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Topology and Localization

Mathematical Aspects of Electrons in
Strongly-Disordered Media

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ABSTRACT

Topological insulators are usually studied in physics under the assumption of translation invariance, which allows for the usage of Bloch decomposition. Mathematically vector bundle theory over the Brillouin zone is employed. Yet more sophisticated (and already deep within the realms of mathematical physics) are the methods of non-commutative geometry, championed by Bellissard and coworkers [BvS94], which dispose of translation invariance. Instead of classifying vector bundles over the Brillouin zone, they classify (Fermi) projections in a non-commutative C-star algebra, which is called the non-commutative Brillouin zone.

Simply put, this dissertation is about what happens when the Fermi projection does not belong to the non-commutative C-star algebra which is the Brillouin torus. This is called *the mobility gap regime*, which happens when there is either no spectral gap at all (strong-disorder) or there is one but the Fermi energy is placed within the localized part of the spectrum. Physically this is the most interesting case (it is the only way to explain the plateaus in the integer quantum Hall effect (IQHE) for example), and mathematically it is the most difficult. Indeed, usually to classify projections in a C-star algebra, one uses (algebraic) K-theory. However, the K-theory groups of the W-star algebras which contain the Fermi projections (as they are defined via the Borel bounded calculus, they belong to the enveloping W-star algebra of a given Hamiltonian) are trivial. This problem is avoided in the spectral gap regime since then the Fermi projection may be deformed into a continuous function of the Hamiltonian, whence it belongs to the C-star algebra generated by a given Hamiltonian, and K-theory applies.

The mobility gap regime has been studied previously, first in the context of the IQHE by the Bellissard school. They defined a non-commutative Sobolev space [BvS94] which is smaller than the aforementioned W-star algebra. [EGS05] provided the first bulk-edge correspondence proof in the mobility gap regime of the IQHE, and also provided a definition for the edge index in this regime. [PS16b] studied the mobility gap regime of chiral systems in all dimensions, although always within the probabilistic ergodic framework.

This work is another step forward in studying the mobility gap regime, this time of chiral one-dimensional systems and Floquet two-dimensional systems. We also study localization for these chiral one-dimensional systems, which is crucial to have a well-defined topology. Beyond the technical achievements, the main message of the present work is the connection between topology and localization, which is established in a quantitative way, and not just in terms of one being a prerequisite for the other.

This dissertation has four chapters as well as a technical appendix. In the introductory chapter we give a brief overview of the field of strongly-disordered topological insulators from our mathematical physics perspective. This is a description of the problem as an algebraic topology of disordered insulating single-particle Hamiltoni-

ans. Our approach is deterministic (it doesn't use ergodicity) though its assumptions are modeled after almost-sure statements one can make about ergodic random operators which exhibit localization.

In the next two chapters, which are based on [GS18b; GS18a] we study the chiral one-dimensional case of the Kitaev table. We prove its complete dynamical localization at all non-zero energies. We connect its topological invariants to localization (via the zero energy Lyapunov exponents), and prove a bulk-edge duality for this system in the strongly-disordered, mobility gap regime.

In the last chapter, which is based on [ST18], we turn to study Floquet systems with no spectral gap but with a mobility gap, prove the bulk-edge duality for systems in this regime, and formulate a new definition for the topological invariant, which is shown to coincide with the old one, but is perhaps easier to understand or calculate than previous definitions in the literature.

CONTENTS

NOTATION

χ_S — The characteristic function $\mathbb{R} \to \mathbb{R}$ equal one on the set $S \subseteq \mathbb{R}$ and zero otherwise.

$X_{\geq t}$ — The set $\{\, x \in X \mid x \geq t \,\}$ for any ordered X and $t \in X$.

Θ — The Heaviside step-function, $\chi_{\mathbb{R}_{\geq 0}}$ or the anti-unitary time-reversal operator as in (1.1).

X_j — The position operator in direction j: $(X_j\psi)(x) \equiv x_j\psi(x)$ for any $\psi \in \ell^2(\mathbb{Z}^d)$ and extended trivially to descendant spaces.

$\mathbb{T}^d$ — the d-torus, e. g., $\mathbb{R}^d/(2\pi\mathbb{Z}^d)$.

$\mathbb{S}^d$ — the d-sphere, that is, $\{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$.

σ_i — the i-th Pauli sigma matrix: $\sigma_0 \equiv \mathbb{1}_{2\times 2}$, and
$$\sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\mathrm{Gr}_k(V)$ — The Grassmannian manifold: the space of all k-dimensional subspaces of the vector space V.

$\pi_k(X)$ — The kth homotopy group of the (pointed) topological space X, namely the set of homotopy classes of (pointed) maps $\mathbb{S}^k \to X$.

$\mathbb{C}^\infty$ — Any separable $\mathbb{C}$-Hilbert space (they are all isomorphic), e. g. $\ell^2(\mathbb{N})$.

$\mathcal{B}(\mathbb{C}^\infty)$ — The C-star algebra of bounded linear operators on $\mathbb{C}^\infty$, norm continuous operators.

$\mathcal{B}(\Omega)$ — The Borel sigma-algebra of a topological measure space Ω.

$\mathcal{K}(\mathbb{C}^\infty)$ — The two-sided ideal of compact operators on $\mathbb{C}^\infty$, norm limits of finite rank operators.

$\mathcal{F}(\mathbb{C}^\infty)$ — The space of Fredholm operators on $\mathbb{C}^\infty$, invertible operators up to compacts.

$\mathcal{U}(\mathbb{C}^\infty)$ — The group of unitary operators on $\mathbb{C}^\infty$.

$C(A; B)$ — The space of continuous maps $A \to B$.

$C(A)$ — $C(A; \mathbb{C})$ (used for brevity).

$\mathrm{End}(S), \mathrm{Aut}(S)$ — The group or monoid of automorphisms or endomorphisms respectively of the object S of some category.

$\mathrm{im}(A)$ — The image of the map $A : X \to Y$, i.e. the set $\{\, A(x) \in Y \mid x \in X \,\}$

$\mathbb{E}[X]$ — The disorder average of a random variable $X : \Omega \to \mathbb{R}$, i.e., $\int_\Omega X(\omega)\, d\,\mathbb{P}(\omega)$.

$\lvert J \rvert$	The Lebesgue measure of the set $J \subseteq R^d$.
$\lvert A \rvert^2$	The absolute value square of an operator A, given by A^*A.
$\lVert A \rVert_1$	The trace norm of a trace class operator A, equal to $\mathrm{tr}(\lvert A \rvert)$.
$\mathcal{J}_1(\mathcal{H})$	The ideal of trace-class operators.
$\lfloor t \rfloor$	The integer part of $t \in \mathbb{R}$.
$Sp_{2N}^*(\mathbb{C})$	The Hermitian symplectic group, as in Definition A.36.
$(\delta_x)_{x \in \mathbb{Z}^d}$	The position of $\ell^2(\mathbb{Z}^d)$, with $(\delta_x)_y \equiv \delta_{xy}$, the latter being the Kronecker delta symbol, equal to one if $x = y$ and zero otherwise.
A_{xy}	For $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$, the linear transformation $\mathbb{C}^N \to \mathbb{C}^N$ between the internal space at y to the one at x, i.e., the $N \times N$ matrix $\langle \delta_x, A\delta_y \rangle$.
$G(x,y;z)$	The Green's function of a Hamiltonian $G(x,y;z) \equiv \langle \delta_x, (H - z\mathbb{1})^{-1}\delta_y \rangle$.

ACRONYMS

PCC	Path-connected component.
IQHE	Integer quantum Hall effect.
FQHE	Fractional quantum Hall effect.
BEC	Bulk-edge correspondence (Principle 1.8).
FMC	Fractional moments condition (Definition 1.21).
FMM	Fractional moments method.
MSA	Multi-scale analysis.
LE	Lyapunov exponents (see (2.17)).
DC	Direct current.
AC	Alternating current.
GOE	Gaussian orthogonal ensemble of random matrix theory.
RHS	Right hand side (of an equation or inequality).
LHS	Left hand side.
WLOG	Without loss of generality.

INTRODUCTION

The important issue implicit in the von Klitzing discovery is not the existence of physical law but rather what physical law is, where it comes from, and what its implications are.
— Robert B. Laughlin

We give a brief introduction to some theoretical aspects of strongly-disordered topological insulators. Our goal here is not to initiate the beginner (for that purpose see e.g. [Ber13; HK10; PS16a; Sha16]) but rather to orient the expert into what kind of questions and problems have driven this work. As a result, our approach is cavalier and schematic. Having setup the context, we finally narrow the focus to a description of the main results of the present work.

1.1 TOPOLOGICAL INSULATORS

Experience with experimental physics implies that the discovery of a physical quantity that is essentially insensitive to many of the microscopic details of the experimental sample is something of a surprise–a macroscopic manifestation of quantum mechanics. Such quantities are what we nowadays call *topological invariants*, which are continuous functions which map a Hamiltonian (the quantum mechanical object which encodes all the information about the material and perhaps the experiment) into an additive group with a single generator (e.g. $\mathbb{Z}$). Being continuous and taking values in a discrete space implies these maps are locally constant, which is the mathematical reason for their aforementioned experimental stability. Moreover the single generator for the group implies the measured quantity will always come in multiples of some constant.

1.1 Definition. A *Hamiltonian H* is a self-adjoint operator on $L^2(X) \otimes \mathbb{C}^N$ (where X is physical space and N is the number of internal degrees of freedom of the particles) such that H is *local*, i.e., its matrix elements in the position basis have rapid off-diagonal decay (a precise condition is given in Definition A.2).

The main effort in explaining such phenomena from a theoretical perspective is thus to define the space of Hamiltonians, its topology, the maps which should be associated with experimentally measurable quantities, prove their continuity and finally that they take values in a discrete space. If one is more industrious one would even like to calculate the set of path-connected components (PCCs henceforth) of the space of Hamiltonians and show that the invariants establish an *isomorphism* between the set of PCCs and the target space; this gives us a picture of the topological phase space. Usually, what one works with is not the space of *all* possible Hamiltonians (which is quite big and probably has just one PCC) but rather focus on a physically interesting

subspace, e. g. the set of all Hamiltonians corresponding to *insulators* in a fixed space dimension.

1.2 Example. Let us consider the space of all *translation invariant* Hamiltonians on the two-dimensional lattice with two internal degrees of freedom; the Hilbert space is $\mathcal{H} := \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$. We further specify to the subspace of *spectrally gapped* Hamiltonians, which corresponds to (non-disordered) insulators. By Fourier transform and translation invariance we are left with Hamiltonians $H(k)$ acting on $\mathbb{C}^2$, i. e. 2×2 Hermitian matrices, depending on k in the Brillouin zone $\mathbb{T}^2$. By Definition 1.1 the entries are smooth functions of k. The gap condition implies the two eigenvalues of these matrices never degenerate for all $k \in \mathbb{T}^2$. Hence the rank-1 projection onto the lower eigenspaces, $P(k)$, is well-defined for all k. Such $2x2$ self-adjoint projections have a convenient representation using the Pauli matrices $\{\sigma_i\}_{i=1}^3$ as $P(k) = \frac{1}{2}(\mathbb{1}_2 - d_i(k)\sigma_i)$ with $d : \mathbb{T}^2 \to \mathbb{S}^2$ smooth and where we use the Einstein summation convention for $i = 1, \ldots, 3$. The Whitney approximation theorem [Lee03] says that for smooth manifolds X and Y (with $\partial Y = \varnothing$), any continuous map $X \to Y$ is (continuously) homotopic to a smooth map $X \to Y$. We conclude that to classify this subspace of Hamiltonians we need to find all homotopy classes of *continuous* maps $\mathbb{T}^2 \to \mathbb{S}^2$. In Lemma A.45 we see that $[\mathbb{T}^2 \to \mathbb{S}^2] \cong \pi_2(\mathbb{S}^2)$, using the fact that the two-sphere is path- and simply-connected. Now we use the fact [Bre93] that $\pi_2(\mathbb{S}^2) \cong \mathbb{Z}$ to conclude that the set of PCCs of our subspace of Hamiltonians is isomorphic to $\mathbb{Z}$. One can prove [ASS83; MS74] that the isomorphism is given by a special map called the Chern number:

$$\text{Chern}(P) := \frac{\mathrm{i}}{2\pi} \int_{\mathbb{T}^2} \text{tr}(P[\partial_1 P, \partial_2 P]) \in \mathbb{Z}\,.$$

Before proceeding we want to formulate the problem in a another way. Consider the Grassmannian manifold $\text{Gr}_1(\mathbb{C}^2)$, which is actually homeomorphic to $\mathbb{S}^2$. What we want to classify are continuous maps $d : \mathbb{T}^2 \to \text{Gr}_1(\mathbb{C}^2)$. The Grassmannians are the classifying spaces for the principal bundle with the unitary group (see [Ati94]), that is, we have the homotopy equivalence $[X \to \text{Gr}_n(\mathbb{C}^\infty)] \cong \text{Vect}_n(X)$ where on the RHS we have the space of equivalence classes of rank n $\mathbb{C}$-vector bundles over the base space X. The point of all this is that apparently we have another way to identify our subspace of Hamiltonians: the space of all line-bundles above $\mathbb{T}^2$. More generally if we want to consider not just two-level gapped systems then we'll classify rank n-vector bundles above $\mathbb{T}^2$.

We note in passing that there is a mathematical framework to classify vector bundles, called *topological K-theory* [Ati94]. What K-Theory does (via the Atiyah-Jänich theorem [BB89]) is replace $[X \to \text{Gr}_n(\mathbb{C}^\infty)]$ with $[X \to \mathcal{F}(\mathbb{C}^\infty)] \cong K_0(C(X))$ where $\mathcal{F}(\mathbb{C}^\infty)$ is the space of Fredholm operators. Since a Fredholm operator is characterized by having finite kernel and cokernel, one can morally think of a Fredholm operator as encoding the difference of two finite subspaces of $\mathcal{H}$. Thus K-Theory does two things coarser: it allows for arbitrary but always finite number of occupied levels (n is not fixed through homotopies), and it calculates a *relative* phase (the Grothendieck construction).
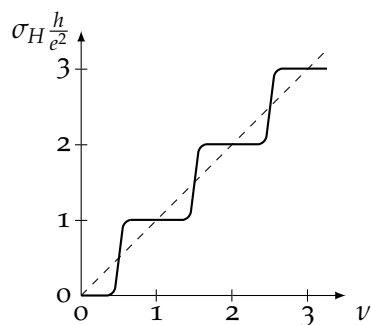
FIGURE 1.1: The plateaus of the IQHE: The dashed line is the expected result of the classical Hall effect, the solid line the (schematic) experimental result of [KDP80].

Historically the first topological invariant was discovered in the context of the integer quantum Hall effect [Gra07] (IQHE henceforth, of which the example above is one particular model) with the ground breaking experimental work of [KDP80] and theoretical work of [Lau81; Tho+82]. In essence the experimental revolution was the discovery that at low temperatures, certain materials (e. g. gallium arsenide heterostructures) that are placed in a Hall setup [AM76], exhibit a very peculiar relationship between the Hall conductivity σ_H and the magnetic field: one finds σ_H at plateaus whose heights are integer multiples of a universal constant (the von Klitzing constant e^2/h with e the charge of the electron and h Planck's constant):

$$\sigma_H \in \frac{e^2}{h}\mathbb{Z}\,.$$

The existence of these plateaus seemed to be insensitive to many choices in the setup; they were obtained when the longitudinal conductivity vanished, so these materials are also *insulators*. The theoretical revolution [Lau81; Tho+82] was the application of Bloch theory in quantum mechanics to identify the space of (translation invariant gapped) Hamiltonians with the space of vector bundles with the quasi-momentum space, i. e. the Brillouin torus, as a base space, as detailed in the example above. Further theoretical work was done by [BvS94] in applying Connes' ideas of non-commutative geometry to the problem. The point was that one could abandon the (unphysical) assumption of translation invariance; instead of classifying vector bundles, consider projections in a certain C-star algebra, which was called the non-commutative Brillouin torus. One then uses *C-star algebra* K-theory (which generalizes the earlier theory of Atiyah [Ati94]), to classify projections in such a non-commutative Brillouin torus. However, K-theory could not deal with the *plateaus* of the IQHE; it could only tell us that the Hall conductivity is integer-valued. Meanwhile the mathematical study of Anderson localization was developing rapidly [FS84] and only with it came a complete explanation of the existence of the plateaus, which one can find for instance in [BvS94, Theorem 14]; see also [AG98, Theorem 5] and the deterministic [EGS05, Proposition 2]. Heuristically, the change in the magnetic field is associated with a change in the Fermi energy, which controls how many states in the system are
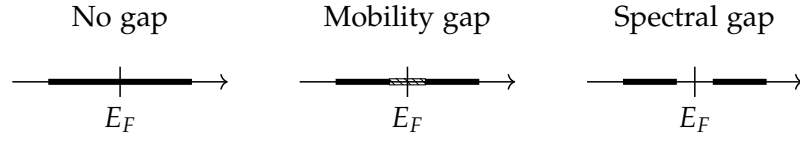
No gap   Mobility gap   Spectral gap

E_F E_F E_F

FIGURE 1.2: A schematic picture of the possible spectrum of the Hamiltonian $\sigma(H)$. Here E_F is the Fermi energy, the solid line refers to any type of spectrum and the stripes to pure point spectrum.

filled. If we fill the system with more states, which are however localized, they cannot contribute to mobility so that the Hall conductivity stays constant for changes that respect this localization. That's why this region of the spectrum of a localized Hamiltonian is referred to as a *mobility gap* (as opposed to the more special spectral gap which is the prerequisite for analysis with K-theory). The main point of this work is to further our topological understanding of mobility gapped Hamiltonians, on which we'll elaborate in the next section.

It is worth pointing out that K-theory does not apply to the mobility gap regime due to the following reason: The K_0 group classifies projections in a given C-star algebra. Usually one takes the C-star algebra that contains the Hamiltonian of interest (or the group of Hamiltonians of interest), $\mathcal{A}_d$, e.g., the C-star algebra generated by the d magnetic shifts in d dimensions on $\ell^2(\mathbb{Z}^d)$, possibly tensored with $C(\Omega)$, the disorder configuration space (see [PS16a]). Now, if we have a spectral gap, the Fermi projection P obeys $P = f(H)$ for some continuous f (a continuous version of the step function, deformed only within the gap). Since the continuous functional calculus stays within the C-star algebra generated by H, $f(H)$, and hence the Fermi projection, lie in $\mathcal{A}_d$. However, in the absence of a spectral gap one must take $f \equiv \chi_{(-\infty, E_F)}$, which is not continuous but merely Borel bounded. The corresponding functional calculus now lies in the W-star algebra generated by H [RS80], whose K-theory groups are trivial [Rør00]. See also the definition of "non-commutative Sobolev spaces" in [BvS94].

To this day the theory of the IQHE as well as its sophisticated relative, the fractional quantum Hall effect (FQHE) remains a very active field of research in solid-state physics. However, around 2005 Charles Kane and co-workers made a groundbreaking discovery [KM05] which develops the field in a perpendicular direction. They realized that, even though without breaking time-reversal there is no non-trivial IQHE, one could still define a topological invariant for 2D systems which do obey a (fermionic) time-reversal symmetry.

1.3 Example. We want to continue Example 1.2 but change it in two ways. First of all let us consider the Hilbert space now with *four* degrees of freedom on each lattice site, $\mathcal{H} := \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$. Second, consider the anti-unitary time-reversal map

$$\Theta : \mathcal{H} \to \mathcal{H} \tag{1.1}$$

and assume it obeys $\Theta^2 = -\mathbb{1}$, which corresponds to fermions. We now want to consider the subspace of all translation invariant Hamiltonians which are gapped in between the second and third level (so, still insulators), and which furthermore are time-reversal invariant, $[H, \Theta] = 0$. Time-reversal invariance implies via Kramers degeneracy a two-fold degeneracy at special points of $\mathbb{T}^2$ [KM05] so that is why to respect this constraint and the gap we must have at least four levels. With the same reasoning as in Example 1.2, this means that now we want to calculate the homotopy classes of smooth maps $\mathbb{T}^2 \to \mathrm{Gr}_2(\mathbb{C}^4)$ subject to certain constraints due to time-reversal symmetry. We omit the explicit formulation of these constraints via the parametrization of $\mathrm{Gr}_2(\mathbb{C}^4)$ in the Plücker embedding and instead just mention in passing that *if* one further specifies to the subspace of 4×4 Hamiltonians which can be written as a linear combination of five Gamma matrices, then $\mathrm{Gr}_2(\mathbb{C}^4)$ reduces to S^4 and the time-reversal constraint becomes the specification of $d : \mathbb{T}^2 \to S^4$ on only *half* of $\mathbb{T}^2$ with special boundary conditions; namely, if we consider half of $\mathbb{T}^2$ as $S^1 \times [0, \pi]$, the boundary conditions are that $d|_{S^1 \times \{0\}}$ and $d|_{S^1 \times \{\pi\}}$ are loops which must start and end on either a north or south pole of S^4. This implies that the set of PCCs of such maps is isomorphic to $\mathbb{Z}_2$ [Sha17]. This invariant is nowadays called the Fu-Kane-Mele $\mathbb{Z}_2$-invariant.

There is another perspective on the $\mathbb{Z}_2$ invariant. By [BvS94, Proposition 9] we know that the Chern number in the IQHE is given by the Fredholm index $\mathrm{Chern}(P) = \mathrm{ind}(F) \equiv \dim \ker F - \dim \mathrm{coker} F$ with the operator

$$F := PUP + P^\perp \tag{1.2}$$

where $P \equiv \chi_{(-\infty, E_F)}(H)$ is the Fermi projection corresponding to the single particle Hamiltonian H and $U \equiv \exp(\mathrm{i} \arg(X_1 + \mathrm{i} X_2))$, X_i being the position operator in direction i. [AG98, (4.14)] also contains a proof that this operator is Fredholm as long as E_F is within the localized part of the spectrum of H. One verifies that when $[H, \Theta] = 0$ with $\Theta^2 = -\mathbb{1}$ and anti-unitary, $\mathrm{ind}(F) = 0$. When $[H, \Theta] = 0$, $[P, \Theta] = 0$, which in turn implies

$$F = -\Theta F^* \Theta. \tag{1.3}$$

Since the Fredholm index obeys the logarithm law $\mathrm{ind}(AB) = \mathrm{ind}(A) + \mathrm{ind}(B)$, and since $\pm \Theta$ is an isomorphism (so it has trivial kernel and co-kernel), and finally since $\mathrm{ind}(A^*) = -\mathrm{ind}(A)$, we find that $\mathrm{ind}(F) = 0$ due to fermionic time-reversal symmetry. In [AS69] the PCCs of Fredholm operators F which obey (1.3) (we call them Θ-*odd Fredholm operators*) is shown to be isomorphic to $\mathbb{Z}_2$, and the isomorphism is given by $\mathrm{ind}_2(F) := \dim \ker F \mod 2$. This index shares many of the properties of the Fredholm index, in particular, it is also stable under compact and norm continuous perturbations. See also [Sch15] and its rephrasing in [KK16].

We note in passing that it remains an open problem to show that this invariant is equivalent to the FKM invariant of quaternionic vector-bundles in the translation invariant case directly, without using homotopy arguments (since both invariants

> are constant on homotopy classes, since there are only two classes, and they agree on the trivial class, the equivalence is abstractly true) as one would like to see the connection explicitly. There is also no good understanding of this invariant in the mobility gap regime for the edge (for the bulk it extends to the mobility gap regime). Perhaps one formula which might prove useful is the half Fedosov formula which applies in the case that the Θ-odd Fredholm operator has $\|F\| \leq 1$ (which is certainly true for $F = PUP + P^{\perp}$, P projection, U unitary):
>
> $$\mathrm{ind}_2(F) = \lim_{n \to \infty} \mathrm{tr}((\mathbb{1} - |F|^2)^n) \mod 2 \,. \tag{1.4}$$

Actually some of the mathematics behind Kane et al's discovery was already known from the late 1960s [AS69; Dup69] (either as Θ-odd Fredholm operators discussed in Example 1.3 or as the theory of characteristic classes of quaternionic vector bundles and the $\mathbb{Z}_2$-valued quaternionic Pontryagin number). Indeed, the constraint of fermionic time-reversal symmetry which is the quaternionic structure on the vector bundle, implies that its Chern characteristic classes are trivial, yet, there is *still* more than one PCC. What makes the discovery of [KM05] even more impressive is that there was no experiment at the time which this work was to explain. Soon after, Ryu and co-workers [Ryu+10; Sch+08] understood that one could consider all possible ten symmetry classes of Altland and Zirnbauer [AZ97] for such systems and in different dimensions they could give topologically non-trivial results. These symmetry classes correspond to the presence or absence of time-reversal or particle-hole symmetry, or their composition, chiral symmetry. In each case, one has to decide if the symmetry operator squares to plus or minus one. It wasn't until Kitaev [Kit09] in 2009, that we understood how to systematically organize all space dimensions and ten possible symmetry classes into a periodic table, called *the Kitaev periodic table*. This table has a periodicity of eight in the space dimension d, which is ultimately traced to the Bott periodicity of K-theory [Karo8] or of Clifford modules [ABS64]. Thus one needs to consider only 80 possible kinds of matter: eight possible dimensions times ten possible symmetry classes. The table determines, in each case, if there will be a non-trivial topology, and if so, what would be the set of PCCs of such symmetric gapped Hamiltonians. On top of the 80 cases of *strong invariants* there are many subcases, specifed by the various *weak invariants*. For example, in 3D there are weak topologically-non-trivial quantum Hall phases despite the zero of that bracket in Table 1.1. It is important to note that this table exists only at the generality of the spectral gap regime with stable classification, since it is inherently a reflection of K-theory and its Bott periodicity. In an unstable classification, one specifies to the space of n-level Hamiltonians which are gapped between the jth and $j + 1$th levels, where n and j are fixed throughout all homotopies. Conversely, the stable classification allows both n and j to vary throughout homotopies (technically this is achieved by embedding a Hamiltonian H in sufficiently large but finite matrix $H \oplus 0$).

> **1.4 Example.** Clearly the stable homotopy is coarser than the unstable one, as the end of Example 1.2 abstractly explains. Here is a concrete manifestation of this difference. If in Example 1.2 we rather consider the space of Hamiltonians on

AZ	S			d							
	Θ	$\mathcal{P}$	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE 1.1: The Kitaev table as shown in [HK10]: Each row refers to a symmetry label of [AZ97], with Θ being time-reversal, $\mathcal{P}$ being particle-hole reversal; either one could square to $\pm\mathbb{1}$ if it is present, and $\Pi \equiv \Theta \circ \mathcal{P}$; the columns are the space dimensions (up to eight due to Bott periodicity); each entry is the (isomorphic) set of PCCs of gapped Hamiltonians in the corresponding case.

the three dimensional lattice, still with two-levels, we get to the space $[\mathbb{T}^3 \to \mathbb{S}^2]$, which is *not* trivial, yet according to Table 1.1, the three-dimensional no symmetry class case has no non-trivial topology. Indeed, the table merely gives us $[\mathbb{T}^3 \to \mathrm{Gr}_r(\mathbb{C}^n)]$ for r, n sufficiently large, which is zero. This distinction between the classification of homotopies of vector bundles vs. K-theory was elaborated on in [DG15]. Yet others [PS16a] say that the stable classification is more experimentally relevant, since the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^n$ is just an approximation for the actual experiment in which there are infinitely many valence and conduction bands.

In conclusion, the Kitaev table is the marriage between the K-theoretic Bott-periodicity and the Altland-Zirnbauer symmetry classification. To understand its most simple interpretation one has to get from the quantum mechanics of translation invariant spectrally gapped insulators to $[\mathbb{T}^d \to \mathcal{F}(\mathbb{C}^\infty)]$ for the first row, as explained in Example 1.2, or an equivariant version of $\mathcal{F}(\mathbb{C}^\infty)$ for all other rows. For the chiral class discussed in Example 1.5, one replaces $\mathcal{F}(\mathbb{C}^\infty)$ with $\mathcal{U}(\mathbb{C}^\infty)$, Bott's unitary group, since the chiral symmetry implies the Fermi projection has a unitary sub-block. For the general case see [AS69] for the definitions of skew-adjoint Fredholm operators (one still has to work out how the symmetry constraints yield other spaces. This was done using different notations in [GS16]).

While the stable vs. unstable and the relative vs. absolute distinction is not particularly crucial for us, the topological question *is*. At least in the spectral gap regime, one can get a handle on this problem: in (1.2) for instance, in the spectral gap regime $\chi_{(-\infty, E_F)}$ may be deformed to a smooth function of H, so that $\|F - F'\|$ is small if

$H - H'$ is small (in operator norm) and local, so Fredholm theory may be applied to obtain $\mathrm{Chern}(P) = \mathrm{Chern}(P')$. To say this we basically worked with the subspace topology of $\mathcal{B}(H)$ on the set of local spectrally gapped Hamiltonians. However, since a crucial part of the IQHE is its plateaus, it is of relevance to find a topological classification which does not rely on a spectral gap, and hence the issue of defining the topology on the space of Hamiltonians becomes harder. What is probably necessary is a notion of proximity between Hamiltonians that makes sure their locality and localization estimates are compatible, but we postpone a serious discussion of this point to another occasion.

> **1.5 Example.** The chiral symmetry class of Hamiltonians may be succinctly defined as operators acting on a Hilbert space which breaks into the direct sum $\mathcal{H} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$ for some other Hilbert space $\tilde{\mathcal{H}}$ and which are written in block form as
>
> $$H = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$
>
> for some $S \in \mathcal{B}(\tilde{\mathcal{H}})$ not necessarily self-adjoint. This implies that the spectrum of H is symmetric about zero and so to obtain an insulator we must require a (spectral or mobility) gap about zero. For the sake of simplicity let us focus on H having a spectral gap. This implies that S is invertible within $\mathcal{B}(\tilde{\mathcal{H}})$ since $\sigma(H) = \sigma(|S|) \cup \sigma(-|S|)$. Thus, it may seem that classifying chiral spectrally gapped Hamiltonians is equivalent to classifying invertible bounded operators on $\tilde{\mathcal{H}}$. However, by Kuiper's theorem [BB89, Theorem I.6.2] all invertible operators are path-connected in the norm topology to $\mathbb{1}$ so that the space of such Hamiltonians would appear to have just one PCC. Table 1.1 says this is *not* the case in one-dimension, for example (our discussion so far has been dimension agnostic so it applies in particular to dimension one). The solution is simply that the norm topology is *not* appropriate for the classification. Indeed, the path that one obtains between S and $\mathbb{1}$ does not preserve the locality property of Hamiltonians (it allows the electron to hop arbitrarily far away on the lattice).

1.2   FLOQUET SYSTEMS

In our discussion of topological insulators so far we have been focused on time-independent systems which have (some sort of) gap. However, the insulator property is just one possible feature for a sub-class of Hamiltonians, as hinted already in the beginning of this chapter. In recent years it has been discovered [Rud+13] that periodically-driven Hamiltonians, that is, *Floquet systems*, can also exhibit topological properties. These Hamiltonians describe independent electrons on a lattice subject to a periodic driving beyond the adiabatic regime. One goal could be to use the driving to "bypass constraints of static systems imposed by the chemistry of the material and by the fabrication process, potentially enabling a larger set of topological phases to be realized in the same experimental setup, by adjusting the properties of an externally applied, periodic driving field" [LF18]. Indeed, most non-trivial cells in Table 1.1

have not yet been experimentally realized (even in $d \leq 3$). The analogous "gap condition" for such systems is not directly related to an electric insulator property, which is why they should probably be called *Floquet topological systems* rather than Floquet topological insulators as in [RH17].

The idea is to take a time-dependent Hamiltonian (still local as in Definition 1.1 point-wise in time) which is periodic, that is, a point

$$H \in C(\mathbb{S}^1; \mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}) \otimes C(\mathbb{S}^1).$$

Such a Hamiltonian induces a unitary evolution $U : [0, 1] \rightarrow \mathcal{U}(H)$, by the Schrödinger equation

$$i \dot{U} = HU \tag{1.5}$$

with initial condition $U(0) = \mathbb{1}$. We note that as a rule, even though H is periodic, U need not be.

For Floquet systems, U, rather than H, is the main object of study. A-priori the solution of (1.5) is on the whole line (if $\mathbb{S}^1$, the domain of H, is interpreted as $\mathbb{R}/\mathbb{Z}$), however, U obeys the semi-group property $U(t) = U(t-1)U(1)$ for all $t \in (1, 2]$ so that obtaining a solution for all $t \in [0, 1]$ generates the solution on the entire line. Indeed, we write a general $t > 0$ as $t = \lfloor t \rfloor + (t - \lfloor t \rfloor)$ in which case $U(t) = U(t - \lfloor t \rfloor) U(1)^{\lfloor t \rfloor}$. We see that $U(1)$ determines the long time dynamics of the system and $U(t)$ for $t \in [0, 1]$ could generate short-term fluctuations.

Since $U(1)$ is unitary, its spectrum is on $\mathbb{S}^1$ (now meant as a subset of $\mathbb{C}$). If it happens that $\exists \lambda \in \mathbb{S}^1 \setminus \sigma(U(1))$ (that is, $U(1)$ has a spectral gap), then one could define a self-adjoint logarithm of $U(1)$ with a branch cut out of the spectrum $H_\lambda := i \log_\lambda(U(1))$. This logarithm may be considered as the effective *static* Hamiltonian which would have generated the same dynamics after one period: $U(1) \equiv \exp(-i H_\lambda)$. If we then let the dynamics generated by H_λ run backwards after having U run, we get an evolution that starts and ends with $\mathbb{1}$. Indeed, the concatenation of two operators $A, B : [0, 1] \rightarrow \mathcal{B}(\mathcal{H})$, which we denote by $A \sharp B : [0, 1] \rightarrow \mathcal{B}(\mathcal{H})$ is given by

$$(A \# B)(t) = \left\{ \begin{array}{ll} A(2t), & (0 \leq t \leq 1/2) \\ B(2(1-t)), & (1/2 \leq t \leq 1) \end{array} \right. . \tag{1.6}$$

We define the *relative* evolution as

$$U^{\text{rel}} := U \sharp \exp(-i \cdot H_\lambda) \tag{1.7}$$

and have by definition $U^{\text{rel}}(0) = U^{\text{rel}}(1) = \mathbb{1}$, that is, $U^{\text{rel}} : \mathbb{S}^1 \rightarrow \mathcal{U}(\mathcal{H})$ is a loop based at $\mathbb{1}$. As such, there is an algebraic topology to classify it.

> **1.6 Example.** In the translation invariant case, we have the task of classifying continuous loops $\mathbb{S}^1 \times \mathbb{T}^d \rightarrow U(N)$ based at $\mathbb{1}$. We wish to compare this with Example 1.5, where it turned out that the translation invariant case reduces to the classification of continuous maps $\mathbb{T}^d \rightarrow U(N)$. It is suggestive to think that the classification of static chiral systems in dimension d is equivalent to that of Floquet systems with no symmetry in dimension $d - 1$, treating time as an extra

space dimension. After all, in Example 1.2 we made the case that the classification of $[\mathbb{T}^2 \to \mathbb{S}^2]$ is equivalent to $[\mathbb{S}^2 \to \mathbb{S}^2]$. It turns out that while this is not quite the case in general, for the *top dimensional* topological invariants, i.e. the so-called *strong* topological invariants (those invariants which remain defined also in the strongly-disordered case, and explore all dimensions of the material), one could identify the static chiral classification at dimension d with the Floquet no-symmetry classification at $d-1$ by identifying the $\mathbb{S}^1$ factor as a suspension and using Bott periodicity, see [SS17] for a discussion. The full classification via K-theory in the translation invariant case proceeds as follows: the space of continuous loops $\mathbb{S}^1 \times \mathbb{T}^d \to \mathcal{U}(\mathbb{C}^\infty)$ based at $\mathbb{1}$ is the K_1 group (which classifies unitaries) of the C-star algebra $SC(\mathbb{T}^d)$, that is, the suspension S of the C-star algebra of continuous maps $\mathbb{T}^d \to \mathbb{C}$ (recall that the suspension of a C-star algebra A is defined as $SA := \{ f \in C([0,1]; A) \mid f(0) = f(1) = \mathbb{1} \}$, see [Rør00]). Hence we have $K_1(SC(\mathbb{T}^d)) \equiv K_2(C(\mathbb{T}^d))$, which by Bott periodicity $K_{n+2}(A) \cong K_n(A)$ is isomorphic to $K_0(C(\mathbb{T}^d)) \cong \mathbb{Z}^{2^{d-1}}$. In [PS16a] it is explained that only if $d \in 2\mathbb{N}$, one of these factors of $\mathbb{Z}$ corresponds to a "top" dimensional invariant. Hence the no symmetry case of Floquet systems should look precisely as the first row of Table 1.1. See also Appendix A.6.1.

In $d = 2$, this invariant is defined for a unitary $V : \mathbb{S}^1 \times \mathbb{T}^2 \to U(N)$ as,

$$W(V) \equiv -\frac{1}{8\pi^2} \int_{\mathbb{S}^1} \int_{\mathbb{T}^2} \operatorname{tr} \dot{V} V^* [V_{,1} V^*, V_{,2} V^*] \quad \in \mathbb{Z}, \qquad (1.8)$$

which has generalizations to the disordered case, see (4.6).

The crucial ingredient is the fact that U^{rel} is local (Definition 1.1). This is guaranteed in the spectral gap regime (see Lemma 4.23) by means of the Combes-Thomas estimate ([CT73] and Proposition 2.35).

One could consider Floquet systems in various dimensions and symmetries and produce an analogue of Table 1.1, see [RH17; YYW17]. It is however not clear how the various symmetries ascend from H to U, for instance, compare the chiral symmetry definition of [Fru16] and [RH17]; the question is whether chiral symmetry should reverse the direction of time as well, as it is, by definition, a composition of time-reversal and particle-hole conjugation.

The physical meaning of the various conditions for the topology to be well-defined as well as the meaning of the invariants remains somewhat less clear than in the static case. For instance, in the static case the spectral gap implies a dynamical consequence: an insulator. In the Floquet case, we do not know what kind of conditions on H imply a spectral gap for $U(1)$, and furthermore, a spectral gap for $U(1)$ does *not* mean that the system is insulating (although $U(1) = \mathbb{1}$ does, since it means that after one period the particles do not move, and hence also at infinite time).

An interesting aspect in the topology of Floquet systems is the fact that the unitary $U(t)$ is always local, uniformly in $t \in [0,1]$ (cf. Proposition 1.23 where the supremum is taken over $t > 0$), and it is rather its self-adjoint logarithm which has to be verified for being local. That is, the solution of the Schrödinger equation, which yields a unitary operator (the propagator) out of a self-adjoint one (the Hamiltonian), is *always*
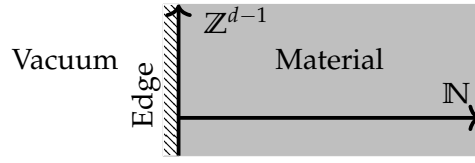
FIGURE 1.3: An illustration for the geometry of the edge.

local if the Hamiltonian is. In contrast, for static topological insulators, the Hamiltonian H is as well always local, and the topological data is also contained in a partial isometry–its polar part sgn $H = \mathbb{1} - 2\chi_{(-\infty,E_F)}(H)$ in the polar decomposition (assuming we set $E_F = 0$ WLOG) which is (almost-surely) unitary. This way of extracting a unitary operator out of a Hamiltonian, however, depends crucially either on a gap or on localization to ensure that the unitary is local too. The difference between the two procedures is ultimately traced back to the fact that taking the polar part of an operator is *not* a continuous operation whereas the Duhamel formula is.

1.3 THE EDGE PICTURE AND THE BULK-EDGE CORRESPONDENCE

The entire discussion so far, in particular the translation invariant Examples 1.2 and 1.3 have implicitly assumed that real space is infinite and has no boundaries–a *bulk* picture. Of course no actual material in the laboratory is truly infinite, and the justification for such a description stems from the different scales in the problem, that is, the size of the electron versus the dimensions of the material. Such a description however fails to capture what inevitably happens at the boundary of the sample, which is described by the corresponding *edge* picture. Hence there are two physical phenomena, one associated to the bulk and another to the edge, and a real sample hosts *both* simultaneously. For example, in the IQHE, there are both chiral currents along the boundary of the sample as well as transversal Hall current in the bulk. Apparently the conductivity of either of them is a topological invariant. More generally, be it bulk or edge description, we are still considering the same material, so we expect the topological classification to ultimately reflect the material and not our description of it (which depends on the choice of bulk vs. edge). This is the point of *the bulk-edge correspondence* (henceforth BEC), which we'll define shortly. The study of the BEC started in the context of the IQHE (before Table 1.1 existed), since from the discovery of the effect there were competing explanations using bulk vs. edge physics [Lau81; Tho+82]. The first BEC proof was given by [Hat93] and more general proofs followed after [EG02; EGS05; SKR00].

To model the edge we take space as half-infinite, for the same reason that justified the bulk description: near one edge we are inevitably far away from all other edges. Even though procedures exist to analyze topology in finite-dimensional Hilbert spaces [LS17], it is the *infinite* dimensional system that provides the idealization in which the topological concepts arise naturally, as the following example illustrates.

1.7 Example. On the Hilbert space $\ell^2(\mathbb{N})$, the unilateral left shift operator $(L\psi)_x \equiv \psi_{x+1}$ has Fredholm index

$$\mathrm{ind}(L) \equiv \dim \ker L - \dim \ker L^* = 1 - 0 = 1.$$

Indeed, the kernel of L is spanned by the state supported on the left most site, and since the adjoint is the right shift, it has no kernel. The non-zero Fredholm index is a signature of non-trivial topology. Conversely, the left shift operator on a finite truncation of $\ell^2(\mathbb{N})$, $\ell^2(\{1,\dots,L\}) = \mathbb{C}^L$ necessarily has zero Fredholm index, due to the rank-nullity theorem which guarantees that any finite square matrix has kernel and cokernel of the same dimension. Concretely one sees on the finite truncation, while the kernel of the left shift has the same spanning vector, the right shift now also has a kernel which is spanned by the right-most site, resulting in a $1 - 1 = 0$ index.

In the bulk, we assume that space is either continuous $\mathbb{R}^d$ or discrete $\mathbb{Z}^d$, and allow the particle to have N internal degrees of freedom $\mathbb{C}^N$ (spin, isospin, chirality, or whatever else), so that the appropriate Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$ or $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$. Since we are most interested in the tight-binding description, let us focus henceforth on the latter Hilbert space. As a first crude approximation, we assume the boundary of the material lies along the d-axis of space, so that for the edge a possible Hilbert space is $\mathcal{H}^\sharp := \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \otimes \mathbb{C}^N$ (we denote all edge objects by a sharp from now on). Using $\mathbb{N} \subseteq \mathbb{Z}$, we have a natural injection $\iota : \mathcal{H}^\sharp \hookrightarrow \mathcal{H}$ which is just extension by zero:

$$\iota(\psi)_x := \begin{cases} \psi_x, & (x_d \in \mathbb{N}) \\ 0, & (x_d \in \mathbb{Z} \setminus \mathbb{N}) \end{cases}.$$

The adjoint $\iota^* : \mathcal{H} \twoheadrightarrow \mathcal{H}^\sharp$ is restriction of a wave-function to the half-space, so that $|\iota|^2 \equiv \iota^*\iota = \mathbb{1}_{\mathcal{H}^\sharp}$ whereas $|\iota^*|^2 \equiv \iota\iota^* = \chi_\mathbb{N}(X_d)$, a projection in $\mathcal{H}$ associated to the half-space; hence ι is an isometry but not unitary, and ι^* is a partial isometry. We denote by ad_ι the induced action on $\mathcal{B}(\mathcal{H})$:

$$\mathrm{ad}_\iota(A) := \iota^* A \iota : \mathcal{H}^\sharp \to \mathcal{H}^\sharp.$$

This gives the truncation of A onto the half-space with *Dirichlet* boundary conditions. More general boundary conditions may be implemented by adding to $\mathrm{ad}_\iota(A)$ a term which decays in the d-direction, as appropriate for boundary effects. Thus we always assume that any truncated operator $A^\sharp$ and the Dirichlet-truncated operator $\mathrm{ad}_\iota(A)$ obey

$$\|(A^\sharp - \mathrm{ad}_\iota(A))_{xy}\| \leq C\,e^{-\mu(x_d + y_d)} \qquad (x, y \in \mathbb{Z}^{d-1} \times \mathbb{N}). \tag{1.9}$$

Schematically, let $\mathfrak{H}$ be the topological space of bulk Hamiltonians on the Hilbert space $\mathcal{H}$ that we want to classify. For instance, it can be the set of all spectrally gapped, local, self-adjoint bounded linear operators (see Definition 1.1) on $\ell^2(\mathbb{Z}^d)$ with the subspace topology from $\mathcal{B}(\ell^2(\mathbb{Z}^d))$. The edge space of Hamiltonians $\mathfrak{H}^\sharp$ is defined as

$$
\begin{array}{ccc}
\mathfrak{H} & \xrightarrow{\ (-)^{\sharp}\ } & \mathfrak{H}^{\sharp} \\
\mathcal{N} \downarrow & \circlearrowleft & \downarrow \mathcal{N}^{\sharp} \\
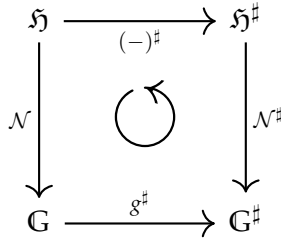\mathbb{G} & \xrightarrow{\ g^{\sharp}\ } & \mathbb{G}^{\sharp}
\end{array}
$$

FIGURE 1.4: The commutative diagram representing the BEC. In principle each choice of boundary conditions defines another diagram, which shows that the edge topological structure is insensitive to the choice of boundary conditions.

follows: A given point $B \in \mathcal{B}(\mathcal{H}^{\sharp})$ is in $\mathfrak{H}^{\sharp}$ iff it obeys Definition 1.1 *and* additionally there is some $A \in \mathfrak{H}$ such that $B - \mathrm{ad}_{\iota}(A)$ decays into the bulk, that is, obeys (1.9). This last point is subtle so let us elaborate: As was seen in the previous examples, usually the definition of $\mathfrak{H}$ will entail an insulator condition, e. g. that its elements have a spectral gap. Once we truncate such insulators into the half-space, they actually might become conductors (as in the archetypical IQHE). Thus the space of edge Hamiltonians is defined as those Hamiltonians which descend from an *insulating* bulk Hamiltonian. More generally there could be other bulk defining properties that define $\mathfrak{H}$ and do not survive the truncation; yet what defines a "topological" edge Hamiltonian is that there is some bulk one from which it descends. Hence one could say that $\mathfrak{H}^{\sharp}$ doesn't have an independent existence without $\mathfrak{H}$.

Now assume that we have a continuous topological invariant $\mathcal{N} : \mathfrak{H} \to \mathbb{G}$ for some discrete group $\mathbb{G}$, and *also* an analogous edge invariant $\mathcal{N}^{\sharp} : \mathfrak{H}^{\sharp} \to \mathbb{G}^{\sharp}$. If $\mathcal{N}, \mathcal{N}^{\sharp}$ ascend to *isomorphisms* on $\pi_0(\mathfrak{H}), \pi_0(\mathfrak{H}^{\sharp})$, (i. e. at the level of the PCCs), then we say we have a *complete* classification of the bulk and edge pictures. Then, the weakest form of the BEC is the statement that $\mathbb{G} \cong \mathbb{G}^{\sharp}$, which says that the two descriptions have the same set of PCCs. For the IQHE e. g. one obtains this rather quickly from a K-theoretic description using the six-term short exact sequence connecting the bulk and edge algebras [PS16a, Prop. 3.2.3], setting aside for the moment the fact that K-theory affords a somewhat coarser description than what one might mean by $\mathfrak{H}$ as explained above, and the fact it only works in the spectral gap regime. But one might conceive of other ways to prove that $\mathbb{G} = \mathbb{G}^{\sharp}$. Yet much more important is the numerical equality which roughly speaking says that truncation commutes with $\mathcal{N}$.

1.8 Principle (The BEC). *For every bulk Hamiltonian $A \in \mathfrak{H}$ and every $B \in \mathfrak{H}^{\sharp}$ such that A and B are compatible (that is, $B - \mathrm{ad}_{\iota}(A)$ decays into the bulk as in (1.9)) we have*

$$
\mathcal{N}(A) = \mathcal{N}^{\sharp}(B) \,. \tag{1.10}
$$

> *If $\sharp : A \mapsto A^\sharp$ is any map providing a truncation of $A \in \mathfrak{H}$ obeying (1.9), then one may write this equality as*
>
> $$g^\sharp \circ \mathcal{N} = \mathcal{N}^\sharp \circ (-)^\sharp, \tag{1.11}$$
>
> *where $g^\sharp : \mathbb{G} \to \mathbb{G}^\sharp$ is an isomorphism, i.e. the diagram in Figure 1.4 commutes.*

We note that such a result is the strongest type possible, since it formulates that we know both $\mathbb{G}, \mathbb{G}^\sharp$ and that $\mathcal{N}, \mathcal{N}^\sharp$ are isomorphisms at the level of the PCCs. Sometimes, however, especially in the mobility gap regime, we work with partial information and a BEC proof still is extremely insightful. For example, in [EGS05] the edge IQHE invariant is defined, however, the only way to know that $\mathcal{N}^\sharp$ is integer-valued (i.e. $\mathbb{G}^\sharp = \mathbb{Z}$) is via the BEC itself, and furthermore, it is not yet proven that $\mathcal{N}, \mathcal{N}^\sharp$ are isomorphisms on the PCCs, since the topologies of $\mathfrak{H}, \mathfrak{H}^\sharp$ have not yet been defined in the mobility gap regime.

Up to this work, in the mobility gap regime Principle 1.8 has only been proven for the IQHE in [EGS05], i.e. only for one cell in Table 1.1. In the spectral gap regime, within the framework of K- or KK-theory, Principle 1.8 has been proven for the *entire* Table 1.1 in [BKR17]; see also [Kub17].

> **1.9 Example.** As we have seen in Example 1.5, bulk chiral one dimensional spectrally gapped insulators can be specified by invertible local (Definition 1.1) operators $S \in \mathcal{B}(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N)$. Since the chiral symmetry acts within the fiber $\mathbb{C}^N$, the truncation is compatible with the block structure so that a compatible edge Hamiltonian for a given S is fixed by $\mathrm{ad}_\iota(S)$ on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ (with the choice of Dirichlet boundary conditions).
>
> We will see below in Lemma 3.16 that in the spectral gap regime, due to the locality of S (as in Definition 1.1), $\mathrm{ad}_\iota(S)$ is necessarily a Fredholm operator and it makes sense to define its invariant as the Fredholm index, $\mathcal{N}^\sharp := \mathrm{ind}$. The bulk topological invariant reduces to a winding number,
>
> $$\mathcal{N}(S) := \mathrm{tr}(U^*[\chi_\mathbb{N}(X), U]) \tag{1.12}$$
>
> where U is the polar part of S. Now the proof of Principle 1.8, in this case, follows by noting that $\mathcal{N}(S) = \mathrm{ind}(U^*\chi_\mathbb{N}(X)U, \chi_\mathbb{N}(X))$, the index of the pair of projections, which is equal to the Fredholm index of $\chi_\mathbb{N}(X)U\chi_\mathbb{N}(X)|_{\mathrm{im}\,\chi_\mathbb{N}(X)}$ as explained in [ASS94]. Their proof uses some algebra of projections and the Fedosov formula [Mur94], that is, the relation between ind and the trace formula:
>
> $$\mathrm{ind}(F) = \mathrm{tr}(FG - GF) \tag{1.13}$$
>
> where G is the parametrix (inverse up to compacts) of F. Since
>
> $$\mathrm{ind}(\chi_\mathbb{N}(X)U\chi_\mathbb{N}(X)|_{\mathrm{im}\,\chi_\mathbb{N}(X)}) = \mathrm{ind}(\mathrm{ad}_\iota(U)),$$

> the proof concludes after verifying that the truncated path between S and U is Fredholm due to the locality and gap condition, using the continuity of the Fredholm index.
>
> Hence here the crucial ingredient is (1.13), which is analogous to a Gauss-Bonnet theorem.

Since our focus is on strongly-disordered topological insulators, we proceed to discuss the theory of Anderson localization, which will allow us to refine the notion of an insulator from a spectral gap to *the mobility gap*.

1.4   LOCALIZATION

The theory of localization started with the ground breaking work of Anderson [And58]. Roughly speaking it says that if electrons are placed in a sufficiently disordered medium–neglecting electron-electron interactions–they will get "stuck" in confined regions rather than flow throughout space (compare this with translation-invariant media where Bloch theorem says that electrons are blind to the crystal structure and flow through it freely). One important consequence is that the DC electrical conductivity at the corresponding Fermi energy is zero, which means we should associate such materials with insulators. Mathematically the first proof of localization appeared in [FS84]; a simpler, different proof appeared in [AM93] which was further developed in [AG98], allowing for the understanding of the role of localization in the plateaus of the IQHE.

Anderson's strategy to understand a disordered material was to toss coins in order to generate a random potential, and make statements which hold *almost surely* with respect to the probability distribution of the coins or alternatively statements about expectations (w.r.t. disorder) of physical quantities. While an actual experiment is performed on *one single* material (and hence corresponding to a deterministic Hamiltonian), the individual macroscopic sample contains in itself many microscopic sub-samples, and hence the averaging. Indeed, the actual process with which disorder is formed in materials is likely described by some probability distribution (ultimately relating to a quantum stochastic process) and our probabilistic model is merely a (gross) simplification of the real one. Another philosophical justification for this approach is via Wigner's random matrix theory. It says that in the absence of better knowledge about the actual physical laws, we pretend the unknown part of the model is given by a collection of random variables. General physical principles (e.g. locality) will then give constraints on these random variables (e.g., their independence). For an introduction to random operators, see [AW15].

Let us describe the so-called Anderson model on which many results are proven. We pick a dimension $d \in \{1, 2, 3, \dots\}$ and the Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z}^d)$, which corresponds to a tight-binding description (appropriate for localization). Let μ be a given probability distribution on $\mathbb{R}$ obeying some regularity conditions. For instance, for some $\tau \in (0, 1]$, μ is a uniformly τ-Hölder continuous measure, as appeared in [AG98, Appendix B] or more recently in [AW15, Def. 2.2].

> **1.10 Definition** (uniform τ-Hölder continuity)**.** Let $\tau \in (0,1]$. The probability measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ is said to be uniformly τ-Hölder continuous iff there is some constant $C > 0$ such that for all intervals $J \subseteq \mathbb{R}$ with $|J| \le 1$ one has
>
> $$\mu(J) \le C|J|^{\tau}.$$
>
> Here $|J|$ is the Lebesgue measure of J.

The measure μ induces a measure $\mathbb{P}$ on "the disorder configuration space", the probability space $(\Omega := \mathbb{R}^{\mathbb{Z}^d}, \mathcal{A}, \mathbb{P})$, where $\mathcal{A}$ is the tensor-product sigma-algebra of the Borel sigma algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, and $\mathbb{P} := \bigotimes_{x \in \mathbb{Z}^d} \mu$, the product measure. Then to every $\omega \in \Omega$ a random multiplication operator V_ω is associated via

$$(V_\omega \psi)_x := \omega(x)\psi_x$$

for all $x \in \mathbb{Z}^d$, for all $\psi \in \mathcal{H}$.

We also have the deterministic discrete Laplacian given by

$$(-\Delta\psi)_x \equiv \sum_{e \in \mathbb{Z}^d : \|e\| = 1} \psi(x + e) \qquad (x \in \mathbb{Z}^d).$$

Its spectrum is absolutely continuous and is given by $\sigma(-\Delta) = [-2d, 2d]$. This is seen by using the (unitary) Fourier map $\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$ given by

$$\mathcal{F}(\psi) := \mathbb{T}^d \ni k \mapsto \sum_{x \in \mathbb{Z}^d} e^{-ik \cdot x} \psi_x. \tag{1.14}$$

Its inverse is $\mathcal{F}^{-1}(\psi) = \mathbb{Z}^d \ni x \mapsto \frac{1}{(2\pi)^d} \int_{k \in \mathbb{T}^d} e^{ik \cdot x} \psi(k) \, dk$. One finds that $\mathcal{F}(-\Delta)\mathcal{F}^*$ is a multiplication operator associated to the function $\mathbb{T}^d \ni k \mapsto 2\sum_{j=1}^d \cos(k_j)$.

> **1.11 Definition** (The Anderson model)**.** The Hamiltonian corresponding to the Anderson model is specified by a uniformly τ-Hölder continuous probability measure μ, a real number $\lambda \in \mathbb{R}$ representing the disorder strength and the space dimension d as follows:
>
> $$H_\omega(\lambda) := -\Delta + \lambda V_\omega. \tag{1.15}$$

We note that most statements below apply to Hamiltonians more general than (1.15). Some questions important for localization, probably regardless of the specific model, are:

1. The space dimension d. Localization seems to depend strongly on whether $d < 3$ or $d \ge 3$, as we shall explain below.

2. The presence of time-reversal symmetry (1.1). We note (1.15) obeys $[\Theta, H] = 0$, but the IQHE does not (it has a magnetic field).

3. At least in $d \ge 2$ on the lattice, some sort of regularity of the probability measure of the randomness. It is believed that this is an artifact of the mathematical tools we use and not the actual physics (contrast this with [BK05] for the continuum).

4. That the inter-dependence of the randomness of the Hamiltonian decays with distance. (1.15) has the i.i.d. property, namely, its randomness is only on the onsite potential, the values of which form a sequence of *independent identically distributed* random variables. Hence, the randomness at site x is independent of site y if $x \neq y$. In general we could also consider randomness in the hopping (as we do in Chapter 2) or even correlated randomness, and most of the results go through, if there is some sort of locality in the dependence of the randomness (this was already presented in [AM93]; see also [Kib05]).

We establish some *minimal* mathematical context [AW15; CL90] to which the Anderson model belongs.

1.12 Definition. Let $(\Sigma, \mathcal{F}, \nu)$ be a given probability space.

1. The measure-space morphism $T : \Sigma \to \Sigma$ is called *measure-preserving* iff $\nu(A) = \nu(T^{-1}(A))$ for all $A \in \mathcal{F}$, where $T^{-1}(A)$ is the pre-image of A under T.

2. Iff T is a measure-preserving morphism on $(\Sigma, \mathcal{F}, \nu)$, the tuple $(\Sigma, \mathcal{F}, \nu, T)$ is called a *measure-preserving dynamical system*.

3. Let G be a group, or monoid, or semigroup and assume T is a G-action. That is, $T : G \to \mathrm{Aut}(\Sigma)$ or $T : G \to \mathrm{End}(\Sigma)$ is a group, monoid or semigroup morphism (if G is a semigroup $\mathrm{End}(\Sigma)$ may be regarded as a semigroup too). Assume further that for all $g \in G$, T_g is measure-preserving. Then the tuple $(\Sigma, \mathcal{F}, \nu, T.)$ is called a *measure-preserving G-dynamical system*.

4. Let $X : \Sigma \to \mathbb{R}$ be a measure-space morphism (i.e. a *random variable*) on $(\Sigma, \mathcal{F}, \nu, T.)$. X is called *invariant* iff $X \circ T_g = X$ for all $g \in G$.

5. A measure-preserving G-dynamical system is called *ergodic* iff every invariant random variable is constant $\mathbb{P}$-almost-surely.

6. The weakly-measurably map $B : \Sigma \to \mathcal{B}(\mathcal{H})$ (i.e. a *random operator* on some Hilbert space $\mathcal{H}$) defined over the ergodic $(\Sigma, \mathcal{F}, \nu, T.)$ is itself *ergodic* iff for all $g \in G$ and for all $\sigma \in \Sigma$, $B(\sigma)$ and $B(T_g(\sigma))$ are unitary conjugates (the unitary may well depend on both σ and g).

Going back to the concrete example of (1.15), since $\mathbb{Z}^d$ may be considered as an additive group, we define $T : \mathbb{Z}^d \to \mathrm{Aut}(\Omega)$ as the lattice shifts: $T_x(\omega) := \omega(\cdot - x)$ for all $x \in \mathbb{Z}^d$. Such shifts are of course measure preserving as $\mathbb{P}$ is just a product of identical copies of μ. In fact this product structure of $\mathbb{P}$ means $(\Omega, \mathcal{A}, \mathbb{P}, T.)$ is an ergodic measure-preserving $\mathbb{Z}^d$-dynamical system (see [EW11, Prop. 2.15] for a sketch of the proof). Then the Hamiltonian for the Anderson model (1.15) is ergodic, with the unitary transformations U_x being the lattice translations in real space $(U_x \psi)_y \equiv \psi_{y-x}$. Then $(V_{T_x \omega} \psi)(y) = \omega(y - x)\psi_y = (U_x V_\omega U_x^* \psi)_y$; the discrete Laplacian has $[-\Delta, U_x] = 0$. The main point about the ergodic property is the fact that spatial averages coincide with random averages (sometimes one says that time averages coincide with random

averages, but for random Schrödinger operators we have space and disorder average). This is stated in the following theorem whose proof may be found in [EW11, Thm. 2.30]:

1.13 Theorem (Birkhoff)**.** *If $(\Sigma, \mathcal{F}, \nu, T.)$ is an ergodic measure-preserving $\mathbb{Z}^d$-dynamical system and $X \in L^1(\Omega, \nu)$ then the following limit exists for ν-almost-all $\sigma \in \Sigma$*

$$\lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{x\in\mathbb{Z}^d:\|x\|_1\leq n} X(T_x\sigma) = \mathbb{E}[X].$$

One very fruitful consequence of the ergodicity of (1.15) is given by Pastur's theorem [Pas80] about the almost-sure spectrum, whose proof may be found in [AW15, Theorem 3.10].

1.14 Theorem (Pastur)**.** *If $(\Sigma, \mathcal{F}, \nu, T.)$ is an ergodic measure-preserving $\mathbb{Z}^d$-dynamical system and $B : \Sigma \to \mathcal{B}(\mathcal{H})$ is an ergodic random self-adjoint operator then there are subsets $s(B), s_{\mathrm{pp}}(B), s_{\mathrm{sc}}(B), s_{\mathrm{ac}}(B) \subseteq \mathbb{R}$ such that for ν-almost-all ξ, we have*

$$\sigma_\sharp(B(\xi)) = s_\sharp(B)$$

with $\sharp \in \{ \ , \mathrm{pp}, \mathrm{sc}, \mathrm{ac} \}$. The sets $s_\sharp$ are called the almost sure spectra of B.

For H of (1.15) one can actually say more, in fact, we have a concrete picture of $s(H(\lambda))$ which stems from the fact $(\omega_x)_{x\in\mathbb{Z}^d}$ is an independent identically distributed sequence of random variables. This was first proven in [KS80].

1.15 Theorem (Kunz/Souillard)**.** $\sigma(H(\lambda)) = [-2d, 2d] + \lambda \operatorname{supp}(\mu)$.

While this statement holds for the whole almost-sure spectrum $s(H(\lambda))$, there is no analogous theorem about the various sub-types of the spectrum. Indeed, the existence of $s_{ac}(H(\lambda))$ in $d \geq 3$ for sufficiently small $\lambda > 0$ is a major open problem in mathematical physics.

The distinction between various spectral types of the almost-sure spectrum is relevant to us due to the following theorem by Ruelle, Amrein, Georgescu and Enss [AG74; Ens78; Rue69], which connects the dynamics of a system with the measure-theoretic spectral types.

1.16 Theorem (RAGE)**.** *Let A be a self-adjoint operator on some Hilbert space $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_c$ where $\mathcal{H}_\sharp \equiv \left\{ \varphi \in \mathcal{H} \ \middle| \ \nu_{(A,\varphi)} \text{ is } \sharp \right\}$; $\nu_{(A,\varphi)} \equiv \langle \varphi, \chi.(A)\varphi \rangle$ is the spectral-measure of (A, φ) and $\sharp \in \{ \mathrm{pp}, \mathrm{c} \}$ is the decomposition of the measure according to the Lebesgue-Radon-Nikodym theorem [Rud87, p. 6.10]. Then one may characterize $\mathcal{H}_\sharp$ also via*

$$\mathcal{H}_{\mathrm{pp}} = \left\{ \varphi \in \mathcal{H} \ \middle| \ \lim_{R\to\infty} \sup_{t\geq 0} \|\chi_{B_R(0)}(X)^\perp \exp(\pm \mathrm{i}\, tA)\varphi\| = 0 \right\}$$

$$\mathcal{H}_{\mathrm{c}} = \left\{ \varphi \in \mathcal{H} \ \middle| \ \forall R > 0, \ \lim_{t\to\infty} \frac{1}{t} \int_0^t \|\chi_{B_R(0)}(X) \exp(\pm \mathrm{i}\, sA)\varphi\|^2 \mathrm{d}\, s = 0 \right\}$$

> *where X is the position operator and $B_R(0) \subseteq \mathbb{R}^d$ is the ball with radius R about the origin.*

See Theorem A.54 for an analogous statement for unitary operators. Thanks to this theorem, it makes sense to make the following definition.

> **1.17 Definition** (Spectral localization)**.** We say that H is spectrally-localized in an energy interval $(a,b) \subseteq \mathbb{R}$ iff
>
> $$\sigma(H) \cap (a,b) = \sigma_{pp}(H) \cap (a,b).$$

Yet another (stronger) form of localization is a statement regarding absence of diffusion. For a given Hamiltonian H define the second moment of the position operator as

$$\mathcal{M}_{ij}(t) := \langle \delta_0, \mathrm{e}^{\mathrm{i}tH} X_i X_j \, \mathrm{e}^{-\mathrm{i}tH} \delta_0 \rangle \qquad (t \in \mathbb{R}, i, j \in \{1, \ldots, d\}).$$

Here δ_0 is chosen arbitrarily since we expect the discussion not to depend on this choice. If H is a random operator, such as the Anderson model, we rather define

$$\mathcal{M}_{ij}(t) := \mathbb{E}[\langle \delta_0, \mathrm{e}^{\mathrm{i}tH} X_i X_j \, \mathrm{e}^{-\mathrm{i}tH} \delta_0 \rangle] \qquad (t \in \mathbb{R}, i, j \in \{1, \ldots, d\}).$$

Note that sometimes it is preferable to define these moments with a time average $\tilde{\mathcal{M}}_{ij}(t) := \frac{1}{t} \int_0^t \mathcal{M}_{ij}(t') \, \mathrm{d}t'$, see e.g. [JSS03]. We then define for each $\alpha > 0$ the limit, $m_{ij}(\alpha) := \limsup_{t \to \infty} \frac{1}{t^\alpha} \mathcal{M}_{ij}(t)$.

> **1.18 Definition.** *Ballistic motion* means
>
> $$\exists m_{ij}(2) \in (0, \infty), \tag{1.16}$$
>
> whereas *diffusive motion* means
>
> $$\exists D_{ij} := \frac{1}{2} m_{ij}(1) \in (0, \infty). \tag{1.17}$$
>
> D_{ij} is called *the diffusion coefficient*.
>   One also defines the diffusion coefficient corresponding to a state ψ with $\|\psi\| = 1$, $D_{ij}(\psi)$ by replacing δ_0 with ψ in the above formula for $\mathcal{M}_{ij}(t)$. For some $E \in \mathbb{R}$, if we then take ψ to be a wave-packet with energy concentrated about E, $\psi = \varphi(H)\delta_0$, $\varphi : \mathbb{R} \to \mathbb{R}$ a bump function concentrated about E such that $\|\psi\| = 1$, then we can define the diffusion coefficient corresponding to an energy E:
>
> $$D(E) := \limsup_{\mathrm{supp}(\varphi) \to \{E\}} \frac{1}{d} \sum_{i=1}^{d} D_{ii}(\psi).$$
>
> where $\limsup$ is over all sequences of functions with supports shrinking on E.

With these notions, a stronger definition of what localization is could be the following.

1.19 Definition (Absence of diffusion). We say that H is localized in the sense that it has absence of diffusion in an energy interval $(a, b) \subseteq \mathbb{R}$ iff

$$D(E) = 0 \qquad \forall E \in (a, b), \, i, j \in \{1, \ldots, d\} \, .$$

This being stronger due to the fact that for reasonable models it would imply pure point spectrum (though in principle it is possible to have vanishing diffusion exponent, but singular continuous spectrum of dimension zero, see [Sim96] and references within).

We may also connect this to the DC conductivity by the Einstein relation, for which we need to define the density of states.

1.20 Definition.   1. The *mean local spectral measure* ν of the random operator H with translation-invariant probability distribution is given by

$$\nu := \mathbb{E}[\langle \delta_0, \chi. (H) \delta_0 \rangle] \, . \tag{1.18}$$

Here the choice of zero was arbitrary (any other choice should give the same measure) due to the fact that the measure of the potential $\mathbb{P}$ is translation-invariant.

2. The *integrated density of states* is a function $n : \mathbb{R} \to [0, \infty)$ given by $n(E) := \nu((-\infty, E))$.

3. If ν is absolutely-continuous w.r.t. the Lebesgue measure $\mathcal{L}$, its Radon-Nikodym derivative is defined as the *local density of states* $\rho := \frac{\mathrm{d}\nu}{\mathrm{d}\mathcal{L}}$. We have then $n(E) = \int_{-\infty}^{E} \rho(E) \, \mathrm{d}\mathcal{L}(E)$.

The Einstein relation is

$$D(E)\rho(E) = \frac{1}{d} \sum_{i=1}^{d} \sigma_{ii}(E) \, , \tag{1.19}$$

where $\sigma_{ij}(E)$ is the DC conductivity as given by the Kubo formula. We note that the existence of ρ is guaranteed by a Wegner estimate [AW15, Theorem 4.1]. The regularity of ν should not be confused with a statement about the almost-sure absolute-continuous spectrum, $s_{ac}(H) \neq \varnothing$. ν is defined *with* disorder averaging, so it is not the spectral measure of the tuple (H, δ_0).

In [Pas80] a relation between the DC conductivity $\sigma_{ij}(E)$ and the Green's function is derived (starting from the definition of $\sigma_{ij}(E)$ via Kubo's linear response formula to an electric field) as

$$\sigma_{ij}(E) = \lim_{\eta \to 0^+} \frac{\eta^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \mathbb{E}[\|G(x, 0; E + \mathrm{i}\eta)\|^2] \tag{1.20}$$

where $G(x, y; z) \equiv \langle \delta_x, (H - z\mathbb{1})^{-1} \delta_y \rangle$ and to obtain this relation one has to assume that the conductivity measure, as defined by Pastur [Pas80], has a density, which

is associated with diffusion or localized motion (i.e. not ballistic). With (1.19), Definition 1.19 becomes a statement about the DC conductivity of the material, that is, localization in this sense implies *electric insulator*. We note in passing that in [KLM07; KM14] Mott's formula and the AC conductivity of the Anderson model has been studied.

There are more (inequivalent) ways to characterize localization (for example via inverse participation ratio, or via the statistics of the eigenvalues: the theory of random matrices distinguishes between *GOE* and *Poisson* statistics; localization corresponds to the latter) but let us jump directly to what is possibly the strongest criterion of localization, which readily implies most others [AG98; Gra94]:

1.21 Definition (The fractional moments condition)**.** The Hamiltonian H is said to obey *the fractional moments condition* (FMC henceforth) in an energy interval $(a, b) \subseteq \mathbb{R}$ iff for any $E \in (a, b)$ there is a fraction $s_E \in (0, 1)$ and constants $C_E < \infty, \mu_E > 0$ such that

$$\mathbb{E}[\|G(x, y; E + \mathrm{i}\eta)\|^s] \leq C_E \, \mathrm{e}^{-\mu_E \|x-y\|} \qquad (x, y \in \mathbb{Z}^d, \eta > 0). \qquad (1.21)$$

In many reasonable models this condition implies the *second moment condition*, which holds iff for any $E \in (a, b)$, there are constants $C_E < \infty, \mu_E > 0$ such that

$$\eta \mathbb{E}[\|G(x, y; E + \mathrm{i}\eta)\|^2] \leq C_E \, \mathrm{e}^{-\mu_E \|x-y\|} \qquad (x, y \in \mathbb{Z}^d, \eta > 0). \qquad (1.22)$$

The implication (1.21) to (1.22) relies on finite rank perturbation theory of the particular model at hand, whereas the converse is not known to be true in a model-agnostic way. The immediate importance of the FMC for topological insulators is its following corollary [AG98, Theorem 2]:

1.22 Proposition. *If the FMC holds for H in an energy interval (a, b) then for all $E_F \in (a, b)$ we have*

$$\mathbb{E}[\|\chi_{(-\infty, E_F)}(H)_{xy}\|] \leq C \, \mathrm{e}^{-\mu \|x-y\|} \qquad (x, y \in \mathbb{Z}^d).$$

It is the second moment condition which is important for dynamical localization.

1.23 Proposition. *If the second moment condition holds for H in an energy interval (a, b) then*

$$\sup_{f \in B_1((a,b))} \mathbb{E}[\|f(H)_{xy}\|] \leq C \, \mathrm{e}^{-\mu \|x-y\|} \qquad (x, y \in \mathbb{Z}^d)$$

where $B_1((a, b))$ is the set of Borel bounded functions $|f| \leq 1$ and which are constant on both $\mathbb{R}_{<a}$ and $\mathbb{R}_{>b}$.

This implies *dynamical* localization by taking $f_t = \exp(\pm \mathrm{i}\,t\cdot)\chi_{(a,b)}$ and the supremum then ranges over all $t > 0$, implying that wave-packets with energy in (a, b) do not spread in space. We note in passing that either the second moment condition or the
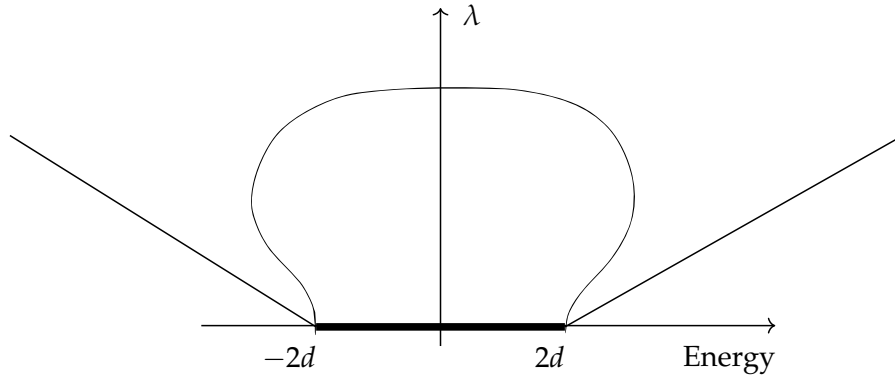
FIGURE 1.5: The phase diagram of the Anderson model (1.15) for $d > 2$ as shown in [AW15, pp. 4]. The thick line denotes $\sigma(-\Delta)$. The area within the bubble is expected to be *delocalized*, with ac-spectrum, whereas the area outside of it is (proven to be) localized. The two diagonal arms denote where the spectrum starts and ends according to Theorem 1.15, so that for a fixed $\lambda > 0$, we start with a localized region, called *the mobility edge*, enter into a conjectured region of continuous spectrum, and then encounter another mobility edge before the spectrum ends.

FMC imply that the DC conductivity is zero within the interval of localization [Gra94, Corollary 4] or [AG98, (E.1)].

Now that we have some mathematical definitions for localization, let us discuss the proofs of it. There are two main mathematical techniques to establish localization for general d: the multi-scale analysis (MSA henceforth) launched in [FS84] was the first rigorous proof, and the fractional moments method (FMM) of [AM93] greatly simplified the path to localization. We will mostly concentrate on the latter, though ultimately both seem to imply equally powerful results as far as what we need for topological study. Indeed, the first proof of the implication from localization to Propositions 1.22 and 1.23, which are the main prerequisite for us, was presented in [AG98] via the FMM; it was later shown to be obtained also via MSA in [DS01], which goes back to [GD98].

So far, Definition 1.21 (1.21) has been proven for (1.15) in the following cases. First, for general d, μ sufficiently regular as above, and one of the following is true:

1. λ is sufficiently large, in which case the entire real axis is localized, or if the probed energy is sufficiently large or small (with $\lambda > 0$ arbitrarily small) [AM93].

2. $\lambda > 0$ is arbitrarily small with the energy probed being arbitrarily near to the absolutely-continuous bands of the unperturbed, deterministic Hamiltonian [Aiz94].

This is summarized in Figure 1.5. The FMC proofs crucially rely on finite rank perturbation theory in order to bound $\mathbb{E}[\|G(x, x; z)\|^s]$ by the disorder average of just one site (site x). This is the so-called *a-priori* bound.

The situation in $d = 1$ (or more generally the strip) for (1.15) is that *any* $\lambda > 0$ brings about complete localization of the entire spectrum, even without μ being regular. This is proven using the MSA in [KLS90]; in this work we finally give a proof (that *does*

rely on regularity of μ–this is crucial for the a-priori bound) via the FMC. Since the mechanism of localization in one dimension is quite special it deserves more words of introduction, but we postpone them to Chapter 2. In $d = 2$ it is mathematically *conjectured* that one has complete localization at all non-zero disorder strengths and all energies based on the physics work [Abr+79].

We note in passing that localization of unitary operators, which is relevant for Chapter 4, is somewhat analogous to the analysis of (1.15), and has been studied in [AM10; HJS09].

For topological insulators, it is important to study localization also for *block* operators (i.e. with orbital degrees of freedom), which is one of the main focuses of Chapter 2 for our chiral model. However this question in general has already been addressed previously [GM13; KMM11], see also [ESS14] using the FMM in the strong coupling regime, or in the weak coupling regime [DDS15].

Finally it has to be mentioned that the fact that for sufficiently small values of λ at $d > 3$, $D_{ij}(E)$ should be finite and strictly positive remains unproven.

We go on to discuss how all of this is relevant to topological insulators.

1.5 THE DETERMINISTIC MOBILITY GAP CONDITION

While there is a systematic way within the framework of K-theory to come up with all topological invariants, namely, using K-homology [BKR17], since this formalism doesn't quite lend itself to the mobility gap situation we are about to describe, we rather stick a poor man's version of it. Within that, let us define

> **1.24 Definition** (Switch function). A switch function [EG02] is any function $\Lambda : \mathbb{Z} \to \mathbb{R}$ such that $\Lambda(n) = 1$ (resp. $= 0$) for n (resp. $-n$) large and positive. We will usually apply this function to the position operator in some direction, $\Lambda(X_i)$, in which case we abbreviate this operator as Λ_i. It defines a (multiplication) operator on $\ell^2(\mathbb{Z}^d)$, which carries naturally to its descendant spaces $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$.

> **1.25 Definition** (Non-commutative derivative and integral). We define the non-commutative derivative in the jth direction, denoted by ∂_j, as $-\mathrm{i}[\Lambda_j, \cdot]$. It satisfies the Leibniz rule. The non-commutative integral is the trace, which obeys $\operatorname{tr} \partial_j A = 0$ if $\partial_j A$ is trace-class.

The reason for this notation is that in the translation invariant case, this derivative becomes the actual derivative in the jth direction with respect to the quasi-momentum (i.e. $\mathcal{F}(-\mathrm{i}[X_j, A])\mathcal{F}^{-1} = M_{\partial_j a}$ for any periodic operator $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ with signature $a : \mathbb{T}^d \to \mathbb{C}$, where M_a is the multiplication operator corresponding to the symbol a), and the trace becomes integration over the Brillouin zone divided by $(2\pi)^d$. In principle the notation for the non-commutative derivative ∂_j should contain Λ as well, but we will see later that the topological invariants do not depend on the choice of Λ and so it makes sense to omit it from the notation.

Schematically, all topological invariants, that is, all conceivable maps $\mathcal{N} : \mathfrak{H} \to \mathbb{G}$ are probably expressible in terms of traces of combinations of *Fermi projections P* of a

given Hamiltonian, and their non-commutative derivatives. As we mentioned already, the more systematic approach to this statement could be found e. g. in the textbook [PS16a]. Here we content ourselves with simply noting the following three cases:

1. In the chiral 1D case, the chiral invariant (1.12) can be written as $\mathcal{N} = 2 \operatorname{tr} \Pi P \partial P$ where $\Pi \equiv \mathbb{1} \oplus (-\mathbb{1})$ is the chiral symmetry operator.

2. In the IQHE, the Chern number may be written as

$$\mathcal{N} = 2\pi \, \mathrm{i} \operatorname{tr} P[\partial_1 P, \partial_2 P] \,.$$

3. In the time-reversal invariant case discussed in Example 1.3,
   $\mathcal{N} = \dim \ker(PUP + P^\perp) \mod 2$ with U defined there. We note that this is *not* a trace formula and we shall comment on it later.

In these three examples, $P \equiv \chi_{(-\infty, E_F)}(H)$ with E_F being the Fermi energy (in the chiral case it must be set to zero). We will see in Corollary A.20 that to prove the expressions in the traces are trace-class (and additionally for $PUP + P^\perp$ to have a parametrix, which guarantees the corresponding Θ-odd Fredholm index is well-defined; see [AG98]) it is sufficient to have some sort of off-diagonal decay of P, that is, $\|P_{xy}\|$ should decay in $\|x - y\|$ sufficiently fast. The intuitive reason is that the off-diagonal decay of an operator A implies that $\partial_j A$ is "relevant" in an infinite cylinder parallel to the j axis about the origin. This property is stable under multiplication, so that $(\partial_1 A) \ldots (\partial_d A)$ is "relevant" mostly in a finite box about the origin, and is hence trace-class.

Assume for a moment that H has a *spectral* gap within the interval $(a, b) \subseteq \mathbb{R}$, that is $\sigma(H) \cap (a, b) = \varnothing$. Then P may be written as a Riesz projection, that is, as an integral over $R(z) \equiv (H - z\mathbb{1})^{-1}$ with z always a finite distance from $\sigma(H)$:

$$P = \frac{\mathrm{i}}{2\pi} \int_\Gamma R(z) \, \mathrm{d}z$$

where Γ is some counter encircling $(-\|H\|, E_F)$ in $\mathbb{C}$. Then the Combes-Thomas estimate Proposition 2.35 guarantees the off-diagonal exponential decay of the matrix elements of $R(z)$, and so the topological invariants are always well-defined in the spectral gap regime, as long as $E_F \in (a, b)$.

The mobility gap is (morally) defined as having *localized* spectrum rather than no spectrum at all, see Figure 1.2. Now there is a philosophical point: as we began our discussion of localization, we granted that we'll need to use randomness in order to be able to make any useful statements. Otherwise, there is simply no grasp on the spatial structure of the model. To study the topology of disordered media, the Bellissard school [BvS94] approach is to take the randomness into account from the beginning and model the topology on top of it. Indeed, the C-star algebra of operators which are generated by magnetic shifts is tensored with $C(\Omega)$, continuous functions on the disorder configuration space, in a covariant way, and the main object of study are thus covariant operators [PS16a]. Topological statements are thus made about ensembles of *random* operators, and results such as Principle 1.8 are proven about expectations of quantities, which however by Theorem 1.13 become averages in space.

While there is probably no dispute that the covariant structure is necessary for localization statements, for topological questions, one might be able to get away without it. Indeed, for *strongly* disordered media, it is insufficient. In [EGS05] the authors proceed without the assumption of translation invariance: neither actual, nor in an ambient probability space. Their (fixed) operator of study is one deterministic realization of a localized ensemble and all statements are made about that one fixed realization without further recourse to probability theory or ergodicity. What *is* assumed then, are almost sure consequence of localization, and so within the ensemble of random operators which obey localization, one focuses on a subset which arises with probability one, and picks *one single* Hamiltonian of this subset.

Because one has to go from conditions about expectations of random variables to almost-sure statements, some of the decay estimates become worse. For instance, we have

1.26 Proposition. *If $\mathbb{E}[\|\chi_{(-\infty,E_F)}(H)_{xy}\|]$ is exponentially decaying in $\|x-y\|$ as in Proposition 1.22 then almost-surely, there are (deterministic) constants $\mu > 0, \nu > d+1$ and a random constant $C < \infty$ such that*

$$\|\chi_{(-\infty,E_F)}(H)_{xy}\| \leq C\,e^{-\mu\|x-y\|}(1 + \|x\|)^{+\nu}. \tag{1.23}$$

This is a special case of Proposition A.1. What is important to observe in this formula is that the off-diagonal decay is non-uniform: there is also the possibility of diagonal explosion.

Hence what defines a *deterministic* insulator are conditions which are true with probability one for *random* insulators. We then take these conditions and plug them into our topological analysis, which results (so far) in *partial* reproduction of the achievements of K-theory.

In addition to the decay of P, two additional almost-sure properties of random Schrödinger operators are the fact that any fixed value of energy is almost-surely not an eigenvalue [AW15], as well as the fact that all eigenvalues within the localized region are almost-surely of finite degeneracy [Sim94].

We finally come to the deterministic condition for the mobility gap:

1.27 Definition (Mobility gap)**.** The (deterministic) Hamiltonian H is said to be mobility gapped about the Fermi energy $E_F \in \mathbb{R}$ iff there is an open interval $(a,b) \ni E_F$ such that all three of the following conditions hold:

1. $\sigma(H) \cap (a,b) = \sigma_{\text{pp}}(H) \cap (a,b)$ and all eigenvalues of H within (a,b) are of finite degeneracy.

2. E_F is not an eigenvalue of H.

3. $\|\chi_{(-\infty,E_F)}(H)_{xy}\|$ obeys (1.23).

Sometimes it is necessary to require a stronger condition as the mobility gap criterion. This comes from using Proposition 1.23 instead of Proposition 1.22, and results in

1.28 Definition (Dynamical mobility gap). The (deterministic) Hamiltonian H is said to be dynamically mobility gapped about the Fermi energy $E_F \in \mathbb{R}$ iff there is an open interval $(a, b) \ni E_F$ such that all three of the following conditions hold:

1. All eigenvalues of H within (a, b) are of finite degeneracy.

2. E_F is not an eigenvalue of H.

3. $\sup_{f \in B_1((a,b))} \|f(H)_{xy}\|$ obeys (1.23), where $B_1((a, b))$ is the set of Borel bounded functions $|f| \leq 1$ which are constant out of (a, b) (with possibly different constants above and below).

Of course, Definition 1.27 is a special case of Definition 1.28 as $\chi_{(-\infty, E_F)} \in B_1((a, b))$. In addition, of course a bona fide spectral gap implies both these definitions via the Combes-Thomas estimate. See also the appropriate definition for Floquet systems Definition 4.4, which is modeled after Definition 1.28.

In conclusion, we expect any operator that obeys Definition 1.27 to have some sort of well-defined topological structure associated with the gap (a, b). We see how the spectral gap condition is just a special case of this definition. The failure of (1.23), or perhaps more generally, of the integrand in a trace formula defining $\mathcal{N}$ to cease being trace class, is a signature of a topological phase transition in the mobility gap regime.

The entire discussion so far focused on bulk systems, which are bona fide insulating. Edge systems will, as a rule, not obey conditions analogous to Definition 1.28, as they are expected to be conductors (in non-trivial topological phases). The edge situation in $d = 1$ for chiral systems is somewhat special, see Section 3.2.2. The point is that the edge is a compact perturbation of the bulk in $d = 1$ (unlike $d \geq 2$) so that (morally) it doesn't modify localization properties.

1.6 ORGANIZATION

This work is divided into three main parts, corresponding to Chapters 2 to 4. Chapters 2 and 3 focus on a typical one dimensional model of class AIII from Table 1.1. The goal here was to study the simplest mobility gap example of some non-standard symmetry as an extension to the IQHE analysis of [EGS05].

In Chapter 2, we study its localization properties for a given random ensemble. The proof here turns out to have some surprises compared to the Anderson model in the same dimension, namely, localization at zero energy is parameter-dependent rather than automatic (as opposed to all non-zero energies where one has the same behavior as the Anderson model). The main result here is a proof of the property stated in Definition 1.21 at all non-zero energies, however, there are several other results of independent interest along the way, for instance, showing the off-diagonal decay of the fractional moments of the Green's functions for the Anderson model on the strip

(so, finally using the FMM rather than the MSA), or extending that decay from real to complex energies by subharmonicity Proposition 2.36, and an implication that any decay implies exponential decay for our model Lemma 2.37.

In Chapter 3 we turn to a *topological* study of the very same model. Here we finally elaborate more on the meaning of the chiral symmetry and its implications, and explain why this nearest-neighbor model is actually rather generic by re-dimerization. We go on to formulate our main result, which is a proof of Principle 1.8 for such systems *in the mobility gap regime*, essentially the only other mobility gap BEC proof after the IQHE proof of [EGS05]. We give also another proof (independent of K-theory) of the BEC in the spectral gap regime which uses the Fredholm properties, as well as elaborate on the translation invariant case. We finish by discussing the extension to more general boundary conditions.

In the middle of this chapter, in Section 3.2.2, there lies the following result: a direct relationship between localization and topology for this chiral one-dimensional model. The connection is brought by formulating the edge invariant as the number of negative Lyapunov exponents of the zero energy Schrödinger equation. Since localization (morally, but also as proven in Chapter 2) holds as long as none of these exponents are zero, this gives a direct description of topological phase transitions in the mobility gap regime, namely, whenever one of the exponents passes through zero, which is precisely when the phase is not well-defined. Of course also in the IQHE there was the connection between localization and topology, in the sense that the topological phases are only well-defined as long as the Fermi energy is placed within the region of localization, but here this relationship is even quantitative in a sense.

In Chapter 4 we change gears from Table 1.1 into Floquet topological systems, analyzing their topological properties in $d = 2$ without any symmetries, again, in the strongly-disordered, mobility gap regime. After verifying that the usual relative evolution scheme goes through for the mobility gap as well, and proving a version of Principle 1.8, we find that in certain systems where the entire circle is a mobility gap, the relative evolution seems to not be necessary at all, as its contribution to the winding is zero. This is related to the magnetization studied in physics with connection to such systems (see references in Chapter 4). We conclude by showing that *any* mobility gapped systems can be reduced to the previous situation where the entire circle is a mobility gap, essentially by stretching the gap onto the entire circle. The invariants from this stretched operator are proven to be equal to the invariants associated with the relative evolution, which goes via a certain continuity result in the mobility gap, perhaps of independent interest.

Appendix A contains many of the technical lemmas referred to in the main text, although there are some sections that could stand alone. One in particular is Appendix A.2 where locality and (deterministic) localization properties are studied from algebraic and (very briefly) analytic perspectives. The main result here is that if the matrix elements of an operator A have (in some weak sense) off-diagonal decay then $(\partial_1 A)\ldots\partial_d A$ is trace-class, which means that the topological invariants associated to A are well-defined. This is of course not new and goes back to [EGS05], the point here was rather to phrase it in a convenient algebraic language, whence the aforementioned

main result really can be stated as a simple Corollary A.20. There is a unitary version of the Helffer-Sjöstrand formula which helps us to make locality statements about the smooth functional calculus of local normal operators. We include a short discussion of some properties of the Hermitian symplectic group Appendix A.5.1 necessary for Chapter 2. The last section Appendix A.9 contains a unitary RAGE theorem.

COMPLETE LOCALIZATION FOR DISORDERED CHIRAL CHAINS

2.1 INTRODUCTION

The Anderson model in one-dimension was long known [KS80] to exhibit *complete* localization, that is, localization regardless of the strength of the disorder or the energy at which the system is probed (cf. with Figure 1.5). The latest advancement in this direction was made in [KLS90], which handles the strip with singular distributions for the onsite randomness. However, as it turns out, additional constraints on the randomness can make *complete* localization fail at some special energy values or ranges, as was demonstrated for the random polymer model [JSS03].

Such special energy values are intimately linked to the existence of rich topological phases. Indeed, this localization question emerged from our study of chiral topological insulators in 1D which we'll present in Chapter 3. A link will be realized between the failure of localization at zero energy and topological phase transitions. This is precisely what makes the Anderson model in 1D topologically trivial–it cannot have any phase transitions, being always completely localized. Thus our main goal now is to show that indeed Chapter 3 does not involve an empty set of models since its assumptions are fulfilled with a probability of either zero or one, as we shall show.

The chiral model dealt with in the present study–a disordered analog of the SSH model [SSH79]–is characterized by having no on-site potential and *alternating* distributions of the nearest-neighbor hopping. It exhibits complete dynamical localization at all non-zero energies (in the sense of the FMC Definition 1.21), as long as the aforementioned distributions are regular enough to prove an a-priori bound and have moments to define Lyapunov exponents. If the Lyapunov spectrum of the system at zero energy does not contain zero, then this localization holds also at zero energy. As noted above if the spectrum does contain zero, then the system could exhibit a topological phase shift, so that it makes sense that localization should fail then (as the topological indices are only defined in the localization regime). It should be noted that [Mon+14] already calculated analytically the Lyapunov spectrum for a particular model and explored the phase transitions numerically.

There is also an independent interest in using the FMM for localization proofs rather than the MSA, since its consequences for dynamical localization seem easier to establish. For this reason it is interesting to note that the 1D Anderson model on the strip can also be endowed with a localization proof using the FMM via our methods (cf. [KLS90]). The two main differences between the Anderson model and our chiral model (which has disorder only in the kinetic energy) are the a-priori bounds and the particular form of transfer matrices, which has consequences on how one applies Furstenberg's theorem, i.e., how to prove irreducibility. In that regard it should be noted that [CS15] already deals with a similar case of disordered hopping, but stays

within the MSA framework of [KLS90]. Studies of the Lyapunov spectrum for various symmetry classes have also been conducted in [ABJ10; LSS13; SS10] among others.

This chapter is organized as follows. After defining the model and formulating our main result, we go on to discuss the transfer matrices, and how a product of n of them is (morally) the Green's function betwen 0 and n, Lemma 2.12. This result together with the Anderson model irreducibility proof from [GM89] and the usual a-priori bound resting on rank-1 perturbation theory implies the FMC localization in one dimension for the *usual* Anderson model. We go on to discuss irreducibility Proposition 2.18 for *our* model, which rests on our different form of the transfer matrices. This implies that at all non-zero energies the Lyapunov spectrum is simple via Furstenberg's theorem. We go on to prove an a-priori bound Proposition 2.29 between the x and $x - 1$ entries of the resolvent, which uses rank *two* perturbation theory (perhaps of independent interest). Since the irreducibility result relies heavily on the Hermitian symplectic structure (Appendix A.5.1) of the transfer matrices, which is only true at *real* energies, this implies the decay of Green's functions at finite volume and real energies (Theorem 2.32). We extend this result to complex energies via the Combes-Thomas estimate using crucially the sub-harmonicity of the Green's function (in the energy variable), which allows us to go to infinite volume, however, with polynomial rather than exponential decay in the distance. For our particular model, though, interestingly, *any* decay implies exponential decay by a sort of decoupling lemma (Lemma 2.37). We finally conclude by analyzing the zero energy situation by an explicit solution at finite volumes.

2.2 THE MODEL AND THE RESULTS

Let $N \in \mathbb{N}_{\geq 1}$ be given, and let α_0, α_1 be two given probability distributions on $GL_N(\mathbb{C})$. We assume for both $i = 1, 2$ the following:

2.1 Assumption. *(Fatness)* supp α_i *contains some subset* $U_i \in Open(GL_N(\mathbb{C}))$.

2.2 Assumption. *(Uniform τ-Hölder continuity) For any $k, l \in \{1, \ldots, N\}$, let $\mu_{R,I}$ be the conditional probability distributions on $\mathbb{R}$ obtained by "wiggling" only the real, respectively imaginary part of $M_{k,l}$, that is,*

$$\mu_{R,I} = \alpha_i(\cdot | (M_{k',l'})_{k',l' \neq k,l}, (M_{kl})_{I,R}).$$

Then we assume that for some $\tau \in (0,1]$, $\mu_{R,I}$ is a uniformly τ-Hölder continuous measure as in Definition 1.10.

2.3 Assumption. *(Regularity) α_i has finite moments in the following sense:*

$$\int_{M \in GL_N(\mathbb{C})} \|M^{\pm 1}\|^2 \, \mathrm{d}\, \alpha_i(M) < \infty. \tag{2.1}$$

2.4 REMARK. These assumptions are not optimal for the proof of localization that shall follow (cf. [KLS90]), and were chosen as a good middle way optimizing simple proofs and strong results. It still need not be that α_i has a density with respect to the Lebesgue measure.

We define a random sequence of independent *alternatingly*-distributed matrices $(T_n)_n$ such that all its even members follow the law of α_0 and all its odd members follow the law of α_1. This sequence $(T_n)_n$ defines a random Hamiltonian H acting on $\psi \in \mathcal{H} := \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ by

$$(H\psi)_n := T_{n+1}^* \psi_{n+1} + T_n \psi_{n-1} \qquad (n \in \mathbb{Z}). \tag{2.2}$$

An important condition to verify about this model is the following

2.5 Definition. We say that the system is *localized at zero* iff α_0 and α_1 are such that the Lyapunov spectrum $\{\gamma_j(0)\}_{j=1}^{2N}$ (see (2.17) for the definition) of the zero-energy Schrödinger equation $H\psi = 0$ does not contain zero:

$$0 \notin \{\gamma_j(0)\}_{j=1}^{2N}. \tag{2.3}$$

The main result of this chapter is a proof of Definition 1.21, (1.21) for the model described above:

2.6 Theorem. *(I) Under Assumptions 2.1 to 2.3, the fractional moments condition holds at all non-zero energies:* $\forall \lambda \in \mathbb{R} \setminus \{0\}, \exists s \in (0,1) : \exists 0 < C, \mu < \infty :$

$$\mathbb{E}[\|G(x,y;\lambda + i\eta)\|^s] \le C\, e^{-\mu|x-y|}, \qquad \forall \eta \ne 0, \forall x, y \in \mathbb{Z}. \tag{2.4}$$

(II) If moreover (2.3) holds, then (2.4) is extended to $\lambda = 0$ as well.

With $P := \chi_{(-\infty,0)}(H)$ the Fermi projection, the theorem implies via Propositions 1.22 and A.1 the following important corollary for the next chapter.

2.7 Corollary. *Under the foregoing assumptions, including (2.3), P is almost-surely weakly-local in the sense of Definition A.4 (that is, Assumption 3.1): With probability one, for some deterministic $\mu, \nu > 0$ and random $C > 0$ we have*

$$\sum_{n,n' \in \mathbb{Z}} \|P(n,n')\|(1 + |n|)^{-\nu} e^{\mu|n-n'|} \le C < +\infty.$$

We also have the second assumption of the next chapter fulfilled almost-surely due to the FMC being satisfied at zero energy:

2.8 Corollary. *Under the foregoing assumptions, including (2.3), zero is almost-surely not an eigenvalue of H (that is, Assumption 3.2).*

Proof. The inequality (2.4) at zero energy implies that almost-surely,

$$\lim_{\eta \to 0^+} \|G(x,y;i\eta)\|^s \le C_\varepsilon'\, e^{-\frac{1}{2}\mu|x-y|+\varepsilon|x|}$$

for some $C'_\varepsilon < \infty$, for all $\varepsilon > 0$ as in Proposition A.1. In particular for $y = x$ we find that almost-surely, $\lim_{\eta \to 0^+} \|G(x, x; i\eta)\|^s$ is bounded, which implies

$$\lim_{\eta \to 0^+} \eta \|\operatorname{Im} G(x, x; i\eta)\| = 0 \,,$$

for any x. However, zero is an eigenvalue of $H(\omega)$ iff

$$\lim_{\eta \to 0^+} \eta \|\operatorname{Im} G_\omega(x, x; i\eta)\| > 0$$

for some x (see [AJP06, Jakšic: Topics in spectral theory]). $\qquad\square$

In conclusion, taking the (full-measure) intersection of the two above sets, we get that with probability one both Assumptions 3.1 and 3.2 hold under the foregoing assumptions, including (2.3). We note in passing that the finite degeneracy condition which is part of Definition 1.27 is not actually necessary for Chapter 3 so that we omit its proof here.

2.3 TRANSFER MATRICES

In this section we will estimate the Green's function in terms of transfer matrices, which arise by looking at the Schrödinger equation (with H acting to the right or to the left),

$$(H - z)\psi = 0, \qquad \varphi(H - z) = 0 \,, \tag{2.5}$$

($z \in \mathbb{C}$), as a second order difference equation, where $\psi = (\psi_n)_{n \in \mathbb{Z}}$, $\varphi = (\varphi_n)_{n \in \mathbb{Z}}$ are (possibly unbounded) sequences with $\psi_n, \varphi_n \in \mathbb{C}^N$ viewed as column, respectively row vectors. Evaluated at $n \in \mathbb{Z}$, eqs. (2.5) read

$$\begin{aligned} T_n \psi_{n-1} + T_{n+1}^* \psi_{n+1} &= z\psi_n \,, \\ \varphi_{n-1} T_n^* + \varphi_{n+1} T_{n+1} &= z\varphi_n \,. \end{aligned} \tag{2.6}$$

For any two sequences ψ, φ of columns and rows respectively, let

$$C_n(\varphi, \psi) := \varphi_n T_{n+1}^* \psi_{n+1} - \varphi_{n+1} T_{n+1} \psi_n$$

be their Wronskian (or Casoratian); if they solve (2.5), then their Wronskian is independent of n, as seen from the identity

$$C_n(\varphi, \psi) - C_{n-1}(\varphi, \psi) = \varphi_n(T_{n+1}^* \psi_{n+1} + T_n \psi_{n-1}) - (\varphi_{n+1} T_{n+1} + \varphi_{n-1} T_n^*)\psi_n.$$

Moreover, the Wronskian may also be expressed as

$$C_n(\varphi, \psi) = \begin{pmatrix} \varphi_{n+1} T_{n+1} & \varphi_n \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} T_{n+1}^* \psi_{n+1} \\ \psi_n \end{pmatrix} \,,$$

which prompts us to associate to any sequence $\psi : \mathbb{Z} \to \mathbb{C}^N$ the sequence $\Psi : \mathbb{Z} \to \mathbb{C}^{2N}$ by

$$\Psi_n := \begin{pmatrix} T_{n+1}^* \psi_{n+1} \\ \psi_n \end{pmatrix} \,. \tag{2.7}$$

The *transfer matrix* is the map

$$A_n(z) : \mathbb{C}^{2N} \to \mathbb{C}^{2N}, \quad \Psi_{n-1} \mapsto \Psi_n$$

defined by the first equation (2.6):

$$\Psi_n = A_n(z)\Psi_{n-1}, \tag{2.8}$$

holds for any $n \in \mathbb{Z}$ such that that equation holds. The transfer matrix is thus given by the square matrix of order $2N$

$$A_n(z) = \begin{pmatrix} zT_n^\circ & -T_n \\ T_n^\circ & 0 \end{pmatrix} \tag{2.9}$$

with the abbreviation $M^\circ := (M^*)^{-1} = (M^{-1})^*$. Since the second equation (2.5) is equivalent to $(H - \bar{z})\varphi^* = 0$, the constancy of the Wronskian implies

$$A_n(\bar{z})^* J A_n(z) = J, \qquad J := \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix},$$

which can of course also be verified directly from (2.9). Another property that can be so verified is that $\det A_n(z)$ is independent of z and moreover

$$|\det A_n(z)| = 1 \qquad (z \in \mathbb{C}, n \in \mathbb{Z}). \tag{2.10}$$

The matrix J defines the symplectic structure of $\mathbb{C}^{2N}$. In particular for $z = \lambda \in \mathbb{R}$ the transfer matrix is *Hermitian symplectic*, $A_n(\lambda) \in Sp_{2N}^*(\mathbb{C})$, where

$$Sp_{2N}^*(\mathbb{C}) \equiv \{A \in Mat_{2N}(\mathbb{C}) | A^* J A = J\} \tag{2.11}$$

is the Hermitian symplectic group (compare with

$$Sp_{2N}(\mathbb{C}) \equiv \{A \in Mat_{2N}(\mathbb{C}) | A^T J A = J\}$$

and see Definition A.36). But we keep $z \in \mathbb{C}$ for now so that $A_n(z) \notin Sp_{2N}^*(\mathbb{C})$ as a rule.

Let $I \subseteq \mathbb{Z}$ be an interval (in the sense of $\mathbb{Z}$) and assume (2.8) for all $n \in I$. Then clearly

$$\Psi_n = B_{n,m}(z)\Psi_{m-1} \tag{2.12}$$

for $n, m \in I$, $n \geq m$ with the matrix of order $2N$

$$B_{n,m}(z) := A_n(z) \cdots A_m(z).$$

2.9 REMARK. The relations between Eqs. (2.5–2.8) trivially extend to matrix solutions ψ and Ψ respectively which are obtained by placing $l = 1, \ldots, 2N$ solutions next to one another in guise of columns. Seen that way they become linear maps $\psi : \mathbb{C}^l \to \mathbb{C}^N$, $\Psi : \mathbb{C}^l \to \mathbb{C}^{2N}$.

2.10 Lemma. *For I, n, m and Ψ as just above we have*

$$|\Psi_{m-1}|^2 \leq \frac{\operatorname{tr}(|\wedge^{l-1}(BP)|^2)}{\operatorname{tr}(|\wedge^l(BP)|^2)}|\Psi_n|^2, \tag{2.13}$$

where we use $|M|^2 \equiv M^*M$, set $B = B_{n,m}(z)$, and where $P : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ is an orthogonal projection of rank l such that $P\Psi_{m-1} = \Psi_{m-1}$.

The proof rests on the following lemma, to be proven below.

2.11 Lemma. *Let $W \subseteq V$ be linear spaces; let V be equipped with an inner product and let $P : V \to V$ be the self-adjoint projection onto W. Let $B : V \to V$ be a linear map and $B' := B|_W : W \to V$ its restriction to W. Then*

$$\operatorname{tr}(|B'|^{-2}) = \frac{\operatorname{tr}(|\wedge^{l-1}(BP)|^2)}{\operatorname{tr}(|\wedge^l(BP)|^2)} \tag{2.14}$$

with $l := \dim W$.

Proof of Lemma 2.10. By (2.12) we have

$$|\Psi_n|^2 = |B\Psi_{m-1}|^2 = |B'\Psi_{m-1}|^2$$

with $B' := B|_W$, $W := \operatorname{im} P$; and thus

$$|\Psi_{m-1}|^2 \leq \operatorname{tr}(|B'|^{-2})|\Psi_n|^2$$

in view of

$$Q \geq \|Q^{-1}\|^{-1}, \qquad \|Q^{-1}\| \leq \operatorname{tr}(Q^{-1})$$

for any $Q > 0$, applied to $Q := |B'|^2$. We conclude by (2.14). $\qquad\square$

Proof of Lemma 2.11. We recall Lemma C.12 from [CS15]: Let $L \geq l \geq 1$ be integers and let $v_1, \cdots, v_l \in \mathbb{C}^L$ be linearly independent. Define $w := v_1 \wedge \cdots \wedge v_l$ and $w_k := v_1 \wedge \cdots \wedge \hat{v}_k \wedge \ldots v_l$ $(k = 1, \ldots, l)$ with $\hat{\ }$ denoting omission and (for $l = 1$) the empty product being $w_1 = 1 \in \wedge^0\mathbb{C}^L = \mathbb{C}$. Let $\mathfrak{G}$ be the Gramian matrix (of order l) for $v_1, \ldots, v_l$, i.e.

$$\mathfrak{G}_{jk} = \langle v_j, v_k \rangle, \qquad (j, k = 1, \ldots, l).$$

Then

$$\text{tr}(\mathfrak{G}^{-1}) = \frac{\sum_{k=1}^{l} \|w_k\|^2}{\|w\|^2}. \tag{2.15}$$

This having been done, let $(e_1, \ldots, e_l)$ be an orthonormal basis of W, so that

$$\varepsilon = e_1 \wedge \cdots \wedge e_l$$
$$\varepsilon_k = e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_l, \qquad (k = 1, \ldots, l)$$

are orthonormal bases of $\wedge^l W$ and $\wedge^{l-1} W$ respectively. We then apply (2.15) with $L = 2N$ to $v_i = B'e_i$, and thus to $w = (\wedge^l B')\varepsilon$, $w_k = (\wedge^{l-1} B')\varepsilon_k$ and to $\mathfrak{G}_{jk} = (|B'|^2)_{jk}$, the result being

$$\text{tr}(|B'|^{-2}) = \frac{\text{tr}_{\wedge^{l-1} W}(|\wedge^{l-1} B'|^2)}{\text{tr}_{\wedge^l W}(|\wedge^l B'|^2)}.$$

Finally, just as we have $\text{tr}_W(|B'|^2) = \text{tr}(|BP|^2)$, so we do

$$\text{tr}_{\wedge^k W}(|\wedge^k B'|^2) = \text{tr}(|\wedge^k (BP)|^2),$$

because $(\wedge^k B)(\wedge^k P) = \wedge^k (BP)$. $\qquad\qquad\square$

The entries $G_{nk} = G_{nk}(z)$ of the Green's function are matrices of order N which may be looked at in their dependence on n at fixed k. So viewed $G_k = (G_{nk})_{n \in \mathbb{Z}}$ satisfies

$$(H - z)G_k = \delta_k$$

with $\delta_k = (\delta_{nk} 1)_{n \in \mathbb{Z}}$. Thus $\psi = G_k$ satisfies (2.5) at sites $n \neq k$ and in the matrix sense of Remark 2.9 (with $l = N$). Likewise,

$$\mathcal{G}_k = (\mathcal{G}_{nk})_{n \in \mathbb{Z}}, \quad \mathcal{G}_{nk} = \begin{pmatrix} T_{n+1}^* G_{n+1,k} \\ G_{nk} \end{pmatrix}$$

satisfies (2.8) with $\Psi = \mathcal{G}_k$ by (2.7). In particular, it does for $n \in I = (-\infty, k-1]$ whence Lemma 2.10 applies to $m \leq n = k - 1$. The result is as follows:

2.12 Lemma. *We have*

$$|G^{(m-1)k}(z)|^2 \leq \frac{\text{tr}(|\wedge^{N-1} (BP)|^2)}{\text{tr}(|\wedge^N (BP)|^2)} |\mathcal{G}_{k-1,k}(z)|^2$$

where $B = B_{k-1,m}(z)$ and P is a projection onto the range of $\mathcal{G}_{m-1k}$.

Proof. Given the preliminaries it suffices to observe that $|G^{(m-1)k}|^2 \leq |\mathcal{G}_{(m-1)k}|^2$.
$\qquad\qquad\square$

In particular, we can pick G^+ as the right-half-space Green's function on $[m-1, \infty)$ and P a projection onto the first N dimensions of $\mathbb{C}^{2N}$. As we'll see below, this will then relate $G_{(m-1)k}^+(z)$ to the Nth Lyapunov exponent of the transfer matrices, times a "constant" (in $|m - k|$) factor $\mathcal{G}_{k-1,k}(z)$.

2.4 THE LYAPUNOV EXPONENTS

In this section we define the Lyapunov spectrum associated to the random sequence of matrices $(A_n(z))_n$. The main result here will be Corollary 2.21 which will show that the smallest positive exponent is *strictly* positive. Since it encodes in it the localization length, at least morally this already implies localization, and we shall show this rigorously using the FMM.

For brevity we define the maps a, S into $Sp_N^*(\mathbb{C})$ via

$$GL_N(\mathbb{C}) \ni X \overset{a}{\mapsto} \begin{pmatrix} X^\circ & 0 \\ 0 & X \end{pmatrix}$$

$$\mathbb{R} \ni \lambda \overset{S}{\mapsto} \begin{pmatrix} \lambda & -\mathbb{1}_N \\ \mathbb{1}_N & 0_N \end{pmatrix}.$$

Then we may factorize the transfer matrix as $A_n(z) = S(z)a(T_n)$, the first factor being deterministic. Note that $(A_n(z))_n$ is not an independent-identically-distributed sequence, since its odd and even elements are distributed differently. However, taking two steps at a time results in $(A_{2n}A_{2n+1})_n$ which is a bona-fide i.i.d. sequence of random matrices.

Let μ_λ be the push forward measure induced by

$$GL_N(\mathbb{C})^2 \ni (X, Y) \mapsto S(\lambda)a(X)S(\lambda)a(Y) \in Sp_N^*(\mathbb{C})$$

where X, Y are distributed with α_0, α_1 respectively. Below we sometimes leave λ implicit in the notation.

2.13 Proposition. *We have $\int \log^+(\|g\|) \, \mathrm{d}\, \mu(g) < \infty$ where $\log^+$ is the positive part of log.*

Proof. We have by definition

$$\int \log^+(\|g\|) \, \mathrm{d}\, \mu(g) = \int \log^+(\|Sa(X)Sa(Y)\|) \, \mathrm{d}\, \alpha_0(X) \, \mathrm{d}\, \alpha_1(Y).$$

Since $\log^+$ is monotone increasing,

$$\log^+(\|Sa(X)Sa(Y)\|) \leq 2\log^+(\|S\|) + \log^+(\|a(X)\|) + \log^+(\|a(Y)\|).$$

Hence it is sufficient to show that

$$\int_{GL_N(\mathbb{C})} \log^+(\|a(X)\|) \, \mathrm{d}\, \alpha_i(X) < \infty. \tag{2.16}$$

We recall that the norm is the largest singular value (denoted by σ_1), and $|a(X)| = \begin{pmatrix} |X|^{-1} & 0_N \\ 0_N & |X| \end{pmatrix}$. We find that $\|a(X)\|$ is equal to the largest between the singular values of X and X^{-1}, that is, $\max(\sigma_1(X), \sigma_N(X)^{-1})$.

We use Jensen's inequality on the concave function $\log^+$ in order to conclude (2.16) from Assumption 2.3. $\qquad\square$

2.14 Corollary. *Using [BL85, pp. 6] we have that the $2N$ Lyapunov exponents (henceforth LE)*

$$\gamma_j(z) \equiv \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\log(\sigma_j(B_n(z)))], \qquad (2.17)$$

where σ_j is the jth singular value of a matrix (ordered such that σ_1 is the largest), are well-defined and take values in $[-\infty, \infty)$. We use the abbreviation $B_n(z) = B_{n,1}(z)$.

2.15 REMARK. While (2.10) alone does not allow us to conclude a symmetry property for the exponents, if we restrict to $z \in \mathbb{R}$, then the Hermitian symplectic condition implies that the exponents are symmetric about zero, that is, $\gamma_j(z) = -\gamma_{2N-(j-1)}(z)$ for all $j \in \{1, \ldots, N\}$.

Proof. Since we are taking the logarithm, the symmetry of the singular values of the Hermitian symplectic matrix $B_n(z)$ about one (as shown in Proposition A.42) implies a symmetry of the exponents about zero. $\qquad\square$

IRREDUCIBILITY

2.16 Lemma. *Let*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2N}(\mathbb{C})$$

with $D \in GL_N(\mathbb{C})$. Then $M \in Sp^*_{2N}(\mathbb{C})$ iff

$$B = RD, \qquad C = DS \qquad (2.18)$$

with $R = R^*$, $S = S^*$ and

$$A = D^\circ(\mathbb{1} + B^*C) \qquad (2.19)$$

Proof. (2.19) is equivalent to the last condition of (A.18) when D is invertible. So by Proposition A.40 we need only verify that A^*C and B^*D are self-adjoint. But $B^*D = D^*RD$ which is self-adjoint by hypothesis on R and $A^*C = (\mathbb{1} + C^*B)D^{-1}DS = S + SD^*RDS$, the last expression being a sum of two terms which are self-adjoint using the assumptions on R and S. $\qquad\square$

Let $\mathscr{S}_0 \subseteq Sp^*_{2N}(\mathbb{C})$ be the open subset (and hence submanifold) given by $\mathscr{S}_0 = \{M \in Sp^*_{2N}(\mathbb{C}) | \det D \neq 0\}$. By Lemma 2.16, the map $\mathscr{S}_0 \to GL_N(\mathbb{C}) \times Herm_N(\mathbb{C}) \times Herm_N(\mathbb{C})$, $M \mapsto (D, R, S)$ is a coordinate chart of $\mathscr{S}_0$; for short $M \cong (D, R, S)$. In particular

$$\dim_\mathbb{R} Sp^*_{2N}(\mathbb{C}) = \dim_\mathbb{R} \mathscr{S}_0 = 2N^2 + N^2 + N^2 = 4N^2.$$

In the following $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is fixed.

Let $\mathscr{S}_1 \subseteq \mathscr{S}_0$ be the submanifold which in terms of the chart consists of matrices $M \cong (D, R = \lambda\mathbb{1}, S) =: (D, S)$. In particular $\dim_{\mathbb{R}} = 3N^2$. Moreover, we consider matrices of the form

$$M = \begin{pmatrix} \lambda T^\circ & -T \\ T^\circ & 0 \end{pmatrix} \tag{2.20}$$

with $T \in GL_N(\mathbb{C})$. Then $M \in Sp^*_{2N}(\mathbb{C})$, as remarked before (2.11), though they are not of the form discussed in Lemma 2.16 because of $D = 0$. However, their products are, $M_1 M_2 \in \mathscr{S}_1$, with

$$M_1 M_2 \cong (D, S) = (-T_1^\circ T_2, -\lambda(T_2^* T_2)^{-1}) \tag{2.21}$$

in terms of the chart. In fact

$$M_1 M_2 = \begin{pmatrix} \star & -\lambda T_1^\circ T_2 \\ \lambda T_1^\circ T_2^\circ & -T_1^\circ T_2, \end{pmatrix}$$

from which the claims about D and R are evident by comparison with (2.18); the one about S follows from $T_2(T_2^* T_2)^{-1} = T_2^\circ$.

Let M be as in (2.20) and $M' \cong (D, S) \in \mathscr{S}_1$. Then $MM' \in \mathscr{S}_0$ with $MM' \cong (\tilde{D}, \tilde{R}, \tilde{S})$, where

$$\tilde{D} = \lambda T^\circ D, \qquad \tilde{R} = \lambda - \lambda^{-1}|T^*|^2, \qquad \tilde{S} = S + \lambda^{-1}|D|^{-2}. \tag{2.22}$$

In fact,

$$\begin{pmatrix} \lambda T^\circ & -T \\ T^\circ & 0 \end{pmatrix} \begin{pmatrix} A & \lambda D \\ DS & D \end{pmatrix} = \begin{pmatrix} \star & (\lambda^2 T^\circ - T)D \\ T^\circ A & \lambda T^\circ D \end{pmatrix}$$

where $A = D^\circ + \lambda DS$ follows from (2.19). Then the claim (2.22) is obvious for $\tilde{D}$ and by (2.18) amounts for the rest to

$$T^\circ A = \tilde{D}\tilde{S}, \qquad (\lambda^2 T^\circ - T)D = \tilde{R}\tilde{D}.$$

The two conditions simplify to $D^{-1}D^\circ + \lambda S = \lambda\tilde{S}$ and to $\lambda^2 - T(T^\circ)^{-1} = \lambda\tilde{R}$, which by $D^{-1}D^\circ = |D|^{-2}$ are satisfied.

2.17 Proposition. *The above product maps*

$$GL_N(\mathbb{C}) \times GL_N(\mathbb{C}) \to \mathscr{S}_1, \qquad (T_1, T_2) \mapsto M_1 M_2 = M',$$
$$GL_N(\mathbb{C}) \times \mathscr{S}_1 \to \mathscr{S}_0, \qquad (T, M') \mapsto MM'$$

are submersions, i.e. their tangent maps have maximal rank.

Proof. We represent domain and codomain of both maps in their charts:

$$(T_1, T_2) \mapsto (D, S), \qquad (T, D, S) \mapsto (\tilde{D}, \tilde{R}, \tilde{S}) \qquad (2.23)$$

with right hand sides given by (2.21, 2.22). As a preparation we observe that the following maps are submersions:

- $GL_N(\mathbb{C}) \circlearrowright, T \mapsto T^{-1}, T^*, TM$ or MT for some fixed $M \in GL_N(\mathbb{C})$,

- $GL_N(\mathbb{C}) \cap Herm_N(\mathbb{C}) \circlearrowright, S \mapsto S^{-1}$

- $GL_N(\mathbb{C}) \to Herm_N(\mathbb{C}), T \mapsto |T|^2$ or $|T^*|^2$

Only the map $T \mapsto |T|^2$ may deserve comment. It has a differentiable right inverse $S \mapsto S^{1/2}$ on the open subset $\{S > 0\} \subseteq Herm_N(\mathbb{C})$, whence the claim.

The first map (2.23) is then written as a concatenation

$$(T_1, T_2) \mapsto (D, T_2), \qquad T_2 \mapsto S$$

of maps that are seen to be submersions. Likewise for the second map:

$$(T, D, S) \mapsto (T, D, \tilde{S}), \qquad (T, D) \mapsto (T, \tilde{D}), \qquad T \mapsto \tilde{R}.$$

$\square$

2.18 Proposition. *Let $\lambda \in \mathbb{R} \setminus \{0\}$. Then the semigroup T_{μ_λ} generated by $\operatorname{supp} \mu_\lambda$ contains an open subset of $Sp^*_{2N}(\mathbb{C})$.*

Proof. By definition, the measure μ_λ is the one induced by α_0, α_1 through (2.20) and the map $(M_1, M_2) \mapsto B := M_2 M_1$. Now T_{μ_λ} contains for sure all matrices $B'B = M_2' M_1' M_2 M_1$. By Assumption 2.1, the proposition and the submersion theorem ([Bre93] Theorem 7.1), the matrices $M_1' M_2 M_1$ cover an open subset of $\mathscr{S}_0$ and hence of $Sp^*_{2N}(\mathbb{C})$, and so does $B'B$. $\square$

2.19 REMARK. For the usual Anderson model on a strip, as in [KLS90; Sim85] for example, the transfer matrix is rather

$$A_x(z) = \begin{pmatrix} z - V_x & -\mathbb{1}_N \\ \mathbb{1}_N & 0_N \end{pmatrix}$$

with $\{V_x\}_{x \in \mathbb{Z}}$ the independent, identically distributed sequence of onsite potentials (which in general should take values in $Herm_N(\mathbb{C})$). Then known results, say, in [KLS90] (and references within) show that the semigroup generated by the support of this transfer matrix contains an open subset of the Hermitian symplectic group (for *all* $z \in \mathbb{C}$). Taking this result (which holds also for $z = 0$) to replace our

Proposition 2.18 and proving an a-priori bound, one could extend the analysis here to the Anderson model on the strip as well.

2.20 Corollary. *If $z \in \mathbb{R}\backslash\{0\}$, then the Lyapunov spectrum is simple: $\gamma_j(z) \neq \gamma_{j'}(z)$ for all $j \neq j'$ in $\{1, \ldots, 2N\}$.*

Proof. We apply [BL85, Proposition IV.3.5], which goes through even though it is applied on $Sp_{2N}(\mathbb{R})$ whereas here we apply it on $Sp_{2N}^*(\mathbb{C})$. □

2.21 Corollary. *If $z \in \mathbb{R}\backslash\{0\}$, then $\gamma_N(z) > 0$.*

Proof. We always have $\gamma_N \geq \gamma_{N+1}$ because this is how we choose the ordering of the labels. By Remark 2.15 we have that $\gamma_N(z) = -\gamma_{N+1}(z)$ and by Corollary 2.20 $\gamma_N(z) \neq \gamma_{N+1}(z)$. □

2.22 REMARK. $\gamma_N(0) = 0$ is possible though generically false.

Proof. When $z = 0$, via (2.8), for even sites:

$$\Psi_{2x} \equiv \begin{pmatrix} T_{2x+1}^* \psi_{2x+1} \\ \psi_{2x} \end{pmatrix} = A_{2x}(0)\Psi_{2x-1}$$

$$= \begin{pmatrix} 0 & -T_{2x} \\ T_{2x}^\circ & 0_N \end{pmatrix} \begin{pmatrix} T_{2x}^* \psi_{2x} \\ \psi_{2x-1} \end{pmatrix} = \begin{pmatrix} -T_{2x}\psi_{2x-1} \\ \psi_{2x} \end{pmatrix}.$$

That is,

$$\psi_{2x+1} = -T_{2x+1}^\circ T_{2x} \psi_{2x-1},$$

and similarly for the odd sites,

$$\psi_{2x+2} = -T_{2x+2}^\circ T_{2x+1} \psi_{2x}.$$

As a result, within its zero eigenspace, H commutes with the operator $(-1)^X$ where X is the position operator (this operator gives the parity of the site) and the problem splits into two independent first-order difference equations, with transfer matrices $\{-T_{2x+1}^\circ T_{2x}\}_x$ and $\{-T_{2x+2}^\circ T_{2x+1}\}_x$ respectively. These transfer matrices are *not* Hermitian symplectic, nor is the absolute value of their determinant equal to one–they are merely elements in $GL_N(\mathbb{C})$, and only their direct sum has these properties.

Hence the theorem insuring the simplicity of the Lyapunov spectrum, [BL85, pp. 78, Theorem IV.1.2] does not help in this case, since the simplicity of the

Lyapunov spectrum of these transfer matrices will not imply that none of the exponents are zero. Instead we are reduced to the more direct question of whether any of the exponents are zero or not.

Here is a (trivial) example where there is a zero exponent: when $N = 1$, there is only one exponent for each (separate) chirality sector, and the hopping matrices are merely complex numbers. Then for the (even) positive chirality e.g. we have

$$\gamma_1^+(0) \equiv \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(|(-T_{2n+1}^\circ T_{2n})(-T_{2n-1}^\circ T_{2n-2}) \ldots (-T_3^\circ T_2)|)]$$

$$= \lim_{n \to \infty} \frac{1}{n} \big(- \sum_{l=3, l \text{ odd}}^{2n+1} \mathbb{E}[\log(|T_l|)] + \sum_{l=2, l \text{ even}}^{2n} \mathbb{E}[\log(|T_l|)]\big)$$

(independence property)

$$= \mathbb{E}_{\alpha_0}[\log(|\cdot|)] - \mathbb{E}_{\alpha_1}[\log(|\cdot|)].$$

We immediately see that when $\alpha_1 = \alpha_0$, the exponent is zero, and when $\alpha_1 \neq \alpha_0$ and both have non-zero log-expectation value (not the same value), then the exponent is non-zero.

In general, as can be seen from the formula for the transfer matrices, if $\operatorname{supp} \alpha_1$ is concentrated within the unit ball of $GL_N(\mathbb{C})$ and $\operatorname{supp} \alpha_0$ is very far outside the unit ball of $GL_N(\mathbb{C})$, then we are guaranteed that none of the exponents will be zero. □

In the rest of this section we establish continuity properties of the exponents and finally connect $\gamma_N(z)$ to the decay rate (in n) of a product of n transfer matrices. This is a simple extension of the analysis in [KLS90, Section 2] to the complex-valued case with off-site randomness. Thus we frequently use the notation and conventions of [KLS90] below sometimes without explicit reference.

We adopt the following viewpoint which is customary in the study of product of random matrices. Since we are analyzing matrices in $Sp_{2N}^*(\mathbb{C})$, we'll be interested in *isotropic* subspaces (the subspaces of $\mathbb{C}^{2N}$ on which J restricts to the zero bilinear form) which are left invariant by $Sp_{2N}^*(\mathbb{C})$. Correspondingly we study the *isotropic* Grassmannian manifold, $\bar{L}_k$, the set of isotropic subspaces of dimension k within $\mathbb{C}^{2N}$ with $k \in \{1, \ldots, N\}$. A convenient way to parametrize such isotropic subspaces is via exterior powers: a simple vector $u_1 \wedge \cdots \wedge u_k \in \wedge^k \mathbb{C}^{2N}$ with $\{u_i\}_i$ linearly independent and $\langle u_i, J u_j \rangle = 0$ for all $i, j \in \{1, \ldots, k\}$ defines a point in $\bar{L}_k$, and one can talk about the action of $Sp_{2N}^*(\mathbb{C})$ on such a point by lifting $g \in Sp_{2N}^*(\mathbb{C})$ to $\wedge^k g$ on $\wedge^k \mathbb{C}^{2N}$.

2.23 Definition. For any $[v] \in \bar{L}_k$ and $z \in \mathbb{C}$ let

$$\Phi_z([v]) := \mathbb{E}\big[\log\big(\frac{\| \wedge^k A_1(z) v\|}{\|v\|}\big)\big].$$

2.24 Proposition. $[v] \mapsto \Phi([v])$ *is continuous.*

Proof. This is [KLS90, Prop. 2.4] or [CL90][V.4.7 (i))] for our model. Via Proposition A.35 and Proposition 2.13 we find that the "sequence" $[v] \mapsto \log(\frac{\|\wedge^k A_1 v\|}{\|v\|})$ is bounded by the integrable function

$$k(2N - 1) \log(\|A_1\|).$$

So using the Lebesgue dominated convergence theorem and the fact that $[v] \mapsto \log(\frac{\|\wedge^k A_1 v\|}{\|v\|})$ is continuous we find our result. $\square$

2.25 Proposition. $z \mapsto \Phi_z([v])$ *is Lipschitz continuous, uniformly in* $[v]$, *as long as* z *ranges in a compact subset.*

Proof. This is [KLS90, Prop. 2.4] or [CL90][V.4.7 (ii))] for our model. First note that since $\|Mu\| = \|ML^{-1}Lu\| \le \|ML^{-1}\| \|Lu\|$ we have for all $[v]$,

$$\Phi_z([v]) - \Phi_w([v]) \le \mathbb{E}[\log(\|A_1(z)(A_1(w))^{-1}\|)].$$

By symmetry,

$$
\begin{aligned}
\Phi_w([v]) - \Phi_z([v]) &\le \mathbb{E}[\log(\|A_1(w)(A_1(z))^{-1}\|)] \\
&\qquad (|\det(A_1(w)(A_1(z))^{-1})| = 1) \\
&= (2N-1)\mathbb{E}[\log(\|A_1(z)(A_1(w))^{-1}\|)],
\end{aligned}
$$

so that

$$|\Phi_z([v]) - \Phi_w([v])| \le (2N-1)\mathbb{E}[\log(\|A_1(z)(A_1(w))^{-1}\|)].$$

Next we have

$$
\begin{aligned}
&= S(z)a(X)S(z)a(Y)[S(w)a(X)S(w)a(Y)]^{-1} \\
&= S(z)a(X)S(z)S(w)^{-1}a(X)^{-1}S(w)^{-1}.
\end{aligned}
$$

We remark that

$$
S(z)S(w)^{-1} = \begin{pmatrix} z & -\mathbb{1}_N \\ \mathbb{1}_N & 0_N \end{pmatrix} \begin{pmatrix} 0_N & \mathbb{1}_N \\ -\mathbb{1}_N & w \end{pmatrix} = \begin{pmatrix} \mathbb{1}_N & (z-w)\mathbb{1}_N \\ 0 & \mathbb{1}_N \end{pmatrix}
$$

$$
= \mathbb{1}_{2N} + (z-w) \begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix},
$$

so that

$$\|S(z)a(X)S(z)S(w)^{-1}a(X)^{-1}S(w)^{-1}\|$$

$$= \|S(z)a(X)[\mathbb{1}_{2N} + (z - w)\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}]a(X)^{-1}S(w)^{-1}\|$$

$$\leq 1 + |z - w| \|\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}\| + |z - w|\|S(z)\|\|a(X)\| \times$$

$$\times \|\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}\|\|a(X)^{-1}\|\|S(w)^{-1}\|.$$

Now $\|\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}\| = 1$, $\|a(X)^{-1}\| = \|a(X)\|$ and $\log(1 + x) \leq x$ for all $x \in \mathbb{R}$ so that we find

$$\log(\|A_1(z)(A_1^w)^{-1}\|) \leq |z - w|(1 + \|S(z)\|\|S(w)^{-1}\|\|a(X)\|^2).$$

Since $z \mapsto \|S(z)\|$ is continuous, z ranges in a compact set, and using Assumption 2.3, we find

$$|\Phi_z([v]) - \Phi_w([v])| \leq |z - w|(1 + \sup_{z'}\|S_{z'}\|^2\mathbb{E}[\|a(T_1)\|^2]),$$

obtaining the claim for $k = 1$; the other cases being an easy generalization. □

▎ **2.26 Corollary.** *The map $z \mapsto \gamma_j(z)$ is continuous for all j.*

Proof. This is ([KLS90] Proposition 2.5 or [CL90] V.4.8.) The same proof as theirs goes through thanks to the claims above. □

We will need [KLS90, Prop. 2.6]:

▎ **2.27 Proposition.** *For each $j \in \{1, \ldots, N\}$ and $[x] \in \bar{L}_j$, we have*

$$\lim_{n \to \infty} \frac{1}{n}\mathbb{E}[\log(\frac{\|\wedge^j B_n(z)x\|}{\|x\|})] = \sum_{l=1}^{j} \gamma_l(z),$$

the limit being uniform *as z ranges in a compact subspace of $\mathbb{R}$ and also* uniform *in $[x]$.*

We also recall [KLS90, Prop. 2.7]:

▎ **2.28 Proposition.** *For any compact $K \subseteq \mathbb{R}$, $j \in \{1, \ldots, N\}$ such that $\gamma_j(z) > 0$ for all $z \in K$, there exist some $s_K \in (0, 1)$, $n_K \in \mathbb{N}$ and $C_{j,K} > 0$ such that*

$$\mathbb{E}[(\frac{\|\wedge^{j-1} B_n(z)y\|}{\|y\|} \frac{\|x\|}{\|\wedge^j B_n(z)x\|})^{s_K}] \leq \exp(-C_{j,K}n)$$

for all $n \in \mathbb{N}_{\geq n_K}$, for all $[x] \in \bar{L}_j$ and for all $[y] \in \bar{L}_{j-1}$.

The proof is presented in Appendix A.7 for the reader's convenience.

2.5   AN A-PRIORI BOUND

In this short section we establish the a-priori boundedness of one-step Green's functions, which is a staple of the FMM. The fact that for our model one uses the one-step Green's function rather than the diagonal one is due to the fact we don't have on-site randomness, which forces the usage of rank-2 perturbation theory.

2.29 Proposition. *For any $s \in (0,1)$ we have some strictly positive constant C_s such that*

$$\mathbb{E}[\|G(x, x-1; z)\|^s] \le C_s < \infty,$$

for all $x \in \mathbb{Z}$ and for all $z \in \mathbb{C}$. So C_s depends on s, α_1 and α_0.

Proof. Using finite-rank perturbation theory, we can find the explicit dependence of the complex number $G(x, x-1; z)_{i,j}$ (for some $(i,j) \in \{1,\ldots,N\}^2$) on the random hopping $(T_x)_{i,j}$. Indeed,

$$(T_x)_{i,j} = \langle \delta_x \otimes e_i, H(T), \delta_y \otimes e_j \rangle .$$

We find that $G(x, x-1; z)_{i,j}$ is equal to the bottom-left matrix element of the 2×2 matrix

$$\left(A + \begin{pmatrix} 0 & \overline{\lambda} \\ \lambda & 0 \end{pmatrix}\right)^{-1}, \tag{2.24}$$

where A is the inverse of the 2×2 matrix

$$\begin{pmatrix} \tilde{G}(x-1, x-1; z)_{j,j} & \tilde{G}(x-1, x; z)_{j,i} \\ \tilde{G}(x, x-1; z)_{i,j} & \tilde{G}(x, x; z)_{i,i} \end{pmatrix},$$

with $\tilde{H}$ being H with $(T_x)_{i,j}$ "turned off", and $\lambda := (T_x)_{i,j}$ for convenience. Thus our goal is to bound the fractional moments with respect to $\lambda \in \mathbb{C}$ of the off-diagonal entry of the matrix in (2.24) where A is some given 2×2 matrix with complex entries. The only thing we know about A is that $\mathbb{Im}\{A\} > 0$ though we won't actually use this.

First note that $\begin{pmatrix} 0 & \overline{\lambda} \\ \lambda & 0 \end{pmatrix} = \mathbb{Re}\{\lambda\}\sigma_1 + \mathbb{Im}\{\lambda\}\sigma_2$ and also that for any 2×2 matrix M we have $\frac{1}{2}tr(M\sigma_1) = \frac{1}{2}(M_{12} + M_{21})$ whereas

$$\frac{1}{2}tr(M\sigma_2) = \frac{i}{2}(M_{12} - M_{21}).$$

Thus by the triangle inequality,

$$\mathbb{E}[|M_{12}|^s] = \mathbb{E}[|\frac{1}{2}tr(M\sigma_1)|^s] + \mathbb{E}[|\frac{1}{2}tr(M\sigma_2)|^s].$$

So, if we get control on each summand separately we could bound $\mathbb{E}[|M_{12}|^s]$.

We write

$$A + \begin{pmatrix} 0 & \bar{\lambda} \\ \lambda & 0 \end{pmatrix} = \tilde{A} + \mathbb{Re}\{\lambda\}\sigma_1$$

and we expand $\tilde{A}$ as $\tilde{A} = a_0 + a_i\sigma_i$ for some $\begin{pmatrix} a_0 \\ a \end{pmatrix} \in \mathbb{C}^4$, so that if $M := (a_0 + a_i\sigma_i + \mathbb{Re}\{\lambda\}\sigma_1)^{-1}$ we find

$$\frac{1}{2}\operatorname{tr}(M\sigma_1) = \frac{-a_1 - \mathbb{Re}\{\lambda\}}{a_0{}^2 - a_2{}^2 - a_3{}^2 - (a_1 + \mathbb{Re}\{\lambda\})^2}.$$

We now apply Proposition A.34 with $z = -a_1 - \mathbb{Re}\{\lambda\}$, $c = a_0{}^2 - a_2{}^2 - a_3{}^2$ to get the result (using the Layer-Cake representation), which relies on Assumption 2.2. The term $\mathbb{E}[|\frac{1}{2}tr(M\sigma_2)|^s]$ is dealt with in precisely the same way. $\square$

2.30 Corollary. *For any $s \in (0,1)$ we have some strictly positive constant C_s (not the same as the one from above) such that*

$$\mathbb{E}[\|G(x,x;z)\|^s] < \frac{1}{|z|}C_s,$$

for all $x \in \mathbb{Z}$ and for all $z \in \mathbb{C}\backslash\{0\}$. C_s depends only on $s \in (0,1)$, α_1 and α_0.

Proof. From the relation $(H - z)R(z) = \mathbb{1}$ we find

$$-zG(x,x) + T_{x+1}^*G(x+1,x;z) + T_xG(x-1,x;z) = \mathbb{1}_N,$$

so that

$$G(x,x;z) = z^{-1}[\mathbb{1}_N - T_{x+1}^*G(x+1,x;z) + T_xG(x-1,x;z)].$$

Hence by the triangle inequality, Hoelder's inequality, Assumption 2.3 and Proposition 2.29 we find the result. $\square$

2.31 REMARK. The last two statements hold equally well if we replace $G(x,x)$ or $G(x-1,x)$ with the half-line Green's function or even a finite-volume Green's function.

2.6 LOCALIZATION AT NON-ZERO ENERGIES

In this section we establish localization for all non-zero energies. We do this in two steps: first at real energies (and hence finite volume) due to the fact that the Fursten-berg analysis requires the transfer matrices to be $Sp_{2N}^*(\mathbb{C})$-valued which needs the

energy to be real. We then extend this exponential decay off the real axis using the harmonic properties of the Green's function to get polynomial decay at complex energies, which in turn implies exponential decay via a decoupling-type lemma. Once localization of finite volume at complex energies is established, the infinite volume result is implied via the strong-resolvent convergence of the finite volume Hamiltonian to the infinite volume one.

2.6.1 *Finite volume and real energy*

2.32 Theorem. *For any $K \in Compact(\mathbb{R})$ such that $0 \notin K$, there is some $s_K \in (0,1)$ and $\mu_K > 0$ such that*

$$\sup_{z \in K} \mathbb{E}[G^{[x,y]}(x,y;z)^{s_K}] < |z|^{-s_K} e^{-\mu_K|x-y|} ,$$

for all $(x,y) \in \mathbb{Z}^2$ with $|x-y|$ sufficiently large. Here $G^{[x,y]}$ is the Green's function of the finite-volume restriction of H to $[x,y] \subseteq \mathbb{Z}$.

Proof. Let $z \in K$ be given and $s \in (0,1)$. By Lemma 2.12 we know that

$$\|G^{[x,y]}(x,y;z)\|^s \le |z|^{-s}C(z)^{\frac{s}{2}} \Big(\sum_{j=1}^{N} \frac{\| \wedge^{N-1} B_{y-1,x-1}(z)u_j\|^2}{\| \wedge^N B_{y-1,x-1}(z)u\|^2} \Big)^{\frac{s}{2}}$$

$$\le |z|^{-s}C(z)^{\frac{s}{2}} \sum_{j=1}^{N} \frac{\| \wedge^{N-1} B_{y-1,x-1}(z)u_j\|^s}{\| \wedge^N B_{y-1,x-1}(z)u\|^s} .$$

Now using Hoelder's inequality we get

$$\mathbb{E}[\|G^{[x,y]}(x,y;z)\|^s] \le |z|^{-s}\mathbb{E}[C(z)^s]^{\frac{1}{2}} \sum_{j=1}^{N} \mathbb{E}\Big[\frac{\| \wedge^{N-1} B_{y-1,x-1}(z)u_j\|^s}{\| \wedge^N B_{y-1,x-1}(z)u\|^s}\Big]^{\frac{1}{2}} .$$

$\mathbb{E}[C(z)^s]^{\frac{1}{2}}$ is bounded uniformly in z using Proposition 2.29, Corollary 2.30 and Assumption 2.3. Now $u_j \in \bar{L}_{N-1}$ for any j and $u \in \bar{L}_N$; $K \notin 0$, $\gamma_N(z) > 0$ for all $z \in K$ so that we may apply Proposition 2.28 to get that for s_K, for appropriate $\infty > C > 0$, for $|x-y| > n_K$

$$\mathbb{E}[\|G^{[x,y]}(x,y;z)\|^{s_K}] < |z|^{-s_K}CN\exp(-\frac{1}{2}C_{j,K}|x-y|) ,$$

which implies the bound in the claim. Note that we have used stationarity of $\mathbb{P}$ to go from $0, x-y$ to y, x. $\qquad\square$
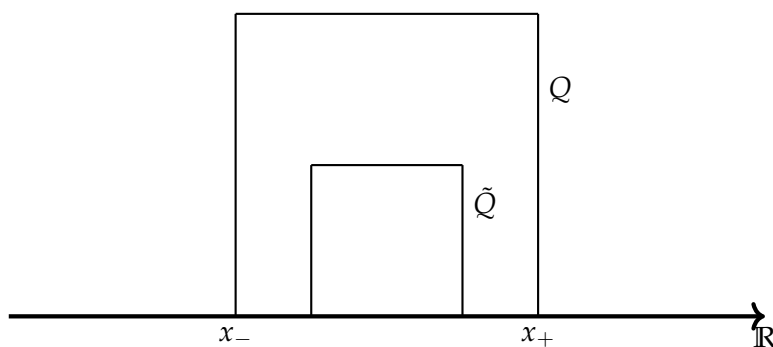
FIGURE 2.1: The square $Q(a)$ in $\mathbb{C}$.

2.6.2 *Infinite volume and complex energy*

Obtaining polynomial decay off the real axis from exponential decay at the real axis was already accomplished in [Aiz+01, Theorem 4.2]. Here we provide another proof of this fact and go on to show that *any* decay implies exponential decay for our one dimensional models.

Let $Q(a) \subseteq \mathbb{C}$ be an open square of side $a > 0$; its lower side is placed on the real axis $\mathbb{R} \subseteq \mathbb{C}$ with endpoints denoted $x_{\pm}$, $(x_+ - x_- = a)$. By $\tilde{Q}(a) \subseteq Q(a)$ we mean the symmetrically placed subsquare $\tilde{Q}(a) = Q(a/2)$, see Figure 2.1.

In this section $f = f(z)$ is a subharmonic function defined on Q. By its boundary values we simply mean

$$f(z) = \limsup_{z' \to z} f(z') \quad (\leq +\infty, z \in \partial Q).$$

We will use the maximum principle in the form

$$f(z) \leq \sup_{w \in \partial Q} f(w), \quad (z \in Q). \tag{2.25}$$

As usual, $0 < s < 1$.

We present two lemmas. The first one says that if f is bounded everywhere on ∂Q, except for some controlled divergence when the real axis is approached, then f is bounded on $\tilde{Q}$ with explicit bounds, i.e. not just by compactness. The second lemma says that if f is small everywhere on ∂Q, except very near the real axis, where it is just bounded, then f is small on $\tilde{Q}$. The two lemmas may be used in concatenation.

2.33 Lemma. *Let $M \geq 0$ and suppose*

$$\begin{aligned}
f(z) &\leq M, \quad (z \in (\partial Q)_h), \\
f(z) &\leq M\left(\frac{a}{\mathbb{Im}\{z\}}\right)^s, \quad (z \in (\partial Q)_v),
\end{aligned} \tag{2.26}$$

where $(\partial Q)_{h/v}$ are the horizontal / vertical parts of ∂Q. Then

$$f(z) \leq M(1 + s^{1-s}), \quad (z \in \tilde{Q}).$$

2.34 Lemma. *Let $m, M \geq 0$ and let $I \subseteq (\partial Q)_v$ be the union of the two verticle intervals of length $b \leq a$ next to $x_{\pm}$. Suppose*

$$\begin{aligned}
f(z) &\leq m, \quad (z \in \partial Q \setminus I), \\
f(z) &\leq M, \quad (z \in I).
\end{aligned} \tag{2.27}$$

Then

$$f(z) \leq 2^{1-s} M \left(\frac{b}{a}\right)^s + m.$$

As a preliminary to the proofs, we consider the function

$$v(z) := -\,\mathbb{Im}\{z^{-s}\}$$

defined on the first quadrant $\{z \in \mathbb{C} \mid \mathbb{Re}\{z\} > 0, \mathbb{Im}\{z\} > 0\}$. It is harmonic and the chosen branch is made clear using polar coordinates $z = re^{i\theta}$, $(r > 0, 0 < \theta < \pi/2)$ as

$$v(z) = r^{-s} \sin(s\theta).$$

In particular, $0 < v(z) < r^{-s} \sin(\frac{\pi}{2}s)$. We also note its boundary values

$$v(x) = 0, \quad v(iy) = \sin\left(\frac{\pi}{2}s\right) y^{-s}, \quad (x, y > 0).$$

The analogous function on the second quadrant is $v(-\bar{z})$.
  We will also make use of the harmonic function

$$h(z) := \frac{C}{\sin(\frac{\pi}{2}s)} \left(v(z - x_-) + v(x_+ - \bar{z})\right), \quad (z \in Q),$$

for some $C > 0$ and boundary values

$$h(z) \geq C(\mathbb{Im}\, z)^{-s}, \quad (z \in (\partial Q)_v).$$

Moreover we have

$$|z - x_{\pm}| \geq \frac{a}{2}, \quad (z \in \tilde{Q}),$$

which yields the upper bound

$$h(z) \leq 2^{1-s} C a^{-s}, \quad (z \in \tilde{Q}). \tag{2.28}$$

Proof of Lemma 2.33. We have

$$h(z) \geq M\left(\frac{a}{\mathbb{Im}\,z}\right)^s, \quad (z \in (\partial Q)_v)$$

for $C := Ma^s$. Since $h(z), M \geq 0$ anyway we have by (2.26)

$$f(z) \leq h(z) + M, \quad (z \in \partial Q).$$

Since the difference of the two sides is still subharmonic, the inequality applies to $z \in Q$ by the maximum principle (2.25). In particular, for $z \in \tilde{Q}$ we have $f(z) \leq 2^{1-s}M + M$ by (2.28) as claimed. □

Proof of Lemma 2.34. We have

$$h(z) \geq M, \quad (z \in I)$$

for $Cb^{-s} := M$. So

$$f(z) \leq h(z) + m, \quad (z \in \partial Q)$$

by (2.27) and, as before,

$$f(z) \leq 2^{1-s}M\left(\frac{b}{a}\right)^s + m, \quad (z \in \tilde{Q}).$$

□

For completeness we also include here the short proof for the well-known Combes-Thomas estimate [CT73]. This proof assumes that α_i have compact support such that there exists some $K < \infty$ with $\sup_x \|T_x\| \leq K$, which is not needed anywhere else. Since we firmly believe the Combes-Thomas estimate should hold without this assumption, we preferred not to add it as an additional assumption in the introduction but rather explain here about this qualification.

2.35 Proposition (Combes-Thomas). *We have*

$$\|G(x, y; E + i\eta)\| \leq \frac{2}{\eta}e^{-\log(1+\frac{\eta}{4K})|x-y|} \tag{2.29}$$

for all $x, y \in \mathbb{Z}$, for all $\eta > 0$ and $E \in \mathbb{R}$.

Proof. Without loss of generality let $E = 0$ and pick some $\mu > 0$ (to be specified later). Also pick some f bounded and Lipschitz such that $|f(x) - f(y)| \leq \mu|x - y|$. We use the notation $R(z) \equiv (H - z\mathbb{1})^{-1}$ for the resolvent and also define $H_f := e^{f(X)}He^{-f(X)}$ with X the position operator and $B := H_f - H$.

Then we have

$$(H_f \psi)(x) = e^{f(x)-f(x+1)} T^*_{x+1} \psi(x+1) + e^{f(x)-f(x-1)} T_x \psi(x-1)$$

so that the matrix elements of B are given by

$$B_{x,y} = (e^{f(x)-f(x+1)} - 1) T^*_{x+1} \delta_{y,x+1} + (e^{f(x)-f(x-1)} - 1) T_x \delta_{y,x-1},$$

whence it follows by Holmgren that

$$\|B\| \leq 2(e^\mu - 1)K =: b(\mu),$$

If we now choose $\mu := \log(1 + \frac{\eta}{4K})$ then evidently $b(\mu) = \frac{1}{2}\eta$ so that $\|(H_f - z\mathbb{1})\psi\| \geq (\eta - \|B\|)\|\psi\|$ and we get $\|(H_f - z\mathbb{1})^{-1}\| \leq \frac{2}{\eta}$. But $\|G(x,y;E+i\eta)\| = |e^{-f(x)+f(y)}| \|\langle \delta_x, (H_f - z\mathbb{1})^{-1} \delta_y \rangle\| \leq |e^{-f(x)+f(y)}| \frac{2}{\eta}$ and we obtain the result by the freedom in choice between f and $-f$. $\qquad \square$

We proceed to obtain the off-diagonal decay of the Green's function uniformly in $\eta \equiv \mathbb{Im}\, z$.

2.36 Proposition. *The finite-volume Green's function* $\mathbb{E}[\|G^{[x,y]}(x,y;z)\|^s]$ *decays in* $|x-y|$ *at any value of* $\mathbb{Im}\, z$ *for all* $\mathbb{Re}\, z \neq 0$.

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$ be given and define $z := \lambda + i\eta$. Define

$$f(z) := \mathbb{E}[\|G^{[x,y]}(x,y;z)\|^s]$$

for fixed x, y and $0 < s < 1$ sufficiently small such that the hypothesis for Theorem 2.32 holds.

Since the Green's function is holomorphic, it follows that $\|G^{[x,y]}(x,y;z)\|^s$ is subharmonic and hence so is f.

Let $0 < a < 4K$ be such that the interval about λ does not include zero: $0 \notin (\lambda - a, \lambda + a)$. Then due to Theorem 2.32 and the basic fact that $\|G(x,y;z)\| \leq |\mathbb{Im}\, z|^{-1}$, we know that there is some $M \geq 0$ such that the assumptions of Lemma 2.33 are fulfilled (M depends on a and the constants provided by Theorem 2.32). Hence we may conclude that $f(z) \leq M(1 + 2^{1-s})$ as z ranges in $\tilde{Q}(a)$.

Now pick any $b < a/2$. With $I := (\tilde{Q}(a))_v \cap \{z| \mathbb{Im}\, z \leq b\}$, Proposition 2.35 and Theorem 2.32 imply that on $f(z) \leq m$ for all $z \in \partial \tilde{Q}(a) \setminus I$, with $m = Cb^{-s}e^{-bd}$ with $d := |x-y|$ large enough, for some constant $C > 0$ (where we have used that $\log(1+\beta) \geq \alpha\beta$ for all $\beta \leq \alpha^{-1}$ and $\alpha < 1$). We also have still from Lemma 2.33 that $f(z) \leq M(1 + 2^{1-s})$ for all $z \in I \subseteq \tilde{Q}(a)$. Hence Lemma 2.34 applies to give us that $f(z) \leq 2^{1+2s} M(1 + 2^{1-s})(\frac{b}{a/2})^s + m$ for all $z \in \tilde{Q}(a/2)$.

Put succinctly, we find $f(z) \leq C(b^s + b^{-s}e^{-bd})$ for all $z \in \tilde{Q}(a/2)$ for some constant $C > 0$ (different constant than before), $0 < s < 1$ and $d \gg 1$, and we

are free to choose $b < a/2$. Our goal is to get decay in d. If we pick $b := d^{-s}$ for example we get the desired decay in d.   □

Finally we are ready to get the *exponential* decay of the infinite volume Green's function, which concludes the proof for the first part of Theorem 2.6.

2.37 Lemma. *For fixed z, uniformly in $|\operatorname{Im}\{z\}|$: assume that $\mathbb{E}[\|G^{[1,n]}(1,n;z)\|^s] \to 0$ as $|n| \to \infty$ for some $s \in (0,1)$. Then $\mathbb{E}[\|G(1,n;z)\|^{s'}] \leq Ce^{-\mu|n|}$ for all $n \in \mathbb{Z}$ sufficiently large, for some $s' \in (0,1)$ sufficiently small, $\mu > 0$.*

Proof. We have $H^{[x,y]} = \sum_{x'=x+1}^{y} \mathcal{T}_{x'}$ with $(\mathcal{T}_j\psi)(x) := \delta_{j,x}T_j\psi_{j-1} + \delta_{j-1,x}T_j^*\psi_j$ as the finite volume Dirichlet restriction of H onto $[x,y] \cap \mathbb{Z}$, $x < y$. Then the resolvent equation yields

$$R = R^{(-\infty,y]} - \sum_{x'=y+1}^{\infty} R^{(-\infty,y]}\mathcal{T}_{x'}R.$$

Taking the (x,y) matrix element yields (suppressing the z variable for the moment)

$$
\begin{aligned}
G(x,y) &= \\
&= G^{(-\infty,y]}(x,y) - \\
&\quad - \sum_{x'=y+1}^{\infty} G^{(-\infty,y]}(x,x')T_{x'}G(x'-1,y) + G^{(-\infty,y]}(x,x'-1)T_{x'}^*G(x',y) \\
&= G^{(-\infty,y]}(x,y)(\mathbb{1} - T_{y+1}^*G(y+1,y)),
\end{aligned}
$$

where the second line follows because the matrix elements of $R^{(-\infty,y]}$ outside of $(-\infty,y]$ are zero. Next we have again by the resolvent equation

$$R^{(-\infty,y]} = R^{[x,y]} - \sum_{x'=-\infty}^{x} R^{(-\infty,y]}\mathcal{T}_{x'}R^{[x,y]},$$

so that taking the (x,y) matrix element we get

$$
\begin{aligned}
G^{(-\infty,y]}(x,y) &= G^{[x,y]}(x,y) - \\
&\quad - \sum_{x'=-\infty}^{x} G^{(-\infty,y]}(x,x')T_{x'}G^{[x,y]}(x'-1,y) + \\
&\quad + G^{(-\infty,y]}(x,x'-1)T_{x'}^*G^{[x,y]}(x',y) \\
&= (\mathbb{1} - G^{(-\infty,y]}(x,x-1)T_x^*)G^{[x,y]}(x,y),
\end{aligned}
$$

where the second line follows again because the matrix elements of $R^{[x,y]}$ outside of $[x,y]$ are zero. So we find

$$\mathbb{E}[\|G(x,y)\|^s] \leq C\mathbb{E}[\|G^{[x,y]}(x,y)\|^{s'}],$$

where $s' = 4s$ for example. To get C one has to invoke the Hoelder inequality twice as well as the a-priori bound which is known for $\mathbb{E}[\|G(x, x + 1; z)\|^s]$ uniformly in z.

The upshot is that we may concentrate on exponential decay of

$$g(n) := \mathbb{E}[\|G^{[1,n]}(1, n)\|^s]$$

in n (by stationarity it does not matter to shift the object by $x - 1$ and call $y - x + 1 =: n$).

Our next procedure is to get a one step bound between $g(n + m)$ and $g(n)g(m)$ for any n, m:

We use again the resolvent identity to get

$$G^{[1,n+m]}(1, n + m) = -G^{[1,n+m]}(1, n)T^*_{n+1}G^{[n+1,n+m]}(n + 1, n + m),$$

(note $G^{[n+1,n+m]}(1, n + m) = 0$) and

$$G^{[1,n+m]}(1, n) = G^{[1,n]}(1, n) - G^{[1,n]}(1, n)T^*_{n+1}G^{[1,n+m]}(n + 1, n),$$

so that

$$\begin{aligned}
G^{[1,n+m]}(1, n + m) = {} & -G^{[1,n]}(1, n)T^*_{n+1}G^{[n+1,n+m]}(n + 1, n + m) + \\
& + G^{[1,n]}(1, n)T^*_{n+1}G^{[1,n+m]}(n + 1, n) \times \\
& \times T^*_{n+1}G^{[n+1,n+m]}(n + 1, n + m).
\end{aligned}$$

Taking the fractional moments expectation value, using the triangle inequality as well as the submultiplicativity of the norm, we find

$$\begin{aligned}
\mathbb{E}[\|G^{[1,n+m]}(1, n + m)\|^s] \leq {} & \\
\leq {} & \mathbb{E}[\|G^{[1,n]}(1, n)\|^s \times \\
& \times \|T^*_{n+1}\|^s\|G^{[n+1,n+m]}(n + 1, n + m)\|^s] + \\
& + \mathbb{E}[\|G^{[1,n]}(1, n)\|^s\|T^*_{n+1}\|^{2s}\|G^{[1,n+m]}(n + 1, n)\|^s \times \\
& \times \|G^{[n+1,n+m]}(n + 1, n + m)\|^s].
\end{aligned}$$

Note that in the first line, the first and last factors in the expectation are actually independent of each other and both independent of T_{n+1}. Hence that expectation factorizes. In the second line, again the first and last factors do not depend on T_{n+1} (yet the middle one does) so that we can perform the integration over T_{n+1} *first*, which would involve integration only over $\|T^*_{n+1}\|^{2s}\|G^{[1,n+m]}(n + 1, n)\|^s$. We use Hoelder once and the a-priori bound on $G(n + 1, n)$, which requires only integration over T_{n+1}. After the bound on that integral the remaining integral factorizes as the two remaining factors are independent of each other. We find

$$\mathbb{E}[\|G^{[1,n+m]}(1, n + m)\|^s] \leq$$

$$\leq C\mathbb{E}[\|G^{[1,n]}(1,n)\|^s]\mathbb{E}[\|G^{[n+1,n+m]}(n+1,n+m)\|^s]$$

with $C := \mathbb{E}[\|T_{n+1}^*\|^s] + \mathbb{E}[\|T_{n+1}^*\|^{4s}]^{\frac{1}{2}}\mathbb{E}[\|G^{[1,n+m]}(n+1,n)\|^{2s}]^{\frac{1}{2}}$. If $s < 1$ is sufficiently small and we assume that there are moments for $\|T_{n+1}^*\|^s$ for such s, then we find $0 < C < \infty$.

The crucial point now is that due to stationarity,

$$\mathbb{E}[\|G^{[1,m]}(1,m)\|^s] = \mathbb{E}[\|G^{[n+1,n+m]}(n+1,n+m)\|^s].$$

The final result is that

$$g(n+m) \leq Cg(n)g(m) \quad \forall n \in \mathbb{N}.$$

Now we use the assumption that g is decaying, which means we could find some $n_0 \in \mathbb{N}$ sufficiently large so that $\beta := Cg(n_0) < 1$. Then for any $n \in \mathbb{N}$, write $n = pn_0 + q$ (for some $p \in \mathbb{N}, q \in \mathbb{N}$ with $0 \leq q < n_0$). We find by iteration, defining $\mu := -\log(\beta) > 0$:

$$\begin{aligned}
g(n) = g(pn_0 + q) &\leq Cg(pn_0)g(q) \\
&\leq (Cg(n_0))^p g(q) \\
&(g \text{ is decaying and } q < n_0) \\
&\leq C'\beta^p = C'\exp(-\mu\frac{n-q}{n_0}) \leq C''\exp(-\mu'n).
\end{aligned}$$

$\square$

2.7 LOCALIZATION AT ZERO ENERGY

The foregoing discussion only worked at non-zero energies. There were two reasons for that:

1. We could not guarantee that the LE at zero energy are all non-zero. This goes back to Proposition 2.18.

2. We could not get an a-priori bound on the diagonal matrix element of the Green's function $G(x, x; z)$ which is uniform as $z \to 0$. This goes back to Corollary 2.30.

In order to deal with that special situation, we have to consider the Schrödinger equation at zero energy and then conclude about slightly non-zero values of the energy. We note that H is not invertible, so an expression for the resolvent like $R(0) \equiv H^{-1}$ does not make sense. Hence we use the finite-volume regularization in this section.

Thus we are considering the operator $H^{[1,L]}$ for some $L \in \mathbb{N}$, which is just H with Dirichlet boundary conditions. It is the finite $L \times L$ "band" matrix of $N \times N$ blocks given as

$$
H^{[1,L]} = \begin{pmatrix}
0 & T_2^* & & & \\
T_2 & 0 & T_3^* & & \\
& T_3 & & \ddots & \\
& & & \ddots & T_L^* \\
& & \ddots & T_L & 0
\end{pmatrix}.
$$

2.38 Proposition. *$H^{[1,2n+1]}$ is not invertible and $H^{[1,2n]}$ is invertible, for all $n \in \mathbb{N}$.*

Proof. Using the left boundary condition we have $\psi_0 = 0$. Then the Schrödinger equation at zero energy implies that the wave function at all even sites is zero, by iteration:

$$
T_2^* \psi_2 + T_1 \psi_0 = 0
$$

and so on. Thus the even sites are all zero by the left boundary condition.

If we consider $H^{[1,2n]}$, then the right boundary condition is $\psi_{2n+1} = 0$, and then again by using the Schrödinger equation the wave function at all odd sites must be zero. Hence $H^{[1,2n]}\psi = 0$ implies $\psi = 0$, that is, $H^{[1,2n]}$ is invertible.

If on the other hand we have $H^{[1,2n+1]}$, then the right boundary condition is $\psi_{2n+2} = 0$, which does not give any new information: it is merely compatible with having the wave function at all even sites zero. Hence, the wave function at odd sites is unconstrained. Once $\psi_1 \in \mathbb{C}^N$ is chosen, we use the equation to obtain the wave function's value at odd sites along the entire chain:

$$
T_3^* \psi_3 + T_2 \psi_1 = 0
$$

and so on. In conclusion $\ker(H^{[1,2n+1]}) \cong \mathbb{C}^N$ and $\ker(H^{[1,2n]}) = \{0\}$.   □

Consequently, it would not make sense to consider the resolvent for odd chain-lengths, and we shall restrict our attention to $H^{[1,2n]}$.

It turns out that it is easy to calculate the matrix elements of $R^{[1,2n]}(0) \equiv (H^{[1,2n]})^{-1}$. We only need to describe the elements of $R^{[1,2n]}(0)$ on the diagonal and above it due to the self-adjointness of $H^{[1,2n]}$.

2.39 Proposition. *The only non-zero matrix elements of $R^{[1,2n]}(0)$ on or above the diagonal are given by*

$$
G^{[1,2n]}(2k, 2l+1; 0) = (-T_{2k}^\circ T_{2k-1}) \ldots (-T_{2l+4}^\circ T_{2l+3}) T_{2l+2}^\circ
$$

for all $(k, l) \in \mathbb{N}^2$ such that $(2k, 2l + 1) \in [1, 2n]^2$ and such that $k > l$.

Proof. Let $l \in \mathbb{N}$ be given such that $2l + 1 \in [1, 2n]$. We start from the left boundary condition, which is that $G^{[1,2n]}(0, 2l + 1; z) \equiv 0$. We then evaluate the Schrödinger equation at zero energy in the left most position to find

$$T_2^* G^{[1,2n]}(2, 2l + 1; 0) = \mathbb{1}\delta_{1,2l+1}.$$

If $l = 0$ then we find $G^{[1,2n]}(2, 1; 0) = T_2^\circ$. Otherwise $G^{[1,2n]}(2, 2l + 1; 0) = 0$. We continue in this fashion to find that $G^{[1,2n]}(2k, 2l + 1; 0) = 0$ as long as $k \leq l$ and the equation above once $k = l + 1$, and then iterate for $k > l + 1$.

We cannot proceed in the same way for $G^{[1,2n]}(1, 2l + 1; 0)$ because the boundary condition on the left does not say anything about it. Instead we must use the boundary condition on the right, which says $G^{[1,2n]}(2n + 1, 2l + 1; z) \equiv 0$. In the same way we use the Schrödinger equation to conclude about $G^{[1,2n]}(2n - 1, 2l + 1; 0) = 0$:

$$T_{2n+1}^* \underbrace{G^{[1,2n]}(2n + 1, 2l + 1; 0)}_{\equiv 0} + T_{2n} G^{[1,2n]}(2n - 1, 2l + 1; 0) = \mathbb{1}\underbrace{\delta_{2n,2l+1}}_{=0}.$$

Since the right hand side will *always* be zero (due to the difference in parity), we find that

$$G^{[1,2n]}(2k + 1, 2l + 1; 0) = 0$$

for all k such that $2k + 1 \in [1, 2n]$.

In a similar way we also find that

$$G^{[1,2n]}(2k, 2l; 0) = 0$$

for all $2k$ and $2l$ within the chain, using the boundary condition on the left and then evolving to the right. $\qquad\square$

Now that we know that all diagonal matrix elements of $R^{[1,2n]}(0)$ are zero, we proceed to get an expression at non-zero energy, but still finite volume:

2.40 Proposition. *If the Lyapunov exponents are all non-zero for $z = 0$ (so we assume more than what Corollary 2.20 automatically gives), then we have*

$$\mathbb{E}[\|G^{[1,2n]}(x, x; z)\|^s] < C$$

for some constant uniformly in z (as $z \to 0$), uniformly in n, and independent of x.

Proof. We use the resolvent identity to get

$$G^{[1,2n]}(x, x; z) = G^{[1,2n]}(x, x; z) - G^{[1,2n]}(x, x; 0)$$

$$= \left\langle \delta_x, [R^{[1,2n]}(z) - R^{[1,2n]}(0)]\delta_x \right\rangle$$

(Resolvent identity)

$$= \left\langle \delta_x, z R^{[1,2n]}(z) R^{[1,2n]}(0)\delta_x \right\rangle$$

$$= \sum_{y=1}^{2n} z G^{[1,2n]}(x, y; z) G^{[1,2n]}(y, x; 0).$$

So

$$\mathbb{E}[\|G^{[1,2n]}(x, x; z)\|^s] \leq \sum_{y=1}^{2n} |z|^s (\mathbb{E}[\|G^{[1,2n]}(x, y; z)\|^{2s}])^{\frac{1}{2}} (\mathbb{E}[\|G^{[1,2n]}(y, x; 0)\|^{2s}])^{\frac{1}{2}}.$$

We now use Lemma 2.37 (namely that the finite volume complex energy Green's function is exponentially decaying) to conclude:

$$\mathbb{E}[\|G^{[1,2n]}(x, x; z)\|^s] \leq \sum_{y=1}^{2n} |z|^s (|z|^{-s} e^{-\mu_s |x-y|})(e^{-\mu'_s |x-y|})$$

$$< C.$$

We note that the bound on $\mathbb{E}[\|G^{[1,2n]}(y, x; 0)\|^{2s}]$ does not include a factor of $|z|^{-s}$ precisely because we know that at zero energy $G^{[1,2n]}(y, y; 0) = 0$. □

Our next goal is to conclude the same bound for the infinite system. By ergodicity instead of working with $[1, 2n]$ we could just as well work with $[-n+1, n]$, which also holds an even number of sites.

2.41 Proposition. *We have*

$$\text{s-lim}_{n \to \infty} R^{[-n+1,n]}(z) = R(z)$$

for all fixed $z \in \mathbb{C} \backslash \mathbb{R}$.

Proof. The operator $H^{[-n+1,n]}$ is defined as

$$H^{[-n+1,n]} = \sum_{j=-n+2}^{n} \mathcal{T}_j$$

with $(\mathcal{T}_j \psi)(x) := \delta_{j,x} T_j \psi_{j-1} + \delta_{j-1,x} T_j^* \psi_j$ so that the resolvent identity gives

$$R^{[-n+1,n]} = R + R(H - H^{[-n+1,n]})R^{[-n+1,n]}$$

$$= R + \sum_{j \in \mathbb{Z} \backslash \{-n+2, \dots, n\}} R \mathcal{T}_j R^{[-n+1,n]}.$$

Now

$$\| \sum_{j \in \mathbb{Z} \setminus \{-n+2,\ldots,n\}} \mathcal{T}_j \psi \|^2$$

$$= \sum_{l \in \mathbb{Z}} \| \sum_{j \in \mathbb{Z} \setminus \{-n+2,\ldots,n\}} \langle \delta_l, \mathcal{T}_j \psi \rangle \|^2$$

$$= \sum_{j \in \mathbb{Z}} \| \sum_{l \in \mathbb{Z} \setminus \{-n+2,\ldots,n\}} (\delta_{j,l} T_l \psi_{l-1} + \delta_{j,l-1} T_l^* \psi_l) \|^2$$

$$\leq \sum_{j \in \mathbb{Z} \setminus \{-n+2,\ldots,n\}} \| T_j \psi_{j-1} \|^2 + \sum_{j \in \mathbb{Z} \setminus \{-n+1,\ldots,n-1\}} \| T_{j+1}^* \psi_{j+1} \|^2 .$$

Next we observe that

$$(R^{[-n+1,n]}(z)\psi)_j = -z^{-1}\psi_j \quad \forall j \in \mathbb{Z} \setminus \{-n+2,\ldots,n\}$$

so that

$$\| R(z) \sum_{j=-n+2}^{n} \mathcal{T}_j R^{[-n+1,n]}(z)\psi \|^2 \leq \| R(z) \|^2 |z|^{-2} \sum_{j \in \mathbb{Z} \setminus \{-n+2,\ldots,n\}} \| T_j \psi_{j-1} \|^2 +$$

$$+ \| R(z) \|^2 |z|^{-2} \sum_{j \in \mathbb{Z} \setminus \{-n+1,\ldots,n-1\}} \| T_{j+1}^* \psi_{j+1} \|^2 .$$

But $\psi \in l^2$ and $\| R(z) \| \leq |\mathrm{Im}\{z\}|^{-1}$, so that the right hand side converges to zero as $n \to \infty$, all at fixed $z \neq 0$. $\qquad\square$

2.42 Corollary. *We have*

$$\lim_{n \to \infty} \| G^{[-n+1,n]}(x,x;z) \| = \| G(x,x;z) \|$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$, so that using Fatou's lemma,

$$\mathbb{E}[\| G(x,x;z) \|^s] \leq \lim_{n \to \infty} \mathbb{E}[\| G^{[-n+1,n]}(x,x;z) \|^s] .$$

But since the bound Proposition 2.40 is uniform in n, we find that $\mathbb{E}[\| G(x,x;z) \|^s]$ is bounded uniformly in $|\mathrm{Im}\{z\}|$ under the same assumptions on the Lyapunov spectrum as in Proposition 2.40.

As a result, we may now go back to the previous section and apply all the proofs there, extending them so that it holds uniformly including in the limit $z \to 0$ as long as the Lyapunov spectrum does not include zero, at all *real energies, including zero energy. This concludes the proof of the second part of Theorem 2.6.*

THE BULK-EDGE CORRESPONDENCE FOR DISORDERED CHIRAL CHAINS

3.1 INTRODUCTION

We now turn to a topological study of the model analyzed in Chapter 2 for localization. This is a fairly general model obeying *chiral symmetry* (class AIII of Table 1.1) in *dimension one*, and which exhibits moreover *strong disorder*. The symmetry of the Hamiltonian is matched by that of the state, which is at half-filling. The prototypical model in the same class, yet lacking disorder, is the Su-Schrieffer-Heeger model of polyacetylene [SSH79]: This is an alternating chain of sites or, in other words, a bipartite lattice, along which electrons hop between sub-lattices, either to the right or to the left, but without experiencing an on-site potential. As a result, the Hamiltonian H and its opposite, $-H$, are unitarily conjugate. In particular the energy zero is special, being the fixed point under the sign flip, and it singles out half-filling. If that energy lies in a spectral gap of H, the model exhibits topological properties which depend on the (constant) ratio of the amplitudes for hopping in the two directions (from a given sub-lattice). How much of this survives when the hopping changes randomly from bond to bond? And what if the disorder is actually so strong as to close the spectral gap about zero? At first sight, disorder seems to induce localization throughout the spectrum, as it certainly is the case for on-site randomness [KS80] which corresponds to the class A, and is topologically trivial. The truth for class AIII however is that localization may fail, but need not, at the one special energy, i.e. zero. This is enough to rescue the topological features; in fact Hamiltonians may be loosely viewed as belonging to a same topological phase as long as they can be deformed while preserving localization (mobility gap) at zero energy. Put differently: The closing of the mobility gap about zero defines the phase boundaries.

More precisely, we will cast the crucial assumption of a mobility gap Definition 1.27. We then consider two quantities associated to the bulk and the edge of the material respectively, and show that they are well-defined and integer-valued, whence they serve as indices. We show that they agree (*bulk-edge correspondence*), that is, we prove a type of Principle 1.8 and finally that the index can be characterized in terms of the Lyapunov spectrum of the time-independent Schrödinger equation.

This chapter is organized as follows. We start in Section 3.2 by describing the mathematical setting, defining chiral symmetry and its features, including the bulk and edge invariants. We make the assumption a mobility gap and state the main result about bulk-edge correspondence in that context. We also reformulate the index in terms of Lyapunov exponents. Section 3.3 is an aside about the more restrictive case of a spectral gap and the resulting simplifications. For completeness the even more special, translation invariant case is addressed there, too. In Section 3.4 we return to the general case by reformulating the edge index, so as to conclude the proof of
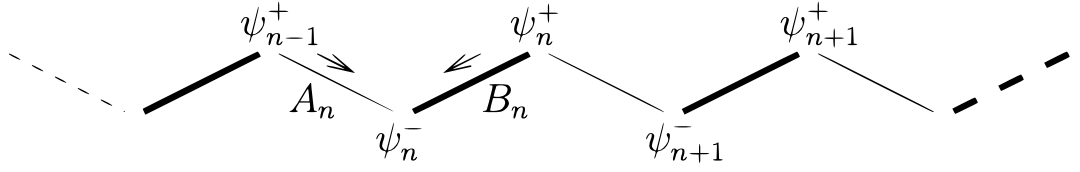
FIGURE 3.1: The lattice underlying the model is an alternating chain. The hopping amplitude s $A_n, B_n \in GL_N(\mathbb{C})$ are in direction of the arrows. In the opposite direction the adjoint matrices apply.

the main result in Section 3.5. In Section 3.6 we extend our result beyond Dirichlet boundary conditions.

We note that related to the framework of stochastically translation invariant Hamiltonians [BvS94]: In [Mon+14] a similar model has been discussed in the strong disorder regime and its phases explored numerically on the basis of the index formula and analytically on the basis of the Lyapunov exponents, and seen to agree; bulk-edge correspondence is shown in [PS16a] for the case of the spectral gap. The appropriate bulk index was introduced in [PS16b] and moreover shown to be well-defined and continuous w.r.t. the Hamiltonian in the case of a mobility gap. Finally we note that in [Bro+98; Ful+11] the role of the Lyapunov exponents at zero energy is addressed, including that of a zero exponent in some model. More precise comments will be made later in connection with the definition of indices.

3.2   THE MODEL AND THE RESULTS

In this section we shall specify the setting of chiral one-dimensional systems, define the relevant indices, and formulate the main result on the bulk-edge correspondence.

3.2.1   *One-dimensional chiral systems*

The lattice underlying the model is an alternating chain, where particles perform nearest-neighbor hopping (see Figure 3.1). The single-particle Hilbert space of a tight-binding model is

$$\mathcal{H} = \mathcal{K} \otimes \mathbb{C}^2 \ni \begin{pmatrix} \psi_n^+ \\ \psi_n^- \end{pmatrix}_{n \in \mathbb{Z}},$$

with $\mathcal{K} := \ell^2(\mathbb{Z}, \mathbb{C}^N)$, where $\mathbb{C}^N$ stands for the internal degrees of freedom of each site and $\mathbb{C}^2$ for their grouping into dimers. The Hamiltonian is

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \tag{3.1}$$

with S acting on $\mathcal{K}$ as

$$(S\psi^+)_n := A_n \psi_{n-1}^+ + B_n \psi_n^+ ; \tag{3.2}$$

hence

$$(S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n\,. \tag{3.3}$$

We assume A_n, $B_n \in GL_N(\mathbb{C})$, whence solutions to $S\psi^+ = 0$ are determined by ψ^+_n for any n. Otherwise, i.e. if some matrices were singular, the corresponding bonds would be effectively broken; put differently, the model would have edges within.

The chiral symmetry

$$\Pi := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

is a symmetry of the Hamiltonian, in the sense that

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0\,.$$

It implies

$$f(H)\Pi = \Pi f(-H) \tag{3.4}$$

for any Borel bounded function $f = f(\lambda)$.

The many-particle state is the Fermi sea at half-filling, meaning that the Fermi level is at $\lambda = 0$. Its single-particle density matrix thus is the Fermi projection $P := \chi_{(-\infty,0)}(H)$, where χ_I is the characteristic function of the set $I \subseteq \mathbb{R}$.

We assume localization for H at the Fermi level and formulate that condition deterministically, as we've seen in Definition 1.27. This means that no further recourse to probabilistic arguments will be made in proofs, and that the indices are properties of the individual system, and not just of the statistical ensemble.

3.1 Assumption. *For some μ, $\nu > 0$ we have*

$$\sum_{n,n'\in\mathbb{Z}} \|P(n,n')\|(1+|n|)^{-\nu}e^{\mu|n-n'|} \leq C < +\infty\,,$$

where $(\delta_n)_{n\in\mathbb{Z}}$ is the canonical (position) basis of $\ell^2(\mathbb{Z})$ and the map $P(n,n') = \langle\delta_n, P\delta_{n'}\rangle :$ $\mathbb{C}^{2N} \to \mathbb{C}^{2N}$ acts between the internal spaces of dimers of n' and n. Here, $\|\cdot\|$ is the trace norm of such maps. That is, we are assuming P is weakly-local in the sense of Definition A.4. Moreover, the same bound applies to the Fermi projections of the edge Hamiltonians introduced below.

3.2 Assumption. $\lambda = 0$ *is not an eigenvalue of H.*

3.3 REMARK. These two assumptions are trivially fulfilled in the spectral gap case. In this work we are rather interested in the mobility gap regime, which is the typical one at large disorder.

In physical terms Assumption 3.2 states that every state is either a particle or a hole state, thus prompting the notation

$$P_- := P\,, \qquad P_+ := \chi_{(0,\infty)}(H)$$

and the rephrasing

$$P_- + P_+ = \mathbb{1}\,, \qquad P_-\Pi = \Pi P_+ \tag{3.5}$$

of the assumption and of the chiral symmetry.

We will define shortly a *bulk index* $\mathcal{N}$ associated to H, as well as an *edge index* $\mathcal{N}_a$ associated to its truncation to the half-lattice to the left of an arbitrary point $a \in \mathbb{Z}$. In this particular case, Principle 1.8 becomes:

> **3.4 Theorem** (Bulk-Edge Correspondence). *Under Assumptions 3.1 and 3.2 we have*
>
> $$\mathcal{N} = \mathcal{N}_a\,.$$

We anticipate that $\mathcal{N}_a$ will be manifestly an integer. Hence so is $\mathcal{N}$, and $\mathcal{N}_a$ is independent of a. In the proof though, we will first establish the independence and then obtain the result by passing to the limit $a \to +\infty$. The two steps will be carried out in Sections 3.4 and 3.5.

RELATION TO THE MODEL FROM CHAPTER 2.   Any realization of the random Hamiltonian (2.2) is of the same (deterministic) type as considered in (3.1), at least up to a unitary map implementing notational changes. To avoid notation conflict let us denote the Hamiltonian of (2.2) by H'. The Hilbert space in Chapter 2 there was given by $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$, where each chirality sector corresponded to either the even or the odd sites of $\ell^2(\mathbb{Z})$. We define the unitary map $U : \mathcal{H} \to \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ via

$$U\left(\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}\right)_n := \begin{cases} \psi_m^+ \,, & (m = 2n+1) \\ \psi_m^- \,, & (m = 2n) \end{cases},$$

so that $UHU^* = H'$ upon setting

$$T_n := \begin{cases} B_m^* \,, & (m = 2n+1) \\ A_m \,, & (m = 2n) \end{cases}.$$

Thus H' is a random version of our current model. The reason we used H' instead of H is that to estimate Green's functions (our ultimate goal in the localization proof of Chapter 2) there is no need to distinguish between two types of sites, and hence no reason to group them into dimers. We note in passing that in the previous chapter, the chirality symmetry operator was $\Pi' := (-1)^X$ with X the position operator, and we still had $\{H', \Pi\} = 0$.

BULK INDEX.   Let $\Sigma := \operatorname{sgn} H$, and recall the the non-commutative derivative from Definition 1.25.

3.5 Definition. The bulk index is

$$\mathcal{N} := \mathrm{i}\,\frac{1}{2}\,\mathrm{tr}(\Pi\Sigma\partial\Sigma)\,. \tag{3.6}$$

The index is well-defined and admits alternative formulations. One is in terms of $\Sigma = P_+ - P_-$ with $P_\pm$ as above, which by the way is the unitary in the (unique) polar decomposition of H: $H = \Sigma|H|$; the other is in terms of the unitary U in the decomposition of S: $S = U|S|$ with $|S| \equiv \sqrt{S^*S}$.

3.6 Lemma.   *(a) The relation between the polar decompositions of H and of S is*

$$\Sigma = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}\,. \tag{3.7}$$

(b) Both $\partial P_\pm$ and ∂U are trace class.

(c) The index can be expressed as

$$\mathcal{N} = \mathrm{i}\,\mathrm{tr}\,U^*\partial U \tag{3.8}$$
$$= -\,\mathrm{i}\,\mathrm{tr}\,\Pi P_+\partial P_- - \mathrm{i}\,\mathrm{tr}\,\Pi P_-\partial P_+\,. \tag{3.9}$$

Eq. (3.8) is equivalent to the definition of $\mathcal{N}$ given in ([PS16b], Proposition 4.2) as the pairing between the class in K_1 defined by U and a Chern character. The index may be interpreted as a non-commutative generalization of the winding number of the unitary U, because $\mathrm{i}[\Lambda, \cdot]$ reduces to the derivative along the unit circle of quasi-momenta in the translation invariant case, cf. Proposition 3.18 below as well as ([AS01],Theorem 7) and references therein. The interpretation is extended in [Mon+14] to systems of class AIII in odd space dimensions.

EDGE INDEX.   The model is truncated to $\mathbb{Z}_a := (-\infty, a] \subset \mathbb{Z}$ with Hilbert space $\mathcal{H}_a := \ell^2(\mathbb{Z}_a, \mathbb{C}^{2N})$. Of course the choice of a ought not be of physical relevance. Here we keep this choice free and explicit in the notation since we shall eventually take the limit $a \to +\infty$ which helps associating edge objects with bulk ones.

The truncation procedure can be recast algebraically as follows. Let $\iota_a : \ell^2(\mathbb{Z}_a) \hookrightarrow \ell^2(\mathbb{Z})$ be the natural injection, whence $\iota_a^* : \ell^2(\mathbb{Z}) \twoheadrightarrow \ell^2(\mathbb{Z}_a)$ is the restriction operator. Thus ι_a is an isometry, but not a unitary: In fact

$$\iota_a^*\iota_a = \mathbb{1}_{\mathcal{H}_a}\,, \qquad \iota_a\iota_a^* = \chi_a\,, \tag{3.10}$$

where χ_a is the projection $\chi_a : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ associated to the subspace $\ell^2(\mathbb{Z}_a) \subseteq \ell^2(\mathbb{Z})$. Thus

$$\chi_a\iota_a = \iota_a\,, \qquad \iota_a^* = \iota_a^*\chi_a$$

and

$$\chi_a A\chi_a : \mathrm{im}\,\chi_a \to \mathrm{im}\,\chi_a = \iota_a^* A\iota_a \tag{3.11}$$

for any operator A acting on $\ell^2(\mathbb{Z})$ or on some of its descendant spaces. In particular, letting

$$S_a := \iota_a^* S \iota_a,\tag{3.12}$$

we then have

$$H_a := \iota_a^* H \iota_a = \begin{pmatrix} 0 & S_a^* \\ S_a & 0 \end{pmatrix}.\tag{3.13}$$

More general boundary conditions will be discussed in Section 3.6.

3.7 REMARK. As $a \to +\infty$ an ever larger portion $\mathbb{Z}_a$ of $\mathbb{Z}$ is retained, resulting in the limit $H_a \to H$ in the strong resolvent sense, as will be seen and used.

From (3.13) we still have

$$\{H_a, \Pi\} = 0\tag{3.14}$$

and its consequence (3.4).

Let $P_{0,a} := \chi_{\{0\}}(H_a)$ be the spectral projection for $\lambda = 0$, i.e. its eigenprojection if it is an eigenvalue. We note that Assumption 3.2 generically fails for the edge system.

3.8 Definition. The edge index is

$$\mathcal{N}_a := \mathrm{tr}(\Pi P_{0,a}).\tag{3.15}$$

3.9 REMARK. Π maps $\mathrm{im}\, P_{0,a} = \ker H_a$ into itself. Indeed, $H_a \psi = 0$ implies $H_a \Pi \psi = 0$ by (3.14). In particular $\mathcal{N}_a \in \mathbb{Z}$ as anticipated, and the index may be written as

$$\mathcal{N}_a = \dim \ker S_a - \dim \ker S_a^*,\tag{3.16}$$

which is finite by (3.2, 3.3). We note in passing that in the spectral gap regime ([PS16a], Section 1.2) has the same definition for $\mathcal{N}_a$ as the pairing between the K_0 class defined by $P_{0,a}$ and a zero-dimensional Chern character. Despite appearances, Eq. (3.16) is not a Fredholm index in general, simply because S_a is not Fredholm in the mobility gap regime of Assumption 3.1. Indeed, $\mathrm{im}\, S_a$ is not closed then.

3.10 Example. Figure 3.1 should be viewed as just one example of a lattice leading to a chiral Hamiltonian (3.1). Other lattices may do so too. An example is shown in Figure 3.2.

The model is of the form (3.1, 3.2) with $N = 2M$ upon grouping amplitudes as bispinors:

$$\psi_n^- = \begin{pmatrix} \varphi_{2n-1}^- \\ \varphi_{2n}^- \end{pmatrix}, \quad \psi_n^+ = \begin{pmatrix} \varphi_{2n}^+ \\ \varphi_{2n+1}^+ \end{pmatrix}.$$

Comparing Figures 3.1 and 3.2 yields

$$A_n = \begin{pmatrix} T_{2n-2}^+ & T_{2n-1} \\ 0 & T_{2n-1}^+ \end{pmatrix}, \quad B_n = \begin{pmatrix} T_{2n}^- & 0 \\ T_{2n} & T_{2n+1}^- \end{pmatrix}.$$

In particular $A_n, B_n \in GL_N(\mathbb{C})$ iff $T_m^\pm \in GL_M(\mathbb{C})$.
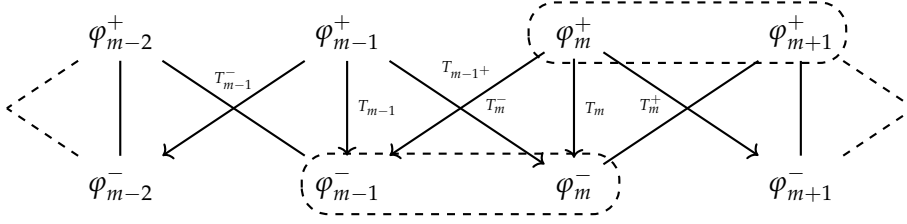

FIGURE 3.2: A lattice with hopping amplitudes $T_m, T_m^\pm \in GL_M(\mathbb{C})$ in direction of the arrows. The blobs indicate a regrouping of the sites relating the lattice to that of Figure 3.1.

3.2.2 *The zero-energy Lyapunov spectrum*

We conclude this section with an alternate formulation of the index. To this end we consider the equation $S\psi^+ = 0$ as a finite difference equation for sequences $\psi^+ : \mathbb{Z} \to \mathbb{C}^N$, foregoing normalizability. By (3.2) and using $A_n \in GL_N(\mathbb{C})$ the equation is solved recursively to the left,

$$S\psi^+ = 0 \qquad \Longleftrightarrow \qquad \psi_{n-1}^+ = T_n \psi_n^+, \qquad (n \in \mathbb{Z}) \tag{3.17}$$

with $T_n := -A_n^{-1} B_n$. (Likewise solvability to the right would call for $B_n \in GL_N(\mathbb{C})$.) The associated transfer matrix is

$$T(n) := T_{n-1} \cdots T_0, \qquad (n < 0).$$

The Lyapunov exponent of a vector $v \in \mathbb{C}^N$ is then given as

$$\chi(v) := \limsup_{n \to -\infty} \frac{1}{|n|} \log \|T(n)v\| \tag{3.18}$$

with $\chi(v) \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\pm\infty\}$ and $\chi(0) = -\infty$. The set

$$V_\chi := \{v \in \mathbb{C}^N | \chi(v) \le \chi\} \tag{3.19}$$

is a linear subspace which is non-decreasing in $\chi \in \bar{\mathbb{R}}$. Let $\chi_N \le \cdots \le \chi_1$ be the values of χ at which $\chi \mapsto \dim V_\chi$ jumps, listed repeatedly according to the jump in dimension.

3.11 Assumption. *Let 0 not be in the Lyapunov spectrum, i.e. $\chi_i \ne 0$, $(i = 1, \dots, N)$.*

3.12 Theorem. *Under Assumption 3.11 the edge index equals the number of negative Lyapunov exponents:*

$$\mathcal{N}_a = \#\{i | \chi_i < 0\} \tag{3.20}$$

for any $a \in \mathbb{Z}$.

The proof, which will be given in Section 3.4, will rest on the vanishing lemma $\ker S_a^* = \{0\}$ for Dirichlet boundary conditions (3.12); more general boundary conditions will be addressed in Section 3.6. In Chapter 2 we gave conditions such that in (3.18) lim sup can be replaced almost surely by lim for all $v \in \mathbb{C}^N$, actually with the limits being finite for $v \neq 0$ and with simple Lyapunov spectrum. Moreover, V_χ is the spectral subspace of the self-adjoint matrix $\Lambda := \lim_{n\to\infty}(T(n)^*T(n))^{1/2n}$ and of eigenvalues $\leq e^\chi$. Finally, in Chapter 2 we shoed that Assumption 3.11 implies Assumption 3.1.

> 3.13 REMARK. As a complement to (3.20), the edge index $\mathcal{N}_a$ may also be expressed in terms of the equation $S^*\psi^- = 0$, in which case it is given by the number of positive Lyapunov exponents. In fact, introducing $\varphi_n^- = B_n^*\psi_n^-$ that equation is $\varphi_{n-1}^- = \tilde{T}_n\varphi_n^-$, where $\tilde{T}_n = T_n^\circ$ and $M^\circ = (M^*)^{-1}$ (using $B_n \in GL_N(\mathbb{C})$, too). Its spaces are
>
> $$\tilde{V}_\chi = V_{-\chi}^\perp, \tag{3.21}$$
>
> provided χ is not in the Lyapunov spectrum. In particular, $\tilde{\chi}_i = -\chi_{N+1-i}$. Eq. (3.21) follows from $\tilde{T}(n) = T(n)^\circ$ and $\tilde{\Lambda} = \Lambda^{-1}$.

> 3.14 REMARK. The usual scenario of a phase transition is that of a spectral gap closing on the Fermi level. Theorem 3.12 gives a different scenario, whereby the Fermi level may not lie in a gap throughout the transition. More precisely, the Lyapunov spectrum associated to $(H - \lambda)\psi = 0$ consists of $2N$ exponents $\{\gamma_i\}_{i=1}^{2N}$ and is even under sign flip $\gamma \mapsto -\gamma$ (counting multiplicity). For $\lambda \neq 0$ the spectrum is moreover simple, implying that 0 is not an exponent and thus localization. For $\lambda = 0$ however the exponents are those of $S\psi^+ = 0$ and their flips, $\{\chi_i, -\chi_i\}_{i=1}^N$. In particular 0 may, but need not be an exponent. If it isn't, Theorem 3.12 applies, but if it becomes one, the localization length diverges at $\lambda = 0$, signaling the topological phase transition.

3.3   THE SPECTRAL GAP CASE

In the case of a spectral gap the analysis simplifies, as was noted in [PS16a] and discussed in terms of K-theory. We present here an equivalent simplification as a contrast to the general case, to be proven later. Until the end of this section we forgo definition (3.2) and allow S to be any operator $S : \mathcal{K} \to \mathcal{K}$ for which $[\Lambda, S]$ is trace class; this being a generalization since the commutator is of finite rank in the former case. The spectral gap condition means that Assumptions 3.1 and 3.2 are now replaced by the stronger condition

$$0 \notin \sigma(H). \tag{3.22}$$

Thus H is Fredholm, and so is S in view of

$$\ker H = \ker S \oplus \ker S^*, \qquad \operatorname{im} H = \operatorname{im} S^* \oplus \operatorname{im} S.$$

We first discuss the bulk index:

Proof of Lemma 3.6. The two kinds of gap (spectral or mobility) naturally affect the proof of part (b), but not at all that of the other claims. We will thus first prove (a) and, assuming (b), also (c); at last we will return to (b).

For (a) we just observe that

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} = \begin{pmatrix} 0 & |S|U^* \\ U|S| & 0 \end{pmatrix} = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \begin{pmatrix} |S| & 0 \\ 0 & U|S|U^* \end{pmatrix} \equiv \Sigma|H|$$

by uniqueness of the polar decomposition. In turn (a) implies

$$[\Lambda, \Sigma] = \begin{pmatrix} 0 & [\Lambda, U^*] \\ [\Lambda, U] & 0 \end{pmatrix},$$

because Λ descends from $\mathcal{K}$, as well as $\Pi\Sigma[\Lambda, \Sigma] = U^*[\Lambda, U] \oplus [\Lambda, U]U^*$. In particular the two parts of (b) are seen to be equivalent by $\Sigma = P_+ - P_-$, and (3.8) follows. As for the other part of (c), we have

$$2\mathcal{N} = \operatorname{tr}\Pi P_+[\Lambda, \Sigma] - \operatorname{tr}\Pi P_-[\Lambda, \Sigma]. \tag{3.23}$$

The first term is the sum of two equal ones,

$$\operatorname{tr}\Pi P_+[\Lambda, \Sigma] = \operatorname{tr}\Pi P_+[\Lambda, P_+] - \operatorname{tr}\Pi P_+[\Lambda, P_-] = -2\operatorname{tr}\Pi P_+[\Lambda, P_-];$$

indeed, by $P_+ = (P_+)^2$, and (3.5) we have

$$\operatorname{tr}\Pi P_+[\Lambda, P_\pm] = \operatorname{tr}\Pi P_+[\Lambda, P_\pm]P_- = \mp\operatorname{tr}\Pi P_+\Lambda P_-.$$

Likewise can be said about the last term in (3.23):

$$\operatorname{tr}\Pi P_-[\Lambda, \Sigma] = 2\operatorname{tr}\Pi P_-[\Lambda, P_+].$$

We obtain (3.9).

Finally we are left with (b). In the mobility gap case this will be done in Appendix A.8. In the spectral gap case, $[\Lambda, U]$ is trace class using the following lemma, because $[\Lambda, S]$ already is. □

3.15 Lemma. *Let $A : \mathcal{K} \to \mathcal{K}$ be Fredholm. If $[A, \Lambda]$ is trace class, then so is $[A|A|^{-1}, \Lambda]$.*

Proof. Let $B : \mathcal{K} \to \mathcal{K}$ with $[B, \Lambda]$ trace class. The commutator property is inherited under taking adjoints and products; and if $B \geq \varepsilon > 0$ also under taking inverses, $[B^{-1}, \Lambda] = -B^{-1}[B, \Lambda]B^{-1}$. In the latter case the property is also passed down to $B^{-1/2}$ because of

$$B^{-1/2} = C \int_0^\infty \lambda^{-1/2}(B + \lambda)^{-1}\,d\lambda$$

$(C^{-1} = \int_0^\infty \lambda^{-1/2}(1+\lambda)^{-1}\,d\lambda)$. In particular, the property applies to $A^*A = |A|^2$ and to $|A|^{-1}$, where we used that A is Fredholm through $A^*A \geq \varepsilon > 0$; finally it applies to $A|A|^{-1}$. □

We notice that the index (3.8) is independent of the choice of the switch function, this being tantamount to the vanishing of the expression when Λ is replaced by a function of compact support. Then, in fact, Λ would already be trace class and the claim seen by expanding the commutator. In particular we may pick $\Lambda = \mathbb{1} - \chi_a$, χ_a being the projection seen in (3.10). We then conclude by ([ASS94], Theorems 6.1, 5.2) that the bulk index is that of a pair of projections:

$$
\begin{aligned}
\mathcal{N} &= -\operatorname{tr}(U^*\chi_a U - \chi_a) \\
&= \operatorname{ind}(\chi_a, U^*\chi_a U) = \operatorname{ind}(\chi_a U \chi_a : \operatorname{im}\chi_a \to \operatorname{im}\chi_a) \\
&= \operatorname{ind}(\iota_a^* U \iota_a).
\end{aligned}
$$

We now turn to the edge index and first state a definition: Let

$$
\sigma_{ess}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda\mathbb{1} \text{ is not Fredholm}\}
$$

be the essential spectrum of a (not necessarily self-adjoint) closed operator A. It enjoys stability under compact perturbations K, i.e. $\sigma_{ess}(A) = \sigma_{ess}(A+K)$, [BB89].

3.16 Lemma. *Suppose $A : \mathcal{K} \to \mathcal{K}$ is such that $[A, \Lambda]$ is compact, where Λ is some (and hence any) switch function. Then*

$$
\sigma_{ess}(\iota_a^* A \iota_a) \subset \sigma_{ess}(A). \tag{3.24}
$$

In particular, if A is Fredholm, then so is $\iota_a^* A \iota_a$.

Proof. We have

$$
A = \Lambda A \Lambda + (\mathbb{1} - \Lambda)A(\mathbb{1} - \Lambda) + K \tag{3.25}
$$

with $K = \Lambda A(\mathbb{1} - \Lambda) + (\mathbb{1} - \Lambda)A\Lambda = [\Lambda, A](\mathbb{1} - \Lambda) + (\mathbb{1} - \Lambda)[A, \Lambda]$ compact.

The injection ι_a is matched by another one, $\tilde{\iota}_a$, corresponding to the complementary half-line $\mathbb{Z} \setminus \mathbb{Z}_a$. For $\Lambda = \chi_a$ the first two terms on the RHS of (3.25) are

$$
\chi_a A \chi_a + (\mathbb{1} - \chi_a)A(\mathbb{1} - \chi_a) \cong \iota_a^* A \iota_a \oplus \tilde{\iota}_a^* A \tilde{\iota}_a
$$

because of (3.11) and the unitarity of $\iota_a \oplus \tilde{\iota}_a$. Eq. (3.24) follows. □

For $A = S$ the lemma yields that S_a is Fredholm by (3.12). Thus $\operatorname{im} S_a$ is closed, $\ker S_a^* = \operatorname{coker} S_a$, and the edge index (3.16) is a Fredholm index,

$$
\mathcal{N}_a = \operatorname{ind} S_a. \tag{3.26}
$$

The proof of the bulk-edge correspondence, $\mathcal{N} = \mathcal{N}_a$, is concluded by

3.17 Lemma.

$$\text{ind } \iota_a^* S \iota_a = \text{ind } \iota_a^* U \iota_a \,. \tag{3.27}$$

Proof. We consider the interpolating family

$$S_t = tU + (1-t)S = U(t\mathbb{1} + (1-t)|S|) \,,$$

$(0 \le t \le 1)$, with $S_0 = S$, $S_1 = U$. The hypothesis of Lemma 3.16 holds true for $t = 0, 1$, as remarked before, and thus true for $0 \le t \le 1$. Moreover, by that lemma, $\iota_a^* S_t \iota_a$ is Fredholm if S_t is. That however is immediate from

$$S_t^* S_t = (t\mathbb{1} + (1-t)|S|)^2 \ge \delta \,,$$

for some $\delta > 0$. Thus (3.27) holds true by the continuity of the index. □

THE TRANSLATION INVARIANT CASE.   We here assume that $S : \mathcal{K} \to \mathcal{K}$ commutes with the shift operator, whence S is of Toeplitz form in the position basis $(\delta_n)_{n \in \mathbb{Z}}$ of $\ell^2(\mathbb{Z})$,

$$\langle \delta_n, S\delta_{n'} \rangle = S_{n-n'} \,,$$

where the maps $S_m : \mathbb{C}^N \to \mathbb{C}^N$ may themselves be viewed as matrices. For simplicity we assume that they rapidly decay in $m \in \mathbb{Z}$.

3.18 Proposition. *The sum*

$$S(z) := \sum_{m \in \mathbb{Z}} S_m z^{-m}$$

is absolutely convergent for $|z| = 1$, i.e. for z on the unit circle $\mathcal{C}$. Then the spectral gap condition (3.22) holds iff $\det S(z)$ vanishes nowhere on $\mathcal{C}$. In that case the index (3.8) is (the negative of) the winding number of $\mathcal{C} \ni z \mapsto \det S(z) \in \mathbb{C}$ (Zak number [Zak89]).

3.19 Example. In the translation invariant case, (3.2) reduces to $S(z) = Az^{-1} + B$. The spectral gap condition requires that

$$T := -A^{-1}B \tag{3.28}$$

has no eigenvalue of unit modulus, or equivalently that 1 is not among its singular values. Since $z \mapsto w = z^{-1}$ reverses the orientation of $\mathcal{C}$, the index equals the winding number of $w \mapsto \det(Aw + B)$ and, by the argument principle, the number of zeroes in $\mathcal{C}$, i.e. the algebraic number of eigenvalues w of T with $|w| < 1$; equivalently it is the number of zeroes of $\det S(z)$ with $|z| > 1$.

Proof of Proposition 3.18. In line with the general assumptions of this section, we first verify that $[\Lambda, S]$ is trace class. This follows from

$$\langle \delta_n, [\Lambda, S]\delta_{n'} \rangle = (\Lambda(n) - \Lambda(n'))S_{n-n'} \,,$$

and from (A.16) by the reasoning used in the proof of Lemma 3.6 (b). Second, we discuss the gap condition (3.22): By Bloch decomposition,

$$S = \int_{\mathcal{C}}^{\oplus} S(z) \frac{\mathrm{d}\, s}{2\pi} \tag{3.29}$$

with $\mathrm{d}\, s = |\,\mathrm{d}\, z| = -iz^{-1}\,\mathrm{d}\, z$ and w.r.t. $\mathcal{K} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$, $\ell^2(\mathbb{Z}) \cong \int_{\mathcal{C}}^{\oplus} \mathbb{C}\,\mathrm{d}\, s/2\pi$. The isomorphism is given by

$$\psi(z) = \sum_{n \in \mathbb{Z}} z^{-n} \psi_n$$

with Parseval identity

$$\sum_{n \in \mathbb{Z}} \varphi_n^* \psi_n = \int_{\mathcal{C}} \varphi(z)^* \psi(z) \frac{\mathrm{d}\, s}{2\pi}. \tag{3.30}$$

Moreover $(S\psi)(z) = S(z)\psi(z)$ is readily verified, proving (3.29). Since $S(z)$ is smooth we have $\sigma(H) = \bigcup_{z \in \mathcal{C}} \sigma(H(z))$. The claim on the spectral gap property now follows, and we assume its validity in the sequel.

Next we compute the index (3.8). The fibers $U(z) = S(z)/|S(z)|$ are smooth as well, whence the sum

$$U(z) =: \sum_{m \in \mathbb{Z}} U_m z^{-m}$$

has rapidly decaying coefficients U_m. Thus

$$\begin{aligned}
\mathcal{N} &= \operatorname{tr} U^*[\Lambda, U] = \sum_{n \in \mathbb{Z}} \operatorname{tr} \langle U\delta_n, [\Lambda, U]\delta_n \rangle \\
&= \sum_{n,m} \operatorname{tr} |U_{m-n}|^2 (\Lambda(m) - \Lambda(n)) = \sum_{n,k} \operatorname{tr} |U_k|^2 (\Lambda(n+k) - \Lambda(n)) \\
&= \sum_k k \operatorname{tr} |U_k|^2,
\end{aligned}$$

where we used $\sum_n \Lambda(n+1) - \Lambda(n) = 1$. Using $-z\partial_z U(z) = \sum_m m U_m z^{-m}$ and (3.30) we obtain

$$\begin{aligned}
\mathcal{N} &= \frac{i}{2\pi} \int_{\mathcal{C}} \mathrm{d}\, z \, \operatorname{tr} U(z)^* \partial_z U(z) \\
&= \frac{i}{2\pi} \int_{\mathcal{C}} \mathrm{d}\, z \, \frac{\frac{\mathrm{d}}{\mathrm{d} z} \det U(z)}{\det U(z)} = \frac{i}{2\pi} \int_{\mathcal{C}} \frac{\mathrm{d}\, \det S(z)}{\det S(z)},
\end{aligned}$$

because $\det |S(z)| > 0$ has no winding.   □

3.4   GENERALIZED STATES OF ZERO ENERGY

Zero energy edge states will be extended to bulk states which are however not ℓ^2 on the other side of the edge, which we refer to as *generalized states*. For that purpose, let us consider the (bulk) equation $S\psi^+ = 0$ as a finite difference equation for $\psi^+ : \mathbb{Z} \to$

$\mathbb{C}^N$. The edge index can be characterized in terms of their behavior at $-\infty$. In fact we have:

3.20 Lemma.

$$\mathcal{N}_a = \dim V\,, \qquad V := \{\psi^+ : \mathbb{Z} \to \mathbb{C}^N | S\psi^+ = 0 \text{ and } \psi_n^+ \text{ is } \ell^2 \text{ at } n \to -\infty\}\,. \tag{3.31}$$

In particular $\mathcal{N}_a$ is seen to be independent of a, independently of Theorem 3.4. Moreover every $\psi \in V$ is uniquely determined by its restriction to $\mathbb{Z}_a$ for any a.

Proof. By (3.16) we are led to determine the null spaces of S_a and S_a^* separately. The two operators act by (3.2, 3.3) with $n \leq a$; (3.2) comes without boundary conditions, whereas (3.3) is supplemented by $\psi_{a+1}^- = 0$. By $B_n \in GL_N(\mathbb{C})$ the difference equation $S\psi^+ = 0$ can be solved recursively to the right (ψ_{n-1}^+ determines ψ_n^+) whereas $S^*\psi^- = 0$ can be solved recursively to the left. Hence $\ker S_a = V$, whereas the Dirichlet boundary condition implies $\ker S_a^* = \{0\}$. $\qquad\square$

We next show that the edge index may be computed using a finite-box truncation.
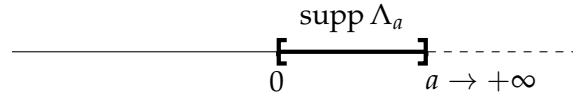

FIGURE 3.3: The finite box used in order to approximate the edge index.

3.21 Lemma. *The common value of $\mathcal{N}_a$, $(a \in \mathbb{Z})$ is*

$$\mathcal{N}^\sharp = \lim_{a \to +\infty} \operatorname{tr}(\Pi\Lambda_a P_{0,a})\,. \tag{3.32}$$

Here we denoted by Λ_a the switch function Λ when viewed as a multiplication operator on $\ell^2(\mathbb{Z}_a)$ and its descendant spaces. We have

$$\iota_a \Lambda_a = \Lambda \iota_a\,, \qquad \Lambda_a \iota_a^* = \iota_a^* \Lambda\,. \tag{3.33}$$

The switch Λ_a roughly restricts states to $n \geq 0$ within $n \leq a$, thereby singling out a finite box growing with a (see Figure 3.3). The lemma asserts that edge states are unaffected by this restriction for $a \to +\infty$, because they are concentrated near the edge $n = a$. Consequently the task is to show $\|(\mathbb{1} - \Lambda_a)P_{0,a}\| \to 0$, $(a \to +\infty)$.

Proof of Lemma 3.21. Let V be the linear space of solutions $\psi = (\psi^+, 0)$ seen in (3.31). By

$$\dim V \leq N < \infty \tag{3.34}$$

all norms on V are equivalent, and we pick one, $\|\cdot\|_V$. For any $b \in \mathbb{Z}$, let $\mathcal{R}_b : V \to \mathcal{H}_b$ be defined by restriction. That map is injective by the conclusion of the previous lemma. Therefore and by (3.34) we have

$$\|\mathcal{R}_b \psi\| \geq c_b \|\psi\| \tag{3.35}$$

for some $c_b > 0$. Elements $\psi \in V$ can not be ℓ^2 at $n \to +\infty$ as well, unless $\psi = 0$, since that would imply a solution of $H\psi = 0$, which is ruled out by Assumption 3.2. We thus have

$$\|\mathcal{R}_b \psi\| \to \infty, \qquad (b \to +\infty). \tag{3.36}$$

For $b < a$ we denote by $\iota_{ab} : \ell^2(\mathbb{Z}_b) \hookrightarrow \ell^2(\mathbb{Z}_a)$ the injection (extension by zero); correspondingly $\iota_{ab}^* : \ell^2(\mathbb{Z}_a) \to \ell^2(\mathbb{Z}_b)$ is the restriction operator. These operators are analogous to those seen in (3.10); in fact $\iota_b = \iota_a \iota_{ab}$. For b large enough we have $(\mathbb{1} - \Lambda)(1 - \chi_b) = 0$ by disjointness of support. Using (3.33) we get $\mathbb{1} - \Lambda = (\mathbb{1} - \Lambda)\iota_b \iota_b^* = \iota_b(\mathbb{1} - \Lambda_b)\iota_b^*$ and thus, by multiplication with ι_a^* and ι_a from left and right,

$$\mathbb{1} - \Lambda_a = \iota_{ab}(\mathbb{1} - \Lambda_b)\iota_{ab}^* \tag{3.37}$$

(b large, $a > b$).

Next we note that $P_{0,a} : \mathcal{H}_a \to \mathcal{H}_a$ induces a natural map $\mathcal{P}_{0,a} : \mathcal{H}_a \to V$, because for any $\psi_a \in \mathcal{H}_a$ the image $P_{0,a}\psi_a$ is the left tail of a solution in V which it fully determines. It satisfies

$$\mathcal{R}_b \mathcal{P}_{0,a} = \iota_{ab}^* P_{0,a}. \tag{3.38}$$

We next claim for any b

$$\|\mathcal{R}_b \mathcal{P}_{0,a}\| \to 0, \qquad (a \to +\infty). \tag{3.39}$$

We have to show that $\|\mathcal{R}_b \mathcal{P}_{0,a}\psi_a\| \to 0$ for every sequence $\psi_a \in \mathcal{H}_a$ with $\|\psi_a\| = 1$. Clearly the sequence at hand is at least bounded in a (as well as in $b > a$) and so is $\|\mathcal{P}_{0,a}\psi_a\|$ because of (3.38) and (3.35). By compactness (dim $V < \infty$) we have $\mathcal{P}_{0,a}\psi_a \to \hat{\psi}$, ($a \to +\infty$) upon passing to a subsequence. Hence $\mathcal{R}_b \mathcal{P}_{0,a}\psi_a$ has a limit $\mathcal{R}_b \hat{\psi}$ as $a \to +\infty$, which inherits the boundedness in b. This contradicts (3.36) unless $\hat{\psi} = 0$, thus proving (3.39).

That in turn implies, by taking b large and using (3.37, 3.38),

$$\|(\mathbb{1} - \Lambda_a)P_{0,a}\| \to 0, \qquad (a \to +\infty).$$

The same then holds in trace class norm $\|\cdot\|_1$ because of Claim A.28 and rank $P_{0,a} = \dim V$. Finally (3.32) follows by taking a (redundant) limit of (3.15). $\qquad\square$

Proof of Theorem 3.12. By (3.19) and the definition of χ_i, as well as by Assumption 3.11, the RHS of (3.20) equals $\dim V_\chi = \dim V_0$ for some $\chi < 0$. We also recall (3.17) and Definition (3.31). We have the inclusions $V_\chi \subseteq V$ for any $\chi < 0$ and $V \subseteq V_0$, since $\ell^2 \subseteq \ell^\infty$ at $-\infty$. The conclusion now follows from Lemma 3.20. $\quad\square$

3.5 PROOF OF THE BULK-EDGE CORRESPONDENCE

3.22 Lemma. *As $a \to +\infty$,*

$$\iota_a P_{\pm,a} \iota_a^* - P_\pm \xrightarrow{\;s\;} 0\,, \tag{3.40}$$

$$[\iota_a P_{\pm,a} \iota_a^* - P_\pm, \Lambda] \xrightarrow{\;t\;} 0\,. \tag{3.41}$$

where s, t denote strong and trace norm convergence respectively, and $P_{\pm,a} := \chi_{(0,\infty)}(\pm H_a)$.

We postpone the proof of this lemma to Appendix A.8 and proceed to that of the main result.

Proof of Theorem 3.4. The operator Λ_a introduced in (3.33) is of finite rank. The basic identity is

$$\mathrm{tr}(\Pi \Lambda_a) = 0\,, \tag{3.42}$$

which follows by evaluating the trace in the position basis and by using $\mathrm{tr}_{\mathbb{C}^{2N}} \Pi = 0$. We insert $\mathbb{1} = P_{0,a} + P_{+,a} + P_{-,a}$ with $P_{\pm,a} \equiv \chi_{(0,\infty)}(\pm H_a)$ and obtain

$$\mathrm{tr}(\Pi \Lambda_a) = \mathrm{tr}(\Pi \Lambda_a P_{0,a}) + \mathrm{tr}(\Pi \Lambda_a P_{+,a}) + \mathrm{tr}(\Pi \Lambda_a P_{-,a})\,. \tag{3.43}$$

The first term tends to $\mathcal{N}^\sharp$ as $a \to +\infty$ by (3.32). The second one is

$$\begin{aligned}
\mathrm{tr}(\Pi \Lambda_a P_{+,a}) &= \mathrm{tr}(\Pi P_{-,a} \Lambda_a P_{+,a}) = \mathrm{tr}(\Pi P_{-,a} [\Lambda_a, P_{+,a}]) \\
&= \mathrm{tr}(\iota_a \Pi P_{-,a} [\Lambda_a, P_{+,a}] \iota_a^*) = \mathrm{tr}(\Pi \iota_a P_{-,a} \iota_a^* [\Lambda, \iota_a P_{+,a} \iota_a^*])\,,
\end{aligned}$$

where we used $\Pi P_{-,a} = P_{+,a} \Pi P_{-,a}$ (see (3.4, 3.14)), $\mathrm{tr}_{\mathcal{H}_a} A = \mathrm{tr}_{\mathcal{H}}(\iota_a A \iota_a^*)$ for any operator A on $\mathcal{H}_a$, as well as (3.10, 3.33). We next use (3.40, 3.41) together with the implication

$$X_a \xrightarrow{\;s\;} X, \; Y_a \xrightarrow{\;t\;} Y \quad \Longrightarrow \quad X_a Y_a \xrightarrow{\;t\;} XY\,, \tag{3.44}$$

(see Claim A.32) to conclude

$$\lim_{a \to +\infty} \mathrm{tr}(\Pi \Lambda_a P_{+,a}) = \mathrm{tr}(\Pi P_- [\Lambda, P_+])\,.$$

We likewise have for the third term in (3.43)

$$\lim_{a \to +\infty} \operatorname{tr}(\Pi \Lambda_a P_{-,a}) = \operatorname{tr}(\Pi P_+ [\Lambda, P_-])$$

and thus find from (3.9, 3.42) that

$$0 = \mathcal{N}^\sharp - \mathcal{N}.$$

□

In comparing the proofs of the cases of spectral and mobility gaps the following may be noted: While in the spectral gap case bulk and edge may be related at any finite a, in the mobility gap case the relation emerges at $a \to +\infty$, and this is made possible by Lemma 3.21.

3.6   MORE GENERAL BOUNDARY CONDITIONS

In this section we generalize Theorem 3.4 to (largely) arbitrary boundary conditions. In the case of a *spectral gap* (see Section 3.3) we implement them by relaxing (3.12) to

$$S_a := \iota_a^* S \iota_a + S_{BC}, \tag{3.45}$$

where S_{BC} is any compact operator. Since the Fredholm index is invariant under compact perturbations, the change does not affect the edge index (3.26) and Theorem 3.4 remains true.

In the *mobility gap* regime and in the context of the model with nearest neighbor hopping Eqs. (3.1–3.3) more general boundary conditions are obtained by allowing S_{BC} to affect only sites a and $a - 1$; we thus allow the hopping matrices A_n, B_n of the boundary $n = a$ to become singular, whereas they remain regular for $n \leq a - 1$. The edge Hamiltonian (3.13) remains defined with S_a as in (3.12). Thus S_a acts as in (3.2) for $n \leq a$; likewise does S_a^* as in (3.3), except for $n = a$ where $(S_a^* \psi^-)_a = B_a^* \psi_a^-$.

3.23 Example. The case of regular A_a, B_a corresponds to the edge Hamiltonian discussed so far. In relation to Figure 3.1 it amounts to breaking the thin bond between dimers a and $a + 1$. To set $A_a = 0$ amounts to further remove one more dimer; to set instead $B_a = 0$ to break the thick bond of the last dimer.

3.24 Proposition. *The edge index $\mathcal{N}_a$ is the same for all boundary matrices (A_a, B_a). In particular it is the same as in (3.31).*

Proof. By (3.16) we have $\mathcal{N}_a = \dim W^+ - \dim W^-$ with

$$W^\pm = \{\psi^\pm : \mathbb{Z} \to \mathbb{C}^N | \psi^\pm \text{ is } \ell^2 \text{ at } n \to -\infty \text{ and}$$
$$\text{satisfies (3.46), resp. (3.47, 3.48)}\},$$
$$A_{n+1} \psi_n^+ + B_{n+1} \psi_{n+1}^+ = 0, \qquad (n \leq a - 1) \tag{3.46}$$
$$A_{n+1}^* \psi_{n+1}^- + B_n^* \psi_n^- = 0, \qquad (n \leq a - 1) \tag{3.47}$$

$$B_a^* \psi_a^- = 0. \tag{3.48}$$

(The first equation is $(S\psi^+)_n = 0$ for $n \le a$ after shifting the index by one.)

Introducing $\varphi_n^- = B_n^* \psi_n^-$, the eqs. (3.46, 3.47) are solved iteratively to the left for $n \le a - 2$ by

$$\psi_n^+ = T_{n+1}\psi_{n+1}^+, \quad \varphi_n^- = T_{n+1}^\circ \varphi_{n+1}^-$$

with $T_n = -A_n^{-1}B_n$ and $M^\circ = (M^*)^{-1}$. In particular,

$$\langle \psi_n^+, \varphi_n^- \rangle = \langle T_{n+1}\psi_{n+1}^+, T_{n+1}^\circ \varphi_{n+1}^- \rangle = \langle \psi_{n+1}^+, \varphi_{n+1}^- \rangle,$$

which means that the LHS is constant in $n \le a - 1$. Actually we have

$$\langle \psi_n^+, \varphi_n^- \rangle = 0, \quad (n \le a - 1) \tag{3.49}$$

because of the ℓ^2-condition, and not by resorting to (3.48). To sum up: Since $A_n, B_n \in GL_N(\mathbb{C})$, $(n \le a - 1)$ the solutions of (3.46, 3.47) with $n \le a - 2$ are bijectively determined by $\psi_{a-1}^+, \varphi_{a-1}^- \in \mathbb{C}^N$; among them, those that are ℓ^2 correspond to subspaces $V^\pm$ (independent of A_a, B_a) that are not only orthogonal, as seen from (3.49), but actually complementary. This follows from (3.21) with $\chi = 0$.

It remains to impose Eqs. (3.46–3.48) for $n = a - 1$. The claim now follows by applying the following lemma by identifying $A_a = A$, $B_a = B$, $\psi_{a-1}^+ = \psi^+$, $\psi_a^+ = \tilde\psi^+$, $B_{a-1}^* \psi_{a-1}^- = \psi^-$, $\psi_a^- = \tilde\psi^-$. $\square$

3.25 Lemma. *Let an orthogonal decomposition $\mathbb{C}^N = V^+ \oplus V^-$ and matrices $A, B \in \mathrm{Mat}_N(\mathbb{C})$ be given. We consider the set of equations*

$$A\psi^+ + B\tilde\psi^+ = 0, \tag{3.50}$$
$$A^*\tilde\psi^- + \psi^- = 0, \quad B^*\tilde\psi^- = 0 \tag{3.51}$$

in the the unknowns $\psi^\pm \in V^\pm$, $\tilde\psi^\pm \in \mathbb{C}^N$. Then

$$\dim\{(\psi^+, \tilde\psi^+) | (3.50)\} - \dim\{(\psi^-, \tilde\psi^-) | (3.51)\} = \dim V^+. \tag{3.52}$$

Proof. Let P be the orthogonal projection onto V^+, whence $\dim \mathrm{im}\, P = \dim V^+$. Then the dimensions on the LHS of (3.52) are unaffected upon supplementing (3.50, 3.51) with $P\psi^+ = \psi^+$ and $P\psi^- = 0$ respectively, while solving for $(\psi^\pm, \tilde\psi^\pm) \in \mathbb{C}^N \oplus \mathbb{C}^N = \mathbb{C}^{2N}$. We are then left computing

$$I := \dim \ker T_+ - \dim \ker T_-$$

with $T_+ : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$, $T_- : \mathbb{C}^{2N} \to \mathbb{C}^{3N}$ given by

$$
T_+ = \begin{pmatrix} \mathbb{1} - P & 0 \\ A & B \end{pmatrix}, \quad T_- = \begin{pmatrix} P & 0 \\ \mathbb{1} & A^* \\ 0 & B^* \end{pmatrix}.
$$

Using that

$$
T_-^* = \begin{pmatrix} P & \mathbb{1} & 0 \\ 0 & A & B \end{pmatrix}, \quad \begin{pmatrix} P & \mathbb{1} - P & 0 \\ 0 & A & B \end{pmatrix}
$$

have the same range, we have

$$
\dim \operatorname{im} T_-^* = \dim V^+ + \dim \operatorname{im} T_+ \,;
$$

using also that $\dim \ker T_- = 2N - \dim \operatorname{im} T_-^*$ we find

$$
I = \dim \ker T_+ + \dim \operatorname{im} T_+ - 2N + \dim V^+ = \dim V^+ \,.
$$

$\square$

STRONGLY-DISORDERED FLOQUET TOPOLOGICAL SYSTEMS

4.1 INTRODUCTION

After having studied the one dimensional AIII cell of Table 1.1 in the mobility gap regime, we embark on a journey outside of this table, and into Floquet topological systems, as described in Section 1.2. As noted above, they were originally designed to induce topological properties on a trivial sample through the periodic driving [OA09], and have recently become a topic of intense study when it was realized that this driving also allowed to engineer new topological phases of matter that have no static counterpart [Rud+13]; some proposals for experimental observation of these phases in cold atoms were recently suggested [Nat+17; Que+17].

So far the main prerequisite to define topological indices in Floquet systems has been the presence of a gap in the spectrum of the unitary Floquet propagator (1.5), describing time evolution in the bulk after one period of driving. In this context Principle 1.8 was first established in clean systems and then extended to weakly disordered samples, for various dimensions and symmetries [Car+15; Fru16; FM16; GT18; RH17; Rud+13; SS17]. By analogy with static systems, the effect of disorder is to progressively fill the spectral gap of the propagator by localized states [HJS09; Tit+16], and all the previous results work only as long as the spectral gap remains open.

This chapter deals with two-dimensional systems with no particular symmetry (class A of [AZ97]). We address the problem of strong disorder, when the gap is completely filled by localized states (see Figure 4.1). This is the mobility gap regime Definition 1.28 adapted to localized unitary operators, and we adopt this (deterministic) approach here as well. The fractional moment condition has been already established for unitary random operators [AM10; HJS09], as well as some numerical evidence of localization in Floquet topological models [Tit+16]. We note in passing that [PS16b] also studied strongly-disordered unitary topological systems in the bulk, however, they used a covariant probabilistic framework.

The zeroth order step here is to verify that the so-called relative construction (1.7), developed in the spectral gap case in dimension two [GT18; Rud+13; SS17], can be extended to the mobility gap regime. This construction reduces the physical unitary evolution to a time periodic propagator in the bulk, which has a well-defined index. This requires a logarithm of the Floquet propagator, that we prove to be well-defined with a branch-cut in the mobility gap. Thanks to the estimates coming from localization, the logarithm is weakly-local Definition A.7–its matrix elements in the position basis have rapid off-diagonal decay, and possible diagonal blowup. With this we can adapt the proof in [GT18] of the bulk-edge correspondence from the spectral gap case, in which the Combes-Thomas estimate was used instead of localization.

The physical implementation of this relative construction is however not straightforward, and one can look for situations where it may be circumvented. For clean

samples, a Floquet system is actually an insulator only when the Floquet propagator is exactly the identity operator $\mathbb{1}$, for which the relative construction is not required. In the spectral and mobility gap cases the system is not insulating anymore and the relative constructions somehow subtracts the other transport contributions from the topological one [Tau18]. In the strongly disordered case the analogue of $\mathbb{1}$ is to consider a Floquet propagator that is completely localized, namely that its entire spectrum is a mobility gap (see Figure 4.1(b) and (b')). It was shown in [Tit+16] that in contrast to the static case [EGS05], such systems may still have edge modes and topological properties. Moreover the indices can be computed without the relative construction and have a nice physical interpretation in terms of quantized orbital magnetization in the bulk [Nat+17] and quantized pumping at the edge [Tit+16]. The first result of this chapter is a rigorous definition of these indices and a proof of their respective bulk-edge correspondence, i.e. Principle 1.8.

Our second result is to show that any mobility gap situation can actually be reduced to a fully localized case, for which the previous indices can be used and circumventing again the relative construction. This reduction is done through the smooth functional calculus with a particular function that stretches the mobility gap onto the entire circle (as in [SS17] who however used this construction only for the edge, in the spectral gap regime).

We finally show that the indices defined in this approach coincide with the ones of the relative construction (which are also defined in this case). To that end we show the continuity of the bulk relative index along a specific path of deformation. We believe this continuity result in the mobility gap regime is important because it joins an extremely short list of results: the deterministic constancy of the quantum Hall conductivity w.r.t. the Fermi energy proven in [EGS05]. Thus Theorem 4.16 opens interesting perspectives in the investigation of the topology of deterministic mobility gapped systems, of which very little is known.

We note in passing that quantum walks, namely finite sequences of unitary operators, can also be seen as discrete-time Floquet systems, for which topological indices have been already defined in clean and weakly disordered models [ABJ17; TD15]. In some cases the Floquet formalism can be applied to quantum walks [DFT17; SS17], so that our result should in principle cover the strongly disordered version of these quantum walks.

The chapter is organized as follows. After describing the setting and stating the results mentioned above in Section 4.2, we detail their respective proofs in Sections 4.3 to 4.5. Appendix A.2 is of particular importance to us, and in particular Corollary A.20, since it is the reason why our invariants are well-defined at all.

4.2   SETTING AND MAIN RESULTS

We re-describe the setting of Section 1.2 in somewhat more detail now, and finally specify to the mobility gap regime.

Let a time-dependent periodic Hamiltonian $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ be given where $\mathcal{H} := \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ is the (bulk) Hilbert space and $N, d \in \mathbb{N}_{\geq 1}$ are the (fixed) number
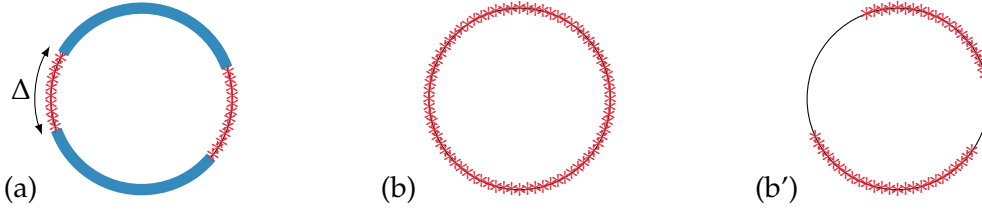
FIGURE 4.1: Examples of spectrum for the Floquet propagator $U(1)$. (a) Typical situation with a mobility gap Δ (possibly several), represented by red crosses. The remaining part of the spectrum, in blue, can be arbitrary. For completely localized operators, the mobility gap is the entire circle (b) or possibly with spectral gaps (special case of a mobility gap) (b'). The relative construction applies to all cases, but can be avoided in (b) and (b') through magnetization and pumping indices. The stretch function construction maps situation (a) to (b) by stretching Δ onto $S^1 \setminus \{1\}$ and mapping the remainder of the spectrum to 1.

of degrees of freedom per lattice site and the space dimension. Here we use $S^1 \cong [0,1]/\{0 \sim 1\}$. We assume the following conditions about H throughout:

4.1 Assumption. *(Continuity) $t \mapsto H(t)$ is strongly-continuous except for a finite number of jump discontinuities.*

Similarly to Definition 1.1, we also assume locality Definition A.2 for this time-dependent Hamiltonian:

4.2 Assumption. *(Locality) There are some constants $C, \mu > 0$ such that for any $t \in S^1$ we have*

$$\|H(t)_{xy}\| < C\,e^{-\mu\|x-y\|} \qquad (x, y \in \mathbb{Z}^d)\,. \tag{4.1}$$

As detailed in Section 1.2, to deal with Floquet systems we consider the unitary propagator $U : [0,1] \to \mathcal{B}(\mathcal{H})$ generated by H, that is, the unique solution to $i\dot{U} = HU$ with initial condition $U(0) = \mathbb{1}$ (it is a fact that even though H is periodic, U need not be and so its domain is a-priori $[0,1]$). A well-defined topological phase exists depending on spectral or dynamical properties that $U(1)$ may or may not satisfy. Such a phase was established (see e.g. [GT18; Rud+13; SS17]) in the presence of a spectral gap:

4.3 Definition (Spectral gap). *$U(1)$ has a spectral gap iff its spectrum is not the entire circle:*

$$\sigma(U(1)) \neq S^1\,. \tag{4.2}$$

Since $\sigma(U(1))$ is a closed subset of S^1, the existence of a point outside it implies the existence of a whole open arc outside of it, which is called *a spectral gap*. In contrast to the parametrization of the domain of H, here we rather use $S^1 \cong \{z \in \mathbb{C} \mid |z| = 1\}$.

The main point of the this chapter is to relax the assumption for the analysis of topological properties of these systems from spectral gap to the following definition, which is essentially Definition 1.28 for localized unitaries.

> **4.4 Definition** (Mobility gap). The arc $\Delta \subseteq \mathbb{S}^1$ is a mobility gap for $U(1)$ iff (1) there is some constant $\mu > 0$ such that for any $\varepsilon > 0$ there is some constant $0 \leq C_\varepsilon < \infty$ such that we have
>
> $$\sup_{g \in B_1(\Delta)} \|g(U(1))_{x,y}\| \leq C_\varepsilon \, e^{-\mu\|x-y\|+\varepsilon\|x\|} \qquad (x,y \in \mathbb{Z}^d) \tag{4.3}$$
>
> where $B_1(\Delta)$ is the space of all Borel maps $g : \mathbb{S}^1 \to \mathbb{C}$ which are constant outside of Δ and obey $|g(z)| \leq 1$ for all $z \in \mathbb{S}^1$; and (2) all eigenvalues of $U(1)$ within Δ are of finite degeneracy.

This definition implies also Definition A.7 of weakly-local operators.

> 4.5 REMARK. The supremum over g implies dynamical localization within Δ by considering the family of functions $\mathbb{S}^1 \ni \lambda \mapsto \lambda^n \chi_\Delta(\lambda)$ indexed by n. Consequently $U(1)$ has pure point spectrum within Δ, due to a RAGE theorem analogue for unitaries [EV83; HJS09]. This is detailed in Appendix A.9.

THE EDGE SAMPLE.   Following Section 1.3 but with slightly different notation, in the edge picture the Hilbert space is $\mathcal{H}_E = \ell^2(\mathbb{N} \times \mathbb{Z}^{d-1}) \otimes \mathbb{C}^N$ describing independent electrons on a half-space. The canonical embedding $\iota : \mathcal{H}_E \hookrightarrow \mathcal{H}$ and truncation $\iota^* : \mathcal{H} \twoheadrightarrow \mathcal{H}_E$ satisfy $|\iota|^2 = \mathbb{1}$ on $\mathcal{H}_E$ and $|\iota^*|^2 = \chi_\mathbb{N}(X_1)$. For A acting on $\mathcal{H}$ we denote the corresponding truncated operator on $\mathcal{H}_E$ by $\widehat{A} := \mathrm{ad}_\iota(A)$. In particular the edge Hamiltonian is

$$H_E(t) := \widehat{H(t)} \tag{4.4}$$

corresponding to Dirichlet boundary condition, although other conditions could be implemented in principle. H_E inherits some properties of H, in particular it satisfies (4.1), and generates a unitary propagator U_E on $\mathcal{H}_E$ through $\mathrm{i}\dot{U}_E = H_E U_E$ and $U_E(0) = \mathbb{1}$. All these properties rely only on the fact that H is local by (4.1) and not on the existence of (any) gap of $U(1)$, so they remain true in the mobility gap regime.

4.2.1 *The relative construction*

Here we finally specify to the case $d = 2$ and no symmetry. For this case, the bulk-edge correspondence is established in [GT18] first when $U(1) = \mathbb{1}$ (so $\mathbb{S}^1 \setminus \{1\}$ is a "special" spectral gap) and then when $U(1)$ has a general spectral gap. The latter case was reduced to the first one by constructing a relative evolution, generated by an effective Hamiltonian. It turns out that the same procedure can be followed in the mobility gap regime. The effective Hamiltonian is defined through a logarithm of the one-period propagator

$$H_\lambda := \mathrm{i}\log_\lambda(U(1)) \tag{4.5}$$

where $\lambda \in \Delta$ is chosen inside the mobility gap and used as a branch cut for the (principal) logarithm. We show in Corollary 4.20 that H_λ is *weakly-local* as a weaker property of being local (i.e. satisfying (4.1)). Note that in the spectral gap case the discontinuity of the logarithm, which is otherwise analytic, maybe ignored since it occurs out of the spectrum of $U(1)$; since analytic functions of local operators are local [AG98, Appendix D], the logarithm is local too. For us, however, H_λ is merely weakly-local via localization. This is enough to define the indices, as we shall see.

For two operators $A, B : [0,1] \to \mathcal{B}(\mathcal{H})$, not necessarily periodic, we define the concatenation in time $A\#B : [0,1] \to \mathcal{B}(\mathcal{H})$ by (1.6). The two operators occur consecutively in time, the second backwards. The relative bulk Hamiltonian is then defined by $H^{\mathrm{rel}} := 2(H\#H_\lambda)$. The effective Hamiltonian being time-independent, its unitary propagator is $U_\lambda(t) = \mathrm{e}^{-\mathrm{i}tH_\lambda}$ and satisfies $U_\lambda(1) = U(1)$ by construction. It follows that the relative evolution generated by H^{rel}, $U^{\mathrm{rel}} = U\#U_\lambda$, satisfies $U^{\mathrm{rel}}(1) = \mathbb{1}$. Similarly, the relative edge Hamiltonian is defined by $H_{\mathrm{E}}^{\mathrm{rel}} := 2(\widehat{H}\#\widehat{H_\lambda}) = \widehat{H^{\mathrm{rel}}}$, which generates $U_{\mathrm{E}}^{\mathrm{rel}}$. Note that $\widehat{H_\lambda}$ is the truncation of H_λ and *not* the logarithm of $U_{\mathrm{E}}(1)$ for which we do not assume a mobility gap to exists. In particular $U_{\mathrm{E}}(1) \neq \mathrm{e}^{-\mathrm{i}\widehat{H_\lambda}}$ so that $U_{\mathrm{E}}^{\mathrm{rel}}$ is not given by their concatenation.

Finally recall the non-commutative three-dimensional winding W defined for a unitary loop $\mathfrak{U} : \mathsf{S}^1 \to \mathcal{B}(\mathcal{H})$ is given by the formula

$$W(\mathfrak{U}) \equiv -\frac{1}{2}\int_0^1 \mathrm{tr}\,\dot{\mathfrak{U}}\mathfrak{U}^*[\mathfrak{U}_{,1}\mathfrak{U}^*, \mathfrak{U}_{,2}\mathfrak{U}^*]. \tag{4.6}$$

Here we use the shorthand notation $A_{,\alpha} \equiv \partial_\alpha A$ with ∂_α the non-commutative derivative as in Definition 1.25. This is the disordered version of (1.8), since now we do not assume the translation invariance of the unitary $\mathfrak{U}$ and so it is an operator on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ rather than a map $\mathbb{T}^d \to \mathrm{Mat}_N(\mathbb{C})$. (4.6) is equivalent to a pairing between the K_1 class defined by $\mathfrak{U}$ and a three-dimensional Chern character, as detailed [PS16b]. The normalization is chosen such that W takes values in $\mathbb{Z}$.

Then the zeroth step in our analysis is:

4.6 Theorem. *Under Assumptions 4.1 and 4.2 and additionally assuming Definition 4.4 holds for $U(1)$: (1) the bulk index $\mathcal{I} \equiv W(U^{\mathrm{rel}})$ is finite, integer-valued, and independent of the choices of switch functions and branch cut $\lambda \in \Delta$; (2) the edge index*

$$\mathcal{I}_{\mathrm{E}} \equiv \mathrm{i}\,\mathrm{tr}\,U_{\mathrm{E}}^{\mathrm{rel}}(1)^*\partial_2 U_{\mathrm{E}}^{\mathrm{rel}}(1) \tag{4.7}$$

is finite, integer-valued and independent of the choice of switch function; (3) the bulk-edge correspondence holds

$$\mathcal{I} = \mathcal{I}_{\mathrm{E}}. \tag{4.8}$$

If H_λ were local, then so would be U_λ, U^{rel} and $U_{\mathrm{E}}^{\mathrm{rel}}$ and this theorem would be already covered by [GT18, Theorem 3.8]. Here instead we need to adapt the proof to weakly-local operators Definition A.7. In particular we need to show that $\partial_j A$ also have a so-called confining property when A is weakly-local, so that expressions involved in (4.6) and (4.7) are trace-class. Apart from that point, the other properties of

$\mathcal{I}$ and $\mathcal{I}_E$ as well as the bulk-edge correspondence follow the same route as in [GT18]. The proof of Theorem 4.6 is detailed in Section 4.3.2.

In conclusion, in contrast to the integer quantum Hall effect (IQHE henceforth) where the mobility gap bulk-edge correspondence is quite different in the spectral and mobility gap regime (cf. [EG02] vs. [EGS05]), the relative construction works similarly for both cases in Floquet topological insulators, once one generalizes from local to weakly-local operators and shows the desired properties of the discontinuous logarithm, Corollary 4.20.

> **4.7 REMARK.** A possible objection to the relative construction is the following. In defining $\mathcal{I}_E$, the truncated generator of the bulk relative propagator, H_E^{rel} (which depends on the logarithm of the *bulk* evolution), and not just the truncated bulk Hamiltonian, H_E, has been used, so that Theorem 4.6 actually connects between $\mathcal{I}$ and an edge index which contains significant information from the bulk. It is thus legitimate to ask for a more independent formulation where bulk and edge indices are strictly separated already at the level of the Hamiltonians, without intertwining their evolutions. The conclusion is however that the relative construction is perfectly valid since the stretch function construction coincides with it in the end, as we show in Theorem 4.16.

> **4.8 REMARK.** Even if the system has a gap, it is of interest to probe the system when placing the branch cut of the logarithm within the localized spectrum, in analogy with the explanation of the plateaus of the IQHE. Thus, if one day we could experimentally determine the position of the branch cut, our results would explain the corresponding plateaus which will be measured in $\mathcal{I}$.

4.2.2   *Completely localized systems*

As preparation for Section 4.2.3, and also of independent interest, we discuss one possible scenario in which the relative construction may be avoided when defining the topological indices.

Interestingly, the formula (4.6) is finite also for unitary maps which are *not* periodic (whose domain is $[0,1]$ rather than $\mathbb{S}^1$), but is an integer (and hence possibly an index) only when the map is periodic [GT18, Proposition 3.3], *e.g.* for the relative evolution U^{rel} which by construction has $U^{\text{rel}}(1) = U^{\text{rel}}(0) = \mathbb{1}$. In this section we propose an alternative definition of the bulk index when the physical evolution is not periodic, $U(1) \neq \mathbb{1}$, avoiding the relative construction.

> **4.9 Definition.** (*Completely localized unitaries*) We call a unitary $\mathfrak{U} \in \mathcal{U}(\mathcal{H})$ *completely localized* iff its whole spectrum is one mobility gap in the sense of Definition 4.4, except possibly for one point of $\mathbb{S}^1$.

Namely the mobility gap arc Δ for $\mathfrak{U}$ as in Definition 4.4 is $\Delta = \mathbb{S}^1$ or $\Delta = \mathbb{S}^1 \setminus \{z_0\}$ for some $z_0 \in \mathbb{S}^1$. In particular $\mathfrak{U}$ has only pure point spectrum. We denote by $\mathcal{E} \subset \mathbb{S}^1$ its (countable) set of eigenvalues and by $P_z \equiv \chi_{\{z\}}(\mathfrak{U})$ the associated spectral projection onto an eigenvalue $z \in \mathcal{E}$. By Definition 4.4 all z's have $\dim \operatorname{im} P_z < \infty$ except possibly

for P_{z_0}. The latter appears in the approach of Section 4.2.3 below, but never carries any topology (Chern number) by construction. In principle we could extend Definition 4.9 to a finite number of infinite degenerate eigenvalues, but we would then have to require that each of them has a trivial topology. This is the so-called anomalous phase [Rud+13; Tit+16]. Here we stick to a single z_0 to avoid extra-technical hypothesis and streamline the presentation.

Inspired by [Nat+17], we define the orbital magnetization corresponding to evolutions U whose endpoints $U(1)$ are completely localized:

4.10 Definition. (*Magnetization*) For the evolution $U : [0,1] \to \mathcal{B}(\mathcal{H})$ (which need not be periodic), define the magnetization operator

$$m(U) := - \int_0^1 \mathbb{Im}(U^* \Lambda_1 H \Lambda_2 U) \,, \qquad (4.9)$$

where $\mathbb{Im}\, A \equiv \frac{1}{2i}(A - A^*)$ is the imaginary part of A, Λ_j the switch function Definition 1.24, and the total orbital magnetization (a number) as

$$M(U) := \sum_{z \in \mathcal{E}} \operatorname{tr} P_z m(U) P_z \,. \qquad (4.10)$$

where P_z are the spectral projections onto the eigenvalues of $U(1)$, and $\mathcal{E}$ is the corresponding set, as above. That this sum is finite is the subject of the next theorem.

Note that the integrand in $m(U)$ can be rewritten

$$- \mathbb{Im}(U^* \Lambda_1 H \Lambda_2 U) = \frac{1}{2}\big(U^* \Lambda_1 U \partial_t (U^* \Lambda_2 U) - 1 \leftrightarrow 2\big). \qquad (4.11)$$

Pretending $\Lambda_i \sim X_i$, the position operator, the latter expression is the third component of $(1/2)\,\mathbf{X}(t) \times \dot{\mathbf{X}}(t)$, which (in natural units) corresponds to the orbital magnetization. The physical aspects of m and M, including a proposal for an experimental realization in cold atoms, were studied in detail in [Nat+17].

4.11 Theorem. *If $U(1)$ is completely localized in the sense of Definition 4.9, the total magnetization $M(U)$ is finite, integer-valued and independent of the choice of switch functions. Moreover $M(U) = \mathcal{I}$. If $U(1) = \mathbb{1}$ then $M(U) = W(U)$ and if H is time-independent then $M(U) = 0$.*

Thus for completely localized systems $U(1)$ the computation of the index $M(U)$ does not require the relative construction, but the price to pay is that operator $m(U)$ is not trace-class anymore. However it is summable in the eigenbasis of $U(1)$, with sum $M(U)$. We emphasize that a mobility gap also applies when $\sigma(U(1)) \neq \mathbb{S}^1$; i.e. for a spectrally gapped system obeying Definition 4.9 the relative construction can also be circumvented using magnetization.

The strategy of the proof is to use the relative construction by choosing an effective Hamiltonian H_λ for an arbitrary $\lambda \in \Delta$. As detailed in Section 4.4 we get

$$\mathcal{I} = M(U) - M(U_\lambda) \,. \qquad (4.12)$$

The effective Hamiltonian being time-independent, its corresponding magnetizations simplifies to

$$M(U_\lambda) = -\sum_{z \in \mathcal{E}} \mathrm{tr}\left(P_z \, \mathbb{Im}(\Lambda_1 H_\lambda \Lambda_2) P_z\right) \tag{4.13}$$

and the task is to show that this expression is well-defined and vanishes. Note that a similar expression already appeared in the context of the IQHE as an extra term required to establish the bulk-edge correspondence of Hall conductivity in the mobility gap regime [EGS05]. An interpretation in terms of magnetization was also proposed there for time-independent Hamiltonians. However in that case the magnetization is not vanishing because the mobility gap is not the entire spectrum.

For completely localized systems, it is also possible to define an edge index without the relative construction, also related to the previous one through the bulk-edge correspondence.

> **4.12 Theorem.** *Let $U : [0,1] \to \mathcal{U}(\mathcal{H})$ be a bulk evolution whose endpoint $U(1)$ is completely localized as in Definition 4.9, and $\mathfrak{U}_E : [0,1] \to \mathcal{U}(\mathcal{H}_E)$ be any edge evolution (e.g. U_E) such that $\sup_{t \in [0,1]} \|\partial_2(\mathfrak{U}_E(t) - \widehat{U}(t))\|_1 < \infty$.*
>
> *The time-averaged charge pumping*
>
> $$P_E(\mathfrak{U}_E(1)) := \lim_{n \to \infty} \lim_{r \to \infty} \frac{1}{n} \, \mathrm{i} \, \mathrm{tr}\left(((\mathfrak{U}_E(1))^*)^n \partial_2 \mathfrak{U}_E(1)^n \Lambda_{1,r}^{\perp}\right). \tag{4.14}$$
>
> *where $\Lambda_{1,r}^{\perp} = \chi_{\leq r}(X_1)$, exists, is finite, integer valued, and independent of the choice of switch function. Moreover the bulk-edge correspondence holds*
>
> $$P_E(\mathfrak{U}_E(1)) = M(U). \tag{4.15}$$

The physical interpretation of P_E is a quantized pumping of charges, counted through $((\mathfrak{U}_E(1))^*)^n \Lambda_2 \mathfrak{U}_E(1)^n - \Lambda_2$, that is confined at the edge [GT18; Tit+16]: if the corresponding $U(1) = \mathbb{1}$, the pumping is quantized within a single cycle, whereas for completely localized $U(1)$, the quantization is true on average over time only, and coincides with magnetization.

4.2.3   *The stretch function construction*

The previous section extends the definition of bulk and edge indices beyond $U(1) = \mathbb{1}$ without using the relative construction. However it only works for completely localized systems. Here we finally give a recipe for reducing the general situation ($\Delta \neq \mathbb{S}^1$) to the one described by Definition 4.9, which completes the story and results in a new approach to define the topology (in both spectrally and mobility gapped cases).

4.13 Definition. (*Stretch function*) Let Ω be an arc in $\mathbb{S}^1 \subseteq \mathbb{C}$. A stretch function $F_\Omega : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth function with $F_\Omega(z) = 1$ for $z \in \mathbb{S}^1 \setminus \Omega$, which winds once:

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{S}^1} F_\Omega(z)^{-1} \, \mathrm{d} \, F_\Omega(z) = 1 \,. \tag{4.16}$$

The role of F is to stretch the arc Ω onto the entire circle except the point at 1, which is the image of $\mathbb{S}^1 \setminus \Omega$. In particular if $\Omega = \mathbb{S}^1 \setminus \{1\}$ then the identity $F_\Omega(z) = z$ is an appropriate stretch function. We think of F_Ω as a function which selects the appropriate (mobility or spectral) gap, analogous to (a smooth deformation of) the function $\chi_{(-\infty, E_F)}$ with $E_F \in \mathbb{R}$ the Fermi energy, for the IQHE. For a given arc Ω we define

$$V(t) := F_\Omega(U(t)), \qquad V_{\mathrm{E}}(t) := F_\Omega(U_{\mathrm{E}}(t)) \qquad (t \in [0,1]) \tag{4.17}$$

via the functional calculus. V and V_{E} are two unitary evolutions on $\mathcal{H}$ and $\mathcal{H}_{\mathrm{E}}$ respectively, that satisfy $V(0) = V_{\mathrm{E}}(0) = \mathbb{1}$.

4.14 Lemma. *If $\Omega = \Delta$ is a mobility gap of $U(1)$ then $V(1)$ is completely localized in the sense of Definition 4.9, with mobility gap $\mathbb{S}^1 \setminus \{1\}$.*

This is just a consequence of Definition 4.4 and Definition 4.13, in particular

$$B_1(\mathbb{S}^1) \circ F_{\Delta'} \subseteq B_1(\Delta) \tag{4.18}$$

for any $\Delta' \subsetneq \Delta$ which is a proper sub-arc. Thus the supremum in (4.3) over the LHS set is bounded by the supremum over the RHS set. Moreover $\{1\}$ is the image of $\mathbb{S}^1 \setminus \Delta$ by F_Δ so it is not a finite degenerate eigenvalue of $V(1)$ in general. As noticed above, this particular point can be present in completely localized systems and is not problematic for the definition of magnetization and pumping indices.

4.15 Corollary. *If Δ is a mobility gap of $U(1)$ and F_Δ is a stretch function, $V(1)$ is completely localized so that*

$$\mathcal{I}' := M(V), \qquad \mathcal{I}'_{\mathrm{E}} := P_{\mathrm{E}}(V_{\mathrm{E}}(1)) \tag{4.19}$$

are well-defined indices according to Theorem 4.11 and Theorem 4.12. In particular the bulk-edge correspondence holds: $\mathcal{I}' = \mathcal{I}'_{\mathrm{E}}$.

Thus the composition of stretch function and magnetization or quantized pumping provides indices for any $U(1)$ with mobility gap Δ and circumvent the relative construction. Note that if Δ is a spectral gap then $V(1) = \mathbb{1}$ so that $\mathcal{I}' = W(V)$ and $\mathcal{I}'_{\mathrm{E}}$ coincides with the edge index definition of [SS17] where a particular stretch function was used.

The proof of Corollary 4.15 is not straightforward as one has to check that the underlying assumptions of Theorem 4.11 and Theorem 4.12 are satisfied for V and V_{E}, namely that all the properties of U and U_{E} are correctly transfered through the stretch function construction. This is done in Section 4.5.1.

It is finally legitimate to ask if the two constructions coincide since the relative indices, $\mathcal{I}$ and $\mathcal{I}_E$, and the ones defined through stretch functions, $\mathcal{I}'$ and $\mathcal{I}'_E$, are both defined in a general mobility gap situation.

> **4.16 Theorem.** *If $U(1)$ has a mobility gap Δ and F_Δ is a stretch function, then*
>
> $$\mathcal{I}' = \mathcal{I}. \tag{4.20}$$
>
> *In particular $\mathcal{I}'$ is independent of the choice of stretch function. Moreover by the respective bulk-edge correspondences one infers $\mathcal{I}'_E = \mathcal{I}_E$.*

The poof is somewhat delicate. By Corollary 4.15 we know that $\mathcal{I}'$ coincides with the relative construction applied to V, namely $M(V) = W(V^{\mathrm{rel}})$. But in order to show that $W(V^{\mathrm{rel}}) = W(U^{\mathrm{rel}}) \equiv \mathcal{I}$ we use a smooth deformation of the stretch function from F_Δ to the identity. Then we have to show that W stays continuous under this deformation. The only other *deterministic* proof of continuity for indices in the mobility gap regime so far was in [EGS05] for the deformation corresponding to tuning the Fermi energy E_F within the mobility gap. Thus the proof of Theorem 4.16 provides another continuity proof for the index W along a different path and paves the way for further development of locally constant indices at strong disorder.

> 4.17 REMARK. It is worth pointing out that it is Theorem 4.16 which implies that $\mathcal{I}$ is independent of the choice of branch cut $\lambda \in \Delta$ (part of item (a) of Theorem 4.6), since $\mathcal{I}'$ is manifestly independent of λ.

SUMMARY   To conclude, the objection at the end of Section 4.2.1 was that the index $\mathcal{I}_E$ from the relative construction is calculated using $U_E^{\mathrm{rel}} = U_E \# e^{-\mathrm{i} \cdot \hat{H}_\lambda}$ that contains a truncation of the effective *bulk* Hamiltonian (Figure 4.2(a)). Instead the stretch function approach removes this intertwining since the index $\mathcal{I}'_E$ is calculated using $V_E = F_\Delta(U_E)$ (Figure 4.2(b)). The only knowledge here from the bulk is the position of the gap $\Delta \subset \mathbb{S}^1$. The latter approach is then more natural for the bulk-edge correspondence, although the two are equivalent by Theorem 4.16.

$$(a) \quad
\begin{array}{ccc}
H & \longrightarrow U \longrightarrow & H_\lambda \\
\downarrow & & \downarrow \\
H_E & \longrightarrow U_E & \hat{H}_\lambda
\end{array}
\qquad\qquad
(b) \quad
\begin{array}{ccc}
H & \longrightarrow U \longrightarrow & V \\
\downarrow & & \\
H_E & \longrightarrow U_E \longrightarrow & V_E
\end{array}$$

FIGURE 4.2: Operator content in the relative construction (a) and stretch function approach (b).

4.3   BULK-EDGE CORRESPONDENCE FOR THE RELATIVE EVOLUTION

The bulk-edge correspondence was established in [GT18] in the case where $U(1)$ has a spectral gap. In that case all the operators involved are local in the sense of Assumption 4.2. In particular U, H_E and U_E are local, uniformly in $t \in [0,1]$ (see [GT18, Proposition 4.7]). These properties are *independent* of the existence of (any) gap of $U(1)$

since they probe the dynamics only in a compact time interval, and hence, at a finite distance from the spectrum, so they remain true also in our setting.

Furthermore when the branch cut of the logarithm is taken inside a spectral gap, H_λ (and thus U_λ) are also local. This is not the case anymore in the mobility gap regime. However the logarithm still has some off-diagonal decay properties that suffice to generalize the proof of the bulk-edge correspondence in the relative construction, as we shall now show.

> 4.18 REMARK. Definition 4.4 entails that the functional calculus of $U(1)$ is weakly-local (in the sense of Definition A.7) uniformly as functions vary in $B_1(\Delta)$. Hence the algebraic properties of Appendix A.2 apply.

4.3.1 The logarithm is weakly-local

In this section we indeed make the mobility gap assumption, that is, we assume that there is some non-empty arc $\Delta \subseteq S^1$ which is a mobility-gap for $U(1)$ in the sense of Definition 4.4. We further assume that $\lambda \in \Delta$, where λ is the position of the branch cut used in the definition of H_λ from (4.5).

> **4.19 Lemma.** $f(U(1))$ is also weakly-local for all bounded $f : S^1 \to \mathbb{C}$ which are smooth outside of Δ and piecewise smooth with a finite number of jump discontinuities within Δ.

Proof. Assume that f has *one* jump discontinuity at some $\lambda_0 \in \Delta$ and is otherwise smooth. Since Δ is an arc, pick some other $\lambda_1 \in \Delta \setminus \{\lambda_0\}$. For each $\Omega \in \{(\lambda_0, \lambda_1), S^1 \setminus (\lambda_0, \lambda_1)\} =: S$, the restriction $f|_\Omega : \Omega \to \mathbb{C}$ is smooth and so has a smooth extension $f_\Omega^s : S^1 \to \mathbb{C}$ (that is $f|_\Omega = f_\Omega^s|_\Omega$). Hence $\chi_\Omega f = \chi_\Omega f_\Omega^s$ and $f = \sum_{\Omega \in S} f \chi_\Omega = \sum_{\Omega \in S} f_\Omega^s \chi_\Omega$. Any smooth function of a local unitary operator is also weakly-local by Corollary A.26. On the other hand, $\chi_\Omega \in B_1(\Delta)$, so the corresponding operator is weakly-local by the assumption entailed in Definition 4.4. Thus Lemma A.11 allows us to conclude about the whole of f. $\square$

> **4.20 Corollary.** Both H_λ and U_λ are weakly-local.

Proof. Since $\lambda \in \Delta$, we get that $\log_\lambda$ is analytic except for a jump discontinuity within Δ as f of Lemma 4.19. Now $U_\lambda(t) = (\mathrm{e}^{-\mathrm{i}t\cdot} \circ \mathrm{i}\log_\lambda)(U(1))$, $\mathrm{e}^{-\mathrm{i}t\cdot}$ is analytic, so that for fixed t, the composition $\mathrm{e}^{-\mathrm{i}t\cdot} \circ \mathrm{i}\log_\lambda$ is again analytic apart from one jump discontinuity within Δ, which is covered by Lemma 4.19. $\square$

4.3.2 The bulk-edge correspondence in the relative construction

The central ingredient of the bulk-edge correspondence is the relation between truncated bulk and edge propagator. For H local, the difference $D = U_{\mathrm{E}} - \widehat{U}$ is local and confined in direction 1, uniformly in $t \in [0, 1]$. This result is also independent of the existence of (any) gap of $U(1)$ (see [GT18, Proposition 4.10]), and is generalized from local to weakly-local operators below.

4.21 Lemma. *Let $\mathfrak{H} : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ be some weakly-local Floquet Hamiltonian, with its associated $\mathfrak{H}_E$, $\mathfrak{U}$ and $\mathfrak{U}_E$ as in Section 4.2. Then $\mathfrak{D} := \mathfrak{U}_E - \widehat{\mathfrak{U}}$ is weakly local and confined in direction 1, uniformly in $t \in [0,1]$.*

Proof. To deal with $\mathfrak{D}$ we recall that $\widehat{\mathfrak{U}} \equiv \iota^*\mathfrak{U}\iota$. Since $\mathfrak{U}_E$ is weakly-local, by Lemma A.11 it suffices to deal with $\mathbb{1} - \iota^*\mathfrak{U}(t)\iota\mathfrak{U}_E(t)^* = \int_{s=0}^{t} \partial_s(\iota^*\mathfrak{U}(s)\iota\mathfrak{U}_E(s)^*)\,\mathrm{d}\,s$. Since all estimates about the weak-locality of the involved operators are uniform in time (the time interval being compact), the weak-locality and confined property of the integrand implies that of the integral. With the shorthand notation ∂ for derivative w.r.t. time and using the defining property $\mathrm{i}\,\partial U = HU$ and the adjoint of this equation, and finally the fact that a Hamiltonian and the semi-group which it generates commute, we have,

$$
\begin{aligned}
\partial(\iota^*\mathfrak{U}\iota(\mathfrak{U}_E)^*) &= \iota^*(\partial\mathfrak{U})\iota(\mathfrak{U}_E)^* + \iota^*\mathfrak{U}\iota\partial(\mathfrak{U}_E)^* \\
&= \iota^*(-\mathrm{i}\,\mathfrak{H}\mathfrak{U})\iota(\mathfrak{U}_E)^* + \iota^*\mathfrak{U}\iota(+\mathrm{i}(\mathfrak{U}_E)^*\iota^*\mathfrak{H}\iota) \\
&= -\mathrm{i}\,\iota^*\mathfrak{U}\mathfrak{H}\iota(\mathfrak{U}_E)^* + \mathrm{i}\,\iota^*\mathfrak{U}\iota\iota^*\mathfrak{H}\iota(\mathfrak{U}_E)^* \\
&= \mathrm{i}\,\iota^*\mathfrak{U}(\iota\iota^* - \mathbb{1})\mathfrak{H}\iota(\mathfrak{U}_E)^*.
\end{aligned}
$$

We note that $\iota\iota^* - \mathbb{1} = -(|\iota^*|^2)^\perp = \Theta(X_1)^\perp$ where Θ is the step-function–a valid choice of switch function. In fact all that matters now is that we found a factor $\Lambda_1^\perp\mathfrak{H}\iota$, and then, using $\Lambda_1^\perp\iota = 0$, this factor equals $[\Lambda_1^\perp, \mathfrak{H}]\iota = -\mathrm{i}(\partial_1\mathfrak{H})\iota$. But $\mathfrak{H}$ is weakly-local and we now invoke Corollary A.19 to assert $\partial_1\mathfrak{H}$ is confined. □

In particular this Lemma applies to $\mathfrak{H} = H_\lambda$ and $\mathfrak{H} = H^{\mathrm{rel}}$.

Proof of Theorem 4.6. The first step is to show that the indices are well-defined and have the claimed properties. This has been the focus of most of the results in this section, and we may now finally put them to use.

By Lemma 4.21 applied to H^{rel} we deduce that $U_E^{\mathrm{rel}}(1) = \mathbb{1} + D^{\mathrm{rel}}(1)$, and consequently $[\Lambda_2, U_E^{\mathrm{rel}}(1)] = \mathrm{i}\,\partial_2 D^{\mathrm{rel}}(1)$ which is trace class by Corollary A.20, so that $\mathcal{I}_E$ is well defined. The invariance under the choice of switch function is a simple computation, and the facts it is integer valued is achieved by choosing $\Lambda_2 = \Theta(X_2)$ and identifying $\mathcal{I}_E$ with an index of a pair of projections. Then by Lemma A.15, Corollary A.19 and Lemma A.17, $[\Lambda_i, U^{\mathrm{rel}}](U^{\mathrm{rel}})^*[\Lambda_j, U^{\mathrm{rel}}] = -\partial_i(U^{\mathrm{rel}})(U^{\mathrm{rel}})^*\partial_j U^{\mathrm{rel}}$ is trace-class for all $t \in [0,1]$ and $i \neq j$ so that $\mathcal{I} = W(U^{\mathrm{rel}})$ is finite. Similarly, the switch-function independence and the integer value follow, with proofs similar to those of [GT18, Proposition 3.3].

The second step is the proof of the bulk-edge correspondence. This is an algebraic computation that involves trace class operators. We comment on how to generalize to our case [GT18, Theorem 3.4] (the bulk-edge duality for the case of *local* unitary evolutions where the bulk is periodic). Passing from local to weakly-local operators does not modify the trace class of the expression, because the switch functions cure the non-uniform off-diagonal decay characteristic of

weakly-local operators, as we have already seen in Lemma 4.21. We first rewrite $\mathcal{I}_E = i \lim_{r \to \infty} \mathrm{tr}(U_E^{\mathrm{rel}}(1)^* \partial_2 U_E^{\mathrm{rel}}(1) \Lambda_{1,r}^\perp)$ where $\Lambda_{1,r}^\perp = \Theta(r - X_1)$ is a cut-off in direction 1 on $\mathcal{H}_E$ for $r \in \mathbb{N}$. At finite r the previous expression becomes trace-class for every $t \in [0,1]$ so we rewrite it as the integral of its derivative. After some algebra we end up with

$$i \, \mathrm{tr}(U_E^{\mathrm{rel}}(1)^* \partial_2 U_E^{\mathrm{rel}}(1) \Lambda_{1,r}^\perp) = W_r(U^{\mathrm{rel}}) + o(r) \tag{4.21}$$

where W_r is given by (4.6) for $\Lambda_1 = \mathbb{1} - \Lambda_{1,r}^\perp$. Since this quantity is independent of the choice of switch function we conclude $W_r(U^{\mathrm{rel}}) = W(U^{\mathrm{rel}})$ so that $\mathcal{I}_E = \mathcal{I}$ in the $r \to \infty$ limit. Equality (4.21) only uses $H_E = \widehat{H}$, Lemma 4.21 and some updated version of it, [GT18, Lemma 5.5], namely that

$$[\Lambda_2, U_E^{\mathrm{rel}}(t)] U_E^{\mathrm{rel}}(t)^* - \iota^* [\Lambda_2, U^{\mathrm{rel}}(t)] U^{\mathrm{rel}}(t)^* \iota \tag{4.22}$$

is trace class for $t \in [0,1]$. All the rest follows by algebraic manipulations and Claim A.32. □

We note that, surprisingly, identity (4.21) remains true when applied to U and U_E instead of the relative evolutions, even if $U(1) \neq \mathbb{1}$. One still has $W_r(U) = W(U)$ but this quantity is not quantized anymore. Yet the left hand side of (4.21) converges to it in the $r \to \infty$ limit but does not coincide with any edge index because the $\partial_2 U_E(1)$ is not anymore trace-class. Although not relevant here, this identity will be used below.

4.4 THE COMPLETELY LOCALIZED CASE

This section is dedicated to the proof of Theorem 4.11 and Theorem 4.12. We start by studying the bulk part M. Let us assume that U is completely localized in the sense of Definition 4.9, and $\lambda \in \mathbb{S}^1 \setminus \{1\} \subseteq \Delta$. According to Theorem 4.6, the bulk index $\mathcal{I} = W(U^{\mathrm{rel}})$ is well defined. By (1.6) and (4.6), W is manifestly additive under concatenation, so we deduce

$$\mathcal{I} = W(U) - W(U_\lambda)$$

where $U_\lambda \equiv e^{-i \cdot H_\lambda}$. Here we have extended W to non-periodic families with the same formula (4.6). Indeed, since both U and U_λ are weakly-local, $W(U)$ and $W(U_\lambda)$ are finite, although they are not separately integers. Before we proceed with U let us rewrite the expression for W, for any weakly-local $\mathfrak{U} : [0,1] \to \mathcal{U}(\mathcal{H})$ and its generator $\mathfrak{H} := i \dot{\mathfrak{U}} \mathfrak{U}^*$

$$W(\mathfrak{U}) = \frac{1}{2i} \int_0^1 \mathrm{tr}\, \varepsilon_{\alpha\beta} \mathfrak{H} (\partial_\alpha \mathfrak{U}) \partial_\beta \mathfrak{U}^*$$

where we have used $\partial_\alpha \mathfrak{U}^* = -\mathfrak{U}^* (\partial_\alpha \mathfrak{U}) \mathfrak{U}^*$ and the $\varepsilon_{\alpha\beta}$ anti-symmetric tensor with $\varepsilon_{12} = 1$ (summation over $\alpha, \beta \in 1, 2$ is understood when indices appear twice). Since $\mathrm{tr}\, \partial_\alpha (\mathfrak{H} \mathfrak{U} \partial_\beta \mathfrak{U}^*) = 0$ and $\varepsilon_{\alpha\beta} \mathfrak{H} \mathfrak{U} \partial_\alpha \partial_\beta \mathfrak{U}^* = 0$ (as the non-commutative derivatives commute) we deduce

$$W(\mathfrak{U}) = \frac{i}{2} \int_0^1 \mathrm{tr}\, \varepsilon_{\alpha\beta} (\partial_\alpha \mathfrak{H}) \mathfrak{U} \partial_\beta \mathfrak{U}^* .$$

Defining $\delta_\alpha^{\mathfrak{U}} := \mathfrak{U}^* \partial_\alpha \mathfrak{U}$ (the logarithmic derivative of $\mathfrak{U}$ in direction α) and noticing that

$$\dot{\delta}_\alpha^{\mathfrak{U}} = \frac{1}{\mathrm{i}} \mathfrak{U}^* (\partial_\alpha \mathfrak{H}) \mathfrak{U} \tag{4.23}$$

we deduce

$$W(\mathfrak{U}) = \frac{1}{2} \int_0^1 \mathrm{tr}\, \varepsilon_{\alpha\beta} \dot{\delta}_\alpha^{\mathfrak{U}} \delta_\beta^{\mathfrak{U}} . \tag{4.24}$$

Consequently for U^{rel} we find,

$$\mathcal{I} = \frac{1}{2} \mathrm{tr} \int_0^1 \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha \delta_\beta - \dot{\delta}_\alpha^\lambda \delta_\beta^\lambda \big)$$

where we use the shorthand notation δ_α (resp. δ_α^λ) for δ_α^U (resp. $\delta_\alpha^{U_\lambda}$). Note that there is no problem to exchange trace and integral here since both $\varepsilon_{\alpha\beta} \dot{\delta}_\alpha \delta_\beta$ and $\varepsilon_{\alpha\beta} \dot{\delta}_\alpha^\lambda \delta_\beta^\lambda$ are trace class for all $t \in [0,1]$. Finally

$$\int_0^1 \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha \delta_\beta - \dot{\delta}_\alpha^\lambda \delta_\beta^\lambda \big) = -\mathrm{i} \int_0^1 \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha U^* \Lambda_\beta U - \dot{\delta}_\alpha^\lambda U_\lambda^* \Lambda_\beta U_\lambda \big) + \mathrm{i} \int_0^1 \varepsilon_{\alpha\beta} \big(\dot{\delta}_\alpha - \dot{\delta}_\alpha^\lambda \big) \Lambda_\beta .$$

The last term vanishes because it is a total derivative, and by the fact that $\delta_\alpha(0) = \delta_\alpha^\lambda(0) = 0$ and $\delta_\alpha(1) = \delta_\alpha^\lambda(1)$ since $U(1) = U_\lambda(1)$ (note however that $\delta_\alpha(1) \neq 0$ in general). Hence, by (4.23) and (4.9),

$$\mathcal{I} = \mathrm{tr}(m(U) - m(U_\lambda)) .$$

This relation is general and does not require $U(1)$ to be completely localized. However it is equivalent to (4.12) only in the latter case. Indeed $m(U)$ and $m(U_\lambda)$ are not separately trace-class, only their difference is. When $U(1)$ is completely localized, the trace of this difference can be computed through its eigenbasis:

$$\mathcal{I} = \sum_{z \in \mathcal{E}} \mathrm{tr}\, P_z \big(m(U) - m(U_\lambda) \big) P_z$$

with $P_z \equiv \chi_{\{z\}}(U(1))$ the projection onto the eigenvalue z. What remains to be shown is that the sum can actually be split into two parts, leading to (4.12).

> **4.22 Proposition.** *If $U(1)$ is completely localized in the sense of Definition 4.9, then the effective evolution magnetization*
>
> $$M(U_\lambda) = -\sum_{z \in \mathcal{E}} \mathrm{tr}\, P_z \int_0^1 \mathbb{Im}(U_\lambda^* \Lambda_1 H_\lambda \Lambda_2 U_\lambda) P_z$$
>
> *is absolutely convergent and vanishes.*

Thus we are left with $\mathcal{I} = M(U)$, so that $M(U)$ is well defined and shares all the properties of $\mathcal{I}$. This proves the main statement of Theorem 4.11. In the particular case where $U(1) = \mathbb{1}$, we have $H_\lambda = 0$ and $\delta_\alpha(1) = 0$, so that $M(U)$ is trace-class by the previous computation, and $\mathcal{I} = W(U) = \mathrm{tr}(m(U)) = M(U)$. Finally the case where H is time-independent is a direct consequence of the latter proposition.

Proof of Proposition 4.22. Since $U_\lambda \equiv e^{-i \cdot H_\lambda}$ and $U_\lambda(1) = U(1)$ is completely localized, then so are H_λ and $U_\lambda(t)$ for $t \in [0, 1]$ with the same eigenbasis as $U(1)$. Thus for a fixed $z \in \mathbb{S}^1$ one has by replacing $U_\lambda(t)P_z = z^t$ and $P_z U_\lambda(t)^* = z^{-t}$ for all $t \in [0, 1]$:

$$P_z \int_0^1 \mathbb{Im}(U_\lambda^* \Lambda_1 H_\lambda \Lambda_2 U_\lambda) P_z = P_z \mathbb{Im}(\Lambda_1 H_\lambda \Lambda_2) P_z. \tag{4.25}$$

By construction H_λ is bounded with a real spectrum that unwinds the circular one of $U(1)$ with respect to the branch cut λ. For each eigenvalue of $U(1)$, $z \in \mathbb{S}^1$, $r := i \log_\lambda(z) \in \mathbb{R}$ is an eigenvalue of H_λ with same eigenprojection P_z. For $x \in \mathbb{R}$ we define the Fermi projection up to x by $P_{<x} := \chi_{(-\infty,x)}(H_\lambda)$, so that $P_{<x} = 0$ for $x < \inf \sigma(H_\lambda)$ and $P_{<x} = \mathbb{1}$ for $x \geq \sup \sigma(H_\lambda)$. We use the following representation of H_λ

$$H_\lambda = C - \int_{\sigma(H_\lambda)} P_{<x} \, \mathrm{d} x \tag{4.26}$$

where $C = \sup(\sigma(H_\lambda)) \in \mathbb{R}$. This representation comes from the following functional equality

$$\int_\Omega \chi_{(-\infty,x)}(y) \, \mathrm{d} x = \int_\Omega \chi_{(y,\infty)}(x) \, \mathrm{d} x = \sup(\Omega) - y \tag{4.27}$$

for some arc Ω and $y \in \Omega$. Inserting (4.26) into (4.25) we get

$$P_z \mathbb{Im}(\Lambda_1 H_\lambda \Lambda_2) P_z = \frac{i}{2} \int_{\sigma(H_\lambda)} \varepsilon_{\alpha\beta} P_z \Lambda_\alpha P_{<x} \Lambda_\beta P_z \, \mathrm{d} x.$$

Consider $z \in \mathcal{E}$ and $x \in \sigma(H)$ fixed, and define $P_{<x}^\perp = \mathbb{1} - P_{<x}$. Then either $i \log_\lambda(z) > x$, in which case $P_z P_{<x} = 0$ and $P_z P_{<x}^\perp = P_z$, or $i \log_\lambda(z) \leq x$, in which case $P_z P_{<x} = P_z$ and $P_z P_{<x}^\perp = 0$. Therefore

$$\frac{i}{2} \varepsilon_{\alpha\beta} P_z \Lambda_\alpha P_{<x} \Lambda_\beta P_z = \frac{i}{2} P_z \varepsilon_{\alpha\beta} P_{<x}^\perp \Lambda_\alpha P_{<x} \Lambda_\beta P_{<x}^\perp P_z - \tag{4.28}$$

$$- \frac{i}{2} P_z \varepsilon_{\alpha\beta} P_{<x} \Lambda_\alpha P_{<x}^\perp \Lambda_\beta P_{<x} P_z$$

$$=: \frac{1}{2} P_z T(x) P_z. \tag{4.29}$$

Moreover, $T(x)$ is trace-class for every $x \in \sigma(H_\lambda)$. Indeed, after some algebra

$$T(x) = -i P_{<x}^\perp [\partial_1 P_{<x}^\perp, \partial_2 P_{<x}^\perp] + i P_{<x} [\partial_1 P_{<x}, \partial_2 P_{<x}] \tag{4.30}$$

and each term is separately trace-class by Corollary A.20. Indeed,

$$P_{<x} = \chi_{(\lambda, e^{-ix})}(U(1))$$

is weakly-local according to Definition 4.4 since $U(1)$ is completely localized (even if the single point of infinite degeneracy z_0 has $z_0 \in (\lambda, \mathrm{e}^{-\mathrm{i}x})$ then $P_{<x} = \mathbb{1} - P_{<x}^\perp$ is weakly local from the fact that $P_{<x}^\perp$ is). Thus for every $z \in \mathcal{E}$, $P_z T(x) P_z$ is trace-class (even if z is infinitely degenerate) and moreover

$$\frac{1}{2} \sum_{z \in \mathcal{E}} \operatorname{tr} P_z T(x) P_z = \frac{1}{2} \operatorname{tr}(T(x)) = \frac{1}{4\pi} \left(\operatorname{Chern}(P_{<x}) - \operatorname{Chern}(P_{<x}^\perp) \right) = \frac{1}{2\pi} \operatorname{Chern}(P_{<x}) .$$

with the sum on the left hand-side that is absolutely convergent due to the trace-class property. The quantity on the right hand side is the Chern number [EGS05] defined by

$$\operatorname{Chern}(P) \equiv 2\pi \, \mathrm{i} \operatorname{tr} P[\partial_1 P, \partial_2 P] = -\operatorname{Chern}(P^\perp)$$

that is well defined and integer-valued for any weakly-local projection P. Since H_λ is bounded we have

$$\frac{1}{2} \int_{\sigma(H_\lambda)} \sum_{z \in \mathcal{E}} |\operatorname{tr} P_z T(x) P_z| \, \mathrm{d}\, x \leq \frac{1}{2} \int_{\sigma(H_\lambda)} \|T(x)\|_1 \, \mathrm{d}\, x$$

$$\leq \frac{1}{2} |\sigma(H_\lambda)| \sup_{x \in \sigma(H_\lambda)} \|T(x)\|_1$$

$$< \infty \tag{4.31}$$

due to the fact that $\sigma(H_\lambda)$ is compact and $x \mapsto \|T(x)\|_1$ is bounded. Indeed, this last fact is a non-trivial consequence of the fact that the estimate of complete localization in the mobility gap, (4.3), includes a supremum over all Borel bounded functions $|f| \leq 1$ (which are constant outside of the mobility gap, but since here the entire circle except one point is a mobility gap, this constraint is vacuously true). One could then include in this supremum the spectral projections $P_{<x}$ with supremum over x since the functions $(\chi_{(\lambda, \mathrm{e}^{\mathrm{i}x})})_{x \in \sigma(H_\lambda)}$ are definitely part of this set of functions. Thus $P_{<x}$ has weakly-local estimates which are uniform in x, which in turn implies via Remark A.21 that $x \mapsto \|T(x)\|_1$ is bounded. (one may be worried about the point of infinite degeneracy in the spectrum of $U(1)$ which is allowed in Definition 4.9, throughout which the estimate (4.3) might fail to hold uniformly, but this is not a problem since we could always just remove this point in the integral before (4.31)).

Hence by Fubini's theorem we may exchange $\sum_z$ and $\int_x$. Putting everything together, we deduce that $M(U_\lambda)$ is defined by an absolutely convergent sum. Moreover

$$M(U_\lambda) = -\int_{\sigma(H_\lambda)} \operatorname{Chern}(P_{<x}) \, \mathrm{d}\, x .$$

It was shown in [EGS05, Prop. 2] that $\operatorname{Chern}(\chi_\Omega(H)) = 0$ for any arc Ω inside the mobility gap of H that contains only finite-multiplicity eigenvalues. Here the entire spectrum of H_λ is a mobility gap, but it might contain one infinite

degenerated eigenvalue corresponding to z_0. However if $z_0 \in (\lambda, e^{-ix})$ then $\mathrm{Chern}(P_{<x}) = -\mathrm{Chern}(P_{<x}^{\perp}) = 0$ as the latter projection only contains finite degenerated eigenvalues. Thus $M(U_\lambda) = 0$. $\qquad \square$

Proof of Theorem 4.12. Let n be a fixed integer. From (4.21) in the proof of Theorem 4.6 and the fact that $U_E(1)^n = U_E(n)$, we have the following identity:

$$\lim_{r \to \infty} \mathrm{i} \, \mathrm{tr} \left((U_E(1)^*)^n \partial_2 U_E(1)^n \Lambda_{1,r}^{\perp} \right) = -\frac{1}{2} \int_0^n \mathrm{tr} \, \dot{U} U^* [(\partial_1 U) U^*, (\partial_2 U) U^*] \, .$$

On the left-hand-side, the expression is trace class for every finite r because of the cut-off $\Lambda_{1,r}^{\perp}$ and confinement in direction 2 through Λ_2. The right-hand-side is expression (4.6) of W but on a time interval $[0, n]$ instead of $[0, 1]$. In partirular it is independent of switch-function Λ_1, which is why the limit $r \to \infty$ is finite. If $U(n) = \mathbb{1}$, W would be quantized and define the bulk index, and the limit on the right would be equal to edge index. Nevertheless the previous equation is true for any pair of bulk and edge operators U and U_E, as long as they are weakly-local and related by Lemma 4.21, although it is not integer-valued. From now we assume $U(1)$ completely localized. Rewriting W as in (4.24)

$$
\begin{aligned}
&-\frac{1}{2} \int_0^n \mathrm{tr} \, \dot{U} U^* [(\partial_1 U) U^*, (\partial_2 U) U^*] \\
&= \frac{1}{2} \int_0^n \mathrm{tr} \, \varepsilon_{\alpha\beta} \dot{\delta}_\alpha \delta_\beta \\
&= \frac{1}{2} \sum_{z \in \mathcal{E}} \mathrm{tr} \Big(P_z \int_0^n \varepsilon_{\alpha\beta} \dot{\delta}_\alpha U^* \Lambda_\beta U P_z \Big) - \frac{1}{2} \sum_{z \in \mathcal{E}} \mathrm{tr} \Big(P_z \int_0^n \varepsilon_{\alpha\beta} \dot{\delta}_\alpha \Lambda_\beta P_z \Big)
\end{aligned}
\qquad (4.32)
$$

where $\delta_\alpha = U^* \Lambda_\alpha U - \Lambda_\alpha$. Since $\dot{\delta}_\alpha \delta_\beta$ is trace class, we permute trace and time integral, and then compute this trace in the eigenbasis of $U(1)$. What remains to be shown is that the two terms in the last formula obtained by splitting δ_β are separately finite, and then study their $n \to \infty$ limit. Note that the eigenbasis of $U(1)$ and $U(n)$ are the same since $U(n) = U(1)^n$, although the eigenvalues are different. The first term in (4.32) is close to magnetization

$$\frac{1}{2} \sum_{z \in \mathcal{E}} \mathrm{tr} \Big(P_z \int_0^n \varepsilon_{\alpha\beta} \dot{\delta}_\alpha U^* \Lambda_\beta U P_z \Big) = -\sum_{z \in \mathcal{E}} \mathrm{tr} \Big(P_z \int_0^n \mathbb{Im}(U^* \Lambda_1 H \Lambda_2 U) P_z \Big)$$

$$=: M_n(U) \, .$$

Then we use the facts that $U(t) = U(t - k)U(1)^k$ for $k \leq t < k + 1$ and $k \in \{0, \dots, n-1\}$ and $U(1)^k P_z = z^k P_z$ for $z \in \mathcal{E} \subset \mathbb{S}^1$. Similarly $U^*(t) = (U(1)^*)^k U^*(t - k)$ and $(U(1)^*)^k P_z = z^{-k} P_z$. Moreover $H(t + k) = H(t)$. Applying these relations to the previous time integral that we cut into n parts, we get, up to a change of variable

$$M_n(U) = n M(U)$$

so that $\mathcal{M}_n(U)$ is finite and shares all the properties of $M(U)$ from Theorem 4.11. Moreover $n^{-1}M_n(U) \to M(U)$ trivially when $n \to \infty$.

The second term of (4.32) is a total derivative and can be simplified to

$$\frac{1}{2}\sum_{z\in\mathcal{E}}\mathrm{tr}(P_z\int_0^n \varepsilon_{\alpha\beta}\dot{\delta}_\alpha\Lambda_\beta P_z) = \frac{1}{2}\sum_{z\in\mathcal{E}}\mathrm{tr}(P_z\varepsilon_{\alpha\beta}U(n)^*\Lambda_\alpha U(n)\Lambda_\beta P_z) \qquad (4.33)$$

since $\delta_\beta(0) = 0$ and $\varepsilon_{\alpha\beta}\Lambda_\alpha\Lambda_\beta = 0$. Note that $U(n) = U(1)^n = \mathrm{e}^{-\mathrm{i}nH_\lambda}$ for H_λ defined in (4.5) and any $\lambda \in \mathbb{S}^1 = \Delta$. Then we use the following functional equality for a continuously differentiable $f : [a,b] \to \mathbb{C}$:

$$f(y) = f(b) - \int_a^b f'(x)\chi_{(a,x)}(y)\,\mathrm{d}x$$

for $y \in [a,b]$, which is a generalization of (4.27), see also [EGS05]. Consequently

$$U(n) = \mathrm{e}^{-\mathrm{i}nH_\lambda} = \mathrm{e}^{-\mathrm{i}nb}\,\mathbb{1} + \mathrm{i}\,n\int_{\sigma(H_\lambda)}\mathrm{e}^{-\mathrm{i}nx}\,P_{<x}\,\mathrm{d}x \qquad (4.34)$$

where $P_{<x} = \chi_{(-\infty,x)}(H_\lambda)$ and $b = \sup(\sigma(H_\lambda))$. When inserting this expression for $U(n)$ in (4.33), the first term vanishes by antisymmetry. In order to show that the second one is finite, we claim that

$$\frac{\mathrm{i}\,n}{2}\int_{\sigma(H_\lambda)}\sum_{z\in\mathcal{E}}\mathrm{e}^{\mathrm{i}nx}\,\mathrm{tr}(P_z\varepsilon_{\alpha\beta}U(n)^*\Lambda_\alpha P_{<x}\Lambda_\beta P_z)\,\mathrm{d}x$$

is absolutely convergent for any fixed n. Indeed since $U(n)^*$ commutes with P_z one has

$$P_z\,\mathrm{i}\,\varepsilon_{\alpha\beta}U(n)^*\Lambda_\alpha P_{<x}\Lambda_\beta P_z = P_zU(n)^*T(x)P_z$$

where $T(x)$ is defined in (4.28). Moreover $T(x)$ is trace class as pointed out in (4.30) so the previous sum over z is absolutely convergent for every $x \in \sigma(H_\lambda)$. The integral is then also absolutely convergent for the same reasons as in (4.31).

Consequently, (4.33) can be rewritten as

$$\frac{1}{2}\sum_{z\in\mathcal{E}}\mathrm{tr}(P_z\varepsilon_{\alpha\beta}U(n)^*\Lambda_\alpha U(n)\Lambda_\beta P_z) =$$

$$= \frac{n}{2}\sum_{z\in\mathcal{E}}z^n\int_{\sigma(H_\lambda)}\mathrm{e}^{\mathrm{i}nx}\,\mathrm{tr}(P_zT(x)P_z)\,\mathrm{d}x$$

with absolute convergence. We finally claim that

$$\lim_{n\to\infty}\sum_{z\in\mathcal{E}}z^n\int_{\sigma(H_\lambda)}\mathrm{e}^{\mathrm{i}nx}\,\mathrm{tr}(P_zT(x)P_z)\,\mathrm{d}x = 0. \qquad (4.35)$$

First for $z \in \mathcal{E}$ denote $g_z(x) := \mathrm{tr}(P_z T(x) P_z)$ that is ℓ^1 on $\sigma(H_\lambda)$ by (4.31). Then

$$\int_{\sigma(H_\lambda)} \mathrm{e}^{\mathrm{i} n x} \, \mathrm{tr}(P_z T(x) P_z) \, \mathrm{d} x = 2\pi (\mathcal{F}^{-1}(g_z))(n) \xrightarrow[n \to \infty]{} 0$$

where $\mathcal{F}$ is the Fourier series defined in (1.14). As such it indeed vanishes in the limit $n \to \infty$ by the Riemann-Lebesgue lemma. Finally $z^n 2\pi (\mathcal{F}^{-1}(g_z))(n)$ is summable in z and vanishes when $n \to \infty$ for fixed z. Moreover

$$|z^n 2\pi (\mathcal{F}^{-1}(g_z))(n)| \leq \int_{\sigma(H_\lambda)} |\mathrm{tr}(P_z T(x) P_z)| \, \mathrm{d} x \,,$$

the RHS is summable in z since $T(x)$ is trace-class, so that we may use it as a dominating function (in z) on the sequence $(z^n 2\pi (\mathcal{F}^{-1}(g_z))(n))_n$ when applying the dominated-convergence lemma to exchange $\lim_n$ and $\sum_z$. This leads to $\sum_z 0 = 0$ which gives (4.35), concluding the proof.   □

4.5 THE STRETCH FUNCTION CONSTRUCTION

4.5.1 *Proof of Corollary 4.15*

Corollary 4.15 is a consequence of Theorems 4.11 and 4.12, that both rely on Theorem 4.6, applied to $V \equiv F_\Delta(U)$ and $V_E \equiv F_\Delta(U_E)$ for a given stretch function F_Δ. By Definition 4.4, $V(1)$ is completely localized since $F_\Delta \in B_1(\Delta)$, but in order to replace U and U_E by V and V_E in the previous theorems we need to show that they satisfy all the required properties concerning locality and confinement. We note that Lemmata 4.23 and 4.24 below are true regardless of the existence of (any) gap of $U(1)$ and moreover all the operators involved are (polynomially) local since H is (exponentially) local by Assumption 4.2, see Corollary A.26.

The existence of the gap only become relevant when we use localization to assert the (weak) locality of the logarithm, which is when we apply Corollary 4.20 to $V(1)$. When we do that, we actually get expressions like $\log_\lambda \circ F_\Delta$ applied to $U(1)$, which, as in Lemma 4.19, gets decomposed to sums of functions such as $g \circ F_\Delta$ with g smooth, which is a smooth function, or $\chi_{[\lambda, \lambda']} \circ F_\Delta$ which is in $B_1(\Delta)$ and so Definition 4.4 applies. The conclusion is that the logarithm of $V(1)$ is also weakly-local so that the relative construction could just as well be applied to V.

> **4.23 Lemma.** *V and V_E are (polynomially) local if U and U_E are (exponentially) local. Moreover the maps $t \mapsto V(t)$ and $t \mapsto V_E(t)$ are strongly differentiable and their respective generators $H_V = \mathrm{i} \dot{V} V^*$ and $H_{V_E} = \mathrm{i} \dot{V}_E V_E^*$ are weakly-local.*

Proof. The first fact is a direct consequence of Corollary A.26, F_Δ being smooth. For the derivatives we compute for $t, s \in [0,1]$, using Lemma A.25 and the resolvent identity,

$$V(s) - V(t) = \frac{1}{2\pi i} \int dz \, d\bar{z} (\partial_{\bar{z}} \tilde{F}_\Delta(z)) R_{U(s)}(z)(U(s) - U(t)) R_{U(t)}(z)$$

where $\tilde{F}_\Delta$ is a quasi-analytic extension of F_Δ and $R_{U(s)}(z) = (U(s) - z)^{-1}$, that is norm-continuous in s. Hence

$$\partial_t V(t) = s - \lim_{s \to t} \frac{V(s) - V(t)}{s - t}$$
$$= \frac{1}{2\pi i} \int dz \, d\bar{z} (\partial_{\bar{z}} \tilde{F}_\Delta(z)) R_{U(t)}(z)(\partial_t U)(t) R_{U(t)}(z).$$

Since $\|R_{U(t)}(z)\| \leq C||z| - 1|^{-1}$ and $|\partial_{\bar{z}} \tilde{F}| \leq C||z| - 1|^N$ for some $N \geq 2$ the integral is convergent. Moreover H and U are local thus so are $\partial_t U = -i HU$ and $R_{U(t)}$, the latter by the Combes-Thomas estimate. Since $\tilde{F}_\Delta$ is compactly supported we deduce that $\partial_t V$ is (polynomially) local, and so is H_V by Lemma A.11. We proceed similarly for V_E. $\qquad\square$

4.24 Lemma. *The differences $V_E - \iota^* V \iota$ and $H_{V_E} - \iota^* H_V \iota$ are weakly-local and confined in direction 1, uniformly in $t \in [0,1]$.*

Proof. This looks like a consequence of Lemma 4.21 (see also [GT18, Prop. 4.10]). However since $\iota^* U \iota$ is not a unitary, it is not obvious how to directly implement functional calculus on it. Instead we should first reformulate this result in the bulk picture. In what follows D denotes an operator that is local and confined in direction 1. We claim that

$$U = \iota U_E \iota^* + j U_- j^* + D \tag{4.36}$$

where $j : \mathcal{H}_- \hookrightarrow \mathcal{H}$ and $j^* : \mathcal{H} \twoheadrightarrow \mathcal{H}_-$ with $\mathcal{H}_- = \ell^2((\mathbb{Z} \setminus \mathbb{N}) \times \mathbb{Z}) \otimes \mathbb{C}^N$ is the left half space. Note that $jj^* = \mathbb{1} - P_1$, $j^* j = \mathbb{1}$, and $j^* i = \iota^* j = 0$. Finally U_- is generated by $H_- := j^* H j$, so that both are local like H_E and U_E are. The proof of (4.36) is completely analogue to Lemma 4.21 and relies on the fact that $[P_1, H]$ is local and confined in direction 1.

Then we consider the unitary $U_d := \iota U_E \iota^* + j U_- j^*$, that satisfies $R_{U_d}(z) = \iota R_{U_E}(z) \iota^* + j R_{U_-}(z) j^*$ where $R_U(z) = (U - z)^{-1}$. By (4.36) and the resolvent identity we deduce

$$R_U(z) - R_{U_d}(z) = -R_U(z) D R_{U_d}(z) \tag{4.37}$$

We compute $F_\Delta(U)$ and $F_\Delta(U_d)$ through quasi-analytic functional calculus, see Lemma A.25, leading to

$$F_\Delta(U) - F_\Delta(U_d) = \frac{1}{2\pi i} \int d z\, d\bar{z}(\partial_{\bar{z}}\tilde{F}_\Delta) R_U(z) D R_{U_d}(z)\,.$$

On the right hand side the integral is convergent because of the decaying behavior of $\partial_{\bar{z}}\tilde{F}_\Delta$ around $\mathbb{S}^1$, similarly to the previous proof. Moreover both resolvents are local by Combes-Thomas estimate so that the integral is weakly-local and confined in direction 1 by Corollary A.26. On the left hand side we have $F_\Delta(U_d) = \iota F(U_E)\iota^* + jF(U_-)j^*$, so that the difference $\iota^*(F_\Delta(U) - F_\Delta(U_d))\iota = \iota^*V\iota - V_E$ has the expected property.

It is then easy to show that $\partial_t V_E - \iota^*\partial_t V\iota$ is also weakly-local and confined in direction 1, by using quasi-analytic functional calculus of Lemma 4.23 and the fact that both $\partial_t U_E - \iota^*\partial_t U\iota$ and $R_{U_E}(z) - \iota^*R_U(z)\iota$ are local and 1-confined, respectively coming from Lemma 4.21 and (4.37). We deduce that $H_{V_E} - \iota^*H_V\iota$ has the expected property. □
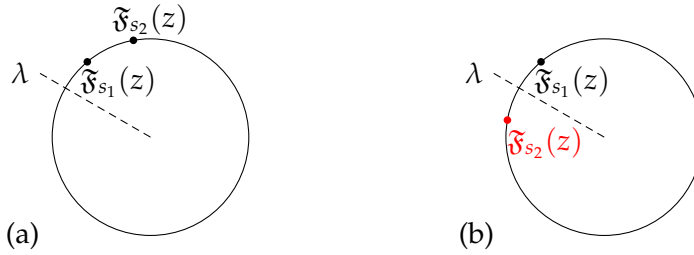
4.5.2 Stretch function invariance

FIGURE 4.3: In the proof, situation (a) happens as a rule and situation (b) never occurs by choice of $\mathfrak{F}$.

Proof of Theorem 4.16. We assume that $F : \mathbb{C} \to \mathbb{C}$ is a stretch function and have $V \equiv F \circ U$. As mentioned we assume F is smooth. Above we have shown that $\mathcal{I}' = W(V^{\mathrm{rel}})$, so that our task now is to show that $W(V^{\mathrm{rel}}) = W(U^{\mathrm{rel}})$. Let $[0, 1] \ni s \mapsto \mathfrak{F}_s(\cdot)$ be a homotopy that interpolates smoothly between the identity map $\mathbb{C} \ni z \mapsto z$ at $s = 0$ and F at $s = 1$. Since F itself is a "stretching" of the mobility gap $\Delta \subseteq \mathbb{S}^1$ onto the entire circle, we pick this interpolation such that it stretches *about* the branch cut $\lambda \in \Delta$. This point is crucial and will be used later on, in that it means no eigenvalue of $\mathfrak{F}_s(U(1))$ crosses λ as s changes. The gist of the argument is as follows. All maps involved are continuous (even smooth) except one, $\log_\lambda$. While this map indeed has a jump discontinuity, the particular form of deformation which we choose doesn't ever cross this point of discontinuity–in other words, λ is a fixed point of the deformation (see Figure 4.3).

The smoothness assumption means that, in particular, for fixed z, $s \mapsto \mathfrak{F}_s(z)$ is differentiable, for all s, $z \mapsto \mathfrak{F}_s(z)$ is smooth (so $\mathfrak{F}_s(U(t))$ is local for all t and it makes sense to take the derivative of $t \mapsto \mathfrak{F}_s(U(t))$) and for fixed z, $s \mapsto \mathfrak{F}_s'(z)$ is differentiable. In addition, because $s \mapsto \mathfrak{F}_s$ interpolates between $\mathbb{1}$ and F, the mobility gap never closes (it only gets stretched from $\Delta \to \mathbb{S}^1 \setminus \{1\}$) for all s, λ is within the mobility gap of $\mathfrak{F}_s(U(1))$ so that $\log_\lambda(\mathfrak{F}_s(U(1)))$ is weakly-local for all s.

Hence $W(\mathfrak{F}_s(U)^{\text{rel}})$ is well-defined and integer valued for all s, so it suffices to prove that $|W(\mathfrak{F}_{s_1}(U)^{\text{rel}}) - W(\mathfrak{F}_{s_2}(U)^{\text{rel}})| < 1$ for any $s_1, s_2 \in [0, 1]$ with $|s_1 - s_2|$ sufficiently small. Recall that $W(\mathcal{V}^{\text{rel}}) = W(\mathcal{V}) - W(\mathcal{V}_\lambda)$ so that by the triangle inequality we can work separately with $|W(\mathfrak{F}_{s_1}(U)) - W(\mathfrak{F}_{s_2}(U))|$ and $|W(\mathfrak{F}_{s_1}(U)_\lambda) - W(\mathfrak{F}_{s_2}(U)_\lambda)|$, though each part is not separately an integer. To probe the smallness of these expressions we use Lemma 4.25.

We will use Claim A.32. This is boosted, using the weakly-local properties, to Lemma A.23 and Lemma A.24. Note that in order to use these lemmas, one must have uniform exponents μ and ν which is certainly not part of the context in Appendix A.2.1. However, this is actually not a problem since the form of weak-locality that is produced by Corollary A.26 gives us the ability to choose the minimal exponents μ once and for all. The exponent ν is actually not even necessary here since the deformation is always applied on $U(1)$ which is honestly local and not just weakly-local, but even if that weren't the case, one can just choose a universal ν which makes $\sum_x (1 + \|x\|)^{-\nu}$ finite.

Since $s \mapsto \mathfrak{F}_s(z)$ is differentiable,

$$\text{s-}\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathfrak{F}_{s+\varepsilon}(U(t)) - \mathfrak{F}_s(U(t))) = \partial_s \mathfrak{F}_s(U(t)),$$

so that $\frac{1}{\varepsilon}T(\mathfrak{F}_{s+\varepsilon}(U(t)) - \mathfrak{F}_s(U(t))) \to T\partial_s\mathfrak{F}_s(U(t))$ in trace-class norm for any trace-class T. Similarly we handle also $\partial_t \mathfrak{F}_s(U(t))) = \mathfrak{F}_s'(U(t)))\dot{U}(t)$ which is also differentiable as a function of s. Since $\mathfrak{F}_s(U(t)))$ is weakly-local for any value of s, we also have similar convergence for the spatial derivatives: $\frac{1}{\varepsilon}T_\alpha\partial_\beta(\mathfrak{F}_{s+\varepsilon}(U(t)) - \mathfrak{F}_s(U(t))) \to T_\alpha\partial_\beta\partial_s\mathfrak{F}_s(U(t))$ in trace-class norm for any T_α which is weakly-local and confined in the α direction. We conclude that $|W(\mathfrak{F}_{s_1}(U)) - W(\mathfrak{F}_{s_2}(U))|$ can be made arbitrarily small as $s_2 \to s_1$.

When dealing with $|W(\mathfrak{F}_{s_1}(U))_\lambda) - W(\mathfrak{F}_{s_2}(U)_\lambda)|$, it might appear that we are stuck, since $(\mathfrak{F}_{s_1}(U))_\lambda(t) \equiv \exp(t \log_\lambda(\mathfrak{F}_{s_1}(U(1))))$ and $\log_\lambda$ is *not* continuous. Furthermore, algebraic laws like $\log(\frac{z_1}{z_2}) = \log(z_1) - \log(z_2)$ only hold mod $2\pi i$ in general, which could introduce jump discontinuities. Since $\mathfrak{F}_{s_1}(U(1))$ and $\mathfrak{F}_{s_2}(U(1))$ are functions of the same operator $U(1)$, they commute and hence have the same diagonalization. Indeed, let P be the projection-valued spectral measure of $U(1)$. Then

$$\log_\lambda(\mathfrak{F}_{s_1}(U(1))) - \log_\lambda(\mathfrak{F}_{s_2}(U(1))) =$$

$$= \int_{z \in S^1} \log_\lambda (\mathfrak{F}_{s_1}(z)) \, \mathrm{d}\, P(z) - \int_{z \in S^1} \log_\lambda (\mathfrak{F}_{s_2}(z)) \, \mathrm{d}\, P(z)$$
$$= \int_{z \in S^1} (\log_\lambda (\mathfrak{F}_{s_1}(z)) - \log_\lambda (\mathfrak{F}_{s_2}(z))) \, \mathrm{d}\, P(z) \,.$$

Now, since λ is a fixed point of the deformation in s and since the deformation is continuous in s, $\mathfrak{F}_{s_1}(z)$ and $\mathfrak{F}_{s_2}(z)$ (for sufficiently small $|s_1 - s_2|$) are sufficiently close on the circle and on the same "side" of the cut so that the algebraic rule of the logarithm holds without the mod $2\pi\mathrm{i}$. Hence

$$\log_\lambda (\mathfrak{F}_{s_1}(U(1))) - \log_\lambda (\mathfrak{F}_{s_2}(U(1))) =$$
$$= \int_{z \in S^1} \log_\lambda (\mathfrak{F}_{s_1}(z)(\mathfrak{F}_{s_2}(z))^{-1}) \, \mathrm{d}\, P(z)$$
$$= \log_\lambda (\mathfrak{F}_{s_1}(U(1))(\mathfrak{F}_{s_2}(U(1)))^{-1}) \,.$$

This gives

$$\mathfrak{F}_{s_1}(U))_\lambda(t) - \mathfrak{F}_{s_2}(U))_\lambda(t) \equiv$$
$$\equiv \exp(t \log_\lambda (\mathfrak{F}_{s_1}(U(1)))) - \exp(t \log_\lambda (\mathfrak{F}_{s_2}(U(1))))$$
$$= \mathfrak{F}_{s_1}(U))^\lambda(t)(\mathbb{1} - e^{t \log_\lambda (\mathbb{1} + (\mathfrak{F}_{s_2}(U(1))(\mathfrak{F}_{s_1}(U(1)))^{-1} - \mathbb{1}))}) \,.$$

We thus find that

$$\text{s-}\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}((\mathfrak{F}_{s+\varepsilon}(U))_\lambda(t) - (\mathfrak{F}_s(U))_\lambda(t)) = t(\mathfrak{F}_s(U))_\lambda(t)(\partial_s \mathfrak{F}_s)(U(1))(\mathfrak{F}_s(U(1)))^{-1} \,,$$

which is weakly-local, as λ always falls within the mobility gap of $\mathfrak{F}_s(U(1))$. For the time derivative we get similar formulas and following the same argument as above, we find that $|W(\mathfrak{F}_{s_1}(U))_\lambda) - W(\mathfrak{F}_{s_2}(U)_\lambda)|$ can also be made arbitrarily small. $\qquad \square$

4.25 Lemma. *For any two unitary maps $A, B : [0,1] \to \mathcal{U}(\mathcal{H})$ which are differentiable, whose derivatives are bounded too, and which are weakly-local, we have*

$$|W(A) - W(B)| \leq \sup_{[0,1]} \|T_1(A - B)\|_1 + \|T_2(\dot{A} - \dot{B})\|_1 +$$
$$+ \sup_{\alpha, \beta} (\|T_{3\alpha}(A - B)_{,\beta}\|_1 + \|(A - B)_{,\alpha}^* T_{4\beta}\|_1 + \qquad (4.38)$$
$$+ \|(A - B)_{,\alpha}^*(A - B)_{,\beta}\|_1)$$

where T_1, T_2 are some (time-dependent) trace class operators depending on A, B their derivatives w.r.t. time and their spatial derivatives, the supremum over α, β is over the two possibilities where $\alpha \neq \beta$. Then $T_{3\alpha}, T_{4\beta}$ is a weakly-local operator confined in the α, β direction.

Proof. We start from (4.6) which says

$$W(A) = -\frac{1}{2} \int_0^1 \text{tr}\, \varepsilon_{\alpha\beta} \dot{A} A^* A_{,\alpha} A^* A_{,\beta} A^*$$

to get

$$|W(A) - W(B)| \leq \frac{1}{2} \sup_{[0,1]} \|\varepsilon_{\alpha\beta}(\dot{A} A^* A_{,\alpha} A^* A_{,\beta} A^* - \dot{B} B^* B_{,\alpha} B^* B_{,\beta} B^*)\|_1$$

$$\leq \frac{1}{2} \sup_{\alpha,\beta,[0,1]} \|\dot{A} A^* A_{,\alpha} A^* A_{,\beta} A^* - \dot{B} B^* B_{,\alpha} B^* B_{,\beta} B^*\|_1$$

$$\leq \frac{1}{2} \sup_{\alpha,\beta,[0,1]} (\|\dot{A} A^* (A_{,\alpha} A^* A_{,\beta} A^* - B_{,\alpha} B^* B_{,\beta} B^*)\|_1 +$$

$$+ \|(\dot{A} A^* - \dot{B} B^*) B_{,\alpha} B^* B_{,\beta} B^*\|_1)$$

$$\leq \frac{1}{2} \sup_{\alpha,\beta,[0,1]} (\|\dot{A}\| \|A_{,\alpha} A^* A_{,\beta} A^* - B_{,\alpha} B^* B_{,\beta} B^*\|_1 +$$

$$+ \|\dot{A}\| \|(A - B) B_{,\alpha} B^* B_{,\beta} B^*\|_1 + \|(\dot{A} - \dot{B}) B^* B_{,\alpha} B^* B_{,\beta}\|_1).$$

The supremum is over all times in $[0,1]$ and all α, β equal to $1, 2$ (without $\alpha = \beta$).

We concentrate on the term $\|A_{,\alpha} A^* A_{,\beta} A^* - B_{,\alpha} B^* B_{,\beta} B^*\|_1$ since the two other terms are in their final desired form. Because A, B are unitary we have $A_{,\alpha} A^* = -A A_{,\alpha}^*$ so that

$$A_{,\alpha} A^* A_{,\beta} A^* - B_{,\alpha} B^* B_{,\beta} B^* = -A A_{,\alpha}^* A_{,\beta} A^* + B B_{,\alpha}^* B_{,\beta} B^*$$

$$= (B - A) A_{,\alpha}^* A_{,\beta} A^* -$$

$$- B(A_{,\alpha}^* A_{,\beta} - B_{,\alpha}^* B_{,\beta}) A^* +$$

$$+ B B_{,\alpha}^* B_{,\beta} (B - A)^*.$$

Only the middle line is not in the form we want, so that we write,

$$A_{,\alpha}^* A_{,\beta} - B_{,\alpha}^* B_{,\beta} = A_{,\alpha}^* (A - B)_{,\beta} + (A - B)_{,\alpha}^* A_{,\beta} - (A - B)_{,\alpha}^* (A - B)_{,\beta}.$$

$\square$

SUMMARY

At the end of this journey we conclude with a few comments about prospects for future research.

TOPOLOGICAL INSULATORS   While the IQHE has been studied extensively, topological insulators have existed in science for less than two decades. There are still many things we do not know. One of the biggest questions is the experimental validity of the bottom eight rows of Table 1.1, i. e. the real class. While some experimental findings seem to support the case for class AII materials in two-dimensions [Kön+07] and three-dimensions [Che+09], other experiments have shown that even with magnetic fields the topological effects persist and perhaps what is more important is reflection symmetry [Du+15], see also discussion in [ZZS14]. In class DIII, an experimental study on topological superconductors [Sas+11] found evidence for three-dimensional non-trivial topology, and [Lev+13] presents another study. In class D in one dimension, the Kitaev chain [Kit01] is a Real example of a non-trivial cell of Table 1.1 with a $\mathbb{Z}_2$ invariant. According to [Fra13] it is still unclear whether there is sufficiently strong experimental evidence for its existence.

As was noted in Section 1.3, while [BKR17; Kub17] contain a rather full classification of Table 1.1 culminating in a proof of Principle 1.8, it is limited to the spectral gap regime. [GS16] extends to the mobility gap regime. On the bulk side this has been done for covariant operators in [BvS94], but for our deterministic mobility gap regime, one would like a proper definition of $\mathfrak{H}, \mathfrak{H}^\sharp$ and most importantly $\mathrm{Open}(\mathfrak{H}), \mathrm{Open}(\mathfrak{H}^\sharp)$. In fact, besides the proof that $\mathcal{N}$ is continuous with respect to the Fermi energy in the IQHE, [EGS05, Prop. 2], there isn't even a (deterministic) proof of its local constancy in general, something we hope to mend in the near future. Proving this would involve extending Fredholm theory by dropping the constraint of essential gap and instead employing a constraint of the form Definition 1.27.

It is also the case that various descriptions of the invariants and their inter-relations have not yet been fully explored. For instance, it is not clear why the Fu-Kane-Mele $\mathbb{Z}_2$ invariant [FK07] is related to the Θ-odd Fredholm index of (1.2). It is also unclear how the IQHE is a "chiral-anomaly" in the sense of the Fujikawa description in effective field theory (as described e. g. in [PS95, pp. 664]), and how this relationship can be made quantitative.

Furthermore, is it possible to make a quantitative connection (at least in one dimension) between the Lyapunov spectrum at zero energy and other symmetry classes of Table 1.1? This has been explored also in [LSS13], and references therein. Since in one dimension one could argue that all edge invariants are defined by some signed number of zero modes, which is what the Lyapunov spectrum counts, this seems like a reasonable conjecture. Can there be such a connection also in higher dimensions? Would it ever be possible to explore the topological phase transition and finally exhibit

a localization-delocalization-localization process as we let some parameter of the system very such that its Chern number passes from 1 to 0 being undefined somewhere in between?

And then there is the question of interactions. The FQHE, which has been studied thoroughly with great success [FS93; Lau83; MR91], but our *topological* understanding of it is lacking, in the sense that going from a microscopic many-body Hamiltonian to Principle 1.8 seems out of reach at the moment. Preliminary studies from the perspective of edge states [FK11] show that in one-dimension for class BDI systems, one has for instance $\mathbb{G} \cong \mathbb{Z}_8$.

RANDOM SCHRÖDINGER OPERATORS   While the theory of localization has been with us for many more decades, some famous long standing conjectures still defy a solution. Perhaps the biggest one is *the extended states conjecture*, which could be phrased in various different ways (some implying others) for the Anderson model (1.15):

- Show $D(E) \in (0, \infty)$ for such E for which Definition 1.21 does *not* hold.

- By (1.19) this is equivalent to the same statement about $\sigma_{ij}(E)$.

- Show that $s_{ac}(H) \neq \varnothing$ for sufficiently small $\lambda \neq 0$.

- Show that for some E and $\lambda \neq 0$ sufficiently small one has

$$\liminf_{\eta \to 0^+} \mathbb{E}[\|G(x, x; E + i\eta)\|^2] < \infty .$$

Some early progress has been made on the Bethe lattice instead of $\mathbb{Z}^d$ [Kle98], and since then this question lay somewhat dormant. Recently a preprint appeared [Ueb16] which seems to suggest that in the special case of Dirac potentials one could say something.

Even more daunting seems to be the problem of complete localization in two-dimensions, since the nature of the localized dynamics could probably only be exhibited in the limit of infinite time. The lattice Bernoulli Anderson model localization remains unsolved, though [Imb17] seems to be getting rather close via the MSA (cf. [BK05]).

What is particularly interesting is the possibility of answering some of these above questions by the usage of topology. For instance, [GKS07] proves that for the random Landau Hamiltonian there is at least one point where the localization length diverges, which should coincide with the (unperturbed) Landau level. In [FGW00] the authors prove $s_{ac}(H) \neq \varnothing$ for the edge sample in the IQHE via Mourre theory, guided by the heuristic fact that the IQHE edge supports current along its edge. See also [BP]. It would be interesting to extend that result to the mobility gap regime for lattice Hamiltonians, in the setting of [EGS05].

APPENDIX

In this section we want to prove Proposition 1.26, or rather a slightly more general version of it.

A.1 Proposition. *Let* $A_i : \Omega \to \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ *be a family of random operators indexed by* $i \in I$ *for some set I. If*

$$\mathbb{E}[\sup_{i \in I} \|\langle \delta_x, A_i \delta_y \rangle\|] \le C\,e^{-\mu|x-y|} \qquad (x, y \in \mathbb{Z}^d) \tag{A.1}$$

for some constants $C < \infty, \mu > 0$, *then almost-surely, for some (random) constant* $C' > 0$, *and any deterministic* $\mu' \in (0, \mu), a \in \ell^1(\mathbb{Z}^d)$ *we have*

$$\sup_{i \in I} \|\langle \delta_x, A_i \delta_y \rangle\| \le C' \frac{1}{|a(x)|}\,e^{-\mu'\|x-y\|}$$

Proof. By Fatou's lemma we can bound

$$\mathbb{E} \sum_{x,y \in \mathbb{Z}^d} \sup_{i \in I} \|\langle \delta_x, A_i \delta_y \rangle\|\,e^{+\mu'\|x-y\|}\,|a(x)|$$

by

$$\liminf_{\Lambda \to \mathbb{Z}^d} \sum_{x,y \in \Lambda} \mathbb{E} \sup_{i \in I} \|\langle \delta_x, A_i \delta_y \rangle\|\,e^{+\mu'\|x-y\|}\,|a(x)|\,.$$

We now use our hypothesis (A.1) to get that this expression is estimated by

$$\liminf_{\Lambda \to \mathbb{Z}^d} \sum_{x,y \in \Lambda} C\,e^{-(\mu-\mu')|x-y|}\,|a(x)|\,.$$

If we interpret $\|x\|$ for $x \in \mathbb{Z}^d$ as the 1-norm, i.e. $\|x\| \equiv \sum_{j=1}^d |x_j|$, and use the formula $\sum_{j \in \mathbb{Z}} e^{-\nu|j|} = \coth(\frac{\nu}{2})$, we find that this last expression is equal to $C(\coth(\frac{\mu-\mu'}{2}))^d \|a\|_1 < \infty$.

Hence $\mathbb{E} \sum_{x,y \in \mathbb{Z}^d} \sup_{i \in I} \|\langle \delta_x, A_i \delta_y \rangle\|\,e^{+\mu'\|x-y\|}\,|a(x)|$ is finite. But an integrable non-negative function must be be finite almost-everywhere, that is, there is some (random) constant $C' < \infty$ such that almost-surely,

$$\sum_{x,y \in \mathbb{Z}^d} \sup_{i \in I} \|\langle \delta_x, A_i \delta_y \rangle\|\,e^{+\mu'\|x-y\|}\,|a(x)| < C'\,.$$

But then,

$$\sup_{i \in I} \| \langle \delta_x, A_i \delta_y \rangle \, \mathrm{e}^{+\mu' \|x-y\|} \, |a(x)| \leq \sum_{x',y' \in \mathbb{Z}^d} \sup_{i \in I} \| \langle \delta_{x'}, A_i \delta_{y'} \rangle \, \mathrm{e}^{+\mu' \|x'-y'\|} \, |a(x')|$$

$$\leq C'.$$

$\square$

A.2   LOCALITY

We define various notions of locality for operators which encode the structure of space (otherwise, all separable Hilbert spaces are isomorphic to $\ell^2(\mathbb{N})$) and the associated dynamics.

Let $d, L \in \mathbb{N}_{\geq 1}$ be given. Define the Hilbert space $\mathcal{H}_{d,L} := \ell^2(\mathbb{Z})^d \otimes \mathbb{C}^L$; the position basis of $\ell^2(\mathbb{Z}^d)$ is denoted by $(\delta_x)_{x \in \mathbb{Z}^d}$. Given an operator $A \in \mathcal{B}(\mathcal{H}_{d,L})$ and two points $x, y \in \mathbb{Z}^d$ we define the linear map $A_{xy} := \langle \delta_x, A \delta_y \rangle : \mathbb{C}^L \to \mathbb{C}^L$ between the internal spaces. The norm $\| \cdot \|$ on such maps is chosen as the matrix trace norm and the norm on $\mathbb{Z}^d$ may be taken as the 1-norm on $\mathbb{R}^d$: $\|x\|_1 \equiv \sum_{j=1}^d |x_j|$.

> **A.2 Definition** (Locality). An operator $A \in \mathcal{B}(\mathcal{H}_{d,L})$ is said to be (exponentially) *local* iff there are some constants $C < \infty, \mu > 0$ such that
>
> $$\|A_{xy}\| \leq C \, \mathrm{e}^{-\mu \|x-y\|} \qquad (x, y \in \mathbb{Z}^d). \tag{A.2}$$
>
> A is said to be *polynomially-local* iff for any $\alpha \in \mathbb{N}$ sufficiently large, there are some constants $C_\alpha < \infty, \mu_\alpha > 0$ such that
>
> $$\|A_{xy}\| \leq C_\alpha (1 + \mu_\alpha \|x - y\|)^{-\alpha} \qquad (x, y \in \mathbb{Z}^d). \tag{A.3}$$

> **A.3 Example** (Finite-hopping). If $A \in \mathcal{B}(\mathcal{H}_{d,L})$ obeys the constraint that $A_{xy} = 0$ for all $\|x - y\| > R$ for some given $R \in \mathbb{N}$ then A is called a *finite-hopping operator of range R*. Finite hopping operators are of course local.

This is a precise way to formulate the expected locality of Hamiltonians in Definition 1.1. Indeed, most concrete models for Hamiltonians that one encounters in physics are of finite hopping. This requirement embodies the principle that physical laws are local in real space. A generic lattice Hamiltonian may be written using multi-index notation as follows. Let R_j be the right-shift operator on the jth axis: $(R_j \psi)_x \equiv \psi_{x-e_j}$ with $(e_j)_{j=1}^d$ the standard basis of $\mathbb{R}^d$. Given a multi-index $\alpha \in \mathbb{N}_{\geq 0}^d$, we define $R^\alpha := \prod_{j=1}^d R_j^{\alpha_j}$, which is unambiguous since $[R_j, R_k] = 0$ for all j, k. Next, we pick for each multi-index α a sequence $a^{(\alpha)} : \mathbb{Z}^d \to \mathrm{Mat}_L(\mathbb{C})$ which induces a multiplication operator $A^{(\alpha)}$ on $\mathcal{H}_{d,L}$. Then

$$H = \sum_\alpha A^{(\alpha)} R^\alpha + (R^\alpha)^* (A^{(\alpha)})^*.$$

If this sum is finite then H is of finite range, and otherwise, in order to obey (A.2), we must have $a^{(\alpha)}$ decay in $|\alpha| \equiv \sum_{j=1}^{d} \alpha_j$.

To calibrate these statements, consider the following. Iff H is translation invariant, all sequences $a^{(\alpha)}$ are constant, iff $A^{(\alpha)}$ commute with all shift operators, in which case the decay in $|\alpha|$ is a statement about the smoothness in quasi-momentum of the function associated to the multiplication operator $\mathcal{F}H\mathcal{F}^{-1}$, with $\mathcal{F}$ as in (1.14).

It is not clear (to us) what kind of physics lies behind the distinction between exponential locality and polynomial locality, and mathematically they seem to imply the same results (w.r.t. topological insulators), with the former easier to handle algebraically.

In accordance with the theory of localization for *random* (local) operators and its almost sure consequence Proposition A.1, we define the notion of weak-locality which is meant for certain Borel bounded functions of a local, localized operator.

A.4 Definition (Weak-locality). An operator $A \in \mathcal{B}(\mathcal{H}_{d,L})$ is called (exponentially) *weakly-local* iff there is a constant $\mu > 0$, an element $a \in \ell^1(\mathbb{Z}^d)$ and a constant $C_a < \infty$ such that

$$\|A_{xy}\| \leq C_a \frac{1}{|a(x)|} \, \mathrm{e}^{-\mu\|x-y\|} \qquad (x,y \in \mathbb{Z}^d). \tag{A.4}$$

A is said to be *polynomially-weakly-local* iff there is an element $a \in \ell^1(\mathbb{Z}^d)$ such that for any $\alpha \in \mathbb{N}$ sufficiently large there are constants $C_{a,\alpha} < \infty$, $\mu_\alpha > 0$, such that

$$\|A_{xy}\| \leq C_{a,\alpha} \frac{1}{|a(x)|} \left(1 + \mu_\alpha\|x-y\|\right)^{-\alpha} \qquad (x,y \in \mathbb{Z}^d). \tag{A.5}$$

The point about the qualifier "weakly" is that while there is (exponential or polynomial) decay in the off-diagonal direction (i.e. in $\|x-y\|$), the estimate allows for explosion in the diagonal direction (i.e. in $\|x\|$). In the language of [Rio+96], if H is SUDL then $\exp(-\mathrm{i}\cdot H)$ is weakly-local. This was introduced first in the context of deterministic strongly disordered topological insulators in [EGS05].

A.5 Example. In applications, one usually takes $a_\varepsilon(x) := \mathrm{e}^{-\varepsilon\|x\|}$ for some $\varepsilon > 0$ or $a_\nu(x) := (1 + \|x\|)^{-\nu}$ for $\nu \geq d+1$.

A.6 REMARK. Clearly locality implies weak-locality, and exponential decay implies polynomial decay.

In this language, (deterministic) localization of an operator in an interval (a,b) is the statement that locality of an operator implies weak-locality of its spectral projections associated with (a,b) as well as with $(-\infty, c)$ for all $c \in (a,b)$. Contrast this with the consequence of a spectral gap in (a,b), which says: Locality of an operator implies locality of its spectral projections associated with (a,b) as well as with $(-\infty, c)$ for all $c \in (a,b)$, which is Proposition 2.35. This is true also for the *holomorphic* functional calculus, which is associated with probing the system at finite distances from the real axis, that is, probing the dynamics at finite time intervals.

The Helffer-Sjöstrand formula Corollary A.26 implies that the *smooth* functional calculus of an (exponentially) local operator is polynomially local. This is associated with probing the dynamics for infinite time, but (morally) in a much less direct and more gentle way than with the Borel bounded functional calculus, which requires the full strength of localization to control.

Sometimes, for example, in Lemma 4.19, we want to get estimates on functions which are piecewise-smooth, where the discontinuities happen in localized regions of the spectrum. When that happens, we get the "worst" of both worlds: we must add the "weakly" qualifier due to (deterministic) localization and also the (polynomial) qualifier due to using the Helffer-Sjöstrand formula.

For topological invariants, we shall now work towards showing that the notion of polynomial weak locality is sufficient to have the invariants well-defined.

A.2.1 *The weakly-local star-algebra*

Here we specify (since this is all what we need) to *a* being polynomial and hence get the following definition (for brevity we drop the polynomial qualifier, and also somewhat simplify (A.5) but in an equivalent way):

A.7 Definition. (*Weakly-local operators*) The operator A is said to be *weakly-local* iff there is some $\nu \geq 0$ such that for any $\mu > 0$ sufficiently large there is some constant $C_\mu < \infty$ with

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu}(1 + \|x\|)^\nu \qquad (x, y \in \mathbb{Z}^d). \tag{A.6}$$

In our application, the sufficiently large value of μ will usually be 2 (Corollary A.26) and fixed throughout for all operators. However, to discuss the algebraic properties we allow this value to be arbitrary.

A.8 REMARK. In the above definition, when $\nu = 0$, the operator is polynomially local.

A.9 REMARK. One could choose various other ways to encode the off-diagonal decay of an operator. Compare with [EGS05, Section 3.3], which illustrates how to encode (exponential) decay either with bounds on matrix elements or by estimates on the operator norm of a space-weighted version of the operator. Here we refrain from reformulating (A.6) in different ways.

A.10 Lemma. *The transpose of a weakly-local operator is again weakly-local.*

Proof. Assume A obeys Definition A.7. Then picking $\mu > \nu$,

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu}(1 + \|x\|)^\nu$$
$$\leq C_\mu (1 + \|x - y\|)^{-(\mu - \nu)}(1 + \|y\|)^\nu (1 + \|y\|)^{-\nu}(1 + \|x - y\|)^{-\nu}(1 + \|x\|)^\nu.$$

But now, $\frac{1+\|x\|}{(1+\|y\|)(1+\|x-y\|)} \leq \frac{1+\|x\|}{1+\|y\|+\|x-y\|}$ and using the reverse triangle inequality, $\|x-y\| \geq \|x\| - \|y\|$ so that this fraction is smaller than or equal to one. So is its νth power.

We find that $\|A_{xy}\| \leq C_{\mu+\nu}(1+\|y-x\|)^{\mu}(1+\|y\|)^{\nu}$ for all μ sufficiently large; in other words, A^T is weakly-local (though with different constants).

$\square$

A.11 Lemma. *The weakly-local operators form a star-algebra.*

Proof. Due to Lemma A.10 the linearity of taking matrix elements and the triangle inequality of the matrix norm, we only verify the product property. Let A, B be two given weakly-local operators with constants C_{μ}^A, C_{μ}^B respectively. Then for any $\mu > 0$ sufficiently large (for both A and B) and $\nu := \max(\{\nu_A, \nu_B\})$ we have,

$$
\begin{aligned}
\|(AB)_{xy}\| &\leq \sum_z \|A_{xz}\|\|B_{zy}\| \\
&\leq \sum_z C_{\mu}^A(1+\|x-z\|)^{-\mu}(1+\|x\|)^{\nu}C_{\mu}^B(1+\|z-y\|)^{-\mu}(1+\|z\|)^{\nu} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.7) \\
&\leq C_{\mu}^A C_{\mu}^B(1+\|x\|)^{\nu}\sum_z(1+\|x-z\|)^{-\mu}(1+\|z-y\|)^{-\mu}(1+\|z\|)^{\nu}.
\end{aligned}
$$

Now note that $(1+\|x-z\|)(1+\|z-y\|) \geq 1+\|x-z\|+\|z-y\| \geq 1+\|x-y\|$ so that

$$
\begin{aligned}
\|(AB)_{xy}\| \leq C_{\mu}^A C_{\mu}^B(1+\|x\|)^{\nu}(1+\|x-y\|)^{-\mu/2} \\
\times \sum_z(1+\|x-z\|)^{-\mu/2}(1+\|z-y\|)^{-\mu/2}(1+\|z\|)^{\nu}.
\end{aligned}
$$

Assume further that $\mu > 2\nu$ has been chosen. Then $(1+\|x-z\|)^{-\mu/2}(1+\|z\|)^{\nu} \leq (\frac{1+\|z\|}{1+\|x-z\|})^{\nu} \leq (1+\|x\|)^{\nu}$ by $(1+\|x\|)(1+\|x-z\|) \geq 1+\|x\|+\|x-z\| \geq 1+\|z\|$. We conclude that

$$
\begin{aligned}
\|(AB)_{xy}\| &\leq C_{\mu}^A C_{\mu}^B(1+\|x\|)^{2\nu}(1+\|x-y\|)^{-\mu/2}\sum_z(1+\|z-y\|)^{-\mu/2} \\
&\leq C_{\mu}^A C_{\mu}^B\Big(\sum_{z\in\mathbb{Z}^d}(1+\|z\|)^{-\mu/2}\Big)(1+\|x-y\|)^{-\mu/2}(1+\|x\|)^{2\nu}.
\end{aligned}
$$

If now we also pick μ large enough so that the sum in the first parenthesis is finite (e.g. $\mu > 2(d+1)$) then we find our result. $\square$

A.2.2 *The weakly-local-and-confined two-sided ideal*

A.12 Definition. (*Weakly-Local-and-Confined Operators*) The operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *weakly-local-and-confined in direction i* for $i = 1, \ldots, d$ iff there is some $\nu > 0$ such that for any $\mu > 0$ sufficiently large there is some constant $C_\mu < \infty$ with

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu}(1 + |x_i|)^{-\mu}(1 + \|x\|)^\nu \qquad (x, y \in \mathbb{Z}^d) \qquad \text{(A.8)}$$

We see that adding the "confined" condition guarantees that the operator has also diagonal decay at least in one direction.

A.13 Lemma. *If A is weakly-local and confined in direction i then so is A^T.*

Proof. Since $\|x\| \geq |x_i|$, we have $(1 + \|x\|)^{-\mu} \leq (1 + |x_i|)^{-\mu}$, and hence,

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu/2}(1 + |x_i - y_i|)^{-\mu/2}(1 + |x_i|)^{-\mu}(1 + \|x\|)^\nu$$
$$\leq C_\mu (1 + \|x - y\|)^{-\mu/2}((1 + |x_i - y_i|)(1 + |x_i|))^{-\mu/2}(1 + \|x\|)^\nu$$

Now, $(1 + |x_i - y_i|)(1 + |x_i|) \geq 1 + |x_i - y_i| + |x_i| \geq 1 + |y_i|$, so that

$$\|A_{xy}\| \leq C_\mu (1 + \|x - y\|)^{-\mu/2}(1 + |y_i|)^{-\mu/2}(1 + \|x\|)^\nu$$

Now we can follow the same procedure as in Lemma A.10 to replace the $(1 + \|x\|)^\nu$ factor with a $(1 + \|y\|)^\nu$ (by worsening the constants). $\square$

A.14 Lemma. *If A is weakly-local-and-confined in direction i, then for all μ sufficiently large and ν as in Definition A.12 we have $\|(1 + \|X\|)^{-\nu}(1 + |X_i|)^\mu A\| < \infty$.*

Proof. We use Holmgren's bound and the assumed bound in Definition A.12 to get

$$\|(1 + \|X\|)^{-\nu}(1 + |X_i|)^\mu A\| \leq \max_{x \leftrightarrow y} \sup_y \sum_x \|((1 + \|X\|)^{-\nu}(1 + |X_i|)^\mu A)_{xy}\|$$
$$\leq \max_{x \leftrightarrow y} \sup_y \sum_x (1 + \|x\|)^{-\nu}(1 + |x_i|)^\mu \|A_{xy}\|$$
$$\leq \max_{x \leftrightarrow y} \sup_y \sum_x (1 + \|x\|)^{-\nu}(1 + |x_i|)^\mu \times$$
$$\times C_\mu (1 + \|x - y\|)^{-\mu}(1 + |x_i|)^{-\mu}(1 + \|x\|)^\nu$$
$$= C_\mu \sum_{x \in \mathbb{Z}^d} (1 + \|x\|)^{-\mu} < \infty,$$

assuming μ is chosen sufficiently large so that this last sum is finite. $\square$

A.15 Lemma. *The space of weakly-local-and-confined in direction i operators forms a star-closed two-sided ideal within the star-algebra of weakly-local operators.*

Proof. The additive subgroup property follows by the linearity of taking matrix elements as well as the triangle inequality of the matrix norm associated to $\mathbb{C}^N$. The star-closure follows due to Lemma A.13.

Let now A be weakly-local and confined in direction i and B be merely weakly-local. Then pick $\mu > 0$ sufficiently large for both A and B and denote $\nu := \max(\{\nu_A, \nu_B\})$, to get

$$
\begin{aligned}
\|(AB)_{xy}\| &\leq \sum_z \|A_{xz}\| \|B_{zy}\| \\
&\leq \sum_z C_\mu^A (1 + \|x - z\|)^{-\mu} (1 + |x_i|)^{-\mu} (1 + \|x\|)^{\nu_A} \times \\
&\quad \times C_\mu^B (1 + \|z - y\|)^{-\mu} (1 + \|z\|)^{\nu_B} \\
&= C_\mu^A C_\mu^B (1 + |x_i|)^{-\mu} \times \\
&\quad \times \sum_z (1 + \|x - z\|)^{-\mu} (1 + \|x\|)^{\nu} (1 + \|z - y\|)^{-\mu} (1 + \|z\|)^{\nu} .
\end{aligned}
$$

Everything after $(1 + |x_i|)^{-\mu}$ is identical to (A.7) (after which we showed that the remainder expression is estimated as weakly-local), so that we find AB is also weakly-local and confined in direction i.

Since $BA = (A^* B^*)^*$, A^* is weakly-local and confined in direction i, B^* is weakly-local, so that by the previous paragraph, $A^* B^*$ belongs to this ideal as well, and hence by the star-closure, BA as well. $\square$

A.16 Lemma. *If A, B are weakly-local-and-confined in direction i, j respectively, then AB is weakly-local and confined in directions i and j simultaneously.*

Proof. Due to Lemma A.13 we may interchange which of the indices of the matrix element we want to represent the confinement. Thus we are allowed to write, for $\mu > 0$ sufficiently large for both A and B, again with $\nu := \max(\{\nu_A, \nu_B\})$

$$
\begin{aligned}
&\|(AB)_{xy}\| \\
&\leq \sum_z \|A_{xz}\| \|B_{zy}\| \\
&\leq \sum_z C_\mu^A (1 + \|x - z\|)^{-\mu} (1 + |x_i|)^{-\mu} (1 + \|x\|)^{\nu} \times \\
&\quad \times C_\mu^B (1 + \|z - y\|)^{-\mu} (1 + |y_j|)^{-\mu} (1 + \|z\|)^{\nu} \\
&\leq C_\mu^A C_\mu^B (1 + |x_i|)^{-\mu} (1 + |y_j|)^{-\mu} \times \\
&\quad \times \sum_z (1 + \|x - z\|)^{-\mu} (1 + \|x\|)^{\nu} (1 + \|z - y\|)^{-\mu} (1 + \|z\|)^{\nu} .
\end{aligned}
$$

Now, by Lemma A.11 we know that the expression from $\sum_z$ and after is estimated by something which is weakly-local. Then we may again use Lemma A.13 to replace the $(1 + |y_j|)^{-\mu}$ factor with $(1 + |x_j|)^{-\mu}$. □

As in [EGS05], multiplying d weakly-local-and-confined operators (each in a distinct direction of all possible directions in $\mathbb{Z}^d$) gives trace class operators. Here our notion of confined is however weaker because we have merely polynomial decay, which changes very little. We denote the trace norm by $\| \cdot \|_1$.

Since we only use $d = 1, 2$ in this work, that's the scope of the lemmas below, whose generalization to arbitrary d is straight-forward.

A.17 Lemma. *If A, B are both weakly-local, A also confined in direction 1 and B in direction 2, then $\|AB\|_1 < \infty$.*

Proof. Assume $\mu > 0$ is sufficiently large for both A and B (with $\nu := \max(\{\nu_A, \nu_B\})$). Using the freedom that Lemma A.13 affords, we may estimate via (A.16) that

$$
\begin{aligned}
\|AB\|_1 \\
\leq \sum_{xyz} \|A_{xz}\| \|B_{zy}\| \\
\leq \sum_{xyz} C_\mu^A (1 + \|x - z\|)^{-\mu} (1 + |z_1|)^{-\mu} (1 + \|z\|)^\nu \times \\
\times C_\mu^B (1 + \|z - y\|)^{-\mu} (1 + |z_2|)^{-\mu} (1 + \|z\|)^\nu .
\end{aligned}
$$

Now, $(1 + |z_1|)(1 + |z_2|) \geq 1 + |z_1| + |z_2| \equiv 1 + \|z\|$ so that we find, by summing first over z and then using translation invariance for the x and y sums,

$$
\|AB\|_1 \leq \Big(\sum_x (1 + \|x\|)^{-\mu}\Big)\Big(\sum_y (1 + \|y\|)^{-\mu}\Big)\Big(\sum_z (1 + \|z\|)^{-(\mu - 2\nu)}\Big) .
$$

If we pick $\mu > 0$ sufficiently large so that all three sums are finite (e.g. $\mu \geq 2\nu + d + 1$) then AB is indeed trace-class. □

A.18 Lemma. *For any switch function Definition 1.24 we have the estimate: for any $\mu > 0$ we have some $C_{\Lambda\mu} < \infty$ such that $|\Lambda(n) - \Lambda(n')| \leq C_{\Lambda\mu}(1 + |n - n'|)^{+\mu}(1 + |n|)^{-\mu}$ for all $n, n' \in \mathbb{Z}$.*

Proof. For large $|n|$ we have $\Lambda(n') = \Lambda(n)$ unless $|n - n'| > |n| / 2$. For such n we have

$$
\big|\Lambda(n) - \Lambda(n')\big| (1 + |n - n'|)^{-\mu} \leq 2\|\Lambda\|_\infty (1 + \frac{1}{2}|n|)^{-\mu}
$$

which implies $|\Lambda(n) - \Lambda(n')| \leq C(1 + |n - n'|)^{+\mu}(1 + |n|)^{-\mu}$ for some $C < \infty$, for all but finitely many n. For the finitely many remaining n we have

$$\left|\Lambda(n) - \Lambda(n')\right| \leq 2\|\Lambda\|_\infty \leq C(1 + |n|)^{-\mu}$$

by adjusting the constant C, and thus the same estimate as the previous one as well. □

A.19 Corollary. *If A is weakly-local then $\partial_i A$ is weakly-local and confined in direction i.*

Proof. We have by the previous estimate on Λ, for any $\mu' > 0$ and $\mu > 0$ sufficiently large for A,

$$\begin{aligned}
\|(\partial_i A)_{xy}\| &= |\Lambda(x_i) - \Lambda(y_i)| \|A_{xy}\| \\
&\leq C_{\Lambda\mu'}(1 + |x_i - y_i|)^{+\mu'}(1 + |x_i|)^{-\mu'}C_{A\mu}(1 + \|x - y\|)^{-\mu}(1 + \|x\|)^\nu \\
&\leq C_{\Lambda\mu'}C_{A\mu}(1 + \|x - y\|)^{-(\mu-\mu')}(1 + |x_i|)^{-\mu'}(1 + \|x\|)^\nu.
\end{aligned}$$

We always take the worst rate of decay to find the form of (A.8) and hence the result. □

A.20 Corollary. *For $d = 1$: If A is weakly-local then $\|\partial A\|_1 < \infty$. For $d = 2$: if A is weakly-local then $\|(\partial_1 A)\partial_2 A\|_1 < \infty$. Moreover if A is also confined in direction i then $\|\partial_j A\|_1 < \infty$ for $j \neq i$.*

A.21 REMARK. If $(A_n)_{n\in\mathbb{N}}$ is a family of weakly-local operators with a uniform estimate (i.e. the constants C, μ and ν in (A.6) do not depend on n) then

$$(\|(\partial_1 A_n)\partial_2 A_n\|_1)_{n\in\mathbb{N}}$$

is a bounded sequence.

Proof. One goes through the entire procedure that leads to Corollary A.20 and verifies that since there is a uniform bound on the xy matrix elements $\|(A_n)_{xy}\|$ (which doesn't depend on n), all estimates are uniform in n, including the final one. □

A.2.3 *Convergence properties of weakly-local operators*

A.22 Lemma. *If $A_n \to A$ strongly within the star-algebra of weakly-local operators (so A is also assumed to be weakly-local) then $\partial_j A_n \to \partial_j A$ strongly within the ideal of weakly-local-and-confined in direction j operators.*

Proof. We already know that $\partial_j A_n$ (for all n) and $\partial_j A$ are weakly-local-and-confined in direction j by the results of Appendix A.2.2. Now let $\psi \in \mathcal{H}$ be given. We have

$$
\begin{aligned}
\|\partial_j A_n \psi - \partial_j A \psi\| &\leq \|\Lambda_j (A_n - A)\psi - (A_n - A)\Lambda_j \psi\| \\
&\leq \|(A_n - A)\psi\| + \|(A_n - A)\Lambda_j \psi\| \\
&\to 0 .
\end{aligned}
$$

$\square$

A.23 Lemma. *If $A_n \to A$ strongly within the ideal of weakly-local-and-confined in direction 1 operators, all having a uniform both ν and sufficiently large μ as in Definition A.12, and T is weakly-local-and-confined in direction 2, then $TA_n \to TA$ in trace-class norm.*

Proof. We have $TA_n = T(1 + |X_1|)^{-\mu}(1 + \|X\|)^{\nu}(1 + \|X\|)^{-\nu}(1 + |X_1|)^{\mu} A_n$. WLOG, we also pick μ such that $T(1 + |X_1|)^{-\mu}(1 + \|X\|)^{\nu}$ is trace-class, and note that $(1 + \|X\|)^{-\nu}(1 + |X_1|)^{\mu} A_n \to (1 + \|X\|)^{-\nu}(1 + |X_1|)^{\mu} A$ strongly. We verify these two statements:

$$
\begin{aligned}
\|T(1 + |X_1|)^{-\mu}(1 + \|X\|)^{\nu}\|_1 &\leq \sum_{xy} \|T_{xy}\|(1 + |y_1|)^{-\mu}(1 + \|y\|)^{\nu} \\
&\leq \sum_{xy} C_\mu^T (1 + \|x - y\|)^{-\mu}(1 + |y_2|)^{-\mu} \\
&\quad (1 + \|y\|)^{\nu}(1 + |y_1|)^{-\mu}(1 + \|y\|)^{\nu} \\
&< \infty .
\end{aligned}
$$

For the second statement, let $C_n := A_n - A$. Then

$$
\begin{aligned}
&\|(1 + \|X\|)^{-\nu}(1 + |X_1|)^{+\mu} C_n \psi\|^2 \\
&\equiv \langle (1 + \|X\|)^{-\nu}(1 + |X_1|)^{+\mu} C_n \psi, (1 + \|X\|)^{-\nu}(1 + |X_1|)^{+\mu} C_n \psi \rangle \\
&= \langle (1 + \|X\|)^{-2\nu}(1 + |X_1|)^{+2\mu} C_n \psi, C_n \psi \rangle \\
&\leq \|(1 + \|X\|)^{-2\nu}(1 + |X_1|)^{+2\mu} C_n \psi\| \|C_n \psi\| \\
&\leq \|(1 + \|X\|)^{-2\nu}(1 + |X_1|)^{+2\mu} C_n\| \|C_n \psi\| .
\end{aligned}
$$

The first norm is finite (for each n) by Lemma A.14 and the second goes to zero because $C_n \to 0$ strongly.

Then we use the result that if S is trace-class and $B_n \to B$ strongly then $SB_n \to SB$ in trace-class norm with $S := T(1 + |X_1|)^{-\mu}(1 + \|X\|)^{\nu}$ and $B_n := (1 + \|X\|)^{-\nu}(1 + |X_1|)^{\mu} A_n$. $\square$

A.24 Lemma. *If $A_n \to A, B_n \to B$ strongly within the ideals of weakly-local-and-confined in direction 1 and 2 respectively, all having a uniform both ν and sufficiently large μ as in Definition A.12, then $A_n B_n \to AB$ in trace-class norm.*

Proof. We again write the factorization

$$A_n B_n = A_n(1+|X_1|)^{\mu}(1+\|X\|)^{-\nu}\cdot$$
$$\cdot (1+|X_1|)^{-\mu}(1+\|X\|)^{2\nu}(1+|X_2|)^{-\mu}\cdot$$
$$\cdot (1+\|X\|)^{-\nu}(1+|X_2|)^{\mu}B_n$$
$$= A_n(1+|X_1|)^{\mu}(1+\|x\|)^{-\nu}\cdot(1+|X_1|)^{-\mu/2}(1+\|X\|)^{\nu}(1+|X_2|)^{-\mu/2}\cdot$$
$$\cdot (1+|X_1|)^{-\mu/2}(1+\|X\|)^{\nu}(1+|X_2|)^{-\mu/2}\cdot(1+\|X\|)^{-\nu}(1+|X_2|)^{\mu}B_n\,.$$

Now if μ is chosen sufficiently large, then the last expression is the product of four factors. The first one converges strongly as shown in the lemma above. The second and third are trace class and the fourth also converges strongly. Thus we conclude the statement based on the properties of products of limits and the previous lemma. $\qquad\square$

A.2.4 *Helffer-Sjöstrand formula for unitary operators*

The Helffer-Sjöstrand formula, which first appeared in [Dyn75], extends holomorphic functional calculus to smooth functions. It was developed for Hermitian operators but can be easily adapted to unitaries, with the simplification that the latter are always bounded. A formula was already proposed in [Mba15] for functions on $\mathsf{S}^1 \setminus \{1\}$ and based on Cayley transformation. Here we provide another proof for any smooth function on S^1 using a conformal mapping.

A.25 Lemma. *Let $f : \mathsf{S}^1 \to \mathbb{C}$ be a smooth function. There exists a quasi-analytic extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$, i.e. $\tilde{f}|_{\mathsf{S}^1} = f$ and $\partial_{\bar{z}}\tilde{f}|_{\mathsf{S}^1} = 0$, such that for any unitary operator U*

$$f(U) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{C}} (\partial_{\bar{z}}\tilde{f}(z))(z-U)^{-1}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{A.9}$$

Moreover $\tilde{f}$ is compactly supported around S^1 and satisfies $|\partial_{\bar{z}}\tilde{f}| \leq C||z|-1|^N$ for any $N \geq 2$.

Proof. Any function $f : \mathsf{S}^1 \to \mathbb{C}$ can be equivalently described by a periodic function $g : \mathbb{R} \to \mathbb{C}$, through the conformal mapping $w \mapsto z = \mathrm{e}^{\mathrm{i}w}$ by $g(w) = f(\mathrm{e}^{\mathrm{i}w})$, satisfying $g(w+2\pi) = g(w)$ by construction. This bijective mapping extends to the the annulus $\mathcal{A}_r$ where $\mathrm{e}^{-r} < |z| < \mathrm{e}^r$ corresponding to the strip $-r < \mathbb{Im}(w) < r$. In both cases the smoothness of f and g are the same. Let $\chi : \mathbb{R} \to \mathbb{C}$ be a smooth function supported in $(-r, r)$ and with $\chi(x) = 1$ near 0. On the real line, we know from Ref. [HS00] that for $N \geq 2$

$$\tilde{g}(\theta, \tau) = \sum_{k=0}^{N-1} g^{(k)}(\theta) \frac{(\mathrm{i}\tau)^k}{k!} \chi(\tau) \tag{A.10}$$

is a quasi-analytic extension of g on the strip, namely $\tilde{g}(\theta,0) = g(\theta)$ and $\partial_{\bar{w}}g|_{\tau=0} = 0$, for $w = \theta + \mathrm{i}\tau$ and $\partial_{\bar{w}} = 1/2(\partial_\theta + \mathrm{i}\partial_\tau)$. Moreover, $|\partial_{\bar{w}}g| \leq C|\tau|^N$. We claim that $\tilde{f}(z = \mathrm{e}^{\mathrm{i}(\theta+\mathrm{i}\tau)}) := \tilde{g}(\theta,\tau)$ is a quasi-analytic extension of f on the annulus. Indeed $\tilde{f}$ coincides with f on $\mathbb{S}^1$ and

$$\partial_{\bar{w}}\tilde{g} = \partial_{\bar{w}}(\overline{\mathrm{e}^{\mathrm{i}\,w}})\partial_{\bar{z}}\tilde{f} = -\,\mathrm{i}\,\bar{z}\partial_{\bar{z}}\tilde{f} \tag{A.11}$$

so that $\partial_{\bar{z}}\tilde{f}|_{\tau=0} = 0$. Moreover on the annulus one has $\mathrm{e}^{-r} < |-\mathrm{i}\,\bar{z}| < \mathrm{e}^r$ and $|\ln x| \leq \mathrm{e}^r|x-1|$ for $x \in (\mathrm{e}^{-r},\mathrm{e}^r)$ applied to $x = |z| = \mathrm{e}^{-\tau}$ we infer $|\tau| \leq C||z|-1|$ so that

$$|\partial_{\bar{z}}\tilde{f}| \leq C||z|-1|^N \tag{A.12}$$

with a different constant C. With the fact that $\|(z-U)^{-1}\| \leq ||z|-1|^{-1}$ for a unitary U we deduce that the integral in (A.9) is absolutely convergent in norm. Then we claim that for $z_0 \in \mathbb{S}^1$

$$f(z_0) = \frac{1}{2\pi}\int_{\mathbb{C}}(\partial_{\bar{z}}\tilde{f}(z))(z-z_0)^{-1}\,\mathrm{d}\,z\,\mathrm{d}\,\bar{z} \tag{A.13}$$

The integral is reduced to the annulus $\mathcal{A}_r$ since $\tilde{f}$ is supported inside it and has to be understood as an improper integral on $\mathcal{A}_r \setminus \mathcal{A}_\varepsilon$ when $\epsilon \to 0$. The equality follows by [Mba15, Cor. 2.3], and (A.9) follows by the functional calculus. □

A.26 Corollary. *The smooth functional calculus of an exponentially local unitary is polynomially local.*

Proof. This is a direct consequence of the Helffer-Sjöstrand formula (A.9), the fact that $\tilde{f}$ is smooth and compactly supported, and Combes-Thomas estimate [CT73] (Proposition 2.35, [HJS09] in the context of unitaries): if U is local then it exists $0 < C < \infty$ such that

$$|R_U(z)|_{x,y} \leq \frac{C}{||z|-1|}\,\mathrm{e}^{-\mu(z)\|x-y\|} \tag{A.14}$$

for $\mu > 0$ small enough. For example one can take $\mu(z) = c||z|-1|$ as in [EG02]. According to Lemma A.25 the quasi-analytic extension of f satisfies $|\partial_{\bar{z}}\tilde{f}(z)| \leq C||z|-1|^N$ for $N \geq 2$ so that

$$|f(U)|_{x,y} \leq \frac{1}{2\pi}\int \mathrm{d}\,z\,\mathrm{d}\,\bar{z}|\partial_{\bar{z}}\tilde{f}(z)||R_U(z)|_{x,y} \leq C_N(1+c\|x-y\|)^{-N} \tag{A.15}$$

□

A.3   FUNCTIONAL ANALYSIS

A.27 Lemma. *We have for T operating on $\ell^2(\mathbb{Z}^d)$*

$$\|T\|_1 \leq \sum_{n,n'} |T(n, n')| . \tag{A.16}$$

The bound is passed down to $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ provided $|\cdot|$ is interpreted as the trace norm of operators on the second factor.

Proof. Let $\mathcal{H}$ be a Hilbert space and $\{\varphi_n\}_n$ an orthonormal basis. Then

$$\|T\|_1 \equiv \sum_{n'} \langle \varphi_{n'}, |T| \varphi_{n'} \rangle \leq \sum_{n'} \| |T| \varphi_{n'} \| = \sum_{n'} \|T\varphi_{n'}\| \leq \sum_{n,n'} |\langle \varphi_n, T\varphi_{n'} \rangle| ,$$

where we used $\|\psi\| \leq \sum_n |\langle \varphi_n, \psi \rangle|$ in the last step. $\square$

A.28 Claim. *If F is of finite rank, then*

$$\|F\|_1 \leq \|F\| \dim (im (F))$$

Proof. Indeed, as F is of finite rank, we can generically write

$$F = \sum_{j=1}^m \alpha_j u_j \langle v_j, \cdot \rangle$$

for some $m \in \mathbb{N}_{\geq 0}$, $\{\alpha_j\}_j \subseteq \mathbb{C}$ (WLOG $\alpha_j \neq 0$ for all j) and two orthonormal bases $\{u_j\}_j$ and $\{v_j\}$ for $\mathcal{H}$. Then using the triangle inequality for $\|\cdot\|_1$, homogeneity, and fact that $\|u_j \langle v_j, \cdot \rangle\|_1 = 1$ we can estimate

$$\|F\|_1 \leq m |\alpha_{j_{max}}|$$

where j_{max} has the obvious meaning. But $\|F\| = |\alpha_{j_{max}}|$ so that we obtain the result. $\square$

The following result may be found e. g. in [EG02, Eq. (56)].

A.29 Claim. *Let $\mathcal{H}$ be any separable Hilbert space and $\{A_n\}_{n\in\mathbb{N}}$ a sequence that converges strongly to some $\tilde{A} \in \mathcal{B}(\mathcal{H})$. If $B \in \mathcal{J}_1(\mathcal{H})$ then $\{A_nB\}_{n\in\mathbb{N}}$ converges in trace-class norm to $\tilde{A}B$.*

We first solve the finite-rank problem:

A.30 Claim. *If $F \in \mathcal{B}(\mathcal{H})$ is of finite rank then $\{ A_n F \}_{n \in \mathbb{N}}$ converges in trace-class norm to $\tilde{A}F$, i.e. we have*

$$\| (A_n - \tilde{A}) F \|_1 \xrightarrow{n \to \infty} 0$$

Proof. Without loss of generality we may write $F = \sum_{j=1}^m \alpha_j u_j \langle v_j, \cdot \rangle$ for some $m \in \mathbb{N}_{\geq 0}$, $\{ \alpha_j \}_j \subseteq \mathbb{C}$ (WLOG $\alpha_j \neq 0$ for all j) and two orthonormal bases $\{ u_j \}_j$ and $\{ v_j \}$. Then

$$\| (A_n - \tilde{A}) F \|_1 = \| (A_n - \tilde{A}) \sum_{j=1}^m \alpha_j u_j \langle v_j, \cdot \rangle \|_1$$

$$\leq \sum_{j=1}^m |\alpha_j| \, \| (A_n - \tilde{A}) u_j \langle v_j, \cdot \rangle \|_1$$

But observe that $\| \varphi \langle \psi, \cdot \rangle \|_1 = \| \varphi \| \| \psi \| = 1$, so that

$$\| (A_n - \tilde{A}) F \|_1 \leq \sum_{j=1}^m |\alpha_j| \underbrace{\| v_j \|}_{=1} \| (A_n - \tilde{A}) u_j \|$$

$$= \sum_{j=1}^m |\alpha_j| \, \| (A_n - \tilde{A}) u_j \|$$

But $A_n \to \tilde{A}$ strongly, which means in particular that $\| (A_n - \tilde{A}) u_j \| \xrightarrow{n \to \infty} 0$. That is, $\forall \varepsilon > 0$ there is some $N(\varepsilon, u_j) \in \mathbb{N}$ such that if $n \in \mathbb{N}_{\geq N(\varepsilon, u_j)}$ then $\| (A_n - \tilde{A}) u_j \| < \varepsilon$. Now let $\varepsilon > 0$ be given. Thus if $n \in \mathbb{N}$ is such that

$$n \geq \max \left(\left\{ N \left(\frac{\varepsilon}{m \, |\alpha_j|}, u_j \right) \, \middle| \, j \in J_m \right\} \right)$$

we have

$$\| (A_n - \tilde{A}) F \|_1 < \sum_{j=1}^m |\alpha_j| \frac{\varepsilon}{m \, |\alpha_j|} = \varepsilon \, .$$

$\square$

Next, we obtain a uniform bound on $\| A_n - \tilde{A} \|$:

A.31 Claim. *We have for some $P < \infty$,*

$$\sup \left(\{ \, \| A_n - \tilde{A} \| \, | \, n \in \mathbb{N} \, \} \right) \leq P \, .$$

Proof. For brevity define $C_n := A_n - \tilde{A}$. Then we have $C_n \to 0$ strongly and want to show that

$$\sup \left(\{ \| C_n \| \, | \, n \in \mathbb{N} \} \right) < \infty$$

Assume the claim is false. Let $M > 0$. Then there is some $m\,(M) \in \mathbb{N}$ such that

$$\|C_{m(M)}\| > M$$

Now,

$$\|C_n\| \equiv \sup\left(\{\ \|\,C_n\psi\|\,|\psi \in \mathcal{H} : \|\psi\| = 1\,\}\right)$$

so that by the approximation property for the supremum we have $\forall \delta > 0$ some $\psi_\delta^n \in \mathcal{H}$ with $\|\psi_\delta^n\| = 1$ and

$$\|C_n\| < \|C_n\psi_\delta^n\| + \delta$$

and since $\|C_{m(M)}\| > M$ we have (now applying the approximation property of the supremum at $n = m\,(M)$):

$$M < \|C_{m(M)}\psi_\delta^{m(M)}\| + \delta$$

and if we pick $\delta := \frac{1}{2}M$ we get

$$\frac{1}{2}M < \|C_{m(M)}\psi_{\frac{1}{2}M}^{m(M)}\|$$

In particular we obtain that the (double) sequence $\{\ \|\,C_n\psi\|\ \}_{n,\psi}$ is not bounded.

But the sequence $\{\ \|\,C_n\psi\|\ \}_{n,\psi}$ converges pointwise to 0 (for any ψ), and is bounded in ψ (for any n) as $C_n \in \mathcal{B}\,(\mathcal{H})$, hence we obtain a contradiction.   □

And finally we are ready for the

Proof of Claim A.29. We know that the finite-rank operators are dense in the trace-class operators with respect to the trace-class norm, so that for all $\varepsilon > 0$ there is some finite rank operator F_ε such that

$$\|F_\varepsilon - B\|_1 < \varepsilon$$

and in addition from the above claim we know that there is some $M\,(\varepsilon, F) \in \mathbb{N}$ such that if $n \in \mathbb{N}_{\geq M(\varepsilon, F)}$ then

$$\|\,(A_n - \tilde{A})\,F\|_1 < \varepsilon$$

for some finite rank operator F.

So that if $n \in \mathbb{N}$ is such that

$$n \geq M\left(\frac{\varepsilon}{2}, F_{\frac{\varepsilon}{2}P}\right)$$

then

$$\|A_n B - \tilde{A}B\|_1 = \|\left(A_n - \tilde{A}\right)\left(B - F_{\frac{\varepsilon}{2}P}\right) + \left(A_n - \tilde{A}\right) F_{\frac{\varepsilon}{2}P}\|_1$$

$$\leq \|\left(A_n - \tilde{A}\right)\left(B - F_{\frac{\varepsilon}{2}P}\right)\|_1 + \underbrace{\|\left(A_n - \tilde{A}\right) F_{\frac{\varepsilon}{2}P}\|_1}_{<\frac{\varepsilon}{2}}$$

$$\leq \|A_n - \tilde{A}\|\|B - F_{\frac{\varepsilon}{2}P}\|_1 + \frac{\varepsilon}{2}$$

$$\leq \sup\left(\{\|A_n - \tilde{A}\| | n \in \mathbb{N}\}\right)\|B - F_{\frac{\varepsilon}{2}P}\|_1 + \frac{\varepsilon}{2}$$

$$< P\underbrace{\|B - F_{\frac{\varepsilon}{2}P}\|_1}_{<\frac{\varepsilon}{2}P} + \frac{\varepsilon}{2}$$

$$< \varepsilon$$

□

A.32 Claim. *Let $\mathcal{H}$ be a separable Hilbert space where $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H})$ is a sequence converging strongly to some $\tilde{A} \in \mathcal{B}(\mathcal{H})$ and $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{J}_1(\mathcal{H})$ is a sequence converging in trace-class norm to some $\tilde{B} \in \mathcal{J}_1(\mathcal{H})$. Then $\{A_n B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{J}_1(\mathcal{H})$ is a sequence converging in trace-class norm to $\tilde{A}\tilde{B} \in \mathcal{J}_1(\mathcal{H})$.*

Proof. We use the same notation as in the proof of A.29:

$$\|A_n B_n - \tilde{A}\tilde{B}\|_1 = \|A_n\|\|B_n - \tilde{B}\|_1 + \|\left(A_n - \tilde{A}\right)\tilde{B}\|_1$$

$$\leq \left(\|A_n - \tilde{A}\| + \|\tilde{A}\|\right)\|B_n - \tilde{B}\|_1 + \|\left(A_n - \tilde{A}\right)\tilde{B}\|_1$$

$$\leq \left(\sup\left(\{\|A_n - \tilde{A}\| | n \in \mathbb{N}\}\right) + \|\tilde{A}\|\right)\|B_n - \tilde{B}\|_1 + \|\left(A_n - \tilde{A}\right)\tilde{B}\|_1$$

$$\leq \left(P + \|\tilde{A}\|\right)\|B_n - \tilde{B}\|_1 + \|\left(A_n - \tilde{A}\right)\tilde{B}\|_1$$

$$\longrightarrow 0$$

where the last line follows by our hypothesis on $\{B_n\}_{n\in\mathbb{N}}$ and by A.29.   □

A.4   PROBABILITY

A.33 Proposition. *We have*

$$\sqrt{\mathbb{E}[(\sum_{j=1}^{n} \log(\|A_j\|))^4]} \leq n^2\sqrt{\mathbb{E}[\log(\|A_1\|)^4]}$$

where $\{A_j\}_j$ are independent variables.

Proof. Using the multinomial theorem we find

$$\mathbb{E}[(\sum_{j=1}^{n} \log(\|A_j\|))^4] = \mathbb{E}[\sum_{k_1+\cdots+k_n=4} \binom{4}{k_1,\ldots,k_n} \prod_{j=1}^{n} \log(\|A_j\|)^{k_j}]$$

$$= \sum_{k_1+\cdots+k_n=4} \binom{4}{k_1,\ldots,k_n} \mathbb{E}[\prod_{j=1}^{n} \log(\|A_j\|)^{k_j}]$$

(Independent Variables)

$$= \sum_{k_1+\cdots+k_n=4} \binom{4}{k_1,\ldots,k_n} \prod_{j=1}^{n} \mathbb{E}[\log(\|A_j\|)^{k_j}]$$

(Independent Variables)

$$= \sum_{k_1+\cdots+k_n=4} \binom{4}{k_1,\ldots,k_n} \prod_{j=1}^{n} \mathbb{E}[\log(\|A_1\|)^{k_j}]$$

(Jensen)

$$= n^4 \mathbb{E}[\log(\|A_1\|)^4].$$

$\square$

A.34 Proposition. *Consider the subset of the complex plane*

$$D_{c,t} := \{z \in \mathbb{C} \,|\, |\frac{z}{z^2 - c^2}| > t\}$$

for $c \in \mathbb{C}$ and $t > 0$ and the line $\mathbb{R} \ni \lambda \mapsto z := \alpha\lambda + \beta$ ($\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$) parametrized by arclength. Then its intersection with $D_{c,t}$ is bounded in Lebesgue measure as

$$|\{\lambda \in \mathbb{R} | z \in D_{c,t}\}| \leq \frac{4}{t}.$$

Proof. Since the statement is invariant w.r.t. rotations of c, α, β about the origin, we may assume $c \geq 0$. We then estimate the measure when $D_{c,t}$ is replaced by its intersection with the right half-plane $\{z \in \mathbb{C} \,|\, \mathbb{Re}\{z\} \geq 0\}$ (and likewise for the left one). Then

$$|\frac{z}{z^2 - c^2}| = \frac{1}{|z - c|} |\frac{z}{z + c}| \leq \frac{1}{|z - c|}$$

because $\mathbb{Re}\{z\} + c \geq \mathbb{Re}\{z\}$ there, which implies $D_{c,t} \cap \{z | \mathbb{Re}\{z\} \geq 0\} \subseteq \{z | |z - c| < t^{-1}\}$. The intersection of that disk with any line is of length $2t^{-1}$ at most. $\square$

A.5 LINEAR ALGEBRA

A.35 Proposition. *([BL85] Lemma III.5.4) If $M \in Mat_{L \times L}(\mathbb{C})$ has $|\det(M)| = 1$ then for any $j \in \{1, \ldots, L\}$ and $v \in \wedge^j \mathbb{C}^L$ we have*

$$|\log(\| \wedge^j M\|)| \leq j(L-1)\log(\|M\|)$$

and

$$|\log(\frac{\| \wedge^j Mv\|}{\|v\|})| \leq j(L-1)\log(\|M\|).$$

Proof. First note that $\| \wedge^j M\|$ is the product of the first j singular values:

$$\| \wedge^j M\| = \sigma_1(M) \ldots \sigma_j(M)$$

so that

$$\| \wedge^j M\| \leq \|M\|^j.$$

Conversely,

$$\| \wedge^j M\|^{-1} \leq \|(\wedge^j M)^{-1}\| = \| \wedge^j M^{-1}\| \leq \|M^{-1}\|^j.$$

For any invertible matrix we have

$$\|M^{-1}\| \leq \frac{\|M\|^{L-1}}{|\det(M)|},$$

so that in our case

$$\| \wedge^j M\|^{-1} \leq \|M\|^{j(L-1)}.$$

We find using the fact that log is monotone increasing,

$$\log(\| \wedge^j M\|) \leq j\log(\|M\|) \leq j(L-1)\log(\|M\|)$$

and

$$-\log(\| \wedge^j M\|) = \log(\| \wedge^j M\|^{-1}) \leq j(L-1)\log(\|M\|).$$

Next, we have

$$\frac{\| \wedge^j Mv\|}{\|v\|} \leq \| \wedge^j M\|,$$

whereas $\|v\| = \|(\wedge^j M)^{-1}(\wedge^j M)v\| \le \|(\wedge^j M)^{-1}\| \| \wedge^j Mv\|$ so that

$$\left(\frac{\| \wedge^j Mv\|}{\|v\|}\right)^{-1} \le \|(\wedge^j M)^{-1}\|$$

which gives the second inequality of the prop.

Finally, note that these inequalities indeed make sense: $1 \le \|M\|\|M^{-1}\| \le \|M\|\|M\|^{L-1} = \|M\|^L$ so that $\log(\|M\|) \ge 0$ always when $|\det(M)| = 1$. $\square$

A.5.1   *The Hermitian symplectic group*

The Hermitian symplectic group, which was mentioned previously in the literature in e. g. [Haroo; RS09] is defined as follows:

A.36 Definition. The (Hermitian) symplectic group $Sp^*_{2N}(\mathbb{C})$ is defined as

$$Sp^*_{2N}(\mathbb{C}) \equiv \{M \in Mat_{2N \times 2N}(\mathbb{C}) | M^* J M = J\}$$

with $J \equiv \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}$ the standard symplectic form. Here we abbreviate $G :=$ $Sp^*_{2N}(\mathbb{C})$. In this section we write M in N-block form as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, and D are unrelatd to their meanings outside of this section.

A.37 Proposition. *G is a group under matrix multiplication.*

Proof. Taking the determinant of

$$M^* J M = J$$

shows that $|\det(M)| = 1$ so that M is invertible if it is in G. From the defining relation we have

$$M = (M^* J)^{-1} J = M^{-1} A^{*-1} J$$

so that using the commutativity of adjoint and inverse, we get:

$$M^{-1} = J^{-1} M^* J = J^{-1}((M^{-1})^{-1})^* J = J^{-1}((M^{-1})^*)^{-1} J = ((M^{-1})^* J)^{-1} J,$$

so that

$$(M^{-1})^* J M^{-1} = J \tag{A.17}$$

and we find that $M^{-1} \in G$ as well. Hence the inverse map restricted to G lands in G. If

$$M_i^* J M_i = J$$

for $i \in \{1, 2\}$ then

$$M_2^* M_1^* J M_1 M_2 = M_2^* J M_2 = J.$$

Hence matrix multiplication map restricted to G^2 lands in G.

Associativity is inherited by the associativity of general matrix multiplication. Finally, the identity matrix is of course conjugate symplectic. $\qquad\square$

A.38 REMARK. Note that G is unitarily equivalent to

$$U(N, N) \equiv \{ M \in \mathrm{Mat}_{2N}(\mathbb{C}) \mid M^*(\mathbb{1}_N \oplus (-\mathbb{1}_N))M = \mathbb{1}_N \oplus (-\mathbb{1}_N) \},$$

the indefinite unitary group. G may also be understood in terms of Krein spaces, as detailed in [GLR05].

A.39 Proposition. *G is closed under adjoint.*

Proof. Let $M \in G$ be given. Then $M^* J M = J$. We want to show that $M J M^* = J$. Taking the inverse of (A.17) we find

$$((M^{-1})^* J M^{-1})^{-1} = J^{-1}.$$

Use the fact that $J^{-1} = -J$ and again the commutativity of adjoint and inverse maps to find:

$$M J M^* = J,$$

so that $M^* \in G$ as desired. $\qquad\square$

A.40 Proposition. *G may be described as*

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{2N}(\mathbb{C}) \,\middle|\, \begin{matrix} (A, B, C, D) \in Mat_N(\mathbb{C})^4 : \\ A^*C \text{ and } B^*D \text{ are S.A.} \\ \text{and } A^*D - C^*B = \mathbb{1}_N \end{matrix} \right\}. \qquad (A.18)$$

Proof. Starting from $M^* J M \overset{!}{=} J$ we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$
= \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}
$$

$$
= \begin{pmatrix} -A^*C + C^*A & -A^*D + C^*B \\ -B^*C + D^*A & -B^*D + D^*B \end{pmatrix}
$$

$$
\stackrel{!}{=} \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}.
$$

$\square$

A.41 REMARK. By the last equation in (A.18), if $D \in GL_N(\mathbb{C})$, then M is specified by only three block entries B, C and D and A may be solved for to satisfy the sympletic constraint.

A.42 Proposition. *If $M \in G$ its eigenvalues are symmetric about $\mathbb{S}^1$ so that its singular values are symmetric about one.*

Proof. We have $\det(J) = 1$ so that $|\det(M)| = 1$. So define $\theta \in \mathbb{S}^1$ via $e^{i\theta} := \det(M^*)^{-1}$. We have $M = -JM^\circ J$ from the definition so that if $p_M(\lambda) \equiv \det(M - \lambda\mathbb{1}_{2N})$ is the characteristic polynomial of M, we find

$$
\begin{aligned}
p_M(\lambda) &= \det(-JM^\circ J - \lambda\mathbb{1}_{2N}) \\
&(\det(J) = 1) \\
&= \det(M^\circ J - \lambda J) \\
&(e^{-i\theta}\det(M^*) = 1) \\
&= e^{-i\theta}\det(J - \lambda M^* J) \\
&(\det(-J) = 1) \\
&= e^{-i\theta}\det(-\mathbb{1}_{2N} + \lambda M^*) = e^{-i\theta}\overline{\det(-\mathbb{1}_{2N} + \overline{\lambda}M)} \\
&= e^{-i\theta}\overline{\lambda}^{2N}\overline{\det(-\mathbb{1}_{2N}\overline{\lambda}^{-1} + M)} = e^{-i\theta}\lambda^{2N}\overline{p_M(\overline{\lambda}^{-1})}.
\end{aligned}
$$

Since $|M|^2$ is also symplectic, this relation holds true also for $p_{|M|^2}$, hence for the singular values. $\square$

A.43 Corollary. *For any $M \in G$, $\|M\| = \|M^{-1}\|$.*

Proof. $\|M\|$ is the largest singular value, $\|M^{-1}\|^{-1}$ is the smallest singular value, but the singular values are symmetric about one. $\square$

A.6   MISCELLANEA

A.44 Proposition. *If $s \in (0,1)$, $n \in \mathbb{N}_{\geq 1}$ and $\{a_i\}_{i=1}^n \subseteq \mathbb{R}_{\geq 0}$ then $(\sum_{i=1}^n a_i)^s \leq \sum_{i=1}^n a_i^s$*

Proof. If $\sum_{i=1}^n a_i = 0$ then the statement holds trivially. Otherwise, define $q := (\sum_{i=1}^n a_i)^{-1}$. Then using the fact that $(\cdot)^s$ is homogeneous, that $\sum_{i=1}^n q a_i = 1$, and that $q a_i \leq (q a_i)^s$ (as $q a_i \leq 1$) we find

$$(\sum_{i=1}^n a_i)^s = \frac{1}{q^s} \sum_{i=1}^n q a_i \leq \frac{1}{q^s} \sum_{i=1}^n (q a_i)^s = \sum_{i=1}^n a_i^s.$$

$\square$

The following lemma has also been investigated in [ASS83; KG15].

A.45 Lemma. *Let X be a path-connected, simply-connected Hausdorff topological space. Then the space of homotopy classes of maps $\mathbb{T}^2 \to X$ is isomorphic to $\pi_2(X)$.*

Proof. Writing $\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$, we follow the strategy of [Gra16]. Since $\mathbb{T}^2$ is a CW-complex, it has a cell-structure, in which there is a 0-cell which is just a point, and $\mathbb{S}^1 \vee \mathbb{S}^1 \equiv \mathbb{S}^1 \sqcup \mathbb{S}^1 / \{\star \sim *\}$ where $\star, *$ are the distinguished points in the two circles respectively, is a one-dimensional sub-complex within the CW-structure of $\mathbb{T}^2$ (it is the figure-eight space). [Bre93, Theorem VII.1.4] implies that the inclusion map $\mathbb{S}^1 \vee \mathbb{S}^1 \hookrightarrow \mathbb{T}^2$ is a cofibration (in the sense of [Bre93, Def. VII.1.2]). The attaching map $a : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ which gives the 2-cell structure results in the following *coexact* sequence

$$\mathbb{S}^1 \xrightarrow{a} \mathbb{S}^1 \vee \mathbb{S}^1 \hookrightarrow \mathbb{T}^2. \tag{A.19}$$

We note that the attaching map $a : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ gives an element in $[a] \in \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$, which is the free group on two generators α and β. The element corresponding to the attaching map is $[a] = \alpha \beta \alpha^{-1} \beta^{-1}$.

Anyway, the sequence (A.19) can be extended *coexactly* by suspension (by [Bre93, Cor. VII.5.5]) to give

$$\mathbb{S}^1 \xrightarrow{a} \mathbb{S}^1 \vee \mathbb{S}^1 \hookrightarrow \mathbb{T}^2 \to \mathbb{S}^2 \xrightarrow{S(a)} \mathbb{S}^2 \vee \mathbb{S}^2 \to \dots \tag{A.20}$$

where the last map $S(a)$ is the suspension of the attaching map. The suspension induces a group morphism $[S] : \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \to \pi_2(\mathbb{S}^2 \to \mathbb{S}^2)$. Since it is a group morphism we find $[S(a)] = [S](\alpha)[S](\beta)([S](\alpha))^{-1}([S](\beta))^{-1}$. Now using the fact that the higher homotopy groups are always Abelian, we obtain that $[S(a)]$ is the identity element.

The co-exactness of (A.20) means that the following sequence of (sets) of pointed homotopy classes is exact, for any space X:

$$\cdots \to [\mathsf{S}^2 \vee \mathsf{S}^2; X] \to [\mathsf{S}^2; X] \to [\mathbb{T}^2; X] \to [\mathsf{S}^1 \vee \mathsf{S}^1; X] \to [\mathsf{S}^1; X].$$

Since $S(a) = 1$, the first map here $[\mathsf{S}^2 \vee \mathsf{S}^2; X] \to [\mathsf{S}^2; X]$ is the zero map. Using the assumption that $\pi_1(X) = 0$, writing already $[\mathsf{S}^2; X] \equiv \pi_2(X)$, and using the fact that $\vee$ is a co-product so that the functor $[-; X]$ respects it (so $[\mathsf{S}^1 \vee \mathsf{S}^1; X] = \pi_1(X) \times \pi_1(X)$):

$$\cdots \to 0 \to \pi_2(X) \to [\mathbb{T}^2; X] \to 0 \to 0.$$

By exactness this implies that $[\mathbb{T}^2; X] \cong \pi_2(X)$. Since X is path-connected, we may replace $[-; X]$ (the set of *pointed* homotopy classes) with the non-pointed one.

$\square$

A.6.1 *An explicit calculation with the Bott map*

The goal here is to write out explicitly the demonstration that $K_1(X) \cong K_0(SX)$ and make sure that the respective isomorphisms to $\mathbb{Z}$ agree (assuming X is s.t. both K-groups are isomorphic to $\mathbb{Z}$ via the 1D winding number and the 2D Chern number respectively).

Thus let $N \in \mathbb{N}_{\geq 1}$ and $U : \mathsf{S}^1 \to \mathcal{U}(N)$ be given. Our goal is to build out of this input a *two-dimensional* projection whose Chern number is equal to the winding of U.

- By Whitehead's lemma [Røroo, p. 2.1.5], we know that that there is a continuous map $z : [0, 1] \to \mathcal{U}(2N)$ from $\mathbb{1}_{2N}$ to $U^* \oplus U$ (both end-points of z have winding number zero). In fact here is an explicit description of z:

  - We know that σ_1 is homotopic to $\mathbb{1}_2$ within the unitaries as follows:

    $$[0, 1] \ni t \mapsto P_{1,+} + P_{1,-}e^{i\pi t} \in \mathcal{U}(2)$$

    where $P_{1,\pm} \equiv \frac{1}{2}(\mathbb{1}_2 \pm \sigma_1)$ is the projection onto the plus or minus eigenspace of σ_1.

  - By tensoring with the identity we get a map $\eta : [0, 1] \to \mathcal{U}(2N)$ interpolating between $\mathbb{1}_{2N}$ and $\sigma_1 \otimes \mathbb{1}_N = \begin{bmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{bmatrix}$.

  - Then note that

    $$U^* \oplus U = U^* \oplus \mathbb{1}_N \begin{bmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{bmatrix} U \oplus \mathbb{1}_N \begin{bmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{bmatrix}$$

    so that

    $$z(t) := U^* \oplus \mathbb{1}_N \eta(t) U \oplus \mathbb{1}_N \eta(t)$$

    fits the bill.

– However, we will not use z explicitly, except for its defining property.

A.46 Definition. Define a continuous map $P : S^1 \times [0, 1] \to \mathrm{Mat}_{2N}(\mathbb{C})$ by

$$P(k_1, k_2) = z(k_2) Q \left[z(k_2)^* \right]$$

where k_1 is to be plugged into the slot of U, with $Q := \mathbb{1}_N \oplus 0_N$.

A.47 Claim. *P is actually a loop also in its second variable, and maps to the self-adjoint idempotents.*

Proof. Since $z_{k_1}(0) \equiv \mathbb{1}$ we have

$$P(k_1, 0) = Q$$

whereas $z_{k_1}(1) = U(k_1)^* \oplus U(k_1)$ gives

$$\begin{aligned}
P(k_1, 1) &= U(k_1)^* \oplus U(k_1) Q \left(U(k_1)^* \oplus U(k_1) \right)^* \\
&= U(k_1)^* \oplus U(k_1) Q U(k_1) \oplus U(k_1)^* \\
&= U(k_1)^* U(k_1) \oplus 0_N \\
&= \mathbb{1}_N \oplus 0_N \\
&\equiv Q
\end{aligned}$$

hence $P(k_1, 0) = P(k_1, 1)$ so that $P(k_1, \cdot)$ is indeed a loop. It is point-wise self-adjoint by construction, and is idempotent, since Q is, too:

$$P^2 = zQ \underbrace{z^* z}_{=\mathbb{1}} Qz^* = zQQz^* = zQz^* \equiv P.$$

$\square$

A.48 Corollary. *Thus P is now a 2D representative of a class in $K_0(\mathbb{T}^2)$ constructed out of the 1D input U which is a representative of a class in $K_1(S^1)$. Note that $K_0(\mathbb{T}^2) = \mathbb{Z}^2$ whereas $K_1(S^1) = \mathbb{Z}$ so that the two groups are not isomorphic, but the other $\mathbb{Z}$ factor in $K_0(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$ corresponds to the rank of a projection, which is a zero-dimensional invariant (the first one being the top invariant, the 2D Chern number).*

A.49 Claim. *We have* $\mathrm{Chern}(P) = \mathrm{Wind}(U)$.

Proof. From [Sha16, p. 8.4.14] e. g., we know that the Chern number of some projection $R : \mathbb{T}^2 \to \mathcal{P}(M)$ can be expressed using the winding of the sewing matrix: We know that R has constant rank, say, n. We were able to show that if we cut the torus into a cylinder $S^1 \times [0, 1]$, then we can find a global section on the frame bundle over the torus, call it $\left\{ \psi_i^R \right\}_{i=1}^n : S^1 \times [0, 1] \to Gr_n(\mathbb{C}^M)$. Then

there is a unitary sewing matrix that sews the phases of $\psi_i^R (k_1, 0)$ with those of $\psi_i^R (k_1, 1)$:

$$\psi_i^R (k_1, 0) = \sum_{j=1}^{n} \psi_j^R (k_1, 1) \left[T_1^R (k_1) \right]_{ji}$$

Then

$$\begin{aligned}
\text{Chern} (R) &= \text{Wind} \left(T_1^R \right) \\
&\equiv \text{Wind} \left(\mathbb{S}^1 \to \mathcal{U} (n) \right) \\
&\equiv \text{Wind} \left(\mathbb{S}^1 \to \mathbb{S}^1; k_1 \mapsto \det \left(T_1^R (k_1) \right) \right)
\end{aligned}$$

For our constant projection Q, note that $n = N$ and the sewing matrix is $T_1^Q (k_1) = \mathbb{1}_N$.

1. The sewing matrix of a unitary conjugation is given by the relation

$$T_1^{vRv^*} (k_1) = v (k_1, 1)^* |_{\text{im}(R)} \, T_1^R (k_1) \, v (k_1, 0) |_{\text{im}(R)}$$

Indeed, we know how the frame $\{ \psi_i^R \}_{i=1}^{n} : \mathbb{S}^1 \times [0, 1] \to Gr_n (\mathbb{C}^M)$ transforms: exactly via v:

$$\psi_i^{vRv^*} (k) = v (k) \psi_i^R (k)$$

We can express $v (k) |_{\text{im}(R)}$ in the basis $\{ \psi_i^R \}_{i=1}^{n}$ as $\left\{ [v (k)]_{ji} \right\}_{j, i=1}^{n}$ so that this becomes:

$$\psi_i^{vRv^*} (k) = \sum_{j=1}^{n} \psi_j^R (k) [v (k)]_{ji}$$

We then find

$$\begin{aligned}
\psi_i^{vRv^*} (k_1, 0) &= \sum_{j=1}^{n} \psi_j^R (k_1, 0) [v (k_1, 0)]_{ji} \\
&= \sum_{j=1}^{n} \sum_{l=1}^{n} \psi_l^R (k_1, 1) \left[T_1^R (k_1) \right]_{lj} [v (k_1, 0)]_{ji} \\
&= \sum_{j=1}^{n} \sum_{l=1}^{n} \psi_l^R (k_1, 1) \underbrace{v (k_1, 1) v (k_1, 1)^*}_{=1} \left[T_1^R (k_1) \right]_{lj} [v (k_1, 0)]_{ji} \\
&\quad \left(\text{Again in the basis } \{ \psi_i^R \}_{i=1}^{n} \right)
\end{aligned}$$

$$= \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{s=1}^{n} \underbrace{\sum_{r=1}^{n} \psi_r^R (k_1, 1) \left[v (k_1, 1) \right]_{rs} \left[v (k_1, 1)^* \right]_{sl}}_{= \psi_s^{vRv^*} (k_1, 1)} \left[T_1^R (k_1) \right]_{lj} \times$$

$$\times \left[v (k_1, 0) \right]_{ji}$$

$$= \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{s=1}^{n} \psi_s^{vRv^*} (k_1, 1) \left[v (k_1, 1)^* \right]_{sl} \left[T_1^R (k_1) \right]_{lj} \left[v (k_1, 0) \right]_{ji}$$

2. By its definition, P is a k-dependent rotation of Q by the unitary z. As a result, we find

$$\text{Chern} (P) =$$

$$= \text{Wind} \left(\mathbb{S}^1 \to \mathbb{S}^1; k_1 \mapsto \det \left(\underbrace{z (k_1, 1)^* |_{\text{im}(Q)}}_{U(k_1)} \underbrace{T_1^Q (k_1)}_{=1} \underbrace{z (k_1, 0) |_{\text{im}(Q)}}_{=1} \right) \right)$$

$$= \text{Wind} (U)$$

$\square$

A.7   PROOFS OF LEMMAS FROM CHAPTER 2

We give the proof of Proposition 2.28 here explicitly for the reader's convenience because in the reference [KLS90, Prop. 2.7] it is merely outlined.

We define the additive co-cycle ζ on $\bar{L}_{j-1} \times \bar{L}_j$, which is an $Sp_{2N}^* (\mathbb{C})$-space, via the formula

$$\zeta(g, [y], [x]) := \log(\frac{\| \wedge^{j-1} gy \|}{\|y\|} \frac{\|x\|}{\| \wedge^j gx \|}) \qquad \forall (g, [y], [x]) \in Sp_{2N}^* (\mathbb{C}) \times \bar{L}_{j-1} \times \bar{L}_j.$$

Indeed, we have

$$\zeta(gh, [y], [x]) = \zeta(g, h[y], h[x]) + \zeta(h, [y], [x]) .$$

Now, we have

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\zeta(B_n(z), [y], [x])] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\| \wedge^{j-1} gy \|}{\|y\|} \frac{\|x\|}{\| \wedge^j gx \|})]$$

$$= \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\| \wedge^{j-1} gy \|}{\|y\|})] - \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\| \wedge^j gx \|}{\|x\|})] ,$$

and then by Proposition 2.27 $\frac{1}{n} \mathbb{E}[\zeta(B_n(z), [y], [x])] \xrightarrow{n \to \infty} -\gamma_j(z)$ uniformly in $z \in K$, $([x], [y]) \in \bar{L}_j \times \bar{L}_{j-1}$. As a result, for any $\varepsilon > 0$ there is some $n_K(\varepsilon)$ such that $\forall n \in \mathbb{N}_{\geq n_K(\varepsilon)}$ we have

$$|\frac{1}{n} \mathbb{E}[\zeta(B_n(z), [y], [x])] + \gamma_j(z)| < \varepsilon$$

$$\downarrow$$

$$\mathbb{E}[\zeta(B_n(z),[y],[x])] < n(-\gamma_j(z)+\varepsilon)$$

$$\downarrow$$

$$\mathbb{E}[\zeta(B_n(z),[y],[x])] < n(-\Gamma_j(K)+\varepsilon),$$

so if we pick $n \geq n_K(\frac{1}{2}\Gamma_j(K))$, we find $\mathbb{E}[\zeta(B_n(z),[y],[x])] < -\frac{1}{2}n\Gamma_j(K)$, with $\Gamma_j(K) := \inf_{z\in K}\gamma_j(z) > 0$ by hypothesis.

We need the intermediate result give the following intermediate

A.50 Proposition. *There is some $n_0(K)$ and some $s_K \in (0,1)$ such that*

$$\mathbb{E}[\exp(s_K\zeta(B_{n_0(K)}(z),[y'],[x']))] < 1 - \varepsilon_K \tag{A.21}$$

for all y', x' and $z \in K$, for some $\varepsilon_K \in (0,1)$.

Proof. This is the strip-analog of [CKM87, Lemma 5.1]. First note that for all $\alpha \in \mathbb{R}$ we have $e^\alpha \leq 1 + \alpha + \alpha^2 e^{|\alpha|}$ so that

$$\exp(s\zeta(B_n(z),[y'],[x']))$$
$$\leq 1 + s\zeta(B_n(z),[y'],[x']) + s^2\zeta(B_n(z),[y'],[x'])^2 e^{|s\zeta(B_n(z),[y'],[x'])|}.$$

Next,

$$|\zeta(B_n(z),[y'],[x'])| \equiv |\log(\frac{\|\wedge^{j-1}B_n(z)y'\|}{\|y'\|}\frac{\|x'\|}{\|\wedge^j B_n(z)x'\|})|$$
$$\leq |\log(\frac{\|\wedge^{j-1}B_n(z)y'\|}{\|y'\|})| + |\log(\frac{\|\wedge^j B_n(z)x'\|}{\|x'\|})|,$$

and via Proposition A.35 we have

$$|\zeta(B_n(z),[y'],[x'])| \leq (j-1)(2N-1)\log(\|B_n(z)\|) + j(2N-1)\log(\|B_n(z)\|)$$
$$\leq 4jN\log(\|B_n(z)\|)$$
$$\leq 4jN\sum_{k=1}^{n}\log(\|A_k(z)\|).$$

Hence by Hölder's inequality and Proposition A.33 we have:

$$\mathbb{E}[\exp(s\zeta(B_n(z),[y'],[x']))] \leq$$
$$\leq 1 + s\mathbb{E}[\zeta(B_n(z),[y'],[x'])] + s^2(\mathbb{E}[\zeta(B_n(z),[y'],[x'])^4])^{\frac{1}{2}} \times$$
$$\times (\mathbb{E}[e^{2|s\zeta(B_n(z),[y'],[x'])|}])^{\frac{1}{2}}$$
(Independence of the variables)
$$\leq 1 + s\mathbb{E}[\zeta(B_n(z),[y'],[x'])]+$$
$$+ s^2 16j^2N^2n^2(\mathbb{E}[\log(\|A_1(z)\|)^4])^{\frac{1}{2}}\mathbb{E}[\|A_1(z)\|^{8sjN}]^{\frac{n}{2}}.$$

We note that $\|A_1(z)\|^{8sjN}$ is integrable by the proof of Proposition 2.13 and Assumption 2.3 if we pick $s \in (0,1)$ sufficiently small.

Now with some choice of constants $C_1(K) := 16j^2N^2 \sup_{z \in K} \sqrt{\mathbb{E}[\log(\|A_1(z)\|^4)]} < \infty$ and $C_2(K) := \sup_{z \in K} \sqrt{\mathbb{E}[\|A_1(z)\|^{8sjN}]}$ we find

$$\mathbb{E}[\exp(s\zeta(B_n(z), [y'], [x']))] \leq 1 + s\mathbb{E}[\zeta(B_n(z), [y'], [x'])] + s^2 n^2 C_1(K) C_2(K)^n.$$

From the above we know that

$$\mathbb{E}[\exp(s\zeta(B_{n_K(\frac{1}{2}\Gamma_j(K))}(z), [y'], [x']))] \leq 1 - s\frac{1}{2} n_K(\frac{1}{2}\Gamma_j(K))\Gamma_j(K) +$$
$$+ s^2 n_K(\frac{1}{2}\Gamma_j(K))^2 C_1(K) C_2(K)^{n_K(\frac{1}{2}\Gamma_j(K))}.$$

Hence there is some $s \in (0,1)$ (which depends on K) so that

$$\varepsilon_K := \frac{1}{2} n_K(\frac{1}{2}\Gamma_j(K))\Gamma_j(K) - s n_K(\frac{1}{2}\Gamma_j(K))^2 C_1(K) C_2(K)^{n_K(\frac{1}{2}\Gamma_j(K))}.$$

is positive. $\qquad\square$

With this setup, we can finally start the

Proof of Proposition 2.28. We note that the object whose expectation we're actually trying to bound is

$$\exp(s\zeta(B_n(z), [y], [x])) \equiv \exp(s\log(\frac{\|\wedge^{j-1} gy\|}{\|y\|} \frac{\|x\|}{\|\wedge^j gx\|}))$$
$$= (\frac{\|\wedge^{j-1} gy\|}{\|y\|} \frac{\|x\|}{\|\wedge^j gx\|})^s,$$

and that

$$\exp(s\zeta(B_{n+m}(z), [y], [x])) \equiv \exp(s\zeta(A_{n+m}(z) \ldots A_{1+m}(z) B_m(z), [y], [x]))$$
$$\text{(cocycle property)}$$
$$= \exp(s(\zeta(A_{n+m}(z) \ldots A_{1+m}(z), B_m(z)[y], B_m(z)[x]) +$$
$$+ \zeta(B_m(z), [y], [x])))$$
$$= \exp(s\zeta(A_{n+m}(z) \ldots A_{1+m}(z), B_m(z)[y], B_m(z)[x])) \times$$
$$\times \exp(s\zeta(B_m(z), [y], [x])).$$

Hence due to the fact that $\{A_n(z)\}_{n \in \mathbb{Z}}$ are independent, we can integrate first only over $\{A_{n+m}(z), \ldots, A_{1+m}(z)\}$, so that in that integration $B_m(z)y$ and $B_m(z)x$ are fixed. Then via (A.21) we find

$$\mathbb{E}[\exp(s_K\zeta(B_{n_0(K)+m}(z), [y], [x]))] \leq (1 - \varepsilon_K)\mathbb{E}[\exp(s_K\zeta(B_m(z), [y], [x]))].$$

So if $n \in \mathbb{N}_{\geq n_0(K)}$ is given, we write it as $n = q_n n_0(K) + r_n$ with $r_n \in \{0, \ldots, n_0(K) - 1\}$. We then have using Hölder's inequality

$$\mathbb{E}[\exp(\frac{1}{2} s_K \zeta(B_n(z), [y], [x]))]$$

$$\leq \mathbb{E} \exp(\frac{1}{2} s_K \zeta(A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z), B_{q_n n_0(K)}(z)[y], B_{q_n n_0(K)}(z)[x])) \times$$

$$\times \exp(\frac{1}{2} s_K \zeta(B_{q_n n_0(K)}(z), [y], [x]))$$

$$\leq \mathbb{E}[\exp(s_K \zeta(A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z), B_{q_n n_0(K)}(z)[y], B_{q_n n_0(K)}(z)[x]))]^{1/2} \times$$

$$\times \mathbb{E}[\exp(s_K \zeta(B_{q_n n_0(K)}(z), [y], [x]))]^{1/2}$$

$$\leq \mathbb{E}[(\| \wedge^{j-1} A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z) \| \times$$

$$\times \|(\wedge^j A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z))^{-1}\|)^{s_K}]^{1/2} (1 - \varepsilon_K)^{\frac{q_n}{2}},$$

and using the proof of Proposition A.35 and the independence condition (assuming again that s_K has to be redefined so that $\|A_1\|^{(j-1)s_K+(2N-1)js_K}$ is also integrable (via Assumption 2.3)) we find

$$\mathbb{E}[\exp(\frac{1}{2} s_K \zeta(B_n(z), [y], [x]))] \leq \mathbb{E}[\|A_1(z)\|^{(j-1)s_K+(2N-1)js_K}]^{\frac{r_n}{2}} (1 - \varepsilon_K)^{\frac{q_n}{2}}$$

$$\leq (\sup_{z \in K} \mathbb{E}[\|A_1(z)\|^{(j-1)s_K+(2N-1)js_K}]^{n_0(K)})(1 - \varepsilon_K)^{\frac{q_n}{2}},$$

which implies the bound in the claim. $\qquad \square$

A.8   PROOFS OF LEMMAS FROM CHAPTER 3

Proof of Lemma 3.6 Item (b) in the mobility gap regime. The operator $T = [\Lambda, P]$ has $\|T\|_1 < \infty$ because Assumption 3.1 implies P is weakly-local in the sense of Definition A.7 so that we may use Corollary A.20.   □

Proof of Lemma 3.22. We first prove (3.40) and claim

$$\iota_a f(H_a)\iota_a^* = f(\iota_a H_a \iota_a^*) + f(0)(\mathbb{1} - \iota_a \iota_a^*)$$

for any (Borel) function f. In fact, let us decompose $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z}_a) \oplus \ell^2(\check{\mathbb{Z}}_a)$, where $\check{\mathbb{Z}}_a = \mathbb{Z} \setminus \mathbb{Z}_a$, as well as any descendant space such as $\mathcal{H}$. The isometries ι_a and $\tilde{\iota}_a$ (similarly defined) provide a partition of unity, $\mathbb{1} = \iota_a \iota_a^* + \tilde{\iota}_a \tilde{\iota}_a^*$, and a block decomposition of

$$\iota_a H_a \iota_a^* = \iota_a H_a \iota_a^* + \tilde{\iota}_a 0 \tilde{\iota}_a^* \equiv H_a \oplus 0.$$

Thus, by the functional calculus,

$$f(\iota_a H_a \iota_a^*) = \iota_a f(H_a) \iota_a^* + \tilde{\iota}_a f(0) \tilde{\iota}_a^*, \tag{A.22}$$

as claimed.

For uniformly bounded operators, like $\iota_a H_a \iota_a^*$ and H, strong resolvent convergence is equivalent to strong convergence (see [RS80], Problem VIII.28). The latter,

$$\iota_a H_a \iota_a^* - H \xrightarrow{s} 0, \qquad (a \to +\infty)$$

is evident, because the LHS vanishes for large but finite a, when applied to any state $\psi \in \mathcal{H}$ from the dense subspace $\{\mathrm{supp}\,\psi \subseteq \mathbb{Z}$ is bounded$\}$. Finally we specialize to $f = \chi_{(-\infty,0)}$. By ([RS80], Theorem VIII.24 (b)) and Assumption 3.2 the strong resolvent convergence implies $f(\iota_a H_a \iota_a^*) - f(H) \xrightarrow{s} 0$, $(a \to +\infty)$. The limit (3.40) now follows from (A.22) by $f(0) = 0$.   □

Proof of (3.41). We write $D_a := \iota_a P_{-,a} \iota_a^* - P_-$ for brevity. As shown in ([EGS05], Eq. (3.20)), Assumption 3.1 implies

$$\|e^{-\mu n} e^{-\varepsilon|n|} P_- e^{\mu n}\| \leq C_\varepsilon, \qquad (\varepsilon > 0),$$

where $g(n)$ denotes the multiplication operator by the namesake function. The same holds true by the same assumption for $\iota_a P_{-,a} \iota_a^*$ instead of P_-, and thus for D_a as well. The same estimate holds for μ replaced by $-\mu$.

We pick a switch function Λ with compactly supported variation and denote by $\Lambda^b(n) = \Lambda(n - b)$ its translate by $b \in \mathbb{N}$. We note that $\Lambda - \Lambda^b$ is of finite rank and that for fixed $\varepsilon \in (0, \mu)$ we have

$$\|(\mathbb{1} - \Lambda)e^{\mu n}e^{\varepsilon|n|}\| \leq C \,,$$
$$\|e^{-\mu n}\Lambda^b\|_1 \leq Ce^{-\mu b} \,.$$

The LHS of (3.41) is

$$[D_a, \Lambda] = (\mathbb{1} - \Lambda)\, D_a\Lambda - \Lambda D_a\, (\mathbb{1} - \Lambda) \tag{A.23}$$

and we claim that in the limit $a \to +\infty$ each term vanishes separately in trace norm. Indeed,

$$(\mathbb{1} - \Lambda)D_a\Lambda = (\mathbb{1} - \Lambda)D_a\Lambda^b + (\mathbb{1} - \Lambda)D_a(\Lambda - \Lambda^b) \,,$$
$$(\mathbb{1} - \Lambda)D_a\Lambda^b = (\mathbb{1} - \Lambda)e^{\mu n}e^{\varepsilon|n|} \cdot e^{-\mu n}e^{-\varepsilon|n|}D_a e^{\mu n} \cdot e^{-\mu n}\Lambda^b$$

Thus $\|(\mathbb{1} - \Lambda)D_a\Lambda^b\|_1$ can be made arbitrarily small, uniformly in a, by first picking b large. Then $\|(\mathbb{1} - \Lambda)D_a(\Lambda - \Lambda^b)\|_1$ will be small for large a by (3.40) (see (3.44)). The other term on the RHS of (A.23) is dealt with similarly. $\qquad\square$

A.9   UNITARY RAGE THEOREM

In this section we prove that our deterministic dynamical localization assumption implies pure point spectrum (so that it's not necessary to also have the latter as an assumption). This entails importing the analysis of the RAGE theorem to the unitary Floquet case. Most of this was already done in [HJS09] but since there it is written for a probabilistic model and we insist in this paper rather on deterministic assumptions and statements (compare our deterministic (A.24) with their probabilistic [HJS09], eqn (3.1)), and also in order to setup the notation for our important Lemma A.56, we included the proof here as succinctly as possible.

Within this section, let a unitary $U \in \mathcal{B}(\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N)$ be given such that it is *localized*. For our purposes it is enough to make the following

A.51 Definition. U is deterministically dynamically localized *in the interval $I \subseteq \mathbb{S}^1$* iff there is some $\mu > 0$ such that for any $\varepsilon > 0$ there is a $0 < C_\varepsilon < \infty$ such that the following holds

$$\sup_{n \in \mathbb{N}} \sum_{x, y \in \mathbb{Z}^d} \|\langle \delta_x, U^n \chi_I(U)\delta_y\rangle\|e^{\mu\|x - y\| - \varepsilon\|x\|} = C_\varepsilon \tag{A.24}$$

A.52 Lemma. *(Discrete Wiener) Let μ be a complex measure on $\mathbb{S}^1$. For $m \in \mathbb{N}$, we define its mth complex moment as $\mu_m := \int_{z \in \mathbb{S}^1} z^m \, \mathrm{d}\,\mu(z)$. Then $\lim_{n \to \infty} \frac{1}{n}\sum_{m=1}^{n} |\mu_m|^2 = \sum_{z \in \mathbb{S}^1} |\mu(\{z\})|^2$ that is, the RHS gives the pure point part of $|\mu(\mathbb{S}^1)|^2$.*

Proof. We have

$$\frac{1}{n} \sum_{m=1}^{n} |\mu_m|^2 \equiv \frac{1}{n} \sum_{m=1}^{n} \int_{z \in \mathbb{S}^1} z^m \, \mathrm{d}\,\mu(z) \int_{w \in \mathbb{S}^1} \overline{w}^m \, \mathrm{d}\,\overline{\mu}(w)$$

$$= \frac{1}{n} \sum_{m=1}^{n} \int_{w \in \mathbb{S}^1} \int_{z \in \mathbb{S}^1} (z\overline{w})^m \, \mathrm{d}\,\mu(z) \, \mathrm{d}\,\overline{\mu}(w)$$

$$= \int_{w \in \mathbb{S}^1} \int_{z \in \mathbb{S}^1} \frac{1}{n} \sum_{m=1}^{n} (z\overline{w})^m \, \mathrm{d}\,\mu(z) \, \mathrm{d}\,\overline{\mu}(w)$$

Note that the sequence of functions

$$\left\{ \mathbb{S}^1 \ni z \mapsto \frac{1}{n} \sum_{m=1}^{n} z^m \right\}_{n \in \mathbb{N}}$$

is uniformly bounded by 1 and converges pointwise to $\delta(\cdot - 1)$. We may thus use the dominated convergence theorem to find

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} |\mu_m|^2 = \int_{w \in \mathbb{S}^1} \int_{z \in \mathbb{S}^1} \delta(z\overline{w} - 1) \, \mathrm{d}\,\mu(z) \, \mathrm{d}\,\overline{\mu}(w)$$

$$= \int_{z \in \mathbb{S}^1} \mathrm{d}\,\mu(z) \overline{\mu(\{z\})}$$

$$= \sum_{z \in \mathbb{S}^1} |\mu(\{z\})|^2 \,.$$

$\square$

A.53 Lemma. *Let U be unitary and K compact. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \|KU^m \psi\|^2 = 0$$

for all $\psi \in \mathcal{H}^c$, the continuous part of the Hilbert space for U.

Proof. This is [AW15, Lemma 2.7] in our setting of discrete rather than continuous time. Assume for the moment that K is finite rank, that is, $K = \rho \otimes \varphi^*$. Then

$$\|KU^n \psi\|^2 = \langle KU^n \psi, KU^n \psi \rangle = \langle \rho \langle \varphi, U^n \psi \rangle, \rho \langle \varphi, U^n \psi \rangle \rangle = \|\rho\|^2 |\langle \varphi, U^n \psi \rangle|^2 \,.$$

Let $\mu_{\varphi, \psi}$ be the complex spectral measure corresponding to the triplet (U, φ, ψ). Then $\langle \varphi, U^n \psi \rangle = \int_{z \in \mathbb{S}^1} z^n \, \mathrm{d}\,\mu_{\varphi, \psi}(z) \equiv (\mu_{\varphi, \psi})_n$, the nth moment as defined in Lemma A.52. Hence by Lemma A.52,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \|KU^m \psi\|^2 = \|\rho\|^2 \sum_{z \in \mathbb{S}^1} |\mu_{\varphi, \psi}(\{z\})|^2 \,.$$

By assumption, ψ is in the continuous part of U, so that $\mu_{\varphi,\psi}(\{z\}) = 0$ for any z (i.e. $\mu_{\varphi,\psi}$ has no pure point part).

Since compact operators are norm limits of finite rank operators, for general K this same argument goes through via an approximation with a finite sum of finite rank operators. $\qquad\square$

A.54 Theorem. *(Unitary RAGE) Let U be unitary and $\{A_L\}_L$ be a sequence of compact operators strongly converging to $\mathbb{1}$. Then*

$$\mathcal{H}^c = \left\{ \psi \in \mathcal{H} \,\middle|\, \lim_{L\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^n \|A_L U^n \psi\|^2 = 0 \right\},$$

and

$$\mathcal{H}^p = \left\{ \psi \in \mathcal{H} \,\middle|\, \lim_{L\to\infty} \sup_{n\in\mathbb{N}} \|(\mathbb{1} - A_L) U^n \psi\| = 0 \right\}.$$

Proof. This is [AW15, Theorem 2.6] in our setting of discrete rather than continuous time, but the same proof goes through with very slight modifications. Let us temporarily denote the right hand sides of the above equations $\mathcal{H}^{c'}$ and $\mathcal{H}^{p'}$ respectively.

Start with $\mathcal{H}^c \subseteq \mathcal{H}^{c'}$. Let $\psi \in \mathcal{H}^c$ be given. Then by Lemma A.53, for *any* L we have $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^n \|A_L U^n \psi\|^2 = 0$ and hence also for the (redundant) limit. We find $\mathcal{H}^c \subseteq \mathcal{H}^{c'}$.

For $\mathcal{H}^p \subseteq \mathcal{H}^{p'}$, let $\psi \in \mathcal{H}^p$ be given. Then either ψ is in the closure of the set of eigenvectors of U, that is, for any $\varepsilon > 0$ we have some $\psi_\varepsilon \in \mathcal{H}$ which is an honest eigenvector of U (say with eigenvalue z_ε) and $\|\psi - \psi_\varepsilon\| < \varepsilon$. Then

$$\begin{aligned}
\|(\mathbb{1} - A_L) U^n \psi\| &= \|(\mathbb{1} - A_L) U^n (\psi_\varepsilon + \psi - \psi_\varepsilon)\| \\
&\leq \|(\mathbb{1} - A_L) U^n \psi_\varepsilon\| + \|(\mathbb{1} - A_L)\| \|U^n\| \varepsilon \\
&\leq \|(\mathbb{1} - A_L) \psi_\varepsilon\| + \varepsilon(1 + \|A_L\|).
\end{aligned}$$

By the uniform boundedness principle, $\sup_L \|A_L\| < \infty$ since $\sup_L \|A_L \varphi\| = \|\varphi\| < \infty$. Additionally, $\|(\mathbb{1} - A_L)\psi_\varepsilon\| \to 0$ as $L \to \infty$ because $A_L \to \mathbb{1}$ strongly. Hence $\lim_{L\to\infty} \sup_{n\in\mathbb{N}} \|(\mathbb{1} - A_L) U^n \psi\| \leq \lim_{L\to\infty} \|(\mathbb{1} - A_L)\psi_\varepsilon\| + \varepsilon(1 + \|A_L\|) \leq C\varepsilon$ for some $C > 0$. Since $\varepsilon > 0$ was arbitrary we find that $\psi \in \mathcal{H}^{p'}$.

Next, we shall show that $\mathcal{H}^{p'} \perp \mathcal{H}^{c'}$: Let $\varphi \in \mathcal{H}^{p'}$ and $\psi \in \mathcal{H}^{c'}$ be given. We want $\langle \varphi, \psi \rangle = 0$. We have

$$|\langle \varphi, \psi \rangle|^2 = \left(\frac{1}{n} \sum_{m=1}^n\right) |\langle \varphi, \psi \rangle|^2$$

$$\text{(By unitarity)}$$

$$= \frac{1}{n} \sum_{m=1}^{n} |\langle U^m \varphi, U^m \psi \rangle|^2$$

$$\leq \frac{1}{n} \sum_{m=1}^{n} |\langle U^m \varphi, (\mathbb{1} - A_L) U^m \psi \rangle|^2 + |\langle A_L^* U^m \varphi, U^m \psi \rangle|^2$$

(Cauchy-Schwarz)

$$\leq \frac{1}{n} \sum_{m=1}^{n} \|U^m \varphi\|^2 \|(\mathbb{1} - A_L) U^m \psi\|^2 + \|A_L^* U^m \varphi\|^2 \|U^m \psi\|^2$$

(Unitarity)

$$\leq \|\varphi\|^2 \frac{1}{n} \sum_{m=1}^{n} \|(\mathbb{1} - A_L) U^m \psi\|^2 + \|\psi\|^2 \frac{1}{n} \sum_{m=1}^{n} \|A_L^* U^m \varphi\|^2$$

$$\leq \|\varphi\|^2 \sup_{m'} \|(\mathbb{1} - A_L) U^{m'} \psi\|^2 + \|\psi\|^2 \frac{1}{n} \sum_{m=1}^{n} \|A_L^* U^m \varphi\|^2$$

Hence taking the (redundant on the LHS and partly redundant for the first term on the RHS) limit $\lim_{L \to \infty} \lim_{n \to \infty}$ we get $|\langle \varphi, \psi \rangle|^2 \leq 0$. But since $\mathcal{H} = \mathcal{H}^p \oplus \mathcal{H}^c$ the result now follows. $\square$

The following theorem and the remark after it are the reason for this section.

A.55 Theorem. *(Deterministic dynamical localization implies spectral localization) If U is deterministically dynamically localized in the interval I then it has pure point spectrum within that interval, that is,*

$$\sigma(U) \cap I = \sigma_{pp}(U) \cap I$$

Proof. Since $\{ \delta_x \}_{x \in \mathbb{Z}^d}$ is an ONB for $\mathcal{H}$, and we want to show that $\chi_I(U)\mathcal{H} \subseteq \mathcal{H}^p$, let $y \in \mathbb{Z}^d$ be given. We claim that $\chi_I(U)\delta_y \in \mathcal{H}^p$. Let A_L be the projection onto a box of total volume $(2L+1)^d$ centered about the origin of $\mathbb{Z}^d$. Using Theorem A.54 it suffices to show

$$\lim_{L \to \infty} \sup_{n \in \mathbb{N}} \|A_L^\perp U^n \chi_I(U)\delta_y\| = 0.$$

By (A.24) we have *for any* $n \in \mathbb{N}$,

$$\sum_{x,y \in \mathbb{Z}^d} \| \langle \delta_x, U^n \chi_I(U)\delta_y \rangle \| e^{\mu\|x-y\| - \varepsilon\|x\|} \leq C_\varepsilon.$$

This in turn implies that

$$\| \langle \delta_x, U^n \chi_I(U)\delta_y \rangle \| e^{\mu\|x-y\| - \varepsilon\|x\|} \leq \sum_{x',y'} \| \langle \delta_{x'}, U^n \chi_I(U)\delta_{y'} \rangle \| e^{\mu\|x'-y'\| - \varepsilon\|x'\|}$$

$$\leq C_\varepsilon$$

since all terms are positive. Hence, $\| \langle \delta_x, U^n \chi_I(U) \delta_y \rangle \| \leq C_\varepsilon e^{-\mu \|x-y\| + \varepsilon \|x\|}$ uniformly in n.

Now we have

$$\|A_L^\perp U^n \chi_I(U) \delta_y\|^2 = \sum_{x \in \mathbb{Z}^d : \|x\| > L} \| \langle \delta_x, U^n \chi_I(U) \delta_y \rangle \|^2$$

$$(\text{Using } \| \langle \delta_x, U^n \chi_I(U) \delta_y \rangle \| \leq 1)$$

$$\leq \sum_{x \in \mathbb{Z}^d : \|x\| > L} \| \langle \delta_x, U^n \chi_I(U) \delta_y \rangle \| \leq \sum_{x \in \mathbb{Z}^d : \|x\| > L} C_\varepsilon e^{-\mu \|x-y\| + \varepsilon \|x\|} .$$

Hence since the square root is monotone increasing and continuous, and using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we find

$$\|A_L^\perp U^n \chi_I(U) \delta_y\| \leq \sqrt{C_\varepsilon} \sum_{x \in \mathbb{Z}^d : \|x\| > L} e^{-\frac{1}{2}\mu \|x-y\| + \frac{1}{2}\varepsilon \|x\|}$$

for any $n \in \mathbb{N}$ so that taking the supremum on both sides (redundant on the RHS) and then the limit $L \to \infty$ we get zero indeed. This follows because (for $\varepsilon < \mu$) $e^{-\frac{1}{2}\mu \|x-y\| + \frac{1}{2}\varepsilon \|x\|}$ is summable in x, and hence taking the limit $L \to \infty$ gives zero. $\qquad \square$

A.56 Lemma. *(The stretch-construction and pure point spectrum) Let U be such that $\sigma(U) \cap I$ is pure point and $\sigma(U) \cap I^c$ is some mixture of pure point and continuous spectrum. Define*

$$V = \chi_I(U) f(U) + \chi_{I^c}(U) .$$

where $f : \mathbb{S}^1 \to \mathbb{S}^1$ has a range which is the entire circle. Then $\sigma(V) = \sigma_{pp}(V)$.

We note that in our application of the stretch-function, strictly-speaking, this lemma could be avoided since $F_\Delta \in B_1(\Delta)$ so that $V(1)$ is actually dynamically-localized as in Definition 4.4 on $\mathbb{S}^1 \setminus \{1\}$, and thus one could invoke Theorem A.55 to conclude $\sigma(V(1)) = \sigma_{pp}(V(1))$. However, the proof below proceeds directly without making an assumption of dynamical localization on U, but rather, only on its spectral type within I.

Proof. We have by Theorem A.54, for any $\psi \in \mathcal{H}$

$$\|(\mathbb{1} - A_L) V^n \psi\| = \|(\mathbb{1} - A_L)(\chi_I(U) f(U) + \chi_{I^c}(U))^n \psi\|$$

$$(\text{By projections being orthogonal})$$

$$= \|(\mathbb{1} - A_L)(\chi_I(U) f(U)^n \psi + \chi_{I^c}(U) \psi)\|$$

$$\leq \|(\mathbb{1} - A_L) f(U)^n \chi_I(U) \psi\| + \|(\mathbb{1} - A_L) \chi_{I^c}(U) \psi\|$$

Now in general we may write $\psi = \psi_1 + \psi_2$ with $\psi_1 \in \chi_I(U)$ and $\psi_2 \in \chi_{I^c}(U)$. Taking the supremum and limit of both sides, using the fact that the supremum of a sum is smaller than the sum of supremums, we find

$$\limsup_{L \to \infty \atop n \in \mathbb{N}} \|(\mathbb{1} - A_L)V^n\psi\| \leq \limsup_{L \to \infty \atop n \in \mathbb{N}} \|(\mathbb{1} - A_L)f(U)^n\psi_1\| + \underbrace{\lim_{L \to \infty} \|(\mathbb{1} - A_L)\psi_2\|}_{=0}$$

We know $\psi_1 \in \mathcal{H}_U^p$ by the assumption on I. That means that either it is an eigenvector of U with eigenvalue λ or it is in the closure of the set of eigenvalues of U. In the former case we have $f(U)\psi_1 = f(\lambda)\psi_1$ whence $\psi_1 \in \mathcal{H}_{f(U)}^p$ so that $\lim_{L \to \infty} \sup_{n \in \mathbb{N}} \|(\mathbb{1} - A_L)f(U)^n\psi_1\| = 0$ by Theorem A.54. Otherwise, for any $\varepsilon > 0$ there is some $\psi_\varepsilon \in \mathcal{H}$ such that ψ_ε is an eigenvector of U (with eigenvalue λ_ε) and $\|\psi_1 - \psi_\varepsilon\| < \varepsilon$. Then

$$\|(\mathbb{1} - A_L)f(U)^n\psi_1\| \leq \|(\mathbb{1} - A_L)f(U)^n\psi_\varepsilon\| + \|(\mathbb{1} - A_L)f(U)^n(\psi_1 - \psi_\varepsilon)\|.$$

When taking the supremum and the limit, the first term will tend to zero as was just remarked. Thus let us concentrate on the second term:

$$\begin{aligned}
\|(\mathbb{1} - A_L)f(U)^n(\psi_1 - \psi_\varepsilon)\| &\leq \varepsilon(1 + \|A_L\|)\|f(U)^n\| \\
&\leq \varepsilon(1 + \|A_L\|)\underbrace{\sup_z |f(z)^n|}_{\leq 1} \\
&\leq 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we find the result. $\qquad\square$

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