lifts to $\pi_0(C(\mathcal{M} \to \mathcal{N})) \to \mathbb{Z}$. C here may be replaced with C^∞, with the same topology, to obtain the same statement. If $\mathcal{N}$ is a sphere then D is a bijection, i.e., $\pi_0(C(\mathcal{M} \to \mathcal{N})) \cong \mathbb{Z}$.

We omit the proof which may be found in [Mil97], but provide the formula for the degree: Since $\mathcal{M}, \mathcal{N}$ are two manifolds of the same degree, any map $f \in C(\mathcal{M} \to \mathcal{N})$ may (after choosing charts and localizing to a neighborhood) be represented as a map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ for $n = \dim(\mathcal{M})$ and let $\mathcal{D}\tilde{f}$ denote the total (Frechet) derivative, i.e., an $n \times n$-matrix-valued function. Pick any point $q \in \mathbb{R}^n$ such that all points in $p \in f^{-1}(\{\, q \,\})$ have $\det\left(\left(\mathcal{D}\tilde{f}\right)(p)\right) \neq 0$ (such points q are called regular values of regular points p). Then

$$D(f) := \sum_{p \in f^{-1}(\{\, q \,\})} \operatorname{sgn}\left(\det\left(\mathcal{D}\tilde{f}\right)\right) \in \mathbb{Z}\,. \tag{3.6}$$

One has to show this definition does not depend on the chart-choices in $\tilde{f}$ nor on the choice of q first.

4 Periodic topological insulators and some algebraic topology

In this brief chapter, we describe how to (topologically) understand translation-invariant (that is, periodic) systems. Our ultimate focus is on systems *not* obeying a periodicity condition, for two reasons: the mathematics is (at least subjectively) easier and the physics is more convincing. Be that as it may, it would be unreasonable not to at least quickly describe the periodic setting. The reason being that the pictures arising from these descriptions are extremely intuitive and suggestive, and bring the underlying geometric notions to life. However, since this is not our main point we skip entirely proofs and remain rather shallow in terms of precision.

4.1 Classification of even dimensional periodic insulators

From everything we have learnt so far, a periodic insulating system associated to a Hamiltonian $H \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N\right)$ (using the Bloch decomposition Definition 1.8 and employing the insulator definition Definition 2.4) is fully specified by a map

$$\hat{H} : \mathbb{T}^d \quad \to \quad \mathrm{Herm}_N(\mathbb{C})$$

such that

1. There is some $\mu \in \mathbb{R}$, the Fermi energy of the system (which determines its filling) such that for all $k \in \mathbb{T}^d$, none of the N eigenvalues of $\hat{H}(k)$ intersect μ.
2. $\mathbb{T}^d \ni k \mapsto \hat{H}(k)$ has a certain regularity which is governed by the locality assumption Definition 1.1 and the Riemann-Lebesgue lemma Theorem 1.14. For simplicity we assume the amount of locality has been constrained so that $\hat{H}$ is: continuous. This is perhaps too lax physically speaking, but mathematically the distinction between continuous and smooth makes little difference thanks to [Lee13, Theorem 10.16].

Example 4.1. (*the integer quantum Hall effect*) Consider the case $d = N = 2$, whence we reduce to studying the space $\mathcal{S}_{d,N,\mu}$ of smooth maps

$$\hat{H} : \mathbb{T}^2 \quad \to \quad \mathrm{Herm}_2(\mathbb{C})$$

whose two eigenvalues (for each $k \in \mathbb{T}^2$) never intersect a given, fixed $\mu \in \mathbb{R}$.

We define an invariant

$$C : \mathcal{S}_{d=2,N=2,\mu} \quad \to \quad \mathbb{Z}$$

(akin to a winding number Theorem 3.18) associated with any such $\hat{H}$ in the following way: if both eigenvalues of $\hat{H}$ are always above (resp. below) μ for any $k \in \mathbb{T}^2$, we define $C\left(\hat{H}\right) = 0$. Otherwise, the insulator condition implies both eigenvalues are always separated by μ and hence are never degenerate, for all $k \in \mathbb{T}^2$. This means that the orthogonal projection matrix onto the lower one,

$$\hat{P} : \mathbb{T}^2 \to \mathrm{Herm}_2(\mathbb{C}) \tag{4.1}$$

is a well-defined continuous function onto 2×2 self-adjoint projection matrices of rank-1. We note that the space $\mathrm{Herm}_2(\mathbb{C})$ is spanned by the Pauli matrices so that any 2×2 Hermitian matrix M may be written as

$$M \quad = \quad h_0\sigma_0 + d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_3$$

where $\sigma_0 \equiv \mathbb{1}_{2\times 2}$, $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $h_0 \in \mathbb{R}$, $d \in \mathbb{R}^3$. If we furthermore require that $M^2 = M$, we get the further constraint that $h_0 = \frac{1}{2}$ and $\|d\| = \frac{1}{2}$. Hence, we may interpret (4.1) as being specified by a continuous map

$$\frac{1}{2}d : \mathbb{T}^2 \to \mathbb{S}^2 \,. \tag{4.2}$$

This 2-sphere is usually called the Bloch sphere, more so in quantum information. Now, Theorem 3.19 says that the space of all such maps is fully classified by the degree. We thus define *the Chern number* as

$$C\left(\hat{H}\right) \quad := \quad \text{Degree of the associated map } \mathbb{T}^2 \to \mathbb{S}^2 \,.$$

In fact, there is another formula (cf. (3.6)) in this case:

$$C\left(\hat{H}, \mu\right) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{T}^2} \mathrm{tr}\left(\hat{P}\left[\partial_1\hat{P}, \partial_2\hat{P}\right]\right) \tag{4.3}$$

although it is far from manifest from this formula that it is, indeed, $\mathbb{Z}$-valued. This formula agrees with the one usually used in the physics literature, which calls for a *Berry curvature*, see e.g. [HK10, (2), (5)] and is also equal to Theorem 2.5.

More generally, one may think of the Grassmannian manifold $\mathrm{Gr}_k\left(\mathbb{C}^N\right)$ of k-dimensional subspaces within $\mathbb{C}^N$ as replacing the Bloch sphere above in case we consider systems which are gapped after k-levels, whence we are interested in homotopy classes of continuous maps

$$\mathbb{T}^2 \quad \to \quad \mathrm{Gr}_k\left(\mathbb{C}^N\right)$$

and for N large enough (whence it is judicious to replace $\mathbb{C}^N$ with any separable Hilbert space, say $\mathbb{C}^\infty$ which stands for $\ell^2(\mathbb{N})$), it is known that for any smooth compact manifold X,

$$[X \to \mathrm{Gr}_k(\mathbb{C}^\infty)] \quad \cong \quad \mathrm{Vect}_k(X)$$

where the RHS is the set of classes of rank-k vector bundles over X [Ati94, Nak03] and the LHS is the topological space of all continuous maps $X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ (this shall be the meaning of the square brackets on spaces of maps from now on) associated to the $\|\cdot\|_\infty$ norm. In fancy language, this equation says that the Grassmannians are the classifying spaces for the principal bundle with the unitary group. It is instructive to compare this formula with the Atiyah-Jänich theorem:

$$[X \to \mathcal{F}(\mathbb{C}^\infty)] \quad \cong \quad K_0(C(X \to \mathbb{C}))$$

here $\mathcal{F}(\mathbb{C}^\infty)$ is the space of Fredholm operators on $\mathbb{C}^\infty$ (see Section 5.1) and K_0 is the zeroth K-theory group associated to the C-star algebra $C(X \to \mathbb{C})$ of continuous functions $X \to \mathbb{C}$, see [RLL00].

The same formula for the Chern number (4.3) works also when $k > 1$ (but the base space X is still $\mathbb{T}^2$).

This example is the classification of (certain classes of) integer quantum Hall systems (or Chern insulators), see [Sha16] for more details and for the physical setup.

This example may be generalized to higher dimensions [RLL00, pp. 234,], to prove that for N sufficiently large,

$$\pi_0(\mathcal{S}_{2d,N,\mu}) \cong \mathbb{Z}\,. \tag{4.4}$$

4.2 Symmetry-protected topological phases

If we consider the previous example in dimension one, then from (4.2), since now $d = 1$, the space of such Hamiltonians corresponds to the space of continuous maps

$$\mathbb{S}^1 \quad \to \quad \mathbb{S}^2\,.$$

However, it is well-known that the sphere is simply-connected, i.e.,

$$\left[\mathbb{S}^1 \to \mathbb{S}^2\right] \quad \equiv \quad \pi_1\left(\mathbb{S}^2\right) \cong \{\,0\,\}\,.$$

For a proof see e.g. [Mun00]. Generalizing this, for N sufficiently large, one may calculate that for all odd d, the classification is trivial, see [RLL00, pp. 234,]. However, even in odd dimensions, if we add a further symmetry constraint on the system we may get a non-trivial classification, as well shall now see.

Historically, symmetry classes defined by Altland and Zirnbauer in the context of random matrices have been most studied [AZ97]. These are three symmetries: time-reversal Θ, particle-hole Ξ, and their composition, chiral symmetry $\Pi \equiv \Theta\Xi$. The first two are anti-unitary anti-$\mathbb{C}$-linear operators on $\mathcal{H}$ whereas the latter one is unitary. To say that a given Hamiltonian H is symmetric w.r.t. Θ, Ξ or Π means that it either commutes with Θ or anti-commutes with Ξ or Π. Furthermore, we may have $\Theta^2 = \pm\mathbb{1}$ as well as $\Xi^2 = \pm 1$. Hence, all together when counting the different possibilities we arrive at ten different symmetry classes, which are called the Altland-Zirnbauer symmetry classes. The ten classes are each associated with one of the ten symmetric spaces. The classification of local insulating Hamiltonians which fall in a certain symmetry class proceeds as above, with the added constraint that Hamiltonians have the given symmetry, which yields further constraints on the map

$$\hat{H} : \mathbb{T}^d \quad \to \quad \mathrm{Gr}_k(\mathbb{C}^\infty)\,.$$

For example, if only time-reversal symmetry Θ is present which is of the type $\Theta^2 = \mathbb{1}$, then Θ in fact defines a "real structure" on the Hilbert space so that instead of classifying $\mathbb{C}$-vector bundles of rank k with base space $\mathbb{T}^d$, one may classify $\mathbb{R}$-vector bundles of rank k with the same base space. If $\Theta^2 = -\mathbb{1}$ one gets a quaternionc structure, and so on. For instance, it is a fact of algebraic topology that vector bundles with quaternionic structure over the 2-torus are isomorphic to $\mathbb{Z}_2$:

$$\mathrm{Vect}_k^{\mathbb{H}}\left(\mathbb{T}^2\right) \cong \mathbb{Z}_2. \tag{4.5}$$

Here, $\mathbb{H}$ is the quaternions.

This entire study *has* been carried out [RSFL10, SRFL08] for all dimensions and all ten Altland-Zirnbauer symmetry classes and then organized by Kitaev into a table (and crucially, associated with K-theory), called the Kitaev table [Kit09]. This table is essentially the same table found in [ABS64] in the context of Clifford modules: each of the ten symmetry classes corresponds also to one of the ten Morita equivalence classes of the real and complex Clifford algebras.

We conclude with a particularly simple example, in fact, much simpler than the integer quantum Hall effect already presented, of a symmetry-protected topological phase.

Symmetry label	Θ^2	Ξ^2	Π	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$AIII$	0	0	$\mathbb{1}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	$\mathbb{1}$	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	$\mathbb{1}$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
$DIII$	$-\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	$-\mathbb{1}$	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	$-\mathbb{1}$	$-\mathbb{1}$	$\mathbb{1}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	$-\mathbb{1}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	$\mathbb{1}$	$-\mathbb{1}$	$\mathbb{1}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Table 1: The Kitaev periodic table of topological insulators. A zero for a symmetry indicates that it is not present, and otherwise it is indicated what it squares to. The cells of the table indicate the set of path-connected components of such spaces of topological insulators. Note that it is periodic, i.e., $d = 9$ is the same as $d = 1$. Also note its diagonal shift structure.

Example 4.2. Consider the space of insulators which are also *chiral* in $d = 1$, $N = 2$ and $\mu = 0$. Chiral symmetry Π is a unitary operator on $\mathcal{H}$ which *anti-commutes* with the Hamiltonian H and commutes with the position operator X. Writing

$$\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \quad \cong \quad \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$$

and using block-operator notation, we thus have

$$\Pi \quad = \quad \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}$$

and from $H = H^*$ and the anti-commutation relation

$$\{ H, \Pi \} \equiv H\Pi + \Pi H \quad = \quad 0$$

we get

$$H \quad = \quad \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$

for some $S \in \mathcal{B}\left(\ell^2(\mathbb{Z})\right)$ which is *not* necessarily self-adjoint. Hence chiral Hamiltonians stand in one-to-one correspondence with non-self-adjoint operators on half the space. The insulator condition on H implies that S must be invertible (verify this) so that employing the periodic constraint as well, we arrive at classifying the space of all continuous maps

$$\hat{S} : \mathbb{S}^1 \quad \to \quad \mathbb{C} \setminus \{ 0 \} .$$

This has been precisely analyzed by the winding number, see Theorem 3.18 and it was calculated that the path-connected components of such systems are bijective with $\mathbb{Z}$. The bijection, which is just the winding number, is physically known as the *Zak phase* [Zak89], and is related to a certain chiral polarization.

5 Disordered topological insulators and some Fredholm theory

In order to understand disordered systems, we will use Fredholm theory. We begin with a crash course on the theory, which requires basic functional analysis. Most of the material in this section can be found in the first part of the accessible textbook [BB89]. One may also wish to consult the short and beautiful paper [Mur94] or the textbook [Sch01, Chapter 5].

5.1 Fredholm theory

In this section, we temporarily forget about the spatial structure of our Hilbert space. Thus, $\mathcal{H}$ is *any* separable Hilbert space, which schematically is usually denoted as $\mathbb{C}^\infty$ as above. If we want a concrete choice it could be $\ell^2(\mathbb{N})$ with ONB $\{ e_j \}_{j=1}^\infty$. On $\mathcal{H}$ we consider $\mathcal{B}(\mathcal{H})$, the C-star algebra of bounded linear operators $\mathcal{H} \to \mathcal{H}$ together with the operator norm

$$\|A\| \equiv \sup(\{ \|A\varphi\| \mid \varphi \in \mathcal{H} : \|\varphi\| = 1 \}) \,.$$

Definition 5.1. An operator $A \in \mathcal{B}(\mathcal{H})$ is called *Fredholm* iff $\ker A$ and $\operatorname{coker} A$ are finite dimensional. These are the two vector spaces defined as

$$\ker A \equiv \{ \varphi \in \mathcal{H} \mid A\varphi = 0 \}$$

and

$$\operatorname{coker} A \equiv \mathcal{H} / \operatorname{im} A$$

with $\operatorname{im} A \equiv \{ \varphi \in \mathcal{H} \mid \exists \psi : A\psi = \varphi \}$. Recall that the quotient vector space is defined as:

$$\varphi \sim \psi \iff \varphi - \psi \in \operatorname{im} A$$

and

$$\begin{aligned} [\varphi] &:= \{ \psi \in \mathcal{H} \mid \varphi \sim \psi \} \\ \operatorname{coker} A &= \{ [\varphi] \mid \varphi \in \mathcal{H} \} \,. \end{aligned}$$

For every Fredholm operator, we define the index, an integer associated with that operator, as

$$\operatorname{index}(A) := \dim \ker A - \dim \operatorname{coker} A \in \mathbb{Z} \,.$$

We denote the space of all Fredholm operators as $\mathcal{F}(\mathcal{H})$.

The kernel of an operator $\ker A$ should not be confused with its *integral kernel*

$$A_{xy} \equiv \langle e_x, A e_y \rangle$$

in the context of PDEs, i.e., its matrix elements in the context of quantum mechanics.

Intuitively speaking, the kernel measures how much an operator deviates from being injective, whereas the cokernel measures how much an operator deviates from being surjective. Hence, Fredholm operators are those that are injective and surjective (and hence invertible) up to some finite dimensional "defect", and the index of the operator measures the severity of this defect, so to speak (in a signed way). The fact that one has a difference of two numbers instead of just one number should be associated with the Grothendiek construction of a group out of a semigroup, or with the construction of $\mathbb{Z}$ out of $\mathbb{N}$ as pairs of naturals which should be identified with the difference.

The following result allows us to speak only of kernels of operators instead of the more mysterious cokernel:

Lemma 5.2. $\operatorname{coker} A$ *is finite-dimensional iff* $\operatorname{im} A \in \operatorname{Closed}(\mathcal{H})$ *and* $\ker A^*$ *is finite dimensional.*

Proof. Assume that $\operatorname{im} A \in \operatorname{Closed}(\mathcal{H})$. Then via Claim A.1,

$$\begin{aligned} (\ker(A^*))^\perp &= \left((\operatorname{im} A)^\perp\right)^\perp \\ &= \overline{\operatorname{im} A} && \text{(Via Claim A.4)} \\ &= \operatorname{im} A \,. && \text{(By hypothesis)} \end{aligned}$$

Now, we always have

$$\begin{aligned} \mathcal{H} &= (\ker A) \oplus \left((\ker A)^\perp\right) = (\ker A^*) \oplus \left((\ker A^*)^\perp\right) \\ &= (\ker A^*) \oplus \operatorname{im} A \,. \end{aligned}$$

Hence

$$\begin{aligned}\mathrm{coker}A &\equiv \mathcal{H}/\mathrm{im}A\\ &\cong (\mathrm{im}A)^{\perp}\\ &= \ker A^{*}\,.\end{aligned}$$

Hence if $\dim\ker A^{*}$ is finite, so is $\dim\mathrm{coker}A$.

Conversely, assume that $\dim\mathrm{coker}A$ is finite. We want to show that $\mathrm{im}A\in\mathrm{Closed}\left(\mathcal{H}\right)$.

Define a map

$$\begin{aligned}\eta:\left(\mathcal{H}/\ker A\right)\oplus(\mathrm{im}A)^{\perp} &\to \mathcal{H}\\ \left(\left[\varphi\right],\psi\right) &\mapsto A\varphi+\psi\,.\end{aligned}$$

It is easy to verify that η is a bounded linear bijection (it is in verifying that η is bounded that we used the fact $\mathrm{coker}A\cong(\mathrm{im}A)^{\perp}$ is finite dimensional). Hence

$$\mathrm{im}A\cong\eta\left(\left(\mathcal{H}/\ker A\right)\oplus\left\{0\right\}\right)\in\mathrm{Closed}\left(\mathcal{H}\right)$$

where the last statement is due to the open mapping them which says that the inverse of η is also continuous, i.e., η is a closed map and hence maps closed sets to closed sets. □

See also [AA02, Corollary 2.17].

The Fredholm index is continuous, as well shall see, but only half of it is *not*:

Lemma 5.3. $\dim\ker:\mathcal{F}\left(\mathcal{H}\right)\to\mathbb{N}_{\geq0}$ *is upper semicontinuous.*

Proof. Decompose $\mathcal{H}=\ker\left(A\right)\oplus\ker\left(A\right)^{\perp}\cong\ker\left(A^{*}\right)\oplus\ker\left(A^{*}\right)^{\perp}$. Since $\mathrm{im}\left(A\right)\cong\ker\left(A^{*}\right)^{\perp}$, we have

$$A = \begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$$

for some isomorphism $a:\ker\left(A\right)^{\perp}\to\mathrm{im}\left(A\right)$. Taking any norm perturbation B of size at most $\left\Vert a^{-1}\right\Vert^{-1}$ will mean that $A+B$ is injective on $\ker\left(A\right)^{\perp}$ and hence $\dim\ker A+B\leq\dim\ker A$. □

To characterizer the the image of an operator being closed, we have

Lemma 5.4. *For $A\in\mathcal{B}\left(\mathcal{H}\right)$, the following are equivalent:*

1. $\mathrm{im}A\in\mathrm{Closed}\left(\mathcal{H}\right)$.
2. $0\notin\sigma\left(\left|A\right|^{2}\right)$ *or zero is an isolated point of* $\sigma\left(\left|A\right|^{2}\right)$.
3. $\exists\varepsilon>0$ *such that*

$$\left\Vert A\varphi\right\Vert \geq \varepsilon\left\Vert \varphi\right\Vert \qquad \left(\varphi\in(\ker A)^{\perp}\right)\,.$$

Proof. ((1)=>(3)): Assume that $\mathrm{im}\left(A\right)\in\mathrm{Closed}\left(\mathcal{H}\right)$. Then $\tilde{A}:\ker\left(A\right)^{\perp}\to\mathrm{im}\left(A\right)$ is a bijection. Since $\mathrm{im}\left(A\right)\in\mathrm{Closed}\left(\mathcal{H}\right)$, $\mathrm{im}\left(A\right)$ is a complete metric space, which implies that $\tilde{A}^{-1}:\mathrm{im}\left(A\right)\to\ker\left(A\right)^{\perp}$ is bounded by the "bounded inverse theorem" [RS80]. I.e., $\left\Vert \tilde{A}^{-1}\right\Vert <\infty$, which is tantamount to saying that $\exists c<\infty$ such that $\left\Vert \tilde{A}^{-1}\varphi\right\Vert <c\left\Vert \varphi\right\Vert$ for all $\varphi\in\mathrm{im}\left(A\right)$. Now if $\psi\in\ker\left(A\right)^{\perp}$, then $A\psi\in\mathrm{im}\left(A\right)$, and so $\tilde{A}^{-1}A\psi\equiv\psi$. Hence

$$\begin{aligned}\left\Vert \psi\right\Vert &\leq c\left\Vert A\psi\right\Vert\\ &\updownarrow\\ \frac{1}{c}\left\Vert \psi\right\Vert &\leq \left\Vert A\psi\right\Vert\,.\end{aligned}$$

((3)=>(1)): Let $\{ \varphi_n \}_n \subseteq \mathrm{im}(A)$ such that $\lim_n \varphi_n = \psi$ for some $\psi \in \mathcal{H}$. Our goal is to show that $\psi \in \mathrm{im}(A)$. If $\psi = 0$ we are finished so assume $\psi \neq 0$. Since $\{ \varphi_n \} \subseteq \mathrm{im}(A)$, $\exists \{ \eta_n \}_n \subseteq \mathcal{H}$ such that $A\eta_n = \varphi_n$. We assume WLOG that $\eta_n \in \ker(A)^\perp$ (if this is false for all η_n then $\varphi_n = 0$, so if necessary take a subsequence of such η_n). Hence, $\|A\eta_n\| \geq \varepsilon \|\eta_n\|$ by hypothesis. We claim $\{ \eta_n \}_n$ is Cauchy. Indeed, $\|\eta_n - \eta_m\| \leq \frac{1}{\varepsilon}\|A(\eta_n - \eta_m)\| = \frac{1}{\varepsilon}\|\varphi_n - \varphi_m\|$. But $\{ \varphi_n \}_n$ converges and is hence Cauchy. Hence $\lim_n \eta_n = \xi$ for some $\xi \in \mathcal{H}$ (because regardless of the status of $\mathrm{im}(A)$, $\mathcal{H}$ certainly *is* complete and hence Cauchy sequence converge). Since A is bounded it is continuous, and so we find that

$$\begin{aligned} A\xi &= A \lim_n \eta_n \\ &= \lim_n A\eta_n \\ &= \lim_n \varphi_n \\ &= \psi\,. \end{aligned}$$

We obtain then that $\psi \in \mathrm{im}(A)$ as desired.

((2)<=>(3)): We have $\ker\left(|A|^2\right)^\perp = \ker(A)^\perp$ using Lemma A.5. Now, since $|A|^2 \geq 0$, (2) is equivalent to $|A|^2 \geq \varepsilon \mathbb{1}$ on $\ker(A)^\perp$. Hence

$$\begin{aligned} |A|^2 - \varepsilon\mathbb{1} \geq 0 &\iff \left\langle \psi, \left(|A|^2 - \varepsilon\mathbb{1}\right)\psi \right\rangle \forall \psi \in \ker(A)^\perp \\ &\iff \|A\psi\|^2 \geq \varepsilon^2 \|\psi\|^2 \forall \psi \in \ker(A)^\perp \end{aligned}$$

□

Thus, we could alternatively define A to be Fredholm when $\ker A, \ker A^*$ are finite dimensional *and* $\mathrm{im}A$ is closed, and we could just as well write for its index

$$\mathrm{index}(A) = \dim\ker A - \dim\ker A^* \in \mathbb{Z}\,.$$

From this formula it is clear that if a self-adjoint operator is Fredholm then its index must be zero.

Example 5.5. Here are a few trivial examples for the notion of a Fredholm operator:

1. The identity operator $\mathbb{1}$ is Fredholm and its index is zero.

2. The zero operator $\mathcal{H} \ni v \mapsto 0$ is *not* Fredholm.

3. Recall the position operator X from (1.5), which is not even bounded. Its inverse $A := X^{-1}$ is however bounded: $\|A\| \leq 1$. It is however *not* Fredholm, even though it is self-adjoint and has an empty kernel. To see that $\mathrm{im}A \notin \mathrm{Closed}\left(\ell^2(\mathbb{N})\right)$, use the second characterization of Lemma 5.4 and note that while zero is not in $\sigma\left(|A|^2\right)$, it is an accumulation point and hence not isolated in $\sigma\left(|A|^2\right)$. What is the cokernel of A? Is it finite dimensional?

4. The right-shift operator R from Theorem 1.10 on $\ell^2(\mathbb{N})$ is Fredholm. Indeed, one checks that

$$|R|^2 = R^*R = \mathbb{1}$$

and hence by the second characterization of Lemma 5.4 it has closed image. Its kernel is empty and the kernel of its adjoint, the left shift operator, is spanned by δ_1 and is hence one dimensional.

$$\mathrm{index}(R) = -1\,.$$

Note that, considered on $\ell^2(\mathbb{Z})$, R is also Fredholm, but it is now invertible and hence has zero index. The right shift operator is the most important example of a Fredholm operator, and in a sense, all other non-zero index operators may be connected to a power of the right shift, as we shall see.

Claim 5.6. If $A \in \mathcal{B}(\mathcal{H}_1 \to \mathcal{H}_2)$ with $\mathcal{H}_1, \mathcal{H}_2$ finite dimensional, then A is Fredholm and its index equals

$$\mathrm{index}(A) = \dim(\mathcal{H}_1) - \dim(\mathcal{H}_2)\,.$$

Proof. The rank-nullity theorem [KK07] states that

$$\dim\left(\mathcal{H}_1\right) = \dim\left(\ker\left(A\right)\right) + \dim\left(\operatorname{im}\left(A\right)\right) .$$

Furthermore, since $\operatorname{coker}\left(A\right) \equiv \mathcal{H}_2/\operatorname{im}\left(A\right)$, we have

$$\dim\left(\operatorname{coker}\left(A\right)\right) = \dim\left(\mathcal{H}_2\right) - \dim\left(\operatorname{im}\left(A\right)\right) .$$

Thus, we have

$$\begin{aligned} \operatorname{index}\left(A\right) &\equiv \dim\left(\ker\left(A\right)\right) - \dim\left(\operatorname{coker}\left(A\right)\right) \\ &= \dim\left(\mathcal{H}_1\right) - \dim\left(\operatorname{im}\left(A\right)\right) - \dim\left(\mathcal{H}_2\right) + \dim\left(\operatorname{im}\left(A\right)\right) \\ &= \dim\left(\mathcal{H}_1\right) - \dim\left(\mathcal{H}_2\right) . \end{aligned}$$

as desired. □

In particular, any square matrix is Fredholm with index zero: finite dimensions are not very interesting for Fredholm theory. Be that as it may some mechanical models in physics have been studied of finite *non*-square matrices, which have a non-zero index.

Definition 5.7. An operator $F \in \mathcal{B}\left(\mathcal{H}\right)$ is called *finite rank* iff $\dim\left(\operatorname{im}\left(F\right)\right) < \infty$ iff it may be written in the form

$$F = \sum_{i=1}^{N} f_i \varphi_i \otimes \psi_i^*$$

where $\left\{ f_i \right\}_{i=1}^{N} \geq 0$ are the singular values of F and $\left\{ \varphi_i \right\}_i$, $\left\{ \psi_i \right\}$ are orthonormal bases of $\mathcal{H}$.

For the next definition, we use the open ball definition

$$B_\varepsilon\left(v\right) \equiv \left\{ u \in \mathcal{H} \mid \|u - v\| < \varepsilon \right\} .$$

Definition 5.8. An operator $K \in \mathcal{B}\left(\mathcal{H}\right)$ is called *compact iff* $\overline{K\left(\overline{B_1\left(0\right)}\right)}$ *is compact in* $\mathcal{H}$ iff it is the operator-norm limit of finite rank operators iff it may be written as

$$K = \lim_{N\to\infty} \sum_{i=1}^{N} f_i \varphi_i \otimes \psi_i^*$$

where the limit is meant in the operator-norm and f_i may accumulate only at zero. The set of all compact operators is denoted as $\mathcal{K}\left(\mathcal{H}\right)$. It is a two-sided ideal within $\mathcal{H}$.

Lemma 5.9. *(Riesz)* $\mathbb{1} - K \in \mathcal{F}\left(\mathcal{H}\right)$ *for all* $K \in \mathcal{K}\left(\mathcal{H}\right)$ *and* $\operatorname{index}\left(\mathbb{1} - K\right) = 0$.

Proof. Write $K = \lim_n F_n$ (in operator norm) where F_n is finite rank. Hence $\mathbb{1} - K + F_n$ is invertible for n sufficiently large since $\|K - F_n\|$ may be made arbitrarily small. Then,

$$\mathbb{1} - K = \left(\mathbb{1} - K + F_n\right)\left(\mathbb{1} - \left(\mathbb{1} - K + F_n\right)^{-1} F_n\right)$$

so that

$$\mathbb{1} - K = G\left(\mathbb{1} - F\right)$$

with G invertible and F finite rank. Hence $\ker\left(\mathbb{1} - K\right) = \ker\left(\mathbb{1} - F\right)$. Now, $v \in \ker\left(\mathbb{1} - F\right)$ iff $v = Fv$ which implies that v is an eigenvector of F with eigenvalue 1. But this if F is finite rank its eigenspaces are finite dimensional. Same for $\mathbb{1}-F^*$. The two kernels are of the same dimension since F is of finite rank. ☐

Theorem 5.10. *(Atkinson) $A \in \mathcal{F}(\mathcal{H})$ iff A is invertible up to compacts, i.e., iff there is some operator $B \in \mathcal{B}(\mathcal{H})$, called* the parametrix of A, *such that,*

$$\mathbb{1}-AB, \mathbb{1}-BA \in \mathcal{K}(\mathcal{H}) .$$

We note that we may have $\mathbb{1}-AB \neq \mathbb{1}-BA$ indeed. Furthermore, $\operatorname{index}(B) = -\operatorname{index}(A)$.

Proof. If $\mathbb{1}-AB, \mathbb{1}-BA \in \mathcal{K}(\mathcal{H})$ then $BA = \mathbb{1}-K$ for some compact K and using Lemma 5.9 we have that BA is Fredholm of index zero. Hence $\ker(BA)$ is finite dimensional. But $\ker(A) \subseteq \ker(BA)$ so that $\ker(A)$ is finite dimensional. Also, $\mathcal{K}(\mathcal{H})$ is closed under adjoint, so that $\ker(A^*)$ is also finite. Finally, let $v \in \ker(A)^\perp$. Then $v \in \ker(BA)^\perp$. Since BA is Fredholm, it has closed image, so that by Lemma 5.4 we have

$$\begin{aligned} \varepsilon\|\varphi\| &\leq \|BA\varphi\| \\ &\leq \|B\|\|A\varphi\| . \end{aligned}$$

Consequently, A has closed range. Thus A is Fredholm. Now, using the logarithmic law Theorem 5.18 further below, since $AB = \mathbb{1}-K$, $0 = \operatorname{index}(AB) = \operatorname{index}(A) + \operatorname{index}(B)$.

Also

$$\begin{aligned} \ker(B) &\subseteq \ker(AB) \\ &= \ker(B^*A^*) \end{aligned}$$

Conversely, assume $A \in \mathcal{F}(\mathcal{H})$. Want to construct two partial inverses: let P, Q be the orthogonal projections onto $\ker(A)$ and $\ker(A^*)$ resp. We claim that $|A|^2+P$ and $|A^*|^2+Q$ are bijections. Indeed, $\ker(A) = \ker\left(|A|^2\right)$ so if $\mathcal{H} \cong \ker\left(|A|^2\right)^\perp \oplus \ker\left(|A|^2\right)$, $|A|^2+P \cong |A|^2\Big|_{\operatorname{im}(P)^\perp} \oplus \mathbb{1}$ and similarly for the other operator. Hence $B := |A|^2+P$ is invertible, and

$$\mathbb{1} = B^{-1}A^*A + B^{-1}P .$$

But now, $B^{-1}P$ is of finite rank and $C := B^{-1}A^*$ is the sought-after parametrix. ☐

Definition 5.11. The *essential spectrum* $\sigma_{\text{ess}}(A)$ of an operator $A \in \mathcal{B}(\mathcal{H})$ is the set of all points $z \in \mathbb{C}$ such that $A - z\mathbb{1}$ is *not* Fredholm.

Claim 5.12. If $A \in \mathcal{F}(\mathcal{H})$ and $\operatorname{index}(A) = 0$ then $A = G + K$ for some G invertible and K compact.

Proof. Since $\operatorname{index}(A) = 0$, $\dim\ker A = \dim\ker A^*$. Thus,

$$\mathcal{H} \cong \ker(A) \oplus \ker(A)^\perp \cong \ker(A^*) \oplus \ker(A^*)^\perp .$$

But we know that since $\dim\ker A = \dim\ker A^*$, there is a natural linear isomorphism $\eta : \ker(A) \to \ker(A^*)$. We also know that $\operatorname{im}(A) \cong \ker(A^*)^\perp$. Hence, $A|_{\ker(A)^\perp}$ is just an isomorphism

$$\ker(A)^\perp \to \operatorname{im}(A) .$$

Hence, the map

$$G := \eta \oplus A|_{\ker(A)^\perp} : \mathcal{H} \to \mathcal{H}$$

is an isomorphism and $\mathcal{H} \ni (v_1, v_2) \overset{K}{\mapsto} (\eta(v_1), 0) \in \mathcal{H}$ is compact and hence the result. ☐

Theorem 5.13. *We have the inclusion*

$$\mathcal{F}(\mathcal{H})+\mathcal{K}(\mathcal{H}) \quad\subseteq\quad \mathcal{F}(\mathcal{H})$$

and the Fredholm index is stable under compact perturbations.

Proof. If $A\in\mathcal{F}(\mathcal{H})$ and $K\in\mathcal{K}(\mathcal{H})$, then by Atkinson Theorem 5.10, there is some parametrix B such that $AB-\mathbb{1}, BA-\mathbb{1}$ is compact. But B will be a parametrix of $A+K$ too:

$$(A+K)B-\mathbb{1} \quad=\quad AB+KB-\mathbb{1}$$

which is compact since KB is compact (as the compacts form an ideal). Hence $A+K$ is Fredholm. We postpone the proof that the index remains stable until the next theorem. □

We see from Atkinson's theorem that the essential spectrum is stable under compact perturbations.

Theorem 5.14. *(Dieudonne)* index : $\mathcal{F}(\mathcal{H})\to\mathbb{Z}$ *is operator-norm-continuous and if* $B\in\mathcal{F}(\mathcal{H})$ *is any parametrix of* $A\in\mathcal{F}(\mathcal{H})$ *then*

$$B_{\|B\|^{-1}}(A) \quad\subseteq\quad \mathcal{F}(\mathcal{H})\,.$$

In particular, $\mathcal{F}(\mathcal{H})\in\operatorname{Open}(\mathcal{B}(\mathcal{H}))$.

Proof. Let $A\in\mathcal{F}(\mathcal{H})$ and B be any parametrix of it. Take any $\tilde{A}\in B_{\|B\|^{-1}}(A)$. We have

$$\left\|B\left(A-\tilde{A}\right)\right\|\leq\|B\|\left\|A-\tilde{A}\right\|<1$$

by assumption, so that $\mathbb{1}-B\left(A-\tilde{A}\right)$ is invertible. We claim that $\left(\mathbb{1}-B\left(A-\tilde{A}\right)\right)^{-1}B$ is a parametrix for $\tilde{A}$. Now,

$$\begin{aligned} 0 &= \operatorname{index}(\mathbb{1})\\ &= \operatorname{index}(\mathbb{1}-K)\\ &= \operatorname{index}\left(\left(\mathbb{1}-B\left(A-\tilde{A}\right)\right)^{-1}B\tilde{A}\right)\\ &= \operatorname{index}\left(B\tilde{A}\right)\end{aligned}$$

Now, via Theorem 5.18 further below we have $\operatorname{index}(B)+\operatorname{index}\left(\tilde{A}\right)$ and since $\operatorname{index}(B)=-\operatorname{index}(A)$ we obtain the result. □

Finally, we finish the proof of Theorem 5.13: if $A\in\mathcal{F}(\mathcal{H})$ and $K\in\mathcal{K}(\mathcal{H})$, the homotopy $[0,1]\ni t\mapsto A+tK\in\mathcal{F}(\mathcal{H})$ interpolates in a norm continuous way between A and $A+K$ and thus the index is constant along this path.

Claim 5.15. A is invertible up to compacts iff it is invertible up to finite ranks.

Proof. Since finite rank operators are compact one direction is trivial. Now, assume that there is some $B\in\mathcal{F}(\mathcal{H})$ with which $\mathbb{1}-AB,\mathbb{1}-BA\in\mathcal{K}(\mathcal{H})$. Let $\{\,F_n\,\}_n$ be a sequence of finite rank operators which converges to $K:=\mathbb{1}-BA$ in operator norm. Then $\|F_n-K\|$ can be made arbitrarily small and hence $W_n:=\mathbb{1}-K+F_n$ is invertible for n sufficiently large. Then,

$$\begin{aligned} BA &= \mathbb{1}-K\\ &= W_n\left(\mathbb{1}-W^{-1}F_n\right)\end{aligned}$$

and hence

$$\mathbb{1} - W_n^{-1} BA \quad = \quad W^{-1} F_n \,.$$

Since finite rank operators form an ideal, $W_n^{-1} F_n$ is finite rank too. This same logic shows that

$$\mathbb{1} - AB\tilde{W}_n^{-1} \quad = \quad \tilde{F}_n \tilde{W}_n^{-1}$$

where now $\tilde{F}_n \tilde{W}_n^{-1}$ is finite rank. So Now, $W_n^{-1} B$ is a partial left inverse and $B\tilde{W}_n^{-1}$is a partial right inverse. Then

$$\begin{aligned} W_n^{-1} BA &= \mathbb{1} - W^{-1} F_n \\ W_n^{-1} BAB\tilde{W}_n^{-1} &= B\tilde{W}_n^{-1} - W^{-1} F_n B \tilde{W}_n^{-1} \\ W_n^{-1} B \left(\mathbb{1} - \tilde{F}_n \tilde{W}_n^{-1}\right) &= B\tilde{W}_n^{-1} - W^{-1} F_n B \tilde{W}_n^{-1} \\ W_n^{-1} B - B\tilde{W}_n^{-1} &= W_n^{-1} B \tilde{F}_n \tilde{W}_n^{-1} - W^{-1} F_n B \tilde{W}_n^{-1} \end{aligned}$$

and since the finite rank operators form an ideal within $\mathcal{B}(\mathcal{H})$, we find that $W_n^{-1} B$ equals $B\tilde{W}_n^{-1}$ up to finite rank operators and hence there is just one parametrix, say, $W_n^{-1} B$.

The following is taken from [Mur94]: ☐

Theorem 5.16. *(Fedosov) If $A \in \mathcal{F}(\mathcal{H})$ and B is* any *parametrix of A such that $\mathbb{1} - AB, \mathbb{1} - BA$ is finite rank, then*

$$\operatorname{index}(A) \quad = \quad \operatorname{tr}([A, B]) \,.$$

Proof. First we note that if B, B' are two such parametrii, then $B' = B + F$ for some F finite rank. Then $[A, B'] = [A, B] + [A, F]$ and since F is finite rank, $\operatorname{tr}([A, F]) = 0$. So we are free to choose *any* such finite rank parametrix.

We claim there is a finite-rank parametrix B such that $A = ABA$. If we can find such a parametrix, then $\mathbb{1} - AB$ and $\mathbb{1} - BA$ are idempotents, and their traces equal the dimension of the cokernel and kernel of A respectively and hence the result.

To find this special B, since A induces an isomorphism $\varphi : \ker(A)^{\perp} \to \operatorname{im}(A)$, by the bounded inverse theorem, φ^{-1} is bounded. Let B be any extension of φ^{-1} to $\mathcal{H}$. Then it fulfills $A = ABA$. ☐

Another useful trace formula is the following:

Claim 5.17. If there is some $n \in \mathbb{N}$ such that $\mathbb{1} - |A|^2$ and $\mathbb{1} - |A^*|^2$ are n-Schatten class, then

$$\operatorname{index}(A) \quad = \quad \operatorname{tr}\left((\mathbb{1} - |A|)^n\right) - \operatorname{tr}\left((\mathbb{1} - |A^*|)^n\right) \,.$$

The proof is left as an exercise to the reader.

Theorem 5.18. *(Logarithmic law) If $A, B \in \mathcal{F}(\mathcal{H})$ then*

$$\begin{aligned} \operatorname{index}(AB) &= \operatorname{index}(A) + \operatorname{index}(B) \\ \operatorname{index}(A \oplus B) &= \operatorname{index}(A) + \operatorname{index}(B) \,. \end{aligned}$$

Proof. The easiest proof is via Fedosov Theorem 5.16: If $\tilde{A}$ is a parametrix for A and $\tilde{B}$ is a parametrix for B then $\tilde{B}\tilde{A}$ is a parametrix for AB. If we let $F := \mathbb{1} - B\tilde{B}$ be finite rank, then some algebra implies

$$\operatorname{index}(AB) \quad = \quad \operatorname{index}(A) + \operatorname{tr}\left(\tilde{A}AB\tilde{B} - \tilde{B}\tilde{A}AB\right)$$

the last trace is seen to equal $\operatorname{index}(B)$ since $\tilde{A}A - \mathbb{1}$ is finite rank.

The statement about the direct sum is trivial. ☐

Theorem 5.19. *(Atiyah-Jähnich) We have* $\pi_0(\mathcal{F}(\mathcal{H})) \cong \mathbb{Z}$.

Proof. We already know that index : $\mathcal{F}(\mathcal{H}) \to \mathbb{Z}$ is continuous and is constant on the path-connected components of $\mathcal{F}(\mathcal{H})$. Thus index lifts to a well-defined map on $\pi_0(\mathcal{F}(\mathcal{H}))$. To see that it's surjective it suffices to consider powers of the right-shift operator. So we only need to show it is injective.

First we claim that if $\operatorname{index}(A) = 0$ then there is a path from A to $\mathbb{1}$: Via Claim 5.12 we have $A = G + K$ for some invertible G and compact K. By Theorem A.6, there is a path γ from $(A-K)^{-1}$ to $\mathbb{1}$, and γA is a path from $\mathbb{1} - \tilde{K}$ to A. From there we can define a further homotopy to reduce $\tilde{K}$ to zero.

Next, we need that if $\operatorname{index}(A) = \operatorname{index}(B)$ then there is a path between them. To that end, let $\tilde{B}$ be the parametrix of B. Then $\operatorname{index}\left(A\tilde{B}\right) = 0$, whence by the above there is a path $\gamma : A\tilde{B} \mapsto \mathbb{1}$. The path $\tilde{\gamma} := \gamma B$ interpolates between $A - K$ and B. Again, a further homotopy brings us to A. ☐

5.2 Chiral 1D topological insulators

The material in this section can be found in [GS18]. As we saw in Theorem 4.2, chiral one dimensional insulators H on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}$ are of the form

$$H = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$

for some S on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ which is not necessarily self-adjoint. Define the projection Λ onto $\mathbb{N}$ within $\mathbb{Z}$:

$$\Lambda\delta_x := \begin{cases} \delta_x & x \geq 1 \\ 0 & \text{else} \end{cases} \qquad (x \in \mathbb{Z}) \,.$$

Claim 5.20. $U := A|A|^{-1}$, the polar part of S, is local too, since S is local and invertible.

Proof. Use the Combes-Thomas estimate Theorem 1.4. ☐

Then we claim that when S is local as in Definition 1.1 and has a gap, $[\Lambda, U]$ is compact where $U := A|A|^{-1}$ is the polar part of S. In this case, it is easy to show that the operator

$$\mathbb{\Lambda}U := \Lambda U \Lambda + \mathbb{1} - \Lambda$$

is Fredholm (indeed, $\mathbb{\Lambda}U$ is its parametrix). It turns out that now

$$\operatorname{index}(\mathbb{\Lambda}U)$$

is the relevant topological index of such disordered chiral systems which generalizes the Zak phase or winding number of Theorem 4.2.

5.3 The integer quantum Hall effect

The material in this section may be found in [EGS05, AG98]. The integer quantum Hall effect, as in Theorem 4.1, is specified by an operator H on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$, and the choice of a Fermi energy μ, with which one defines the Fermi projection

$$P := \chi_{(-\infty,\mu)}(H) \,.$$

Claim 5.21. If μ is in a spectral gap of H and H is local, then P is local.

Proof. Use the Combes-Thomas estimate Theorem 1.4. ☐

Now, using [BSS21, Lemma A1], which is

Lemma 5.22. *If P is local and has $\|P\| \leq 1$, and $f \in \ell^\infty\left(\mathbb{Z}^2\right)$ such that*

$$|f(x) - f(y)| \quad \leq \quad D\frac{\|x-y\|}{1+\|x\|}$$

then $[P, f(X)]$ is Schatten-3.

The function $u : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ given by $z \mapsto \frac{z}{|z|}$ obeys this estimate, and again as above we may thus immediately see that the operator

$$\mathbb{P}u(X) \quad := \quad Pu(X)P + \mathbb{1} - P$$

is Fredholm, with X the position operator (the parametrix is $\mathbb{P}u(X)^*$). It turns out that the Chern number now equals

$$\operatorname{index}(\mathbb{P}u(X))$$

which is just the generalization of (4.3) and also equals Theorem 2.5. This formulation of the Hall conductivity dates back to Laughlin.

6 Epilogue

In concluding the notes that correspond to these *three* lectures, one must comment on everything that was excluded from this presentation–a lot!

6.1 The bulk-edge correspondence

The most glaring omission is the bulk-edge correspondence, which is considered by many as a defining feature of topological insulators and in a sense is a kin to an Atiyah-Singer index theorem or to a holographic duality principle from string theory. This is the statement that half-infinite systems (which we entirely ignored–they model what happens near the boundary of the material, and are typically modeled via the Hilbert space $\ell^2\left(\mathbb{Z}^{d-1}\times\mathbb{N}\right)$) also have topological classification and indices, and that if one takes a doubly infinite (bulk) system H, then calculating its index commutes with truncating it to the half-space (edge) $\hat{H}$ and then calculating edge indices. This was first studied in the context of the integer quantum Hall effect by Hatsugai [Hat93], but see also [SBKR99, EG02, EGS05, FSS+20, BKR17].

6.2 The mobility gap regime

In Section 2 we completely ignored random operators and Anderson localization. However, for certain aspects of the integer quantum Hall effect which we have not explained (namely, their plateaus), one needs strong disorder, see e.g. [Sha16]. To define this regime a discussion of Anderson localization needs to take place, but roughly speaking, one drops the spectral gap assumption and instead employs a dynamical assumption on H, which makes life somewhat harder and precludes the usage of the Combes-Thomas estimate Theorem 1.4.

By now there are various works on in this regime for various cells of the Kitaev table Table 1: In [BvS94] the IQHE bulk index was first defined in this regime, in [EGS05] the IQHE bulk-edge correspondence was established and the stability of the index as μ ranges in this regime was proven. In [Sha20] the stability of the index with respect to the Hamiltonian was proven. In [ST19, GS18, BSS21] the bulk-edge correspondence was proven in this regime for Floquet (periodically time dependent systems), chiral 1D and $\mathbb{Z}_2$ time-reversal invariant systems.

A big open problem in topological insulators is to define an appropriate ambient topology for Hamiltonians in the mobility gap regime so that π_0 of these Hamiltonians is isomorphic to $\mathbb{Z}$, say, in the integer quantum Hall effect. Another important problem is whether the edge system of bulk-mobility-gapped IQHE systems has ac-spectrum or resonances.

6.3 The continuum regime

We have used exclusively the tight-binding regime. In the integer quantum Hall effect, it was shown in [SW22] that the tight-binding limit connects continuum with discrete for the dynamics of low-energy states and and the topological indices agree. It remains to study other cells of Table 1, and to resolve whether really all cells are topological in the continuum, see e.g. [SW21].

6.4 Interacting systems

Since relatively early in the history of the integer quantum Hall effect, the *fractional* quantum Hall effect was experimentally discovered and since Nobel prizes have been awarded for its description. It is generally believed that to understand this phenomenon, interactions must be taken into account. Our topological understanding of these interacting systems is not yet at the same level as that of the single-particle picture. We do not have the analog of Table 1. Be that as it may, recently some progress has been achieved, see e.g. [GMP17, BBDRF20] and references within.

A Some linear algebra or functional analysis lemmas

Claim A.1. $\ker(A^*) = (\operatorname{im} A)^\perp$ for any operator $A : \mathcal{H} \to \mathcal{H}$.

Proof. We have the following sequence of equivalent statements for any $v \in \mathcal{H}$:

1. $v \in \ker(A^*)$.
2. $A^* v = 0$.
3. $\langle u, A^* v\rangle = 0$ for all $u \in \mathcal{H}$.
4. $\langle Au, v\rangle = 0$ for all $u \in \mathcal{H}$.
5. $v \in (\operatorname{im} A)^\perp$.

□

Claim A.2. $W^\perp \in \operatorname{Closed}(\mathcal{H})$ for any subspace $W \subseteq \mathcal{H}$.

Proof. Write

$$\begin{aligned} W^\perp &= \bigcap_{v \in W} \{\, u \in \mathcal{H} \mid \langle v, u\rangle = 0 \,\} \\ &= \bigcap_{v \in W} \langle v, \cdot\rangle^{-1}(\{\, 0 \,\}) \,. \end{aligned} \tag{A.1}$$

But $\{\, 0 \,\} \in \operatorname{Closed}(\mathbb{C})$ and $\langle v, \cdot\rangle^{-1} : \mathcal{H} \to \mathbb{C}$ is continuous, so that $\langle v, \cdot\rangle^{-1}(\{\, 0 \,\}) \in \operatorname{Closed}(\mathcal{H})$. But now, arbitrary intersections of closed subsets are again closed (cf. Definition 3.1). □

Claim A.3. For any subspace $W \subseteq \mathcal{H}$, $\left(\overline{W}\right)^\perp = W^\perp$.

Proof. Since $W \subseteq \overline{W}$, $\left(\overline{W}\right)^\perp \subseteq W^\perp$ via (A.1). Conversely, let $v \in W^\perp$. WTS $v \in \left(\overline{W}\right)^\perp$, i.e., that for all $w \in \overline{W}$, $\langle v, w\rangle = 0$. Let $\{\, w_n \,\}_n \subseteq W$ such that $\lim_n w_n = w$. Then

$$\begin{aligned} \langle v, w\rangle &= \left\langle v, \lim_n w_n \right\rangle \\ &= \lim_n \langle v, w_n\rangle && (\langle v, \cdot\rangle \text{ is continuous}) \\ &= \lim_n 0 = 0 \,. \end{aligned}$$

□

Claim A.4. For any subspace $W \subseteq \mathcal{H}$, $\left(W^\perp\right)^\perp = \overline{W}$.

Proof. Let $w \in \overline{W}$. Then $\langle w, v \rangle = 0$ for all $v \in \left(\overline{W}\right)^\perp$, which, via Claim A.3, implies that $\langle w, v \rangle = 0$ for all $v \in W^\perp$, which is equivalent to saying that $w \in \left(W^\perp\right)^\perp$.

Conversely, by [Rud91, 12.4], for any closed subspace,

$$\begin{aligned} \mathcal{H} &= \overline{W} \oplus \left(\overline{W}\right)^\perp \\ &= \overline{W} \oplus W^\perp . && \text{(Via Claim A.3)} \end{aligned}$$

Now since $W^\perp \in \mathrm{Closed}\,(\mathcal{H})$ via Claim A.2, we may also write

$$\mathcal{H} = W^\perp \oplus \left(W^\perp\right)^\perp .$$

Hence we learn that

$$W^\perp \oplus \left(W^\perp\right)^\perp \quad = \quad W^\perp \oplus \overline{W} .$$

Now if $\overline{W}$ were a proper subspace of $\left(W^\perp\right)^\perp$, this line would lead to a contradiction. ☐

Lemma A.5. *We have* $\ker(A) = \ker\left(|A|^2\right)$ *with* $|A|^2 \equiv A^* A$.

Proof. We have the chain of implications

$$\begin{aligned} \varphi \in \ker(A) \quad &\Longleftrightarrow \quad A\varphi = 0 \\ &\Longleftrightarrow \quad A^* A \varphi = |A|^2 \varphi = 0 \\ &\Longrightarrow \quad v \in \ker\left(|A|^2\right) . \end{aligned}$$

Conversely,

$$\begin{aligned} \varphi \in \ker\left(|A|^2\right) \quad &\Longleftrightarrow \quad |A|^2 \varphi = 0 \\ &\Longleftrightarrow \quad \left\langle |A|^2 \varphi, \psi \right\rangle = 0 \forall \psi \in \mathcal{H} \\ &\Longleftrightarrow \quad \langle A\varphi, A\psi \rangle = 0 \forall \psi \in \mathcal{H} . \end{aligned}$$

In particular, choose $\psi = \varphi$ to get $\|A\varphi\| = 0$ which implies $A\varphi = 0$ and so $\varphi \in \ker(A)$ as desired. ☐

Theorem A.6. *(Kuiper's theorem) The invertible operators are contractible within* $\mathcal{B}(\mathcal{H})$.

Proof. Let $A \in \mathcal{B}(\mathcal{H})$. We wish to find a continuous path to $\mathbb{1}$. Using the polar decomposition, write

$$A \quad = \quad |A|\, U$$

where $|A| \equiv \sqrt{A^* A}$ and $U = A\,|A|^{-1}$ is unitary. By the Hille-Yosida theorem [RS80], there is some self-adjoint operator H such that $U = \mathrm{e}^{\mathrm{i}H}$. Define the map

$$\gamma : [0, 1] \quad \to \quad \mathrm{GL}\,(\mathcal{B}(\mathcal{H}))$$

via

$$\gamma(t) \quad = \quad ((1-t)\,|A| + t\mathbb{1})\, \mathrm{e}^{\mathrm{i}(1-t)H} .$$

Then $\gamma(0) = A$ and $\gamma(1) = \mathbb{1}$, γ is norm continuous and $\gamma(t)$ is invertible because $|A| > 0$. ☐

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