

RICCI DEFECTS OF MICROLOCALIZED EINSTEIN METRICS

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ABSTRACT. This is the third and last in our series of papers concerning rough solutions of the Einstein vacuum equations expressed relative to wave coordinates. In this paper we prove an important result, concerning Ricci defects of microlocalized solutions, stated and used in the proof the crucial Asymptotics Theorem in [Kl-Ro2].

1. INTRODUCTION

This is the third and last in our series of papers concerning rough solutions of the Einstein vacuum equations expressed relative to wave coordinates. More precisely we are concerned with solutions of the Einstein vacuum equations,

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0 \tag{1}$$

expressed¹ relative to wave coordinates x^α ,

$$\square_{\mathbf{g}} x^\alpha = \frac{1}{|\mathbf{g}|} \partial_\mu (\mathbf{g}^{\mu\nu} |\mathbf{g}| \partial_\nu) x^\alpha = 0. \tag{2}$$

The solutions we consider here have a limited degree of differentiability, we only assume that in a time slab $[0, T] \times \mathbb{R}^3$ we control the the first derivatives of \mathbf{g} in the energy norm $L_t^\infty(H_x^{1+\gamma})$, $\gamma > 0$, as well as in the mixed Strichartz norm $L_t^2(L_x^\infty)$. More precisely,

Metric Hypothesis:

$$\|\partial\mathbf{g}\|_{L_{[0,T]}^\infty H^{1+\gamma}} + \|\partial\mathbf{g}\|_{L_{[0,T]}^2 L_x^\infty} \leq B_0, \tag{3}$$

for some fixed $\gamma > 0$ arbitrarily small.

This condition was introduced in section 2 of [Kl-Ro1] as the main bootstrap assumption in the proof of our main theorem concerning $H^{2+\gamma}$ solutions, $\gamma > 0$ arbitrarily small, of (1)–(2).

Microlocalization is an essential technique in dealing with rough solutions of non-linear wave equations, see [Kl-Ro1] and the references therein. By a microlocalized

1991 *Mathematics Subject Classification.* 35J10.

¹In wave coordinates the Einstein equations take the reduced form

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g})$$

with N quadratic in the first derivatives $\partial\mathbf{g}$ of the metric.

rough Einstein metric, at cut-off parameter $\lambda \geq 1$, we understand, essentially, the low frequency part (frequency $< \lambda$) of a given Einstein metric (1)–(2). To explain this in more details we recall below the definition of the Littlewood -Paley projections,

$$P_{<\lambda} = \sum_{\mu < \frac{1}{2}\lambda} P_\mu$$

$$P_\mu f(x) = \int e^{ix \cdot \xi} \chi(\mu^{-1}\xi) \hat{f}(\xi) d\xi$$

where $\chi \in C_0^\infty(\mathbb{R})$ supported in $\frac{1}{2} \leq |\xi| \leq 2$ and $\sum_{\mu \in 2^{\mathbb{Z}}} \chi(\mu^{-1}\xi) = 1$. The operators P_μ are the standard Littlewood -Paley *dyadic projections* corresponding to the frequencies $\mu \in 2^{\mathbb{Z}}$.

Consider a fixed solution \mathbf{g} of (1) satisfying the metric hypothesis (3) relative to the fixed system of wave coordinates (2). Consider also a fixed dyadic parameter $\lambda \in 2^{\mathbb{Z}^+}$ and define the microlocalized rescaled metric,

$$H(t, x) = H_{(\lambda)}(t, x) = (P_{<\lambda} \mathbf{g})(\lambda^{-1}t, \lambda^{-1}x) \quad (4)$$

Observe that $H_{(\lambda)}$ is the low frequency part of the rescaled metric, i.e. $H_{(\lambda)} = P_{<1}(G_{(\lambda)})$ where,

$$G_{(\lambda)}(t, x) = \mathbf{g}(\lambda^{-1}t, \lambda^{-1}x) \quad (5)$$

In the rescaled variables we restrict ourselves to the slab $[0, t_*] \times \mathbb{R}^3$ with $t_* \approx \lambda^{1-8\epsilon_0}$ for some small ϵ_0 , in fact $5\epsilon_0 < \gamma$. In this region we define the optical function u to be the solution of the eikonal equation,

$$H^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0 \quad (6)$$

verifying the initial condition

$$u(\Gamma_t) = t \quad (7)$$

where Γ_t is the timelike geodesic passing through the origin of and orthogonal (with respect to H) to the initial hypersurface Σ_0 . We denote by Σ_t the spacelike level hypersurface generated by the time function $t = x^0$. We denote by C_u the level hypersurfaces of u and by $S_{t,u}$ their intersection with Σ_t . In [Kl-Ro2] we show that the null hypersurfaces C_u form a proper foliation of the domain $\Omega_* = \mathcal{I}_-^+ \cap ([0, t_*] \times \mathbb{R}^3)$. Here \mathcal{I}_-^+ denotes the future domain (domain of influence) of the point $\Gamma_{-1} \in \Sigma_{-1}$.

To each point $p \in \Omega_*$ we associate the canonical null pair,

$$L = T + N, \quad \underline{L} = T - N \quad (8)$$

where T is the future unit normal to Σ_t and N is the outward unit normal to the surface $S_{t,u}$ passing through p . Observe that L is proportional to the null geodesic generator $L' = -H^{\alpha\beta} \partial_\beta u \partial_\alpha$ of C_u .

A null frame $e_1, e_2, e_3 = \underline{L}, e_4 = L$ consists of the null pair L, \underline{L} together with an arbitrary choice of vectors $(e_A)_{A=1,2}$ tangent to $S_{t,u}$ such that $H(e_A, e_B) = \delta_{AB}$.

Relative to such a null frame the metric H has the form,

$$H_{34} = -2, \quad H_{33} = H_{44} = H_{3A} = H_{4A} = 0, \quad H_{AB} = \delta_{AB}. \quad (9)$$

The null components of the inverse metric are therefore,

$$H^{34} = -\frac{1}{2}, \quad H^{33} = H^{44} = H^{3A} = H^{4A} = 0, \quad H^{AB} = \delta^{AB}. \quad (10)$$

While the rescaled spacetime metric $G = G_{(\lambda)}$ verifies the Einstein equations $\mathbf{R}_{\mu\nu}(G) = 0$ this is certainly not true for the microlocalized metric $H = H_{(\lambda)}$.

Definition 1.1. We call $\mathbf{Ric}(H)$ the *Ricci defect* of the microlocalized metric $H = H_{(\lambda)}$.

The Ricci defect of H plays a fundamental role in the proof of the Asymptotics Theorem, see Theorem 4.5 in [Kl-Ro1] or Theorem 2.5 in [Kl-Ro2]. More precisely it appears as a source term in the null structure equations, see section 3 of [Kl-Ro2]. For example the trace of the null second fundamental form $\chi_{AB} = H(\mathbf{D}_{e_A} L, e_B)$ $\text{tr}\chi = \delta^{AB}\chi_{AB}$ verifies an equation, roughly, of the form

$$\frac{d}{ds}\text{tr}\chi = -\mathbf{R}_{44}(H) + \dots \quad (11)$$

where $\mathbf{R}_{44} = \mathbf{Ric}(L, L) = L^\alpha L^\beta \mathbf{R}_{\alpha\beta}$ and s the affine parameter of the vectorfield L , i.e. $L(s) = 1$. Ignoring all other terms on the right hand side of (11) we see that $\text{tr}\chi$ can be controlled pointwise by the mixed $L_t^1(L_x^\infty)$ norm of the Ricci defect. In [Kl-Ro1] we have shown, using the metric hypothesis (3) and the fact that H arises(see (4)) from an Einstein metric \mathbf{g} , that,

$$\|\mathbf{Ric}(H)\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-1-8\epsilon_0}. \quad (12)$$

In the Asymptotics Theorem 9.1. in [Kl-Ro2] the proof of the estimates (118-121) was heavily dependent on (12). However we also need $L^2(S_{t,u})$ estimates for some derivatives of $\text{tr}\chi$, in particular the angular derivatives $\nabla\text{tr}\chi$. Differentiating the equation (11) we see that $\|\nabla\text{tr}\chi\|_{L^2(S_{t,u})}$ depends on,

$$\int_u^t \|\nabla\mathbf{R}_{44}(H)\|_{L^2(S_{\tau,u})} d\tau$$

To establish such an estimate we need first to compare the Ricci defect $\mathbf{Ric}(H)$ with $\mathbf{Ric}(G) = 0$ and then take advantage of energy estimates for derivatives of H along the null hypersurfaces C_u . Here we encounter a substantial difficulty as the 2-surfaces $S_{t,u}$ as well as the null hypersurfaces C_u have been constructed relative to the approximate metric H . This leads to significant differences² between the C_u - energy estimates for derivatives of H and the corresponding ones for G , see proposition 7.7 in [Kl-Ro2] and proposition 2.2 here.

In this paper we use the specific structure of the component \mathbf{R}_{44} relative to the wave coordinates and overcome this difficulty. We prove the following:

²The estimates for the second derivatives of the higher frequencies of G do in fact diverge badly.

Theorem 1.2. *On any null hypersurface C_u ,*

$$\int_u^t \|\nabla \mathbf{R}_{44}(H)\|_{L^2(S_{\tau,u})} d\tau \lesssim \lambda^{-1} \quad (13)$$

This result, stated without proof in theorem 8.1 [Kl-Ro2], played an essential role in the proof of the asymptotics theorem. The asymptotics theorem itself is a crucial step in the proof of our main theorem, see [Kl-Ro1]. The main goal of this paper is to prove theorem 1.2.

2. PRELIMINARIES

2.1. Background estimates. We start by writing down estimates for the rescaled metric $G(t, x) = \mathbf{g}(\lambda^{-1}t, \lambda^{-1}x)$. These are immediate consequences of the metric hypothesis (3) and the choice of the restricted time interval $[0, t_*]$.

$$\|\partial G\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (14)$$

$$\|\partial^2 G\|_{L_t^\infty L_x^2} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad (15)$$

It is also easy to derive the following estimate for G in $L^2(S_{t,u})$ norm.

$$\|\partial G\|_{L^2(S_{t,u})} \lesssim \lambda^{-4\epsilon_0} \quad (16)$$

This estimate follows by virtue of Hölder and the trace inequality (see theorem ?? in [Kl-Ro2]) on $S_{t,u}$ from (15).

We also recall the estimates for H derived in [Kl-Ro1] and [Kl-Ro2]. They are summarized in section 7 of [Kl-Ro2]. We list below only the ones which we need in this paper. Morally, since $H = P_{<1}G$ they follow from the corresponding estimates for G .

$$\|\partial H\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (17)$$

$$\|\partial^2 H\|_{L_t^\infty L_x^2} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad (18)$$

$$\|\partial H\|_{L^2(S_{t,u})} \lesssim \lambda^{-4\epsilon_0} \quad (19)$$

$$\|\mathbf{Ric}(H)\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-1-8\epsilon_0}, \quad (20)$$

We also have the following cone estimates(see section 7 of [Kl-Ro2]), which play an essential role in the proof of theorem 1.2:

Proposition 2.2. *The following estimates hold in the region $\Omega_* = \mathcal{I}_{-1}^+ \cap [0, t_*] \times \mathbb{R}^3$ and $1 \lesssim \lambda, \mu$.*

$$\|D_* \partial H\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}}, \quad \|D_* H\|_{L^2(C_u)} \lesssim \lambda^{\frac{1}{2}} \quad (21)$$

$$\|D_* \partial(P_\mu G)\|_{L^2(C_u)} \lesssim \mu^{\frac{1}{2}-4\epsilon_0} \lambda^{-\frac{1}{2}-4\epsilon_0},$$

$$\|D_*(P_\mu G)\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \mu^{-\frac{1}{2}-4\epsilon_0} \quad (22)$$

We shall also need estimates for the derivatives of the null vectorfield L in Ω_* ,

$$|\nabla L| \lesssim \Theta + r^{-1} \quad (23)$$

where $r = r(t, u)$ is defined by $\text{Area}(S_{t,u}) = 4\pi r^2$ and Θ verifies the following estimates,

$$\|\Theta\|_{L_t^2 L_\infty^\infty} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad (24)$$

$$\|\Theta\|_{L^2(S_{t,u})} \lesssim \lambda^{-2\epsilon_0} \quad (25)$$

By the comparison arguments proved in section 6.4 of [Kl-Ro2] we have

$$r \approx t - u \quad (26)$$

. We also have,

$$\|\Theta\|_{L^2(D_{t,u})} \lesssim \lambda^{-2\epsilon_0} \quad (27)$$

where,

Definition 2.3. The annulus $D_{t,u}$ is defined by $D_{t,u} = \cup_{u \leq u' \leq u+1} S_{t,u'}$ is the annulus on Σ_t of thickness 1 and outer boundary $S_{t,u}$.

Observe that,

$$\|\nabla L\|_{L^2(S_{t,u})} \lesssim 1 \quad (28)$$

Clearly we also have,

$$\|\nabla L\|_{L^2(D_{t,u})} \lesssim 1.$$

For a proof of the estimates (23)–(25) we refer to section 9 of [Kl-Ro2].

2.4. Set-up and error terms.

Definition 2.5. We denote by P the projection on the frequencies of size < 1 and by \bar{P} the projection on the frequencies of size ≤ 2 such that $\bar{P}P = P$.

Definition 2.6. We define

$$\begin{aligned} H(t, x) &= PG(t, x) \\ h(t, x) &= \sum_{\mu > 1} P_\mu G(t, x) \end{aligned}$$

Clearly,

$$G = H + h \quad (29)$$

Also, for the inverse metric,

$$G^{-1} = (H + h)^{-1} = (I + H^{-1}h)^{-1}H^{-1} = H^{-1} - H^{-1}hH^{-1} + O(h^2) \quad (30)$$

Therefore,

$$G_{\alpha\beta} = H_{\alpha\beta} + h_{\alpha\beta} \quad (31)$$

$$G^{\alpha\beta} = H^{\alpha\beta} - h^{\alpha\beta} + O(h^2) \quad (32)$$

where the indices of h are raised according to the matrix H .

In view of the fact that $\mathbf{R}_{\mu\nu}(G) = 0$ we infer that,

$$\mathbf{R}_{\mu\nu}(H) = \mathbf{R}_{\mu\nu}(H) - P \mathbf{R}_{\mu\nu}(G).$$

This is the starting point of our lengthy calculations which are presented in the following sections. In the process we are going to generate a large number of error terms. To better keep track of them we will systematize them in the following subsection.

2.7. Error terms.

We start with some basic commutator estimates which we shall need below.

Lemma 2.8. *Let Q be one of the Littlewood-Paley projections $Q = P, \overline{P}, P_\mu$ with $\mu > 1$. We may assume (see remark below) that the support of the integral kernel $Q(x)$ of the projection Q is localized to the unit ball centered at the origin in the case $Q = P, \overline{P}$, and the ball of radius μ^{-1} if $Q = P_\mu$.*

We denote

$$|Q| = \sup_{x: Q(x) \neq 0} |x|^{-1},$$

Then for all $p \in [1, \infty]$ and arbitrary functions u, w, v such that $\nabla w, \nabla f \in L_x^\infty$ and $v \in L_x^p$,

$$\|[Q, w]v\|_{L^p(D_{t,u})} \lesssim \frac{1}{|Q|} \|\nabla w\|_{L^\infty(D_{t,u})} \|v\|_{L^p(D_{t,u})}, \quad (33)$$

$$\|[Q, w]v\|_{L^p(D_{t,u})} \lesssim \frac{1}{|Q|} \|\nabla w\|_{L^p(D_{t,u})} \|v\|_{L^\infty(D_{t,u})}, \quad (34)$$

$$\|[Q, \nabla w]v\|_{L^p(D_{t,u})} \lesssim \|\nabla w\|_{L_x^\infty} \|v\|_{L_x^p}, \quad (35)$$

$$\|[Q, w], f\|_{L^p(D_{t,u})} v \lesssim \frac{1}{|Q|^2} \|\nabla w\|_{L^\infty(D_{t,u})} \|\nabla f\|_{L^\infty(D_{t,u})} \|v\|_{L^p(D_{t,u})}. \quad (36)$$

Remark 2.9. The assumptions made on the supports of the integral kernels of $Q = P, \overline{P}, P_\mu$ are essentially true³. Consistent with the uncertainty principle we can show that the kernels of Q are rapidly decaying outside the ball of radius one for P, \overline{P} and μ^{-1} for P_μ .

Proof The proof of the lemma is standard. For completeness we show below how to derive estimates (33) and (34). We have

$$\begin{aligned} [Q, w]v &= \int_{\Sigma_t} Q(x-y)(w(y) - w(x))v(y) dy \\ &= - \int_0^1 \int_{\Sigma_t} Q(x-y)(x-y)^i \partial_i w(\tau y + (1-\tau)x)v(y) dy d\tau \end{aligned} \quad (37)$$

Therefore, since the support of $Q(x)$ belongs to the unit ball centered at the origin,

$$\|[Q, w]v\|_{L^p(D_{t,u})} \lesssim \frac{1}{|Q|} \|\nabla w\|_{L^\infty(D_{t,u})} \|v\|_{L^p(D_{t,u})}, \quad (38)$$

³strictly speaking they are incompatible with the compact support assumption of the Littlewood-Paley projections in Fourier space.

where the annuli $D_{t,u}$ on the right hand side of (38) are perhaps twice as large as the original annulus. This proves (33).

To obtain (34) we proceed as follows. Using (37) we obtain

$$\begin{aligned} \|[Q, w]v\|_{L^p(D_{t,u})} &\lesssim \frac{1}{|Q|} \left\| \int_0^1 \int_{\Sigma_t} |Q(z)| |\nabla w(x + \tau z)| |v(x - z)| dz d\tau \right\|_{L^p(D_{\tau,u})} \\ &\lesssim \frac{1}{|Q|} \int_0^1 \int_{\Sigma_t} |Q(z)| \|\nabla w(\cdot + \tau z)v(\cdot - z)\|_{L^p(D_{\tau,u})} dz d\tau \\ &\lesssim \frac{1}{|Q|} \|\nabla w\|_{L^p(D_{\tau,u})} \|v\|_{L^\infty(D_{\tau,u})}. \end{aligned}$$

as desired. Here we once again used that the support of $Q(z)$ belongs to the unit ball centered at the origin. \blacksquare

Definition 2.10. Given functions f, v, w in $L^\infty(\Omega_*)$ we introduce the following:

- We denote by $[f]$ any operator with the property that for any function v in Ω_* and any $t \in [0, t_*]$, $u \geq -1$:

$$\|[f] \cdot v\|_{L^2(D_{t,u})} \lesssim \min \left\{ \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^2(D_{t,u})}, \|f\|_{L^2(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \right\} \quad (39)$$

- We denote by $\pi(f, v; w)$ any function in Ω_* which satisfies the inequality:

$$\|\pi(f, v; w)\|_{L^2(D_{t,u})} \lesssim \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \quad (40)$$

Definition 2.11. Given two operators A and B we say that $A \lesssim B$ if for any function v

$$\|Av\|_{L^2(D_{t,u})} \lesssim \|Bv\|_{L^2(D_{t,u})} \quad (41)$$

We also say that $\pi(f, v; w) \lesssim \pi(g, v; w)$ if

$$\|\pi(f, v; w)\|_{L^2(D_{t,u})} \lesssim \|g\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \quad (42)$$

Remark 2.12. The expression $[f]$ verifies the following trivial property

$$[af] \lesssim \|a\|_{L^\infty} [f].$$

The same holds true for $\pi(f, g; v)$ with respect to all entries.

For the Littlewood-Paley projection Q let (Qf) be the result of the application of Q to f . We also denote by Qf the operator whose action on functions is defined by

$$Qf(v) := Q(fv)$$

The typical examples of expressions of type $[f]$ are listed in the following lemma.

Lemma 2.13.

- For the projections $Q = I, P, \bar{P}, P_\mu$ with dyadic $\mu > 1$, we have $Qf \lesssim [f]$ and $(Qf) \lesssim [f]$.
- For $Q = P, \bar{P}, P_\mu$, we have, $[Q, f] \lesssim \frac{1}{|Q|} [\nabla f]$ and $[\nabla Q, f] \lesssim [\nabla f]$.

Proof To verify that $Qf \lesssim [f]$ we estimate

$$\begin{aligned} \|Qf(v)\|_{L^2(D_{t,u})} &\lesssim \|Q(fv)\|_{L^2(D_{t,u})} \lesssim \|fv\|_{L^2(D_{t,u})} \\ &\lesssim \min \left\{ \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^2(D_{t,u})}, \|f\|_{L^2(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \right\} \end{aligned}$$

A similar argument shows that $(Qf) \lesssim [f]$. We now verify that $[Q, f] \lesssim \frac{1}{|Q|}[\nabla f]$. Using the commutator estimates (33) and (34) we obtain

$$\|[Q, f]v\|_{L^2(D_{t,u})} \lesssim \frac{1}{|Q|} \min \left\{ \|\nabla f\|_{L^\infty(D_{t,u})} \|v\|_{L^2(D_{t,u})}, \|\nabla f\|_{L^2(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \right\}$$

as desired. The proof of the estimate $[\nabla Q, f] \lesssim [\nabla f]$ is similar. It uses the commutator estimate (35). \blacksquare

We also record some similar properties of the triple expressions $\pi(f, g; h)$.

Lemma 2.14.

- For $Q_i = I, P, \overline{P}, P_\mu$ with some dyadic $\mu \geq 1$ and $i = 1, \dots, 3$, we have

$$(Q_1 f)(Q_2 v)(Q_3 w) \lesssim \pi(f, v; w). \quad (43)$$

- With the same choice of Q_1, Q_2 ,

$$[f] \left((Q_1 v)(Q_2 w) \right) \lesssim \pi(v, w; f), \quad (44)$$

$$[f] \left((Q_1 v)(Q_2 w) \right) \lesssim \pi(f, v; w). \quad (45)$$

- With the same choice of Q

$$[f][v](Qw) \lesssim \pi(f, v; w), \quad (46)$$

$$[f][v](Qw) \lesssim \pi(v, w; f). \quad (47)$$

- If $\|f\|_{L^\infty} \lesssim \|g\|_{L^\infty}$ then

$$\pi(f, v; w) \lesssim \pi(g, v; w) \quad (48)$$

Proof The proof of (43) follows immediately from the definition of $\pi(f, v; w)$ and the properties of the projection Q_i . Indeed

$$\begin{aligned} \|(Q_1 f)(Q_2 v)(Q_3 w)\|_{L^2(D_{t,u})} &\lesssim \|(Q_1 f)\|_{L^\infty(D_{t,u})} \|(Q_2 v)\|_{L^\infty(D_{t,u})} \|(Q_3 w)\|_{L^2(D_{t,u})} \\ &\lesssim \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \end{aligned}$$

To obtain (44) we estimate using definition (39) for $[]$,

$$\begin{aligned} \|[f] \left((Q_1 v)(Q_2 w) \right)\|_{L^2(D_{t,u})} &\lesssim \|f\|_{L^\infty(D_{t,u})} \|(Q_1 v)(Q_2 w)\|_{L^2(D_{t,u})} \\ &\lesssim \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \end{aligned}$$

The alternative estimate in (39) for $[f]$ similarly leads to (45).

We also derive

$$\begin{aligned} \|[f][v](Qw)\|_{L^2(D_{t,u})} &\lesssim \|f\|_{L^\infty(D_{t,u})} \|[v](Qw)\|_{L^2(D_{t,u})} \\ &\lesssim \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \end{aligned}$$

as claimed in (46). The estimate (47) once again is obtained by using the alternative term in (39) in the estimates for $[f]$ and $[v]$.

Finally, if $\|f\|_{L^\infty(D_{t,u})} \lesssim \|g\|_{L^\infty(D_{t,u})}$, then,

$$\begin{aligned} \|\pi(f, v; w)\|_{L^2(D_{t,u})} &\lesssim \|f\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \\ &\lesssim \|g\|_{L^\infty(D_{t,u})} \|v\|_{L^\infty(D_{t,u})} \|w\|_{L^2(D_{t,u})} \end{aligned}$$

Thus according to definition 2.11, $\pi(f, v; w) \lesssim \pi(g, v; w)$ as desired in (48). \blacksquare

3. WAVE COORDINATE CONDITION

In what follows we shall rely crucially on the fact that our standard coordinates x^α , $\alpha = 0, \dots, 3$ satisfy the wave coordinate condition (2) relative to the metric G . Recall that the wave coordinate condition has the form:

$$\begin{aligned} 0 &= \frac{1}{\sqrt{|G|}} \partial_\alpha (G^{\alpha\beta} \sqrt{|G|}) \\ &= \partial_\alpha G^{\alpha\beta} + \frac{1}{2} G^{\alpha\beta} G^{\gamma\delta} \partial_\alpha G_{\gamma\delta} \end{aligned}$$

or, in view of $\partial(G^{\alpha\beta} G_{\beta\sigma}) = 0$,

$$G^{\alpha\beta} \partial_\alpha G_{\beta\sigma} = \frac{1}{2} G^{\alpha\beta} \partial_\sigma G_{\alpha\beta}. \quad (49)$$

Next we shall review some basic notation connected to our standard null frame $L = e_4$, $\underline{L} = e_3$, e_A , $A = 1, 2$. When L, \underline{L} are applied to scalar quantities we also use the notation $L = \partial_4$, $\underline{L} = \partial_3$. Recall that the null components of the metric H are given by,

$$H_{34} = -2, \quad H_{33} = H_{44} = H_{3A} = H_{4A} = 0, \quad H_{AB} = \delta_{AB}.$$

The null components of the inverse metric are therefore,

$$H^{34} = -\frac{1}{2}, \quad H^{33} = H^{44} = H^{3A} = H^{4A} = 0, \quad H^{AB} = \delta^{AB}.$$

Given a vectorfield $X = X^\alpha \partial_\alpha$ we decompose relative to the null frame as follows:

$$\begin{aligned} X &= -\frac{1}{2} \langle L, X \rangle \underline{L} - \frac{1}{2} \langle \underline{L}, X \rangle L + \langle e_A, X \rangle e_A \\ &= -\frac{1}{2} X_4 \underline{L} - \frac{1}{2} X_3 L + X_A e_A \end{aligned} \quad (50)$$

or, using upper indices,

$$X = X^3 \underline{L} + X^4 L + X^A e_A \quad (51)$$

where

$$X^3 = -\frac{1}{2} X_4, \quad X^4 = -\frac{1}{2} X_3, \quad X^A = X_A.$$

In view of this we shall use the following notation,

Definition 3.1. For an arbitrary spacetime tensor $M^{\alpha\beta}$,

$$\begin{aligned} M^{3\beta} &:= -\frac{1}{2}M^{\alpha\beta}L_\alpha = -\frac{1}{2}M^{\alpha\beta}H_{\alpha\gamma}L^\gamma = -\frac{1}{2}M^{\alpha\beta}H_{\alpha 4}, \\ M^{4\beta} &:= -\frac{1}{2}M^{\alpha\beta}\underline{L}_\alpha = -\frac{1}{2}M^{\alpha\beta}H_{\alpha\gamma}\underline{L}^\gamma = -\frac{1}{2}M^{\alpha\beta}H_{\alpha 3}, \\ M^{A\beta} &:= M^{\alpha\beta}e_{A\alpha} = M^{\alpha\beta}H_{\alpha\gamma}e_A^\gamma \end{aligned}$$

In particular

$$H^{3\alpha} = H^{34}L^\alpha = -\frac{1}{2}L^\alpha.$$

Definition 3.2. Given a scalar function f we shall denote by D_*f any function for which we have an estimate of the form

$$|D_*f| \lesssim |L(f)| + \left(\sum_{A=1,2} |e_A(f)|^2 \right)^{\frac{1}{2}} = |L(f)| + |\nabla f|$$

Given a tensorfield U with components U_α^β relative to our standard coordinates x^α we denote by D_*U a scalar quantity which can be estimated by,

$$|D_*U| \lesssim \sum_{\alpha,\beta} |D_*U_\alpha^\beta|.$$

Given two tensors U, V we denote by UD_*V a scalar quantity which can be estimated by

$$|UD_*V| \lesssim |U||D_*V|.$$

For example, consider the coordinate vectorfield ∂_α and decompose it relative to the null frame L, \underline{L}, e_A according to (50). We shall write the decomposition formula in the form,

$$\partial_\alpha = -\frac{1}{2}L_\alpha \underline{L} + D_*. \quad (52)$$

Using the above notation we are now ready to state the main result of this section.

Lemma 3.3. *The following identities⁴ are consequences of the wave coordinate condition (49):*

$$2H^{3\alpha}\partial_3(QG)_{\alpha\sigma} = H^{\alpha\beta}\partial_\sigma(QG)_{\alpha\beta} + GD_*(QG) + Err \quad (53)$$

$$L^\mu L^\nu \partial_3(QG)_{\mu\nu} = G \cdot D_*(QG) + Err, \quad (54)$$

$$L^\alpha e_A^\sigma \partial_3(QG)_{\alpha\sigma} = G \cdot D_*(QG) + Err, \quad (55)$$

$$Err = h\partial(QG) + \frac{1}{|Q|}[\partial G]\partial G \quad (56)$$

In particular,

$$L^\alpha L^\sigma \partial_3 H_{\alpha\sigma} = G \cdot D_*H + Err \quad (57)$$

⁴They are in fact approximate identities. The terms on the right hand side are schematically. What we mean is that the terms on the left can be estimated by the quantities appearing on the right.

We also have,

$$\begin{aligned} L^\sigma L^\beta \partial_\alpha \partial_\gamma (QG_{\beta\sigma}) &= H \cdot D_* \partial(QG) + Err \\ Err &= \frac{1}{|Q|} \left([\partial G] \cdot \partial^2 G + [\partial^2 G] \partial G \right) + h \cdot \partial^2(QG) + \partial G \cdot \partial(QG) \end{aligned} \quad (58)$$

Proof We start by projecting (49):

$$Q(G^{\alpha\beta} \partial_\alpha G_{\beta\sigma}) = \frac{1}{2} Q(G^{\alpha\beta} \partial_\sigma G_{\alpha\beta}).$$

In view of the fact that $Q(u \cdot v) = u \cdot Qv + \frac{1}{|Q|} [\nabla u]v$ we derive

$$G^{\alpha\beta} \partial_\alpha (QG_{\beta\sigma}) = \frac{1}{2} G^{\alpha\beta} \partial_\sigma (QG_{\alpha\beta}) + \frac{1}{|Q|} [\partial G] \partial G \quad (59)$$

Expanding (59) relative to the null frame, we have

$$G^{3\beta} \partial_3 (QG_{\beta\sigma}) + G^{4\beta} \partial_4 (QG_{\beta\sigma}) + G^{A\beta} \partial_A (QG_{\beta\sigma}) = \frac{1}{2} G^{\gamma\delta} \partial_\sigma (QG_{\gamma\delta}) + \frac{1}{|Q|} [\partial G] \partial G,$$

whence, for any σ ,

$$G^{3\beta} \partial_3 (QG_{\beta\sigma}) = \frac{1}{2} G^{\gamma\delta} \partial_\sigma (QG_{\gamma\delta}) + GD_*(QG) + \frac{1}{|Q|} [\partial G] \partial G$$

Writing $G^{3\beta} = H^{3\beta} - h^{3\beta} + O(h^2)$ we derive,

$$2H^{3\alpha} \partial_3 (QG_{\alpha\sigma}) = H^{\alpha\beta} \partial_\sigma (QG_{\alpha\beta}) + GD_*(QG) + Err \quad (60)$$

$$Err = h \partial(QG) + h^2 \partial(QG) + \frac{1}{|Q|} [\partial G] \partial G \quad (61)$$

Remark 3.4. Since $\|h\|_{L^\infty} \lesssim 1$, the error term $h^2 \partial(QG)$ can be treated in the same way as $h \partial(QG)$ and we shall ignore it. In what follows we shall often drop terms like this without further mentioning.

We thus derive the desired approximate identity (53).

Contracting (60) with L^σ we obtain,

$$2L^\sigma H^{3\alpha} \partial_3 (QG_{\alpha\sigma}) = GD_*(QG) + HD_*(QG) + Err$$

As HD_*G can be estimated exactly in the same way as the more difficult term GD_*G we shall drop it. We shall later absorb similar terms into related, more difficult terms, without further mentioning.

We now recall that $H^{3\alpha} = -\frac{1}{2} L^\alpha$. Henceforth,

$$-L^\nu L^\mu \partial_3 (QG_{\mu\nu}) = GD_*(QG) + Err,$$

which gives (54).

We can also contract (60) with e_A^σ to obtain

$$e_A^\sigma H^{3\alpha} \partial_3 (QG_{\alpha\sigma}) = GD_*(QG) + Err.$$

Using again the relation $H^{3\alpha} = -\frac{1}{2}L^\alpha$, (55) immediately follows. We shall now prove (58). Differentiating (49), we find,

$$G^{\alpha\beta}\partial_\gamma\partial_\alpha G_{\beta\sigma} = \frac{1}{2}G^{\alpha\beta}\partial_\gamma\partial_\sigma G_{\alpha\beta} + \partial G \cdot \partial G. \quad (62)$$

We manipulate the left hand side of (62) schematically as follows:

$$\begin{aligned} Q(G \cdot \partial^2 G) &= H \cdot \partial^2(QG) + Q(G \cdot \partial^2 G) - H \cdot \partial^2(QG) \\ &= H \cdot \partial^2(QG) + Q(G \cdot \partial^2 G) - G \cdot \partial^2(QG) + h \cdot \partial^2(QG) \quad (\text{as } G = H - h) \\ &= H \cdot \partial^2(QG) + [Q, G]\partial^2 G + h \cdot \partial^2(QG) \\ &= H \cdot \partial^2(QG) + \frac{1}{|Q|}[\partial G]\partial^2 G + h \cdot \partial^2(QG) \end{aligned}$$

Therefore, proceeding in the same way on the right hand side of (62),

$$\begin{aligned} H^{\alpha\beta}\partial_\gamma\partial_\alpha(QG_{\beta\sigma}) &= \frac{1}{2}Q\left(G^{\alpha\beta}\partial_\gamma\partial_\sigma G_{\alpha\beta}\right) + \text{Err} \\ &= \frac{1}{2}H^{\alpha\beta}\partial_\gamma\partial_\sigma(QG_{\alpha\beta}) + \text{Err} \quad (63) \\ \text{Err} &= \frac{1}{|Q|}[\partial G] \cdot \partial^2 G + h \cdot \partial^2(QG) + Q(\partial G \cdot \partial G) \\ &= \frac{1}{|Q|}[\partial G] \cdot \partial^2 G + h \cdot \partial^2(QG) + \frac{1}{|Q|}[\partial^2 G]\partial G + \partial G \cdot \partial(QG) \end{aligned}$$

We now contract (63) with L^σ .

$$\begin{aligned} L^\sigma H^{\alpha\beta}\partial_\gamma\partial_\alpha(QG_{\beta\sigma}) &= \frac{1}{2}L^\sigma H^{\alpha\beta}\partial_\sigma\partial_\gamma(QG_{\alpha\beta}) + \text{Err} \\ &= H \cdot D_*\partial(QG) + \text{Err} \end{aligned}$$

Therefore, expressing $H^{\alpha\beta}\partial_\alpha$ relative to the null frame L, \underline{L}, e_A ,

$$L^\sigma L^\beta\partial_\alpha\partial_\gamma(QG_{\beta\sigma}) = H \cdot D_*\partial(QG) + \text{Err}$$

■

4. FIRST REDUCTION

In this section we show how to reduce the statement of Theorem 1.2 to the following:

$$\int_{u+1}^t \|\mathbf{R}_{44}(H)\|_{L^2(D_{\tau,u})} d\tau \lesssim \lambda^{-1} \quad (64)$$

where, see definition 2.3, $D_{\tau,u} = \cup_{u \leq u' \leq u+1} S_{\tau,u'}$ is the annulus on Σ_τ of thickness 1 and outer boundary $S_{\tau,u}$. Throughout this and the remaining sections we denote $\mathbf{R}_{44} = \mathbf{R}_{44}(H)$ and $\mathbf{Ric} = \mathbf{Ric}(H)$.

Step 1 Take care of $\int_u^{u+1} \|\nabla \mathbf{R}_{44}\|_{L^2(S_{\tau,u})} d\tau$.

We start with formula

$$\nabla \mathbf{R}_{44} = \nabla L^\mu L^\nu \mathbf{R}_{\mu\nu} = 2(\nabla L) \cdot L \cdot \mathbf{Ric} + L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}. \quad (65)$$

Recall that, see (23),

$$\nabla L \lesssim r^{-1} + \Theta$$

with Θ verifying the estimates (24)–(25). Clearly, $\|(\Theta + r^{-1})\|_{L^2(S_{\tau,u})} \lesssim 1$. Also observe that, in view of (26), $r \leq 1$ as τ varies between u and $u+1$. We infer that

$$\begin{aligned} \int_u^{u+1} \|\nabla \mathbf{R}_{44}\|_{L^2(S_{\tau,u})} d\tau &\lesssim \int_u^{u+1} \left(\|(\Theta + r^{-1})\|_{L^2(S_{\tau,u})} \|\mathbf{Ric}\|_{L^\infty} + r \|\nabla \mathbf{Ric}\|_{L^\infty} \right) d\tau \\ &\lesssim \|\mathbf{Ric}\|_{L_t^1 L_x^\infty} + \|\nabla \mathbf{Ric}\|_{L_t^1 L_x^\infty}. \end{aligned}$$

It remains to observe that the frequencies of $\mathbf{Ric}(H)$ are essentially ≤ 2 and therefore, $\|\nabla \mathbf{Ric}\|_{L_x^\infty} \lesssim \|\mathbf{Ric}\|_{L_x^\infty}$. Henceforth, in view of the background estimate (20),

$$\int_u^{u+1} \|\nabla \mathbf{R}_{44}\|_{L^2(S_{\tau,u})} d\tau \lesssim \|\mathbf{Ric}\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-1-4\epsilon_0} \quad (66)$$

Step 2 Take care of $\int_{u+1}^t \|\nabla \mathbf{R}_{44}\|_{L^2(S_{\tau,u})} d\tau$.

We start as in Step 1 with formula (65). To estimate the first term on the right hand side of (65) we use $\|\nabla L\|_{L^2(S_{\tau,u})} \lesssim 1$, see (28).

Therefore, using also (20)

$$\int_{u+1}^t \|\nabla L \cdot L \mathbf{Ric}\|_{L^2(S_{\tau,u})} d\tau \lesssim \sup_{u+1 \leq \tau \leq t} \|\nabla L\|_{L^2(S_{\tau,u})} \|\mathbf{Ric}\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-1-4\epsilon_0}$$

as desired. In other words,

$$\int_{u+1}^t \|\nabla \mathbf{R}_{44}\|_{L^2(S_{\tau,u})} d\tau \lesssim \lambda^{-1-4\epsilon_0} + \int_{u+1}^t \|L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(S_{\tau,u})} d\tau \quad (67)$$

It remains to estimate the second term in (67). Using the simple estimate:

$$\|f\|_{L^2(S_{\tau,u})}^2 \lesssim \|\nabla f\|_{L^2(D_{\tau,u})} \|f\|_{L^2(D_{\tau,u})}$$

where $D_{\tau,u} = \cup_{u \leq u' \leq u+1} S_{\tau,u'}$ is the annulus on Σ_τ of thickness 1 and outer boundary $S_{\tau,u}$.

$$\begin{aligned} \|L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(S_{\tau,u})} &\lesssim \|\nabla L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})}^{\frac{1}{2}} \|L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})}^{\frac{1}{2}} \\ &\lesssim \|\nabla L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} + \|L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \end{aligned}$$

Now, using $\|\nabla L\|_{L^2(D_{\tau,u})} \lesssim 1$,

$$\begin{aligned} \|\nabla L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} &\lesssim \|\nabla L \cdot \nabla \mathbf{Ric}\|_{L^2(D_{\tau,u})} + \|L^\mu L^\nu \nabla^2 \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \\ &\lesssim \|\nabla L\|_{L^2(D_{\tau,u})} \|\nabla \mathbf{Ric}\|_{L_x^\infty} + \|L^\mu L^\nu \nabla^2 \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \\ &\lesssim \|\nabla \mathbf{Ric}\|_{L_x^\infty} + \|L^\mu L^\nu \nabla^2 \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \end{aligned} \quad (68)$$

Now, since $\mathbf{R}_{\mu\nu} \approx \bar{P} \mathbf{R}_{\mu\nu}(H)$, and $\|\nabla^m \bar{P} f\|_{L^2(D_{\tau,u})} \lesssim \|f\|_{L^2(D_{\tau,u})}$ with perhaps a slightly larger annulus $D_{\tau,u}$,

$$\begin{aligned} \|L^\mu L^\nu \nabla^2 \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} &\lesssim \|\nabla^2 \bar{P} L^\mu L^\nu \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} + \|[L^\mu L^\nu, \nabla^2 \bar{P}] \mathbf{Ric}\|_{L^2(D_{\tau,u})} \\ &\lesssim \|L^\mu L^\nu \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} + \|[L^\mu L^\nu, \nabla^2 \bar{P}] \mathbf{Ric}\|_{L^2(D_{\tau,u})} \end{aligned}$$

To treat the second term we shall use the following commutation lemma, see lemma 2.8,

$$\|[f, \nabla^k \bar{P}]g\|_{L^2(D_{\tau,u})} \lesssim \|\nabla f\|_{L^2(D_{\tau,u})} \|g\|_{L_x^\infty} \quad (69)$$

with a possible larger annulus $D_{\tau,u}$ on the right hand side.

Therefore,

$$\|[L^\mu L^\nu, \nabla^2 \bar{P}] \mathbf{Ric}\|_{L^2(D_{\tau,u})} \lesssim \|\nabla L\|_{L^2(D_{\tau,u})} \|\mathbf{Ric}\|_{L_x^\infty} \lesssim \|\mathbf{Ric}\|_{L_x^\infty}$$

Therefore, back to (68),

$$\|\nabla L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \lesssim \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} + \|\mathbf{Ric}\|_{L_x^\infty} + \|\nabla \mathbf{Ric}\|_{L_x^\infty}$$

or, since $\mathbf{Ric} = \mathbf{Ric}(H) \approx \bar{P} \mathbf{Ric}$,

$$\|\nabla L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \lesssim \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} + \|\mathbf{Ric}\|_{L_x^\infty} \quad (70)$$

Also, clearly,

$$\|L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(D_{\tau,u})} \lesssim \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} + \|\mathbf{Ric}\|_{L_x^\infty}.$$

Therefore,

$$\|L^\mu L^\nu \nabla \mathbf{R}_{\mu\nu}\|_{L^2(S_{\tau,u})} \lesssim \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} + \|\mathbf{Ric}\|_{L_x^\infty} \quad (71)$$

whence,

$$\begin{aligned} \int_{u+1}^t \|\nabla \mathbf{R}_{44}\|_{L^2(S_{\tau,u})} d\tau &\lesssim \int_{u+1}^t \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} d\tau + \|\mathbf{Ric}\|_{L_t^1 L_x^\infty} + \lambda^{-1-4\epsilon_0} \\ &\lesssim \int_{u+1}^t \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} d\tau + \lambda^{-1-4\epsilon_0} \end{aligned}$$

Combining this with (66) we obtain,

$$\int_u^t \|\nabla \mathbf{R}_{44}(H)\|_{L^2(S_{\tau,u})} d\tau \lesssim \int_{u+1}^t \|\mathbf{R}_{44}\|_{L^2(D_{\tau,u})} d\tau + \lambda^{-1-4\epsilon_0} \quad (72)$$

as desired.

5. THE ALGEBRAIC STRUCTURE OF $\mathbf{R}_{\mu\nu}(H)$

We start with the formula,

$$\mathbf{R}_{\mu\nu}(H) = \mathbf{R}_{\mu\nu}(H) - P \mathbf{R}_{\mu\nu}(G) \quad (73)$$

Recall the expression of the Ricci tensor relative to local coordinates:

$$\begin{aligned}
\mathbf{R}_{\mu\nu}(H) &= \mathbf{R}_{\mu\nu}^{(1)}(H) + \mathbf{R}_{\mu\nu}^{(2)}(H) \\
\mathbf{R}_{\mu\nu}^{(1)}(H) &= \frac{1}{2}H^{\alpha\beta} \left(H_{\alpha\nu, \beta\mu} + H_{\beta\mu, \alpha\nu} - H_{\alpha\beta, \mu\nu} - H_{\mu\nu, \alpha\beta} \right) \\
&= \frac{1}{2}H^{\alpha\beta} H_{[\alpha\beta\mu\nu]} = \frac{1}{2}H^{\alpha\beta} (H_{[[\alpha\beta\mu\nu]]} - H_{\mu\nu, \alpha\beta}) \\
\mathbf{R}_{\mu\nu}^{(2)}(H) &= H^{\alpha\beta} H_{\gamma\delta} \left(\Gamma_{\mu\beta}^{\gamma}(H) \Gamma_{\alpha\nu}^{\delta}(H) - \Gamma_{\mu\nu}^{\gamma}(H) \Gamma_{\alpha\beta}^{\delta}(H) \right)
\end{aligned} \tag{74}$$

where

$$\Gamma_{\alpha\beta}^{\gamma}(H) = \frac{1}{2}H^{\gamma\sigma} \left(H_{\sigma\beta, \alpha} + H_{\alpha\sigma, \beta} - H_{\alpha\beta, \sigma} \right).$$

To calculate $\mathbf{R}_{\mu\nu}(H) - P \mathbf{R}_{\mu\nu}(G)$ we use (31) and (32),

$$\begin{aligned}
G_{\alpha\beta} &= H_{\alpha\beta} + h_{\alpha\beta} \\
G^{\alpha\beta} &= H^{\alpha\beta} - h^{\alpha\beta} + [h \cdot h].
\end{aligned}$$

Therefore, using the notation in (74) and the fact that $H = P G$, we find,

$$\begin{aligned}
\mathbf{R}_{\mu\nu}^{(1)}(H) - P \mathbf{R}_{\mu\nu}^{(1)}(G) &= \frac{1}{2} \left(H^{\alpha\beta} H_{[\alpha\beta\mu\nu]} - P (G^{\alpha\beta} G_{[\alpha\beta\mu\nu]}) \right) \\
&= \frac{1}{2} \left(h^{\alpha\beta} H_{[\alpha\beta\mu\nu]} + G^{\alpha\beta} H_{[\alpha\beta\mu\nu]} - P (G^{\alpha\beta} G_{[\alpha\beta\mu\nu]}) \right) + h^2 \partial^2 H \\
&= \frac{1}{2} \left(h^{\alpha\beta} H_{[\alpha\beta\mu\nu]} + G^{\alpha\beta} P G_{[\alpha\beta\mu\nu]} - P (G^{\alpha\beta} G_{[\alpha\beta\mu\nu]}) \right) + h^2 \partial^2 H \\
&= \frac{1}{2} \left(h^{\alpha\beta} H_{[\alpha\beta\mu\nu]} + [G^{\alpha\beta}, P] G_{[\alpha\beta\mu\nu]} \right) + \pi(h, h; \partial^2 H)
\end{aligned}$$

For convenience we shall introduce the following notation,

Definition 5.1. Given two scalar functions v, w we define

$$\{v, w\}' = [v, P] \cdot w.$$

Therefore,

$$\mathbf{R}_{\mu\nu}^{(1)}(H) - P \mathbf{R}_{\mu\nu}^{(1)}(G) = \frac{1}{2} \left(h^{\alpha\beta} H_{[\alpha\beta\mu\nu]} + \{G^{\alpha\beta}, G_{[\alpha\beta\mu\nu]}\}' \right) + \pi(h, h; \partial^2 H) \tag{75}$$

Remark 5.2. Observe that,

$$\begin{aligned}
[P, v](I - \bar{P})w &= P(v(I - \bar{P})w) - vP(I - \bar{P})w \\
&= \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} P(v^{\lambda_1} w^{\lambda_2}) \\
&= \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} [P, v^{\lambda_1}] w^{\lambda_2}
\end{aligned} \tag{76}$$

Thus, writing $w = \bar{P}w + (I - \bar{P})w$,

$$\{v, w\}' = [v, P]\bar{P}w + \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} [v^{\lambda_1}, P] w^{\lambda_2} \tag{77}$$

To compute the contribution to (73) of the quadratic terms $\mathbf{R}_{\mu\nu}^{(2)}(H)$ we start $\Gamma_{\alpha\beta}^\gamma(H)$ which we write in the form

$$\Gamma_{\alpha\beta}^\gamma(H) = \frac{1}{2}H^{\gamma\sigma} \left(P G_{\sigma\beta, \alpha} + P G_{\alpha\sigma, \beta} - P G_{\alpha\beta, \sigma} \right).$$

Now commuting P with H , and using (32),

$$\Gamma_{\alpha\beta}^\gamma(H) = P \Gamma_{\alpha\beta}^\gamma(G) + [\partial H] \partial G + [h \partial G]$$

Therefore, using that h and ∂H are bounded and the definition of the error term π , we infer that

$$\begin{aligned} \mathbf{R}_{\mu\nu}^{(2)}(H) &= H^{\alpha\beta} H_{\gamma\delta} \left((P \Gamma_{\mu\beta}^\gamma(G)) (P \Gamma_{\alpha\nu}^\delta(G)) - (P \Gamma_{\mu\nu}^\gamma(G)) (P \Gamma_{\alpha\beta}^\delta(G)) \right) \\ &\quad + \pi(\partial H, \partial H; \partial G) + \pi(h, \partial G; \partial G) \end{aligned} \quad (78)$$

On the other hand, using first the formulae (31), (32) and then commuting P with H ,

$$\begin{aligned} P \mathbf{R}_{\mu\nu}^{(2)}(G) &= P \left(G^{\alpha\beta} G_{\gamma\delta} \left(\Gamma_{\mu\beta}^\gamma(G) \Gamma_{\alpha\nu}^\delta(G) - \Gamma_{\mu\nu}^\gamma(G) \Gamma_{\alpha\beta}^\delta(G) \right) \right) \\ &= H^{\alpha\beta} H_{\gamma\delta} P \left(\Gamma_{\mu\beta}^\gamma(G) \Gamma_{\alpha\nu}^\delta(G) - \Gamma_{\mu\nu}^\gamma(G) \Gamma_{\alpha\beta}^\delta(G) \right) \\ &\quad + \pi(\partial H, \partial G, \partial G) + \pi(h, \partial G, \partial G) \end{aligned} \quad (79)$$

Thus, combining (78) with (79),

$$\begin{aligned} \mathbf{R}_{\mu\nu}^{(2)}(H) - P \mathbf{R}_{\mu\nu}^{(2)}(G) &= -H^{\alpha\beta} H_{\gamma\delta} \left(P \left(\Gamma_{\mu\beta}^\gamma(G) \Gamma_{\alpha\nu}^\delta(G) \right) - (P \Gamma_{\mu\beta}^\gamma(G)) (P \Gamma_{\alpha\nu}^\delta(G)) \right) \\ &\quad - P \left(\Gamma_{\mu\nu}^\gamma(G) \Gamma_{\alpha\beta}^\delta(G) \right) + (P \Gamma_{\mu\nu}^\gamma(G)) (P \Gamma_{\alpha\beta}^\delta(G)) \\ &\quad + \pi(\partial H, \partial H, \partial G) + \pi(h, \partial G, \partial G) \end{aligned} \quad (80)$$

To simplify the expression above we introduce the following,

Definition 5.3. Given two functions v and w we introduce their *modified⁵ paradifferential product* $\{v, w\}$.

$$\{v, w\} := P(v \cdot w) - P v \cdot P w \quad (81)$$

Remark 5.4. Observe that,

$$\begin{aligned} \{v, w\} &= P \sum_{\frac{1}{2} < \lambda_1 \leq 4} v^{\lambda_1} P_{\leq \frac{1}{2}} w + P \sum_{\lambda_1 > \frac{1}{2}, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} v^{\lambda_1} w^{\lambda_2} \\ &\quad + P \sum_{\frac{1}{2} < \lambda_2 \leq 4} P_{\leq \frac{1}{2}} v w^{\lambda_2} + P \sum_{\lambda_2 > \frac{1}{2}, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} v^{\lambda_1} w^{\lambda_2} \\ &\quad - \sum_{\frac{1}{2} < \lambda_1 \leq 1} v^{\lambda_1} P_{\leq \frac{1}{2}} w - \sum_{\frac{1}{2} < \lambda_2 \leq 1} P_{\leq \frac{1}{2}} v w^{\lambda_2} \\ &\quad - \sum_{\frac{1}{2} < \lambda_1, \lambda_2 \leq 1} v^{\lambda_1} w^{\lambda_2} \end{aligned} \quad (82)$$

⁵It differs from the standard paradifferential product. In our definition we have removed the low-low interactions.

With this definition we can write

$$\begin{aligned} \mathbf{R}_{\mu\nu}^{(2)}(H) - P \mathbf{R}_{\mu\nu}^{(2)}(G) &= -H^{\alpha\beta} H_{\gamma\delta} \left(\{ \Gamma_{\mu\beta}^\gamma(G), \Gamma_{\alpha\nu}^\delta(G) \} \right. \\ &\quad \left. - \{ \Gamma_{\mu\nu}^\gamma(G), \Gamma_{\alpha\beta}^\delta(G) \} \right) + \text{Err} \end{aligned} \quad (83)$$

with the error term of the form

$$\text{Err} = \pi(\partial H, \partial H, \partial G) + \pi(h, \partial G, \partial G)$$

Thus, taking into account (75) and (83), we rewrite (73) in the form,

$$\mathbf{R}_{\mu\nu}(H) = I_{\mu\nu} + II_{\mu\nu} + III_{\mu\nu} + \text{Err} \quad (84)$$

$$I_{\mu\nu} = \frac{1}{2} h^{\alpha\beta} H_{[\alpha\beta\mu\nu]} \quad (85)$$

$$II_{\mu\nu} = [G^{\alpha\beta}, P] G_{[\alpha\beta\mu\nu]} \quad (86)$$

$$III_{\mu\nu} = -H^{\alpha\beta} H_{\gamma\delta} \left(\{ \Gamma_{\mu\beta}^\gamma, \Gamma_{\alpha\nu}^\delta \} - \{ \Gamma_{\mu\nu}^\gamma, \Gamma_{\alpha\beta}^\delta \} \right) \quad (87)$$

$$\text{Err} = \pi(\partial H, \partial H, \partial G) + \pi(h, \partial G, \partial G) + \pi(h, h; \partial^2 H) \quad (88)$$

Remark 5.5. Recalling the definition of π and using the fact that the frequency range of h is included in $|\xi| \geq 1$ we have

$$\begin{aligned} \int_{u+1}^t \|\pi(h, \partial G, \partial G)\|_{L^2(D_{\tau,u})} &\lesssim \|h\|_{L_t^2 L_\infty^\infty} \cdot \|\partial G\|_{L_t^2 L_\infty^\infty} \cdot \sup_\tau \|\partial G\|_{L^2(D_{\tau,u})} \\ &\lesssim \|\partial h\|_{L_t^2 L_\infty^\infty} \cdot \|\partial G\|_{L_t^2 L_\infty^\infty} \cdot \sup_\tau \|\partial G\|_{L^2(D_{\tau,u})} \\ &\lesssim \|\partial G\|_{L_t^2 L_\infty^\infty} \cdot \|\partial G\|_{L_t^2 L_\infty^\infty} \cdot \sup_\tau \|\partial G\|_{L^2(D_{\tau,u})} \end{aligned}$$

We can thus replace $\pi(h, \partial G, \partial G)$ by $\pi(\partial G, \partial G; \partial G)$. By a similar argument, taking into account the frequency support of H , we can also replace $\pi(h, h; \partial^2 H)$ by $\pi(\partial G, \partial G; \partial G)$. Finally, by a trivial argument, we can also replace $\pi(\partial H, \partial H, \partial G)$ by $\pi(\partial G, \partial G; \partial G)$. Therefore the error term in (88) can be simplified to

$$\text{Err} = \pi(\partial G, \partial G; \partial G).$$

6. THE STRUCTURE OF I_{44} AND II_{44}

6.1. Structure of I_{44} . Contracting the formula (85) with $L^\mu L^\nu$ we obtain, see also the definition of $[[\]]$ in (74),

$$I_{44} = \frac{1}{2} L^\mu L^\nu \left(h^{\alpha\beta} H_{[[\alpha\beta\mu\nu]]} - h^{\alpha\beta} H_{\mu\nu, \alpha\beta} \right) \quad (89)$$

Observe that

$$L^\mu L^\nu H_{[[\alpha\beta\mu\nu]]} = L^\mu L^\nu \left(H_{\alpha\nu, \beta\mu} + H_{\beta\mu, \alpha\nu} - H_{\alpha\beta, \mu\nu} \right) \approx L(\partial H)$$

It remains to consider the term $L^\mu L^\nu h^{\alpha\beta} H_{\mu\nu, \alpha\beta}$. Observe that,

$$L^\mu L^\nu \partial_\alpha \partial_\beta H_{\mu\nu} = D_* \partial H + [\partial G \partial G] + [\partial G][\partial G] + [\nabla L] \partial G + [\nabla L \partial G].$$

This is obvious if $\alpha = 1, 2, 4$ and follows from (58) of Lemma 3.3 if $\alpha = 3$. Therefore,

$$L^\mu L^\nu h^{\alpha\beta} H_{\mu\nu, \alpha\beta} = h D_* \partial H + \pi(h, \partial G; \partial G) + \pi(h, \partial G, \nabla L)$$

Appealing to remark 5.5 we can summarize our results above in the following

Proposition 6.2. *We can write,*

$$I_{44} = h D_* \partial H + Err,$$

where

$$Err = \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \nabla L)$$

6.3. Structure of II_{44} . Recall that

$$II_{\mu\nu} = \{G^{\alpha\beta}, G_{[\alpha\beta\mu\nu]}\}' = [G^{\alpha\beta}, P] G_{[\alpha\beta\mu\nu]}$$

For technical reasons we also introduce the following,

Definition 6.4. Given scalar functions f, v, w we define,

$$\{v, f \circ w\}' = [v, P] f \bar{P} w + \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} [v^{\lambda_1}, P] f w^{\lambda_2} \quad (90)$$

Lemma 6.5.

$$f\{v, w\}' = \{v, f \circ w\}' + \pi(\nabla f, \nabla v; w) \quad (91)$$

Proof Using representation (77) for $\{, \}'$, the commutation lemma 2.8 and the definition of π (see definition 2.10) we infer that

$$\begin{aligned} f\{v, w\}' &= [v, P] f \bar{P} w + \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} [v^{\lambda_1}, P] f w^{\lambda_2} \\ &+ [f, [v, P]] \bar{P} w + \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} [f, [v^{\lambda_1}, P]] w^{\lambda_2} \quad (92) \\ &= \{v, f \circ w\}' + \pi(\nabla f, \nabla v; w) \end{aligned}$$

■

We also define,

$$\begin{aligned} \{v, L \circ w\}' &:= \{v, L^\mu \circ \partial_\mu w\}' \\ \{v, e_A \circ w\}' &:= \{v, e_A^\mu \circ \partial_\mu w\}' \end{aligned}$$

Definition 6.6. We denote by $\{v, D_* \circ w\}'$ a scalar quantity which can be estimated as follows

$$|\{v, D_* \circ w\}'| \lesssim |\{v, L \circ w\}'| + \left(\sum_{A=1,2} |\{v, e_A \circ w\}'|^2 \right)^{\frac{1}{2}}$$

We now proceed with the estimate for II_{44} .

$$\begin{aligned} II_{44} &= L^\mu L^\nu \{G^{\alpha\beta}, G_{[\alpha\beta\mu\nu]}\}' \\ &= L^\mu L^\nu \left(\{G^{\alpha\beta}, G_{[[\alpha\beta\mu\nu]]}\}' - \{G^{\alpha\beta}, G_{\mu\nu, \alpha\beta}\}' \right) \end{aligned}$$

We start again with the term containing $G_{[[\alpha\beta\mu\nu]]}$. According to the definition 6.4 of $\{v, f \circ w\}'$ and the relation (91), we obtain

$$L^\mu L^\nu \{G^{\alpha\beta}, G_{[[\alpha\beta\mu\nu]]}\}' = \{G^{\alpha\beta}, L^\mu L^\nu \circ G_{[[\alpha\beta\mu\nu]]}\}' + \pi(\nabla L, \partial G; \partial^2 G)$$

Using also definition (6.6), we infer that

$$L^\mu L^\nu \{G^{\alpha\beta}, G_{[[\alpha\beta\mu\nu]]}\}' = \{G, D_* \circ \partial G\}' + \pi(\nabla L, \partial G; \partial^2 G)$$

It remains to consider $L^\mu L^\nu \{G^{\alpha\beta}, G_{\mu\nu, \alpha\beta}\}'$. Proceeding as above we obtain

$$L^\mu L^\nu \{G^{\alpha\beta}, G_{\mu\nu, \alpha\beta}\}' = \{G^{\alpha\beta}, L^\mu L^\nu \circ G_{\mu\nu, \alpha\beta}\}' + \pi(\nabla L, \partial G, \partial^2 G).$$

According to the wave coordinate condition (58),

$$\begin{aligned} L^\mu L^\nu \partial_\alpha \partial_\beta (QG_{\mu\nu}) &= H \cdot D_* \partial (QG) \\ &+ \frac{1}{|Q|} \left([\partial G] \cdot \partial^2 G + [\partial^2 G] \cdot \partial G \right) + h \cdot \partial^2 (QG) + \partial G \cdot \partial (QG) \end{aligned}$$

for any projection $Q = I, P, P_{\lambda_1}$ with $\lambda_1 > 1$. Therefore, in view of definition (90),

$$\{G^{\alpha\beta}, L^\mu L^\nu \circ \partial_\alpha \partial_\beta G_{\mu\nu}\}' = \{G, H \cdot D_* \partial \circ G\}' \quad (93)$$

$$+ \{G, h \circ \partial^2 G\}' + \{G, \partial G \circ \partial G\}' + E \quad (94)$$

where the error term E has the form,

$$\begin{aligned} E &= [G, P] \bar{P} \left([\partial G] \partial^2 G + [\partial^2 G] \partial G \right) \\ &+ \sum_{\lambda_1 > 1, \lambda_2 > 2, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} \frac{1}{\lambda_2} [P_{\lambda_1} G, P] P_{\lambda_2} \left([\partial G] \partial^2 G + [\partial^2 G] \partial G \right) \end{aligned}$$

Observe that the infinite sum above is controlled by the presence of the factor λ_2^{-1} and therefore E is of the form

$$E = \pi(\partial G, \partial G; \partial^2 G).$$

Observe also that the error terms in (94) can also be written in the form,

$$\{G, h \circ \partial^2 G\}' = \pi(\partial G, h; \partial^2 G)$$

$$\{G, \partial G \circ \partial G\}' = \pi(\partial G, \partial G; \partial G)$$

Finally, according to lemma 6.5 the principal term in (93)

$$\{G, H \cdot D_* \partial \circ G\}' = H \cdot \{G, D_* \partial \circ G\}' + \pi(\nabla H, \partial G, pr^2 G)$$

We summarize these calculations in the following.

Proposition 6.7. *We can write*

$$II_{44} = \{G, D_* \partial \circ G\}' + H \cdot \{G, D_* \partial \circ G\}' + Err, \quad (95)$$

where the error term

$$Err = \pi(\nabla L, \partial G; \partial^2 G) + \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \partial^2 G)$$

7. THE STRUCTURE OF III_{44}

Recall (83),

$$\begin{aligned} III_{44} &= -L^\mu L^\nu H^{\alpha\beta} H_{\gamma\delta} \left(\{ \Gamma_{\mu\beta}^\gamma(G), \Gamma_{\alpha\nu}^\delta(G) \} - \{ \Gamma_{\mu\nu}^\gamma(G), \Gamma_{\alpha\beta}^\delta(G) \} \right) \\ &= \mathcal{E}_1 - \mathcal{E}_2 \end{aligned} \quad (96)$$

with $\{ , \}$ denoting the modified paradifferential product introduced in definition 5.3.

Remark 7.1. We note here the following simple property of $\{ , \}$:

$$f\{v, w\} = \{fv, w\} + \pi(v, w; \nabla f) = \{v, fw\} + \pi(v, w; \nabla f).$$

Recalling remark 5.4 we shall now introduce the following expression closely related to $fg\{v, w\}$.

Definition 7.2. Given scalars v, w, f, g we introduce

$$\begin{aligned} \{f \circ v, g \circ w\} &= P \sum_{\frac{1}{2} < \lambda_1 \leq 4} f v^{\lambda_1} \cdot g P_{\leq \frac{1}{2}} w + P \sum_{\lambda_1 > \frac{1}{2}, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} f v^{\lambda_1} \cdot g w^{\lambda_2} \\ &\quad + P \sum_{\frac{1}{2} < \lambda_2 \leq 4} f P_{\leq \frac{1}{2}} v \cdot g w^{\lambda_2} + P \sum_{\lambda_2 > \frac{1}{2}, |\ln(\lambda_1 \lambda_2^{-1})| \leq 2} f v^{\lambda_1} \cdot g w^{\lambda_2} \\ &\quad - \sum_{\frac{1}{2} < \lambda_1 \leq 1} f v^{\lambda_1} \cdot g P_{\leq \frac{1}{2}} w - \sum_{\frac{1}{2} < \lambda_2 \leq 1} f P_{\leq \frac{1}{2}} v \cdot g w^{\lambda_2} \\ &\quad - \sum_{\frac{1}{2} < \lambda_1, \lambda_2 \leq 1} f v^{\lambda_1} \cdot g w^{\lambda_2} \end{aligned} \quad (97)$$

Lemma 7.3. *We have,*

$$f\{v, w\} = \{f \circ v, w\} + \pi(v, w; \nabla f) = \{v, f \circ w\} + \pi(v, w; \nabla f) \quad (98)$$

Proof ■

We also define,

$$\begin{aligned} \{L \circ v, w\} &:= \{L^\mu \circ \partial_\mu v, w\} \\ \{e_A \circ v, w\} &:= \{e_A^\mu \circ \partial_\mu v, w\} \end{aligned}$$

In view of the lemma we have

$$L^\mu \{ \partial_\mu v, w \} = \{L \circ v, w\} + \pi(\partial v, w; \nabla L) \quad (99)$$

Definition 7.4. We denote by $\{D_* \circ v, w\}$ a scalar quantity which can be estimated as follows

$$|\{D_* \circ v, w\}| \lesssim |\{L \circ v, w\}| + \left(\sum_{A=1,2} |\{e_A \circ v, w\}|^2 \right)^{\frac{1}{2}}$$

In the calculation below we shall use the notation $\Gamma_{\alpha\beta}^\gamma = G^{\gamma\sigma}\Gamma_{\sigma|\alpha\beta}$ where,

$$\Gamma_{\sigma|\alpha\beta} = \frac{1}{2}(G_{\alpha\sigma,\beta} + G_{\beta\sigma,\alpha} - G_{\alpha\beta,\sigma}).$$

The term $\mathcal{E}_1 = -H^{\alpha\beta}H_{\gamma\delta}L^\mu L^\nu \left\{ \Gamma_{\mu\beta}^\gamma(G), \Gamma_{\alpha\nu}^\delta(G) \right\}$:

Using remark 7.1 then expressing $G^{\alpha\beta} = H^{\alpha\beta} - h^{\alpha\beta} + O(h^2)$ and applying the definition of π we derive,

$$\begin{aligned} -\mathcal{E}_1 &:= H^{\alpha\beta}H_{\gamma\delta}L^\mu L^\nu \left\{ \Gamma_{\mu\beta}^\gamma(G), \Gamma_{\alpha\nu}^\delta(G) \right\} & (100) \\ &= H^{\alpha\beta}H_{\gamma\delta}L^\mu L^\nu \left\{ G^{\gamma\rho}\Gamma_{\rho|\mu\beta}, G^{\delta\sigma}\Gamma_{\sigma|\alpha\nu} \right\} \\ &= H^{\alpha\beta}H_{\gamma\delta}G^{\gamma\rho}G^{\delta\sigma}L^\mu L^\nu \left\{ \Gamma_{\rho|\mu\beta}, \Gamma_{\sigma|\alpha\nu} \right\} + \text{Err} \\ &= H^{\alpha\beta}H^{\delta\sigma}L^\mu L^\nu \left\{ \Gamma_{\delta|\mu\beta}, \Gamma_{\sigma|\alpha\nu} \right\} + \text{Err} \\ &= H^{\alpha\beta}H^{\delta\sigma} \left\{ L^\mu \circ \Gamma_{\delta|\mu\beta}, L^\nu \circ \Gamma_{\sigma|\alpha\nu} \right\} + \text{Err} \quad (\text{using (98)}) \end{aligned}$$

with the final expression⁶ for the error term

$$\text{Err} = \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \nabla L)$$

Consider now the bilinear term $\{L^\mu \circ \Gamma_{\delta|\mu\beta}, L^\nu \circ \Gamma_{\sigma|\alpha\nu}\}$. As we start manipulating the left hand side we consider $\{L^\mu \circ \Gamma_{\delta|\mu\beta}, w\}$ for a fixed w . As w remains unchanged in the calculations below we shall drop the bracket and simply write $\{L^\mu \circ \Gamma_{\delta|\mu\beta}, w\} = L^\mu \circ \Gamma_{\delta|\mu\beta}$. Thus instead of,

$$\left\{ L^\mu \circ \Gamma_{\delta|\mu\beta}, w \right\} = \frac{1}{2} \left\{ L^\mu \circ \left(G_{\mu\delta,\beta} + G_{\beta\delta,\mu} - G_{\beta\mu,\delta} \right), w \right\} = \dots$$

we write,

$$\begin{aligned} L^\mu \circ \Gamma_{\delta|\mu\beta} &= \frac{1}{2}L^\mu \circ \left(G_{\mu\delta,\beta} + G_{\beta\delta,\mu} - G_{\beta\mu,\delta} \right) = \frac{1}{2}L^\mu \circ \left(G_{\mu\delta,\beta} - G_{\beta\mu,\delta} \right) + D_* \circ G \\ &= \frac{1}{2} \left(L^\mu \partial_\beta \circ G_{\mu\delta} - L^\mu \partial_\delta \circ G_{\beta\mu} \right) + D_* \circ G, \end{aligned}$$

where we have used that ∂ commutes with \circ , i.e. $\{f \circ \partial v, w\} = \{f \partial v, w\}$. Recall that, see (52), $\partial_\alpha = -\frac{1}{2}L_\alpha \underline{L} + D_*$. Therefore,

$$L^\mu \circ \Gamma_{\delta|\mu\beta} = -\frac{1}{4} \left(L_\beta L^\mu \partial_3 \circ G_{\mu\delta} - L_\delta L^\mu \partial_3 \circ G_{\beta\mu} \right) + D_* \circ G \quad (101)$$

According to (53) of Lemma 3.3 and the formula $H^{3\alpha} = -\frac{1}{2}L^\alpha$, we have

$$-L^\alpha \partial_3 (QG)_{\alpha\sigma} = H^{\alpha\beta} \partial_\sigma (QG)_{\alpha\beta} + GD_* (QG) + h \partial (QG) + \frac{1}{|Q|} [\partial G] \partial G$$

⁶Use also remark 5.5, lemma 7.3, and the boundedness of $\|G, H, h\|_{L^\infty}$.

with Q any of the projections $Q = I, P, P_{\lambda_1}$, with $\lambda_1 > 1$, appearing in the definition of $\{ \cdot, \cdot \}$ and \circ . Therefore,

$$-L^\alpha \partial_3 \circ G_{\alpha\sigma} = H^{\alpha\beta} \partial_\sigma \circ G_{\alpha\beta} + GD_* \circ G + h \circ \partial G + [\partial G] \partial G$$

Therefore, from (101), and expanding $\partial_\delta, \partial_\beta$ relative to the null frame,

$$\begin{aligned} L^\mu \circ \Gamma_{\delta|\mu\beta} &= \frac{1}{4} (L_\beta H^{\mu\sigma} \partial_\delta \circ G_{\mu\sigma} - L_\delta H^{\mu\sigma} \partial_\beta \circ G_{\mu\sigma}) + D_* \circ G \\ &\quad + G \cdot D_* \circ G + h \circ \partial G + [\partial G] \partial G \\ &= -\frac{1}{8} (L_\beta L_\delta H^{\mu\sigma} \partial_3 \circ G_{\mu\sigma} - L_\delta L_\beta H^{\mu\sigma} \partial_3 \circ G_{\mu\sigma}) \\ &\quad + D_* \circ G + G \cdot D_* \circ G + h \circ \partial G + [\partial G] \partial G \\ &= D_* \circ G + G \cdot D_* \circ G + h \circ \partial G + [\partial G] \partial G \end{aligned}$$

Similarly we have

$$\begin{aligned} L^\nu \circ \Gamma_{\alpha|\nu\gamma} &= \frac{1}{2} L^\nu \circ \left(G_{\nu\gamma, \alpha} + G_{\alpha\gamma, \nu} - G_{\alpha\nu, \gamma} \right) \\ &= -\frac{1}{4} \left(L^\nu L_\alpha \partial_3 \circ G_{\nu\gamma} - L^\nu L_\gamma \partial_3 \circ G_{\alpha\nu} \right) + D_* \circ G \\ &= \frac{1}{4} \left(L_\alpha H^{\nu\sigma} \partial_\gamma \circ G_{\nu\sigma} - L_\gamma H^{\nu\sigma} \partial_\alpha \circ G_{\nu\sigma} \right) \\ &\quad + D_* \circ G + G \cdot D_* \circ G + h \circ \partial G + [\partial G] \partial G \\ &= D_* \circ G + G \cdot D_* \circ G + h \circ \partial G + [\partial G] \partial G \end{aligned}$$

Thus, going back to (100),

$$\begin{aligned} \mathcal{E}_1 &= H \cdot H \left\{ (D_* \circ G + G \cdot D_* \circ G), (D_* \circ G + G \cdot D_* \circ G) \right\} \\ &\quad + \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \nabla L) \end{aligned} \tag{102}$$

The term $\mathcal{E}_2 = H^{\alpha\beta} H_\gamma \delta L^\mu L^\nu \left\{ \Gamma_{\mu\nu}^\gamma(G), \Gamma_{\alpha\beta}^\delta(G) \right\}$:

Using remarks 7.1 and 7.3,

$$\begin{aligned} \mathcal{E}_2 &= H^{\alpha\beta} H_\gamma \delta L^\mu L^\nu \left\{ H^{\gamma\epsilon} \Gamma_{\epsilon|\mu\nu}, H^{\delta\sigma} \Gamma_{\sigma|\alpha\beta} \right\} + \pi(h, \partial G; \partial G) \\ &= H^{\delta\sigma} \left\{ L^\mu L^\nu \circ \Gamma_{\delta|\mu\nu}, H^{\alpha\beta} \circ \Gamma_{\sigma|\alpha\beta} \right\} + \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \nabla L) \end{aligned}$$

Observe that according to (54), Lemma 3.3,

$$L^\mu L^\nu \partial_\delta (QG)_{\mu\nu} = G \cdot D_* (QG) + h \partial (QG) + \frac{1}{|Q|} [\partial G] \partial G$$

for any $Q = I, P, P_{\lambda_1}$ with $\lambda_1 > 1$. Therefore,

$$L^\mu L^\nu \partial_\delta \circ G_{\mu\nu} = G \cdot D_* \circ G + h \circ \partial G + [\partial G] \partial G$$

Thus

$$\begin{aligned}
L^\mu L^\nu \circ \Gamma_{\delta|\mu\nu} &= L^\mu L^\nu \circ \left(G_{\mu\delta, \nu} + G_{\nu\delta, \mu} - G_{\mu\nu, \delta} \right) \\
&= D_* \circ G - L^\mu L^\nu \partial_\delta \circ G_{\mu\nu} = D_* \circ G + G \cdot D_* \circ G \\
&+ h \circ \partial G + [\partial G] \partial G
\end{aligned} \tag{103}$$

Using (83) we have,

$$H^{\alpha\beta} \partial_\sigma \circ G_{\alpha\beta} = 2H^{3\alpha} \partial_3 \circ G_{\alpha\sigma} + GD_* \circ G + h \circ \partial G + [\partial G] \partial G.$$

Therefore,

$$\begin{aligned}
H^{\alpha\beta} \circ \Gamma_{\sigma|\alpha\beta} &= H^{\alpha\beta} \circ \left(G_{\alpha\sigma, \beta} + G_{\beta\sigma, \alpha} - G_{\alpha\beta, \sigma} \right) \\
&= H^{\alpha\beta} \circ \left(2G_{\alpha\sigma, \beta} - G_{\alpha\beta, \sigma} \right) \\
&= 2H^{\alpha\beta} \partial_\beta \circ G_{\alpha\sigma} - 2H^{3\alpha} \partial_3 \circ G_{\alpha\sigma} + GD_* \circ G + h \circ \partial G + [\partial G] \partial G \\
&= GD_* \circ G + h \circ \partial G + [\partial G] \partial G
\end{aligned}$$

Therefore, similar to (102), we derive

$$\begin{aligned}
\mathcal{E}_2 &= H \left\{ (D_* \circ G + G \cdot D_* \circ G), (D_* \circ G + G \cdot D_* \circ G) \right\} \\
&+ \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \nabla L)
\end{aligned} \tag{104}$$

We now observe that according to the remark 7.1

$$\{G \cdot D_* \circ G, f\} = G\{D_* \circ G, f\} + \pi(\partial G, f; \nabla G)$$

Therefore returning to (96), using (102), (104), and the boundedness of H and G we infer the following

Proposition 7.5. *We can write*

$$III_{44} = \{D_* \circ G, D_* \circ G\} + Err,$$

where

$$Err = \pi(\partial G, \partial G; \partial G) + \pi(\partial G, \partial G; \nabla L)$$

8. ESTIMATES FOR I_{44} , II_{44} , AND III_{44}

According to the reduction (64) and the representation (84),

$$\mathbf{R}_{44} = I_{44} + II_{44} + III_{44} + Err$$

with

$$Err = \pi(\partial H, \partial H; \partial G) + \pi(h, \partial G; \partial G) + \pi(h, h; \partial^2 H).$$

Therefore we need to show that

$$\begin{aligned}
\int_{u+1}^t \|I_{44}\|_{L^2(D_{\tau,u})} d\tau &+ \int_{u+1}^t \|II_{44}\|_{L^2(D_{\tau,u})} d\tau \\
&+ \int_{u+1}^t \|III_{44}\|_{L^2(D_{\tau,u})} d\tau + \int_{u+1}^t \|Err\|_{L^2(D_{\tau,u})} d\tau \lesssim \lambda^{-1}
\end{aligned}$$

We start with error terms accumulated above and in the lemmas 6.2, 6.7, 7.5.

8.1. Estimates for the error terms. According to the property (48) of π ,

$$\pi(\partial H, \partial H; \partial G) \leq \pi(\partial G, \partial G; \partial G).$$

We then estimate, with the help of the estimates (14)–(16) for G ,

$$\begin{aligned} \int_{u+1}^t \|\pi(\partial G, \partial G; \partial G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \|\partial G\|_{L_t^2 L_x^\infty}^2 \sup_\tau \|\partial G\|_{L^2(D_{\tau,u})} \\ &\lesssim \lambda^{-1-8\epsilon_0} \sup_{\tau,u} \|\partial G\|_{L^2(S_{\tau,u})} \lesssim \lambda^{-1-10\epsilon_0} \end{aligned}$$

Since the frequencies of h are restricted to the region $|\xi| \geq 1$, $h = (I - P)G$, we also have

$$\begin{aligned} \int_{u+1}^t \|\pi(h, \partial G; \partial G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \|h\|_{L_t^2 L_x^\infty} \|\partial G\|_{L_t^2 L_x^\infty} \sup_\tau \|\partial G\|_{L^2(D_{\tau,u})} \\ &\lesssim \lambda^{-\frac{1}{2}-6\epsilon_0} \|\partial h\|_{L_t^2 L_x^\infty}^2 \lesssim \lambda^{-1-10\epsilon_0} \end{aligned}$$

In addition, using the background estimates (17)–(20),

$$\begin{aligned} \int_{u+1}^t \|\pi(h, h; \partial^2 H)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \|h\|_{L_t^2 L_x^\infty}^2 \sup_\tau \|\partial^2 H\|_{L^2(D_{\tau,u})} \\ &\lesssim \|\partial h\|_{L_t^2 L_x^\infty}^2 \sup_\tau \|\partial^2 H\|_{L^2(\Sigma_\tau)} \lesssim \lambda^{-\frac{3}{2}-8\epsilon_0} \end{aligned}$$

Estimating the error terms generated in proposition 6.2, and using the estimate (28) for ∇L

$$\int_{u+1}^t \|\pi(\partial G, \partial G; \nabla L)\|_{L^2(D_{\tau,u})} d\tau \lesssim \|\partial G\|_{L_t^2 L_x^\infty}^2 \sup_\tau \|\nabla L\|_{L^2(D_{\tau,u})} \lesssim \lambda^{-1-8\epsilon_0}$$

To bound the error term $\pi(\nabla L, \partial G; \partial^2 G)$ in proposition 6.7 we use the inequality (23), $|\nabla L| \lesssim (\Theta + r^{-1})$ and

$$\begin{aligned} \left(\int_{u+1}^t \|(\Theta + r^{-1})\|_{L^\infty(D_{\tau,u})}^2 d\tau \right)^{\frac{1}{2}} &\lesssim \|\Theta\|_{L_t^2 L_x^\infty} + \left(\int_{u+1}^t \frac{d\tau}{(\tau - u)^2} \right)^{\frac{1}{2}} \\ &\lesssim \lambda^{-\frac{1}{2}-2\epsilon_0} + 1, \end{aligned}$$

which follows from the comparison $r \approx \tau - u$, see (26). Thus,

$$\begin{aligned} \int_{u+1}^t \|\pi(\nabla L, \partial G; \partial G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \left(\int_{u+1}^t \|\nabla L\|_{L^\infty(D_{\tau,u})}^2 d\tau \right)^{\frac{1}{2}} \|\partial G\|_{L_t^2 L_x^\infty} \sup_\tau \|\partial^2 G\|_{L^2(D_{\tau,u})} \\ &\lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \sup_\tau \|\partial^2 G\|_{L^2(\Sigma_\tau)} \lesssim \lambda^{-1-4\epsilon_0} \end{aligned}$$

Finally,

$$\begin{aligned} \int_{u+1}^t \|\pi(\partial G, \partial G; \partial^2 G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \|\partial G\|_{L_t^2 L_x^\infty}^2 \sup_\tau \|\partial^2 G\|_{L^2(D_{\tau,u})} \\ &\lesssim \lambda^{-1-8\epsilon_0} \sup_\tau \|\partial^2 G\|_{L^2(\Sigma_\tau)} \lesssim \lambda^{-\frac{3}{2}-8\epsilon_0} \end{aligned}$$

The error terms in proposition 7.5 are the same as considered above.

8.2. Estimates for the principal terms. These estimates depend decisively on the $L^2(C_u)$ estimates for the tangential derivatives of G and H derived in proposition 7.7 of [Kl-Ro2], see also proposition 2.2. For convenience we recall the result here.

$$\|D_*\partial H\|_{L^2(C_u)} \lesssim \lambda^{-\frac{1}{2}}, \quad \|D_*H\|_{L^2(C_u)} \lesssim \lambda^{\frac{1}{2}} \quad (105)$$

Also,

$$\begin{aligned} \|D_*\partial(P_\mu G)\|_{L^2(C_u)} &\lesssim \mu^{\frac{1}{2}-4\epsilon_0} \lambda^{-\frac{1}{2}-4\epsilon_0}, \\ \|D_*(P_\mu G)\|_{L^2(C_u)} &\lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \mu^{-\frac{1}{2}-4\epsilon_0} \end{aligned} \quad (106)$$

We start with the principal term $hD_*\partial H$ appearing in proposition 6.2.

$$\begin{aligned} \int_{u+1}^t \|hD_*\partial H\|_{L^2(D_{\tau,u})} d\tau &\lesssim \int_{u+1}^t \|h(\tau)\|_{L_x^\infty} \|D_*\partial H\|_{L^2(D_{\tau,u})} d\tau \\ &\lesssim \|h\|_{L_t^2 L_x^\infty} \left(\int_{u+1}^t \|D_*\partial H\|_{L^2(D_{\tau,u})}^2 d\tau \right)^{\frac{1}{2}} \\ &\lesssim \|\partial G\|_{L_t^2 L_x^\infty} \sup_{u \leq u' \leq u+1} \|D_*\partial H\|_{L^2(C_{u'})} \\ &\lesssim \lambda^{-1-4\epsilon_0} \quad (\text{using (105)}) \end{aligned}$$

as desired.

We now estimate the principal terms $\{G, D_*\partial \circ G\}'$ and $H \cdot \{G, D_*\partial \circ G\}'$ appearing in proposition 6.7. Since H is bounded it clearly suffices to treat the first term. Recall that

$$\{G, D_*\partial \circ G\}' = [G, P] D_*\partial(\overline{P}G) + \sum_{\nu > 1, \mu > 2, |\ln(\nu\mu^{-1})| \leq 2} [P_\nu G, P] D_*\partial(P_\mu G)$$

We estimate the first term as follows:

$$\begin{aligned} \int_{u+1}^t \|[G, P] D_*\partial(\overline{P}G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \int_{u+1}^t \|\partial G\|_{L_x^\infty} \|D_*\partial(\overline{P}G)\|_{L^2(D_{\tau,u})} d\tau \\ &\lesssim \|\partial G\|_{L_t^2 L_x^\infty} \left(\int_{u+1}^t \|D_*\partial(\overline{P}G)\|_{L^2(D_{\tau,u})}^2 d\tau \right)^{\frac{1}{2}} \\ &\lesssim \|\partial G\|_{L_t^2 L_x^\infty} \sup_{u \leq u' \leq u+1} \|D_*\partial(\overline{P}G)\|_{L^2(C_{u'})} \\ &\lesssim \lambda^{-1-4\epsilon_0} \quad (\text{using (105)}) \end{aligned} \quad (107)$$

We estimate the high-high interaction as follows

$$\begin{aligned} \int_{u+1}^t \|[P_\nu G, P] D_*\partial(P_\mu G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \int_{u+1}^t \|P_\nu G\|_{L_x^\infty} \|D_*\partial(P_\mu G)\|_{L^2(D_{\tau,u})} d\tau \\ &\lesssim \frac{1}{\nu} \int_{u+1}^t \|P_\nu \nabla G\|_{L_x^\infty} \|D_*\partial(P_\mu G)\|_{L^2(D_{\tau,u})} d\tau \\ &\lesssim \frac{1}{\nu} \|\partial G\|_{L_t^2 L_x^\infty} \sup_{u \leq u' \leq u+1} \|D_*\partial(P_\mu G)\|_{L^2(C_{u'})} \\ &\lesssim \nu^{-1} \mu^{\frac{1}{2}-4\epsilon_0} \lambda^{-1-8\epsilon_0} \quad (\text{using (106)}) \end{aligned}$$

Therefore,

$$\int_{u+1}^t \|[P_\nu G, P] D_* \partial(P_\mu G)\|_{L^2(D_{\tau,u})} d\tau \lesssim \nu^{-1} \mu^{\frac{1}{2}-4\epsilon_0} \lambda^{-1-8\epsilon_0} \quad (108)$$

Combining (107) and (108) we conclude that

$$\begin{aligned} \int_{u+1}^t \|\{G, D_* \partial \circ G\}'\|_{L^2(D_{\tau,u})} d\tau &\lesssim \lambda^{-1-4\epsilon_0} + \lambda^{-1-8\epsilon_0} \sum_{\nu>1, \mu>2, |\ln(\nu\mu^{-1})|\leq 2} \nu^{-1} \mu^{\frac{1}{2}-4\epsilon_0} \\ &\lesssim \lambda^{-1-4\epsilon_0} \end{aligned}$$

It remains to estimate the principal term $\{D_* \circ G, D_* \circ G\}$ in Proposition 7.5. We recall from definition 7.2 that

$$\begin{aligned} \{D_* \circ G, D_* \circ G\} &= P \sum_{\frac{1}{2}<\nu\leq 4} D_*(P_\nu G) \cdot D_*(P_{\leq \frac{1}{2}} G) \\ &+ P \sum_{\nu>\frac{1}{2}, |\ln(\nu\mu^{-1})|\leq 2} D_*(P_\nu G) \cdot D_*(P_\mu G) \\ &+ P \sum_{\frac{1}{2}<\mu\leq 4} D_*(P_{\leq \frac{1}{2}} G) \cdot D_*(P_\mu G) \\ &+ P \sum_{\mu_2>\frac{1}{2}, |\ln(\nu\mu^{-1})|\leq 2} D_*(P_\nu G) \cdot D_*(P_\mu G) \\ &- \sum_{\frac{1}{2}<\nu\leq 1} D_*(P_\nu G) \cdot D_*(P_{\leq \frac{1}{2}} G) \\ &- \sum_{\frac{1}{2}<\mu\leq 1} D_*(P_{\leq \frac{1}{2}} G) \cdot D_*(P_\mu G) \\ &- \sum_{\frac{1}{2}<\nu, \mu\leq 1} D_*(P_\nu G) \cdot D_*(P_\mu G) \end{aligned}$$

By symmetry and similarity it suffices to estimate the first 2 terms in the expression above. We have

$$\begin{aligned} \int_{u+1}^t \|D_*(P_\nu G) \cdot D_*(P_{\leq \frac{1}{2}} G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \int_{u+1}^t \|D_*(P_\nu G)\|_{L^2(D_{\tau,u})} \|D_*(P_{\leq \frac{1}{2}} G)\|_{L^\infty} d\tau \\ &\lesssim \|\partial G\|_{L_t^2 L_x^\infty} \sup_{u\leq u'\leq u+1} \|D_*(P_\nu G)\|_{L^2(C_{u'})} \\ &\lesssim \nu^{-\frac{1}{2}-4\epsilon_0} \lambda^{-1-8\epsilon_0} \quad (\text{by (106).}) \end{aligned}$$

Thus

$$\begin{aligned} \int_{u+1}^t \|P \sum_{\frac{1}{2}<\nu\leq 4} D_*(P_\nu G) \cdot D_*(P_{\leq \frac{1}{2}} G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \sum_{\frac{1}{2}<\nu\leq 4} \int_{u+1}^t \|D_*(P_\nu G) \cdot D_*(P_{\leq \frac{1}{2}} G)\|_{L^2(D_{\tau,u})} \\ &\lesssim \lambda^{-1-8\epsilon_0} \end{aligned}$$

Consider now the high-high interaction term

$$J = P \sum_{\nu>\frac{1}{2}, |\ln(\nu\mu^{-1})|\leq 2} D_*(P_\nu G) \cdot D_*(P_\mu G)$$

Clearly,

$$\begin{aligned}
\int_{u+1}^t \|D_*(P_\nu G) \cdot D_*(P_\mu G)\|_{L^2(D_{\tau,u})} d\tau &\lesssim \int_{u+1}^t \|D_*(P_\nu G)\|_{L^2(D_{\tau,u})} \|D_*(P_\mu G)\|_{L^\infty} d\tau \\
&\lesssim \|\partial G\|_{L_t^2 L_x^\infty} \sup_{u \leq u' \leq u+1} \|D_*(P_\nu G)\|_{L^2(C_{u'})} \\
&\lesssim \lambda^{-1-8\epsilon_0} \nu^{-\frac{1}{2}-4\epsilon_0}
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{u+1}^t \|J\|_{L^2(D_{\tau,u})} d\tau &\lesssim \sum_{\nu > \frac{1}{2}, |\ln(\nu\mu^{-1})| \leq 2} \int_{u+1}^t \|D_*(P_\nu G) \cdot D_*(P_\mu G)\|_{L^2(D_{\tau,u})} d\tau \\
&\lesssim \lambda^{-1-8\epsilon_0} \sum_{\nu > \frac{1}{2}, |\ln(\nu\mu^{-1})| \leq 2} \nu^{-\frac{1}{2}-4\epsilon_0} \lesssim \lambda^{-1-8\epsilon_0}
\end{aligned}$$

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