

# ROUGH SOLUTIONS OF THE EINSTEIN-VACUUM EQUATIONS

SERGIU KLAINERMAN AND IGOR RODNIANSKI

ABSTRACT. This is the first in a series of papers in which we initiate the study of very rough solutions to the initial value problem for the Einstein vacuum equations expressed relative to wave coordinates. By very rough we mean solutions which cannot be constructed by the classical techniques of energy estimates and Sobolev inequalities. Following [Kl-Ro] we develop new analytic methods based on Strichartz type inequalities which results in a gain of half a derivative relative to the classical result. Our methods blend paradifferential techniques with a geometric approach to the derivation of decay estimates. The latter allows us to take full advantage of the specific structure of the Einstein equations.

## 1. INTRODUCTION

We consider the Einstein Vacuum equations,

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0 \tag{1}$$

where  $\mathbf{g}$  is a four dimensional Lorentz metric and  $\mathbf{R}_{\alpha\beta}$  its Ricci curvature tensor. In wave coordinates  $x^\alpha$ ,

$$\square_{\mathbf{g}} x^\alpha = \frac{1}{|\mathbf{g}|} \partial_\mu (\mathbf{g}^{\mu\nu} |\mathbf{g}| \partial_\nu) x^\alpha = 0, \tag{2}$$

the Einstein vacuum equations take the reduced form, see [Br], [H-K-M].

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial \mathbf{g}) \tag{3}$$

with  $N$  quadratic in the first derivatives  $\partial \mathbf{g}$  of the metric. We consider the initial value problem along the spacelike hyperplane  $\Sigma$  given by  $t = x^0 = 0$ ,

$$\nabla \mathbf{g}_{\alpha\beta}(0) \in H^{s-1}(\Sigma), \quad \partial_t \mathbf{g}_{\alpha\beta}(0) \in H^{s-1}(\Sigma) \tag{4}$$

with  $\nabla$  denoting the gradient with respect to the space coordinates  $x^i$ ,  $i = 1, 2, 3$  and  $H^s$  the standard Sobolev spaces. We also assume that  $\mathbf{g}_{\alpha\beta}(0)$  is a continuous Lorentz metric and

$$\sup_{|x|=r} |\mathbf{g}_{\alpha\beta}(0) - \mathbf{m}_{\alpha\beta}| \longrightarrow 0 \quad \text{as } r \longrightarrow \infty, \tag{5}$$

where  $|x| = (\sum_{i=1}^3 |x^i|^2)^{\frac{1}{2}}$  and  $\mathbf{m}_{\alpha\beta}$  the Minkowski metric.

The following local existence and uniqueness result (well posedness) is well known (see [H-K-M] and the previous result of Ch. Bruhat [Br] for  $s \geq 4$ .)

---

1991 *Mathematics Subject Classification.* 35J10.

**Theorem 1.1.** *Considered the reduced equation (3) subject to the initial conditions (4) and (5) for some  $s > 5/2$ . Then there exists a time interval  $[0, T]$  and unique (Lorentz metric) solution  $\mathbf{g} \in C^0([0, T] \times \mathbb{R}^3)$ ,  $\partial \mathbf{g}_{\mu\nu} \in C^0([0, T]; H^{s-1})$  with  $T$  depending only on the size of the norm  $\|\partial \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}}$ . In addition condition (5) remains true on any spacelike hypersurface  $\Sigma_t$ , i.e. any level hypersurface of the time function  $t = x^0$ .*

We establish a significant improvement of this result bearing on the issue of minimal regularity of the initial conditions:

**Main Theorem** Consider a classical solution of the equations (3) for which (1) also holds<sup>1</sup>. We show<sup>2</sup> that the time  $T$  of existence depends in fact only on the size of the norm  $\|\partial \mathbf{g}_{\mu\nu}(0)\|_{H^{s-1}}$ , for any fixed  $s > 2$ .

*Remark 1.2.* Theorem 1.1 implies the classical local existence result of [H-K-M] for asymptotically flat initial data sets  $\Sigma, g, k$  with  $\nabla g, k \in H^{s-1}(\Sigma)$  and  $s > \frac{5}{2}$ , relative to a fixed system of coordinates. Uniqueness can be proved for additional regularity  $s > 1 + \frac{5}{2}$ . We recall that an initial data set  $(\Sigma, g, k)$  consists of a three dimensional complete Riemannian manifold  $(\Sigma, g)$ , a 2-covariant symmetric tensor  $k$  on  $\Sigma$  verifying the constraint equations:

$$\nabla^j k_{ij} - \nabla_i \text{tr} k = 0$$

$$R - |k|^2 + (\text{tr} k)^2 = 0$$

where  $\nabla$  is the covariant derivative,  $R$  the scalar curvature of  $(\Sigma, g)$ . An initial data set is said to be asymptotically flat (AF) if there exists a system of coordinates  $(x^1, x^2, x^3)$  defined in a neighborhood of infinity<sup>3</sup> on  $\Sigma$  relative to which the metric  $g$  approaches the Euclidean metric and  $k$  approaches zero<sup>4</sup>

*Remark 1.3.* The Main Theorem ought to imply existence and uniqueness<sup>5</sup> for initial conditions with  $H^s$ ,  $s > 2$ , regularity. To achieve this we only need to approximate a given  $H^s$  initial data set (i.e.  $\nabla g \in H^{s-1}(\Sigma), k \in H^{s-1}(\Sigma)$ ,  $s > 2$ ) for the Einstein vacuum equations by classical initial data sets, i.e.  $H^{s'}$  data sets with  $s' > \frac{5}{2}$ , for which theorem 1.1 holds. The Main Theorem allows us to pass to the limit and derive existence of solutions for the given, rough, initial data set. We don't know however if such an approximation result for the constraint equations exists in the literature.

<sup>1</sup>In other words for any solution of the reduced equations (3) whose initial data satisfy the constraint equations, see [Br] or [H-K-M]. The fact that our solutions verify (1) plays a fundamental role in our analysis.

<sup>2</sup>We assume however that  $T$  stays sufficiently small, e.g.  $T \leq 1$ . This a purely technical assumption which one should be able to remove.

<sup>3</sup>We assume, for simplicity, that  $\Sigma$  has only one end. A neighborhood of infinity means the complement of a sufficiently large compact set on  $\Sigma$ .

<sup>4</sup>Because of the constraint equations the asymptotic behavior cannot be arbitrarily prescribed. A precise definition of asymptotic flatness has to involve the ADM mass of  $(\Sigma, g)$ . Taking the mass into account we write  $g_{ij} = (1 + \frac{2M}{r})\delta_{ij} + o(r^{-1})$  as  $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \rightarrow \infty$ . According to the positive mass theorem  $M \geq 0$  and  $M = 0$  implies that the initial data set is flat. Because of the mass term we cannot assume that  $g - e \in L^2(\Sigma)$ , with  $e$  the 3D Euclidean metric.

<sup>5</sup>Properly speaking uniqueness holds, with  $s > 2$ , only for the reduced equations. Uniqueness for the actual Einstein equations requires one more derivative, see [H-K-M].

For convenience we shall also write the reduced equations (3) in the form

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \phi = N(\phi, \partial\phi) \quad (6)$$

where  $\phi = (\mathbf{g}_{\mu\nu})$ ,  $N = N_{\mu\nu}$  and  $\mathbf{g}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}(\phi)$ .

Expressed relative to the wave coordinates  $x^\alpha$  the spacetime metric  $\mathbf{g}$  takes the form:

$$\mathbf{g} = -\mathbf{n}^2 dt^2 + g_{ij}(dx^i + \mathbf{v}^i dt)(dx^j + \mathbf{v}^j dt) \quad (7)$$

where  $g_{ij}$  is a Riemannian metric on the slices  $\Sigma_t$ , given by the level hypersurfaces of the time function  $t = x^0$ ,  $\mathbf{n}$  is the lapse function of the time foliation, and  $\mathbf{v}$  is a vector-valued shift function. The components of the inverse metric  $\mathbf{g}^{\alpha\beta}$  can be found as follows:

$$\mathbf{g}^{00} = -\mathbf{n}^{-2}, \quad \mathbf{g}^{0i} = \mathbf{n}^{-2} \mathbf{v}^i, \quad \mathbf{g}^{ij} = g^{ij} - \mathbf{n}^{-2} \mathbf{v}^i \mathbf{v}^j.$$

In view of the Lorentzian character of  $\mathbf{g}$  and the spacelike character of the hypersurfaces  $\Sigma_t$ ,

$$c|\xi|^2 \leq \mathbf{g}_{ij} \xi^i \xi^j \leq c^{-1} |\xi|^2, \quad c \leq \mathbf{n}^2 - |\mathbf{v}|_g^2 \quad (8)$$

for some  $c > 0$ .

The classical local existence result for systems of wave equations of type (6) is based on energy estimates and the standard  $H^s \subset L^\infty$  Sobolev inequality. Indeed using energy estimates and simple commutation inequalities one can show that,

$$\|\partial\phi(t)\|_{H^{s-1}} \leq E \|\partial\phi(0)\|_{H^{s-1}} \quad (9)$$

with a constant  $E$ ,

$$E = \exp\left(C \int_0^t \|\partial\phi(\tau)\|_{L_x^\infty} d\tau\right) \quad (10)$$

By the classical Sobolev inequality,

$$E \leq \exp\left(Ct \sup_{0 \leq \tau \leq t} \|\partial\phi(\tau)\|_{H^{s-1}} d\tau\right)$$

provided that  $s > \frac{5}{2}$ . The classical local existence result follows by combining this last estimate, for a small time interval, with the energy estimates (9).

This scheme is very wasteful. To do better one would like to take advantage of the mixed  $L_t^1 L_x^\infty$  norm appearing on the right hand side of (10). Unfortunately there are no good estimates for such norms even when  $\phi$  is simply a solution of the standard wave equation

$$\square\phi = 0 \quad (11)$$

in Minkowski space. There exist however improved regularity estimates for solutions of (11) in the mixed  $L_t^2 L_x^\infty$  norm. More precisely, if  $\phi$  is a solution of (11) and  $\epsilon > 0$  arbitrarily small,

$$\|\partial\phi\|_{L_t^2 L_x^\infty([0, T] \times \mathbb{R}^3)} \leq CT^\epsilon \|\partial\phi(0)\|_{H^{1+\epsilon}}. \quad (12)$$

Based on this fact it was reasonable to hope that one can improve the Sobolev exponent in the classical local existence theorem from  $s > \frac{5}{2}$  to  $s > 2$ . This can be easily done for solutions of semilinear equations, see [Po-Si]. In the quasilinear case, however, the situation is far more difficult. One can no longer rely on the Strichartz inequality (12) for the flat D'Alembertian in (11); we need instead its extension to the operator  $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$  appearing in (6). Moreover, since the metric  $\mathbf{g}^{\alpha\beta}$  depends on the solution  $\phi$ , it can have only as much regularity as  $\phi$  itself. This means that we have to confront the issue of proving Strichartz estimates for wave operators  $g^{\alpha\beta}\partial_\alpha\partial_\beta$  with *very rough coefficients*  $\mathbf{g}^{\alpha\beta}$ . This issue was recently addressed in the pioneering works of Smith[Sm], Bahouri-Chemin [Ba-Ch1], [Ba-Ch2] and Tataru [Ta1], [Ta2], we refer to the introduction in [Kl1] and [Kl-Ro] for a more thorough discussion of their important contributions.

The results of Bahouri-Chemin and Tataru are based on establishing a Strichartz type inequality, *with a loss*, for wave operators with *very rough coefficients*<sup>6</sup>. The optimal result<sup>7</sup> in this regard, due to Tataru, see [Ta2], requires a loss of  $\sigma = \frac{1}{6}$ . This leads to a proof of local well posedness for systems of type (6) with  $s > 2 + \frac{1}{6}$ .

To do better than that one needs to take into account the nonlinear structure of the equations. In [Kl-Ro] we were able to improve the result of Tataru by taking into account not only the expected regularity properties of the coefficients  $g^{\alpha\beta}$  in (6) but also the fact that they are themselves solutions to a similar system of equations. This allowed us to improve the exponent  $s$ , needed in the proof of well posedness of equations of type<sup>8</sup> (6), to  $s > 2 + \frac{2-\sqrt{3}}{2}$ . Our approach was based on a combination of the paradifferential calculus ideas, initiated in [Ba-Ch1] and [Ta2], with a geometric treatment of the actual equations introduced in [Kl1]. The main improvement was due to a gain of conormal differentiability for solutions to the Eikonal equations

$$H^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0 \tag{13}$$

where the background metric  $H$  is a properly microlocalized and rescaled version of the metric  $\mathbf{g}^{\alpha\beta}$  in (6). That gain could be traced down to the fact that a certain component of the Ricci curvature of  $H$  has a special form. More precisely denoting by  $L'$  the null geodesic vectorfield associated to  $u$ ,  $L' = -H^{\alpha\beta}\partial_\beta u\partial_\alpha$ , and rescaling it in an appropriate fashion<sup>9</sup>,  $L = bL'$ , we found that the  $\mathbf{R}_{LL} = \mathbf{Ric}(H)(L, L)$ , verifies the remarkable identity:

$$\mathbf{R}_{LL} = L(z) - \frac{1}{2}L^\mu L^\nu (H^{\alpha\beta}\partial_\alpha\partial_\beta H_{\mu\nu}) + e \tag{14}$$

where  $z \leq O(|\partial H|)$  and  $e \leq O(|\partial H|^2)$ . Thus, apart from  $L(z)$  which is to be integrated along the null geodesic flow generated by  $L$ , the only terms which depend

<sup>6</sup>The derivatives of the coefficients  $\mathbf{g}$  are required to be bounded in  $L_t^\infty H_x^{s-1}$  and  $L_t^2 L_x^\infty$  norms, with  $s$  compatible with the regularity required on the right hand side of the Strichartz inequality one wants to prove.

<sup>7</sup>Recently Smith-Tataru [Sm-Ta] have shown that the result of Tataru is indeed sharp.

<sup>8</sup>The result in [Kl-Ro] applies to general equations of type (6) not necessarily tied to (1). In [Kl-Ro] we have also made the simplifying assumptions  $\mathbf{n} = 1$  and  $\mathbf{v} = 0$ .

<sup>9</sup>such  $\langle L, T \rangle_H = 1$  with  $T$  is the unit normal to the level hypersurfaces  $\Sigma_t$  associated to the time function  $t$ ,

of the second derivatives of  $H$  appear in  $H^{\alpha\beta}\partial_\alpha\partial_\beta H$  and can therefore be eliminated with the help of the equations (6).

In this paper we develop the ideas of [Kl-Ro] further by taking full advantage of the Einstein equations (1) in wave coordinates (6). An important aspect of our analysis here is that the term  $L(z)$  appearing on the right hand side of (14) vanishes identically. We make use of both the vanishing of the Ricci curvature of  $\mathbf{g}$  and the wave coordinate condition (2). The other important new features are the use of energy estimates along the null hypersurfaces generated by the optical function  $u$  and a more efficient use of the conormal properties of the null structure equations.

Our work is divided in three parts. In this paper we give all the details in the proof of the Main Theorem with the exception of those results which concern the asymptotic properties of the Ricci coefficients( the Asymptotics Theorem), the isoperimetric and trace inequalities on 2-surfaces. We give precise statements of these results in section 4. Our second paper [Kl-Ro2] is dedicated to the proof of the Asymptotics Theorem. The isoperimetric and trace inequalities together with some other results needed in [Kl-Ro2] are proved in our third paper [Kl-Ro3].

We strongly believe that the result of our main theorem is not sharp. The critical Sobolev exponent for the Einstein equations is  $s_c = \frac{3}{2}$ . A proof of well posedness for  $s = s_c$  will provide a much stronger version of the *global stability of Minkowski space* than that of [Ch-Kl]. This is completely out of reach at the present time. A more reasonable goal, at the present time, is to prove the  *$L^2$ -curvature conjecture*, see [Kl2], corresponding to the exponent  $s = 2$ .

## 2. REDUCTION TO DECAY ESTIMATES

The proof of the main theorem can be reduced to a microlocal decay estimate. The reduction is standard<sup>10</sup>; we quickly review here the main steps. The precise statements and their proofs are given in section 8.

- *Energy estimates*

Assuming that  $\phi$  is a solution<sup>11</sup> of (6) on  $[0, T] \times \mathbb{R}^3$  we have the apriori energy estimate:

$$\|\partial\phi\|_{L_{[0,T]}^\infty \dot{H}^{s-1}} \leq C \|\partial\phi(0)\|_{\dot{H}^{s-1}} \quad (15)$$

with a constant  $C$  depending only on  $\|\phi\|_{L_{[0,T]}^\infty L_x^\infty}$  and  $\|\partial\phi\|_{L_{[0,T]}^1 L_x^\infty}$ .

- *Strichartz estimate* To prove our Main Theorem we need, in addition to (15) an estimate of the form:

$$\|\partial\phi\|_{L_{[0,T]}^1 L_x^\infty} \leq C \|\partial\phi(0)\|_{H^{s-1}}$$

---

<sup>10</sup>see [Kl-Ro] and the references therein

<sup>11</sup>i.e. a classical solution according to theorem 1.1.

for any  $s > 2$ . We accomplish it by establishing a Strichartz type inequality of the form,

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \leq C\|\partial\phi(0)\|_{H^{1+\gamma}} \quad (16)$$

with any fixed  $\gamma > 0$ . We achieve this with the help of a bootstrap argument. More precisely we make the assumption,

*Bootstrap Assumption*

$$\|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} + \|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \leq B_0, \quad (17)$$

and use it to prove the better estimate;

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \leq C(B_0)T^\delta. \quad (18)$$

for some  $\delta > 0$ . Thus, for sufficiently small  $T > 0$ , we find that (16) holds true.

- *Proof of the Main Theorem*

This can be done easily by combining the energy estimates with the Strichartz estimate stated above.

- *Dyadic Strichartz Estimate*

The proof of the Strichartz estimate can be reduced to a dyadic version for each  $\phi^\lambda = P_\lambda\phi$ ,  $\lambda$  sufficiently large<sup>12</sup>, where  $P_\lambda$  is the Littlewood-Paley projection on the space frequencies of size  $\lambda \in 2^{\mathbb{Z}}$ .

$$\|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} \leq C(B_0)c_\lambda T^\delta \|\partial\phi\|_{H^{1+\gamma}},$$

with  $\sum_\lambda c_\lambda \leq 1$ .

- *Dyadic linearization and time restriction*

Consider the new metric  $\mathbf{g}_{<\lambda} = P_{<\lambda}\mathbf{g} = \sum_{\mu < 2^{-M_0}\lambda} P_\mu\mathbf{g}$ , for some sufficiently large constant  $M_0 > 0$ , restricted to a subinterval  $I$  of  $[0, T]$  of size  $|I| \approx T\lambda^{-8\epsilon_0}$  with  $\epsilon_0 > 0$  fixed such that  $\gamma > 5\epsilon_0$ . Without loss of generality<sup>13</sup> we can assume that  $I = [0, \bar{T}]$ . Using an appropriate (now standard, see [Ba-Ch1], [Ta2], [K11], [Kl-Ro]) paradifferential linearization together with the Duhamel principle we can reduce the proof of the dyadic Strichartz estimate mentioned above to a homogeneous Strichartz estimate for the equation

$$\mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta \psi = 0,$$

with initial conditions at  $t = 0$  verifying,

$$(2^{-10}\lambda)^m \leq \|\nabla^m \partial\psi(0)\|_{L^2_x} \leq (2^{10}\lambda)^m \|\partial\psi(0)\|_{L^2_x}.$$

There exists a sufficiently small  $\delta > 0$ ,  $5\epsilon_0 + \delta < \gamma$ , such that

$$\|P_\lambda \partial\psi\|_{L^2_x L^\infty_x} \leq C(B_0)\bar{T}^\delta \|\partial\psi(0)\|_{\dot{H}^{1+\delta}} \quad (19)$$

- *Rescaling*

Introduce the rescaled metric<sup>14</sup>

$$H_{(\lambda)}(t, x) = \mathbf{g}_{<\lambda}(\lambda^{-1}t, \lambda^{-1}x)$$

<sup>12</sup>The low frequencies are much easier to treat.

<sup>13</sup>In view of the translation invariance of our estimates.

<sup>14</sup> $H_{(\lambda)}$  is a Lorentz metric for  $\lambda \geq \Lambda$  with  $\Lambda$  sufficiently large. See the discussion following (135) in section 8.

and consider the rescaled equation

$$H_{(\lambda)}^{\alpha\beta} \partial_\alpha \partial_\beta \psi = 0$$

in the region  $[0, t_*] \times \mathbb{R}^3$  with  $t_* \leq \lambda^{1-8\epsilon_0}$ . Then, with  $P = P_1$ ,

$$\|P \partial \psi\|_{L^2_t L^\infty_x} \leq C(B_0) t_*^\delta \|\partial \psi(0)\|_{L^2}$$

would imply the estimate (19).

- *Reduction to an  $L^1 - L^\infty$  decay estimate*

The standard way to prove a Strichartz inequality of the type discussed above is to reduce it, by a  $TT^*$  type argument, to an  $L^1 - L^\infty$  dispersive type inequality. The inequality we need, concerning the initial value problem

$$\square_{H_{(\lambda)}} \psi = \frac{1}{\sqrt{|H_{(\lambda)}|}} \partial_\alpha (H_{(\lambda)}^{\alpha\beta} \sqrt{|H_{(\lambda)}|} \partial_\beta \psi) = 0,$$

with data at  $t = t_0$  has the form,

$$\|P \partial \psi(t)\|_{L^\infty_x} \leq C(B_0) \left( \frac{1}{(1 + |t - t_0|)^{1-\delta}} + d(t) \right) \sum_{k=0}^m \|\nabla^k \partial \psi(t_0)\|_{L^1_x}$$

for some integer  $m \geq 0$ .

- *Final reduction to a localized  $L^2 - L^\infty$  decay estimate*

We state this as the following theorem:

**Theorem 2.1.** *Let  $\psi$  be a solution of the equation,*

$$\square_{H_{(\lambda)}} \psi = 0 \tag{20}$$

*on the time interval  $[0, t_*]$  with  $t_* \leq \lambda^{1-8\epsilon_0}$ . Assume that the initial data is given at  $t = t_0 \in [0, t_*]$ , supported in the ball  $B_{\frac{1}{2}}(0)$  of radius  $\frac{1}{2}$  centered at the origin. We fix a large constant  $\Lambda > 0$  and consider only the frequencies  $\lambda \geq \Lambda$ . There exists a function  $d(t)$ , with  $t_*^{\frac{1}{q}} \|d\|_{L^q([0, t_*])} \leq 1$  for some  $q > 2$  sufficiently close to 2, an arbitrarily small  $\delta > 0$  and a sufficiently large integer  $m > 0$  such that for all  $t \in [0, t_*]$ ,*

$$\|P \partial \psi(t)\|_{L^\infty_x} \leq C(B_0) \left( \frac{1}{(1 + |t - t_0|)^{1-\delta}} + d(t) \right) \sum_{k=0}^m \|\nabla^k \partial \psi(t_0)\|_{L^2_x}. \tag{21}$$

*Remark 2.2.* In view of the proof of the Main Theorem presented above, which relies on the final estimate (18), we can in what follows treat the bootstrap constant  $B_0$  as a universal constant and bury the dependence on it in the notation  $\lesssim$  we introduce below.

**Definition 2.3.** We use the notation  $A \lesssim B$  to express the inequality  $A \leq CB$  with a universal constant, which may depend on  $B_0$  and various other parameters depending only on  $B_0$  introduced in the proof.

The proof of theorem 2.1 relies on a generalized Morawetz type energy estimate which will be presented in the next section. We shall in fact construct a vector-field, analogous to the Morawetz vectorfield in the Minkowski space, which depends heavily on the ‘‘background metric’’  $H = H_{(\lambda)}$ . In the next proposition we display most of the main properties of the metric  $H$  which will be used in the following section.

**Proposition 2.4** (Background estimates). *We fix the region  $[0, t_*] \times \mathbb{R}^3$ , with  $t_* \leq \lambda^{1-8\epsilon_0}$ , where the original Einstein metric<sup>15</sup>  $\mathbf{g} = \mathbf{g}(\phi)$  verifies the bootstrap assumption (17). The metric*

$$H(t, x) = H_{(\lambda)}(t, x) = P_{<\lambda} \mathbf{g}(\lambda^{-1}t, \lambda^{-1}x) \quad (22)$$

can be decomposed relative to our spacetime coordinates.

$$H = -n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt) \quad (23)$$

where  $n$  and  $v$  are related to  $\mathbf{n}, \mathbf{v}$  according to the rule (22). The metric components  $n, v$ , and  $h$  satisfy the conditions

$$c|\xi|^2 \leq h_{ij}\xi^i\xi^j \leq c^{-1}|\xi|^2, \quad n^2 - |v|_h^2 \geq c > 0, \quad |n|, |v| \leq c^{-1} \quad (24)$$

In addition, the derivatives of the metric  $H$  verify the following:

$$\|\partial^{1+m} H\|_{L^1_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-8\epsilon_0}, \quad (25)$$

$$\|\partial^{1+m} H\|_{L^2_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (26)$$

$$\|\partial^{1+m} H\|_{L^\infty_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (27)$$

$$\|\nabla^{\frac{1}{2}+m}(\partial H)\|_{L^\infty_{[0, t_*]} L^2_x} \lesssim \lambda^{-m} \quad \text{for} \quad -\frac{1}{2} \leq m \leq \frac{1}{2} + 4\epsilon_0 \quad (28)$$

$$\|\nabla^{\frac{1}{2}+m}(\partial^2 H)\|_{L^\infty_{[0, t_*]} L^2_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad \text{for} \quad -\frac{1}{2} + 4\epsilon_0 \leq m \quad (29)$$

$$\|\nabla^m(H^{\alpha\beta}\partial_\alpha\partial_\beta H)\|_{L^1_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}, \quad (30)$$

$$\|\nabla^m(\nabla^{\frac{1}{2}} \mathbf{Ric}(H))\|_{L^\infty_{[0, t_*]} L^2_x} \lesssim \lambda^{-1}, \quad (31)$$

$$\|\nabla^m \mathbf{Ric}(H)\|_{L^1_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}. \quad (32)$$

### 3. GENERALIZED ENERGY ESTIMATES AND THE BOUNDEDNESS THEOREM

Consider the Lorentz metric  $H = H_{(\lambda)}$  as in (22) verifying, in particular, the properties of proposition 2.4 in the region  $[0, t_*] \times \mathbb{R}^3$ ,  $t_* \leq \lambda^{1-8\epsilon_0}$ . We denote by  $D$  the compatible covariant derivative and by  $\nabla$  the induced covariant differentiation on  $\Sigma_t$ . We denote by  $T$  the future oriented unit normal to  $\Sigma_t$  and by  $k$  the second fundamental form.

Associated to  $H$  we have the energy momentum tensor of  $\square_H$ ,

$$Q_{\mu\nu} = Q[\psi]_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} H_{\mu\nu} (H^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi). \quad (33)$$

The energy density associated to an arbitrary timelike vectorfield  $K$  is given by  $Q(K, T)$ . We consider also the modified energy density,

$$\bar{Q}(K, T) = \bar{Q}[\psi](K, T) = Q[\psi](K, T) + 2t\psi T(\psi) - \psi^2 T(t). \quad (34)$$

and the total conformal energy,

$$\mathcal{Q}[\psi](t) = \int_{\Sigma_t} \bar{Q}[\psi](K, T). \quad (35)$$

<sup>15</sup>recall that in fact  $\mathbf{g}$  is  $\phi^{-1}$ . Thus, in view of the non degenerate Lorentzian character of  $\mathbf{g}$  the bootstrap assumption for  $\phi$  reads as an assumption for  $\mathbf{g}$ .



We recall below the statement of the main generalized energy estimate upon which we rely.

**Proposition 3.1.** *Let  $K$  be an arbitrary vectorfield with deformation tensor*

$${}^{(K)}\pi_{\mu\nu} = \mathcal{L}_K H_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu$$

and  $\psi$  a solution of  $\square_H \psi = 0$ . Then

$$\mathcal{Q}[\psi](t) = \mathcal{Q}[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbb{R}^3} Q^{\alpha\beta} {}^{(K)}\bar{\pi}_{\alpha\beta} + \frac{1}{4} \int_{[t_0, t] \times \mathbb{R}^3} \psi^2 \square_H \Omega \quad (36)$$

where

$${}^{(K)}\bar{\pi} = {}^{(K)}\pi - \Omega H \quad (37)$$

and  $\Omega$  an arbitrary function.

*Remark 3.2.* In the particular case of the Minkowski spacetime we can choose  $K$  to be the conformal timelike Killing vectorfield

$$K = \frac{1}{2} \left( (t+r)^2 (\partial_t + \partial_r) + (t-r)^2 (\partial_t - \partial_r) \right).$$

In his case we can choose  $\Omega = 4t$  and obtain the total conservation law,

$$\mathcal{Q}[\psi](t) = \mathcal{Q}[\psi](t_0).$$

This conservation law can be used to get the desired decay estimate for the free wave equation, see [Kl1].

As in [Kl-Ro] we construct a special vectorfield  $K$  whose modified deformation tensor  ${}^{(K)}\bar{\pi}$  is such that we can control the error terms

$$\int_{[t_0, t] \times \mathbb{R}^3} Q^{\alpha\beta} {}^{(K)}\bar{\pi}_{\alpha\beta} + \frac{1}{4} \int_{[t_0, t] \times \mathbb{R}^3} \psi^2 \square_H \Omega.$$

As in [Kl-Ro] we set<sup>16</sup>

$$K = \frac{1}{2} n(u^2 \underline{L} + \underline{u}^2 L) \quad (38)$$

with  $u, \underline{u}, L, \underline{L}$  defined as follows:

- *Optical function  $u$*

This is an outgoing solution of the Eikonal equation

$$H^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0 \quad (39)$$

with initial conditions  $u(\Gamma_t) = t$  on the time axis. The time axis is defined as the integral curve of the forward unit normal  $T$  to the hypersurfaces  $\Sigma_t$ . The point  $\Gamma_t$  is the intersection between  $\Gamma$  and  $\Sigma_t$ . The level surfaces of  $u$ , denoted  $C_u$  are outgoing null cones with vertices on the time axis. Clearly,

$$T(u) = |\nabla u|_h \quad (40)$$

where  $h$  is metric induced by  $H$  on  $\Sigma_t$ ,  $|\nabla u|_h^2 = \sum_{i=1}^3 |e_i(u)|^2$  relative to an orthonormal frame  $e_i$  on  $\Sigma_t$ .

<sup>16</sup>Observe that this definition of  $K$  differs from the one in [Kl-Ro] by an important factor of  $n$ .

- *Canonical null pair  $L, \underline{L}$*

$$L = bL' = T + N, \quad \underline{L} = 2T - L = T - N \quad (41)$$

with  $L' = -H^{\alpha\beta}\partial_\beta u \partial_\alpha$  the geodesic null generator of  $C_u$ ,  $b$  the lapse of the null foliation (or shortly null lapse) defined by

$$b^{-1} = -\langle L', T \rangle = T(u), \quad (42)$$

and  $N$  exterior unit normal, along  $\Sigma_t$ , to the surfaces  $S_{t,u}$ , i.e. the surfaces of intersection between  $\Sigma_t$  and  $C_u$ . We shall also use the notation

$$e_3 = \underline{L}, \quad e_4 = L$$

- *The function  $\underline{u} = -u + 2t$ .*
- *The  $S_{t,u}$  foliation*

The intersection between the level hypersurfaces<sup>17</sup> and  $u$  form compact 2- Riemannian surfaces denoted by  $S_{t,u}$ . We define  $r(t,u)$  by the formula  $\text{Area}(S_{t,u}) = 4\pi r^2$ . We denote by  $\nabla$  the induced covariant derivative on  $S_{t,u}$ . A vectorfield  $X$  is called  $S$ -tangent if it is tangent to  $S_{t,u}$  at every point. Given an  $S$ -tangent vectorfield  $X$  we denote by  $\nabla_N X$  the projection on  $S_{t,u}$  of  $\nabla_N X$ .

With the help of these constructions the proof of the  $L^2 - L^\infty$  decay estimate stated in theorem 2.1 can be reduced to the following:

**Theorem 3.3** (Boundedness Theorem). *Consider the Lorentz metric  $H = H_{(\lambda)}$  as in (22) verifying, in particular, the properties of proposition 2.4 in the region  $[0, t_*] \times \mathbb{R}^3$ ,  $t_* \leq \lambda^{1-8\epsilon_0}$ . Let  $\psi$  be a solution of the wave equation*

$$\square_H \psi = \frac{1}{\sqrt{|H|}} \partial_\alpha (H^{\alpha\beta} \sqrt{|H|} \partial_\beta \psi) = 0 \quad (43)$$

with initial data  $\psi[t_0]$ , at  $t = t_0 > 2$ , supported in the geodesic ball  $B_{\frac{1}{2}}(0)$ . Let  $\mathcal{D}_u$  be the region determined by  $u > u'$  in the slab  $[0, t_*] \times \mathbb{R}^3$ . For all  $t_0 \leq t \leq t_*$ ,  $\psi(t)$  is supported in  $\mathcal{D}_{t_0-1} \subset \mathcal{D}_0$  and

$$\mathcal{Q}[\psi](t) \lesssim \mathcal{Q}[\psi](t_0).$$

We consider also the auxiliary energy type quantity,

$$\mathcal{E}[\psi](t) = \mathcal{E}^{(i)}[\psi](t) + \mathcal{E}^{(e)}[\psi](t) \quad (44)$$

where,

$$\begin{aligned} \mathcal{E}^{(i)}[\psi](t) &= \int_{\Sigma_t} (1 - \zeta)(t^2 |\partial\psi|^2 + \psi^2) \\ \mathcal{E}^{(e)}[\psi](t) &= \int_{\Sigma_t} \zeta (\underline{u}^2 (L\psi)^2 + u^2 (\underline{L}\psi)^2 + \underline{u}^2 |\nabla\psi|^2 + \psi^2). \end{aligned}$$

with  $\zeta$  is a smooth cut-off function equal to 1 in the wave zone region  $u \leq \frac{t}{2}$ .

In the proof of theorem 3.3 we need the following comparison between the quantity  $\mathcal{Q}(t)$  and the auxiliary norm  $\mathcal{E}(t) = \mathcal{E}[\psi](t)$ .

<sup>17</sup>The level hypersurfaces of  $u$  are outgoing null cones  $C_u$  with vertices on the time axis  $\Gamma_t$ .

**Theorem 3.4** (Comparison Theorem). *Under the same assumptions as in theorem 3.3 we have, for any  $1 \leq t \leq t_*$ ,*

$$\mathcal{E}[\psi](t) \lesssim \mathcal{Q}[\psi](t).$$

#### 4. ASYMPTOTICS THEOREM AND OTHER GEOMETRIC TOOLS

In this section we record the crucial properties of all the important geometric objects associated to our spacetime foliations  $\Sigma_t$ ,  $C_u$  and  $S_{t,u}$  introduced above. Most of the results of this section will be proved only in the second part of this work.

We start with some simple facts concerning the parameters of the foliation  $\Sigma_t$  relative to the spacetime geometry associated to the metric  $H = H_\lambda$ .

*The  $\Sigma_t$  foliation* Recall, see (23), that the parameters of the  $\Sigma_t$  foliation are given by  $n, v$ , the induced metric  $h$  and the second fundamental form  $k_{ij}$ , according to the decomposition,

$$H = -n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt), \quad (45)$$

with  $h_{ij}$  the induced Riemannian metric on  $\Sigma_t$ ,  $n$  the lapse and  $v = v^i \partial_i$  the shift of  $H$ . Denoting by  $T$  the unit, future oriented, normal to  $\Sigma_t$  and  $k$  the second fundamental form  $k_{ij} = -\langle \mathbf{D}_i T, \partial_j \rangle$  we find,

$$\begin{aligned} \partial_t &= nT + v, & \langle \partial_t, v \rangle &= 0 \\ k_{ij} &= -\frac{1}{2} \mathcal{L}_T H_{ij} = -12n^{-1}(\partial_t h_{ij} - \mathcal{L}_v h_{ij}) \end{aligned} \quad (46)$$

with  $\mathcal{L}_X$  denoting the Lie derivative with respect to the vectorfield  $X$ . We also have the following, see (8), (24), and (137) in section 8:

$$c|\xi|^2 \leq h_{ij} \xi^i \xi^j \leq c^{-1}|\xi|^2, \quad c \leq n^2 - |v|_h^2 \quad (47)$$

for some  $c > 0$ . Also

$$n, |v| \lesssim 1 \quad (48)$$

$$|\partial n| + |\partial v| + |\partial h| + |k| \lesssim |\partial H| \quad (49)$$

*$S_{t,u}$ -foliation* We define the Ricci coefficients associated to the  $S_{t,u}$  foliation and null pair  $L, \underline{L}$ .

**Definition 4.1.** Using an arbitrary orthonormal frame  $(e_A)_{A=1,2}$  on  $S_{t,u}$  we define the following tensors on the surfaces  $S_{t,u}$

$$\begin{aligned} \chi_{AB} &= \langle \mathbf{D}_A e_4, e_B \rangle, & \underline{\chi}_{AB} &= \langle \mathbf{D}_A e_3, e_B \rangle, \\ \eta_A &= \frac{1}{2} \langle \mathbf{D}_3 e_4, e_A \rangle, & \underline{\eta}_A &= \frac{1}{2} \langle \mathbf{D}_4 e_3, e_A \rangle, \\ \underline{\xi}_A &= \frac{1}{2} \langle \mathbf{D}_3 e_3, e_A \rangle. \end{aligned} \quad (50)$$

Using the parameters  $n, v, k$  of the  $\Sigma_t$  foliation we find(see [Kl-Ro2] and [Kl-Ro]),

$$\begin{aligned}\underline{\chi}_{AB} &= -\chi_{AB} - 2k_{AB} \\ \underline{\eta}_A &= -k_{AN} + n^{-1}\nabla_A n \\ \underline{\xi}_A &= k_{AN} - \eta_A + n^{-1}\nabla_A n \\ \eta_A &= b^{-1}\nabla_A b + k_{AN}\end{aligned}$$

Thus all the Ricci coefficients can be expressed in terms of  $k_{ij}$ ,  $n$ , the scalar function  $b$  and, most important, the Ricci coefficients  $\chi$  and  $\eta$ .

We shall also denote by  $\theta_{AB} = \langle \nabla_A N, e_B \rangle$  the second fundamental form of  $S_{t,u}$  relative to  $\Sigma_t$ . It is easy to check that

$$\chi_{AB} = -k_{AB} + \theta_{AB}.$$

We consider the parameters  $b$ ,  $\text{tr}\chi$ ,  $\hat{\chi}$  and  $\eta$  associated to the  $S_{t,u}$  foliation according to (42) and (50). For convenience we shall introduce the quantity:

$$\Theta = |\text{tr}\chi - \frac{2}{r}| + |\text{tr}\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta| \quad (51)$$

*Remark 4.2.* Strictly speaking we need only one of the two quantities  $|\text{tr}\chi - \frac{2}{r}|$ ,  $|\text{tr}\chi - \frac{2}{n(t-u)}|$  in the expression above. Indeed we show in [Kl-Ro2] that these two are comparable.

*Remark 4.3.* Simple calculations based on the definition 4.1, see also Ricci equations in section 2 of [Kl-Ro2], allow us to derive the following:

$$|DL|, |D\underline{L}|, |\nabla N| \lesssim r^{-1} + \Theta + |\partial H| \quad (52)$$

*Remark 4.4.* We shall make use of the following simple commutation estimates, see lemma 3.5 in [Kl-Ro2],

$$|(\nabla_N \nabla - \nabla \nabla_N)f| \lesssim (r^{-1} + \Theta + |\partial H|)|\nabla f| \quad (53)$$

We state below the crucial theorem which establishes the desired asymptotic behavior of these quantities relative to  $\lambda$ .

**Theorem 4.5** (Asymptotics Theorem). *In the spacetime region  $\mathcal{D}_0$  ( see theorem 3.3) the quantities  $b$ ,  $\Theta$  satisfy the following estimates:*

$$|b - n| \lesssim \lambda^{-4\epsilon_0} \quad (54)$$

$$\|\Theta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}, \quad (55)$$

$$\|\Theta\|_{L^q(S_{t,u})} \lesssim \lambda^{-3\epsilon_0}. \quad (56)$$

*In addition, in the exterior region  $u \leq t/2$ ,*

$$\|\Theta\|_{L^\infty(S_{t,u})} \lesssim t^{-1}\lambda^{-\epsilon_0} + \lambda^\epsilon \|\partial H(t)\|_{L_x^\infty}. \quad (57)$$

*for an arbitrarily small  $\epsilon > 0$ .*

We also have the following estimates for the derivatives of  $\text{tr}\chi$ :

$$\| \sup_{u \leq \frac{t}{2}} \|\underline{L}(\text{tr}\chi - \frac{2}{r})\|_{L^2(S_{t,u})} \|_{L_t^1} + \| \sup_{u \leq \frac{t}{2}} \|\underline{L}(\text{tr}\chi - \frac{2}{n(t-u)})\|_{L^2(S_{t,u})} \|_{L_t^1} \leq \lambda^{-3\epsilon_0}, \quad (58)$$

$$\| \sup_{u \leq \frac{t}{2}} \|\nabla \text{tr}\chi\|_{L^2(S_{t,u})} \|_{L_t^1} + \| \sup_{u \leq \frac{t}{2}} \|\nabla(\text{tr}\chi - \frac{2}{n(t-u)})\|_{L^2(S_{t,u})} \|_{L_t^1} \leq \lambda^{-3\epsilon_0} \quad (59)$$

In addition we also have weak estimates of the form,

$$\sup_{u \leq \frac{t}{2}} \|(\nabla, \underline{L})(\text{tr}\chi - \frac{2}{n(t-u)})\|_{L^\infty(S_{t,u})} \lesssim \lambda^C \quad (60)$$

for some large value of  $C$ .

We also have the following comparison between the functions  $r$  and  $t - u$ ,

$$c^{-1} \leq \frac{r}{t-u} \leq c \quad (61)$$

The proof of the Asymptotics Theorem is truly at the heart of this work and it is quite involved. Our second paper [Kl-Ro2] is almost entirely dedicated to it.

*Remark 4.6.* Observe that the estimate (55) holds true also for  $\partial H$ . We shall show, see [Kl-Ro2] proposition 7.4, that the  $\partial H$  also verifies the estimate (56). Thus we can incorporate the term  $|\partial H|$  in the definition (51) of  $\Theta$ .

$$\Theta = |\text{tr}\chi - \frac{2}{r}| + |\text{tr}\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta| + |\partial H| \quad (62)$$

We shall do this freely throughout this paper.

The proof of the next proposition will be delayed to [Kl-Ro3], see also [Kl-Ro].

**Proposition 4.7.** *Let  $S_{t,u}$  be a fixed surface in  $\Sigma_t \cap \mathcal{D}_0$ .*

**i.) Isoperimetric inequality** *For any smooth function  $f : S_{t,u} \rightarrow \mathbb{R}$  we have the following isoperimetric inequality:*

$$\left( \int_{S_{t,u}} |f|^2 \right)^{\frac{1}{2}} \lesssim \int_{S_{t,u}} (|\nabla f| + |\text{tr}\theta||f|). \quad (63)$$

**ii.) Sobolev Inequality** *For any  $\delta \in (0, 1)$  and  $p$  from the interval  $p \in (2, \infty]$*

$$\sup_{S_{t,u}} |f| \lesssim r^{\frac{\epsilon(p-2)}{2p+\delta(p-2)}} \left( \int_{S_{t,u}} (|\nabla f|^2 + r^{-2}|f|^2) \right)^{\frac{1}{2} - \frac{\delta p}{2p+\delta(p-2)}} \left[ \int_{S_{t,u}} (|\nabla f|^p + r^{-p}|f|^p) \right]^{\frac{2\delta}{2p+\delta(p-2)}}, \quad (64)$$

**iii.) Trace Inequality** *For an arbitrary function  $f : \Sigma_t \rightarrow \mathbf{R}$  such that  $f \in$*

$H^{\frac{1}{2}+\epsilon}(\mathbf{R}^3)$  we have,

$$\|f\|_{L^2(S_{t,u})} \leq \|\partial^{\frac{1}{2}+\epsilon} f\|_{L^2(\Sigma_t)} + \|\partial^{\frac{1}{2}-\epsilon} f\|_{L^2(\Sigma_t)}. \quad (65)$$

More generally, for any  $q \in [2, \infty)$

$$\|f\|_{L^q(S_{t,u})} \leq \|\partial^{\frac{3}{2}-\frac{2}{q}+\epsilon} f\|_{L^2(\Sigma_t)} + \|\partial^{\frac{3}{2}-\frac{2}{q}-\epsilon} f\|_{L^2(\Sigma_t)}. \quad (66)$$

Also, considering the region  $Ext_t = \Sigma_t \cap \{0 \leq u \leq \frac{t}{2}\}$ , we have the following:

$$\|f\|_{L^2(S_{t,u})}^2 \leq \|N(f)\|_{L^2(Ext_t)} \|f\|_{L^2(Ext_t)} + \frac{1}{t} \|f\|_{L^2(Ext_t)}. \quad (67)$$

We shall make use of the following, see lemma 6.3 in [Kl-Ro].

**Proposition 4.8.** *The following inequality holds for all  $t \in [1, t_*]$  and  $2 < p < \infty$ :*

$$\int_{\Sigma_t} V^2 w^2 \leq t^{\frac{2}{p}} \sup_u \|V\|_{L^{2p'}(S_{t,u})}^2 \int_{\Sigma_t} (|\nabla w|^2 + r^{-2}|w|^2). \quad (68)$$

where  $p'$  is the exponent dual to  $p$ .

We shall also make use of the form,

$$\int_{\Sigma_t} V^2 w^2 \leq t^{\frac{2}{p}} \|V\|_{L_x^\infty}^{\frac{2}{p}} \sup_u \|V\|_{L^2(S_{t,u})}^{\frac{2}{p'}} \int_{\Sigma_t} (|\nabla w|^2 + r^{-2}|w|^2). \quad (69)$$

In particular, if  $\|V\|_{L_x^\infty}$  is bounded by some positive power of  $\lambda$ , and we restrict ourselves to the exterior region  $Ext_t$ , we deduce that for every  $\varepsilon > 0$  and some constant  $C$

$$\int_{Ext_t} V^2 w^2 \leq t^{-2} \lambda^{C\varepsilon} \sup_{0 \leq u \leq t/2} \|V\|_{L^2(S_{t,u})}^{2-\varepsilon} \mathcal{E}[w](t). \quad (70)$$

**Proof** The proof is straightforward and relies only on the isoperimetric inequality (63), see also 6.1. in [Kl-Ro].  $\blacksquare$

## 5. PROOF OF THE BOUNDEDNESS THEOREM

We first calculate the components of the modified<sup>18</sup> deformation tensor  $\bar{\pi} = {}^{(K)}\bar{\pi} = {}^{(K)}\pi - 4tH$  of our vectorfield  $K = \frac{1}{2}n(\underline{u}^2 L + u^2 \underline{L})$ . Recall that  $\underline{u} = 2t - u$  and  $\underline{L} = -L + 2T$ , thus

$$\begin{aligned} \underline{L}(u^2) &= 4u b^{-1}, \\ L(\underline{u}^2) &= 4\underline{u} n^{-1}, \\ \underline{L}(\underline{u}^2) &= 4\underline{u} (n^{-1} - b^{-1}). \end{aligned}$$

<sup>18</sup>corresponding to the choice  $\Omega = 4t$ .

Proceeding as in section 6.1 of [Kl-Ro] we calculate the null components of  $\bar{\pi} = {}^{(K)}\bar{\pi}$  relative<sup>19</sup> to  $e_4 = L, e_3 = \underline{L}$  and  $(e_A)_{A=1,2}$  an arbitrary orthonormal frame on  $S_{t,u}$  find,

$$\begin{aligned}
\bar{\pi}_{44} &= 2u^2 n(\bar{k}_{NN} - n^{-1}e_4(n)), \\
\bar{\pi}_{34} &= 4un(n^{-1} - b^{-1}) + \underline{u}^2 n(\bar{k}_{NN} - n^{-1}e_4(n)) + u^2 n(\bar{k}_{NN} - n^{-1}e_3(n)), \\
\bar{\pi}_{33} &= -8\underline{u}n(n^{-1} - b^{-1}) - 2\underline{u}^2 n(\bar{k}_{NN} + n^{-1}e_3(n)), \\
\bar{\pi}_{3A} &= \underline{u}^2 n(\eta_A + k_{AN} - n^{-1}\nabla_A n) + u^2 n\underline{\xi}_A, \\
\bar{\pi}_{4A} &= u^2 n(\underline{\eta}_A - k_{AN} - n^{-1}\nabla_A n), \\
\bar{\pi}_{AB} &= 2tn(t-u)\left(\text{tr}\chi - \frac{2}{n(t-u)}\right)\delta_{AB} + 4tn(t-u)\hat{\chi}_{AB} - 2u^2 nk_{AB}
\end{aligned} \tag{71}$$

The following proposition concerning the behavior of the null components of  $\bar{\pi}$  is an immediate consequence of the above formulae and the Asymptotics Theorem stated above.

**Proposition 5.1.**

$$\begin{aligned}
\|u^{-2}\bar{\pi}_{44}\|_{L_t^1 L_x^\infty} &\lesssim \lambda^{-3\epsilon_0}, & \|(u\underline{u})^{-1}\bar{\pi}_{34}\|_{L_t^1 L_x^\infty} &\lesssim \lambda^{-3\epsilon_0}, \\
\|(\underline{u})^{-2}\bar{\pi}_{33}\|_{L_t^1 L_x^\infty} &\lesssim \lambda^{-3\epsilon_0}, & \|(\underline{u})^{-2}\bar{\pi}_{3A}\|_{L_t^1 L_x^\infty} &\lesssim \lambda^{-3\epsilon_0}, \\
\|(u)^{-2}\bar{\pi}_{4A}\|_{L_t^1 L_x^\infty} &\lesssim \lambda^{-3\epsilon_0}, & \|(\underline{u})^{-2}\bar{\pi}_{AB}\|_{L_t^1 L_x^\infty} &\lesssim \lambda^{-3\epsilon_0}.
\end{aligned}$$

The proof of the Boundedness theorem relies on the generalized energy identity (36) with  $K = \frac{1}{2}n(u^2\underline{L} + \underline{u}^2L)$  and  $\Omega = 4t$ . Thus,

$$\begin{aligned}
\mathcal{Q}[\psi](t) &= \mathcal{Q}[\psi](t_0) - \frac{1}{2} \int_{[t_0, t] \times \mathbb{R}^3} Q^{\alpha\beta} {}^{(K)}\bar{\pi}_{\alpha\beta} + \int_{[t_0, t] \times \mathbb{R}^3} \psi^2 \square_H t \\
&= \mathcal{Q}[\psi](t_0) - \frac{1}{2} \mathcal{J} + \mathcal{Y}
\end{aligned} \tag{72}$$

Observe that we can decompose:

$$\begin{aligned}
\mathcal{J} &= \int_{[t_0, t] \times \mathbb{R}^3} Q^{\alpha\beta} [\psi] \bar{\pi}_{\alpha\beta} = \int_{[t_0, t] \times \mathbb{R}^3} \left( \frac{1}{4} \bar{\pi}_{33} (L\psi)^2 + \frac{1}{4} \bar{\pi}_{44} (\underline{L}\psi)^2 + \frac{1}{2} \bar{\pi}_{34} |\nabla\psi|^2 \right. \\
&\quad \left. - \bar{\pi}_{4A} \underline{L}\psi \nabla_A \psi - \bar{\pi}_{3A} L\psi \nabla_A \psi + \bar{\pi}_{AB} \nabla_A \psi \nabla_B \psi + \text{tr}\bar{\pi} \left( \frac{1}{2} \underline{L}\psi L\psi - |\nabla\psi|^2 \right) \right).
\end{aligned}$$

Consider, for example,  $I = \int_{[t_0, t] \times \mathbb{R}^3} \bar{\pi}_{4A} \underline{L}\psi \nabla_A \psi$ . We can estimate it as follows :

$$\begin{aligned}
I &\leq \frac{1}{2} \int_{[t_0, t] \times \mathbb{R}^3} |(u\underline{u})^{-1} \bar{\pi}_{4A}| \left( u^2 (\underline{L}\psi)^2 + \underline{u}^2 (\nabla_A \psi)^2 \right) \\
&\leq \int_{t_0}^t \|(u\underline{u})^{-1} \bar{\pi}_{4A}\|_{L_x^\infty} \mathcal{E}[\psi](\tau) d\tau
\end{aligned}$$

Making use of the comparison theorem and the estimate  $\|(u\underline{u})^{-1} \bar{\pi}_{4A}\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-3\epsilon_0}$  we infer that,

$$I \leq \int_{t_0}^t \|(u\underline{u})^{-1} \bar{\pi}_{4A}\|_{L_x^\infty} \mathcal{Q}[\psi](\tau) d\tau \lesssim \lambda^{-3\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau)$$

<sup>19</sup>We say that  $(e_i)_{i=1,2,3,4}$  forms a null frame.

We can proceed in the same manner with all the terms of  $\mathcal{J}$  with the exception of  $\int_{[t_0, t] \times \mathbf{R}^3} \text{tr} \bar{\pi} \underline{L}\psi L\psi$ . Observe that<sup>20</sup>

$$\text{tr} \bar{\pi} = \delta^{AB} \bar{\pi}_{AB} = 2tn(t-u) \left( \text{tr} \chi - \frac{2}{n(t-u)} \right) - 2u^2 \text{tr} k$$

$$\int_{[t_0, t] \times \mathbf{R}^3} |u^2 \text{tr} k L\psi \underline{L}\psi| \leq \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} |\text{tr} k| (u^2 (L\psi)^2 + u^2 (\underline{L}\psi)^2) \lesssim \int_{t_0}^t \|\partial H\|_{L_x^\infty} \mathcal{E}[\psi](\tau) d\tau$$

Since  $\|\partial H\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-4\epsilon_0}$ , this term can be treated in the same manner as  $I$ . We are thus left with the integral

$$\mathcal{B} = \int_{[t_0, t] \times \mathbf{R}^3} 2tn(t-u) \left( \text{tr} \chi - \frac{2}{n(t-u)} \right) \underline{L}\psi L\psi$$

All other terms  $\mathcal{J} - \mathcal{B}$  can be estimated in precisely the same manner, using the comparison theorem and the estimates of theorem 5.1, by

$$\mathcal{J} - \mathcal{B} \lesssim \lambda^{-3\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (73)$$

To estimate the remaining term  $\mathcal{B}$  requires a more involved argument. In fact we shall need more information concerning the geometry of the null cones  $C_u$  and surfaces  $S_{t,u}$ .

Denote  $Ext_t$  the exterior region  $Ext_t = \{0 \leq u \leq t/2\}$ . Let  $\zeta$  be a smooth cut-off function with support in  $Ext_t$ . Observe that

$$\int_{\Sigma_t} (t^2 (\partial\psi)^2 + \psi^2) (1 - \zeta) \lesssim \int_{\Sigma_t} (1 - \zeta) \bar{Q}[\psi](t) \quad (74)$$

We can split the remaining integral

$$\begin{aligned} \mathcal{B} &= \mathcal{B}^i + \mathcal{B}^e \\ \mathcal{B}^i &= \int_{[t_0, t] \times \mathbf{R}^3} 2tn(t-u) \left( \text{tr} \chi - \frac{2}{n(t-u)} \right) L\psi \underline{L}\psi (1 - \zeta) \\ \mathcal{B}^e &= \int_{[t_0, t] \times \mathbf{R}^3} 2tn(t-u) \left( \text{tr} \chi - \frac{2}{n(t-u)} \right) L\psi \underline{L}\psi \zeta \end{aligned}$$

With the help of (74) the first integral can be estimated as follows:

$$\begin{aligned} \mathcal{B}^i &\lesssim \int_{[t_0, t] \times \mathbf{R}^3} \left| \text{tr} \chi - \frac{2}{n(\tau-u)} \right| \tau^2 (\partial\psi)^2 (1 - \zeta) \\ &\lesssim \int_{t_0}^t \left\| \text{tr} \chi - \frac{2}{n(\tau-u)} \right\|_{L_x^\infty} \bar{Q}[\psi](\tau) d\tau \end{aligned}$$

In view of the estimate  $\left\| \text{tr} \chi - \frac{2}{n(t-u)} \right\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-3\epsilon_0}$ , given by the Asymptotics Theorem (4.5) we infer that,

$$\mathcal{B}^i \lesssim \lambda^{-3\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau)$$

Therefore, it remains to estimate  $\mathcal{B}^e$ .

<sup>20</sup>We use  $tr$  here to denote the trace relative to the surfaces  $S_{t,u}$ . Thus  $\text{tr} k = \delta^{AB} k_{AB}$ . We use  $\text{Tr} k = h^{ij} k_{ij}$  to denote the usual trace of  $k$  with respect to  $\Sigma_t$ .



According to the Asymptotics Theorem the quantity  $z = \text{tr}\chi - \frac{2}{n(t-u)}$  verifies the following estimates:

$$\|z\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}, \quad \|z\|_{L^2(S_{t,u})} \lesssim \lambda^{-2\epsilon_0}, \quad (75)$$

$$\|\sup_{u \leq \frac{t}{2}} \|\nabla z\|_{L^2(S_{t,u})}\|_{L_t^2} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}, \quad \|\sup_{u \leq \frac{t}{2}} \|\underline{L}z\|_{L^2(S_{t,u})}\|_{L_t^2} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}. \quad (76)$$

*Remark 5.2.* The same estimates hold true if we replace  $\text{tr}\chi - \frac{2}{n(t-u)}$  by  $\text{tr}\chi - \frac{2}{r}$ .

It would therefore suffice to prove the following result. Using the estimates (75)–(76) we shall prove that:

$$\mathcal{B}^e = \int_{[t_0, t] \times \mathbf{R}^3} 2tn(t-u)zL\psi\underline{L}\psi\zeta \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (77)$$

To prove (77) we need to rely on the fact that  $\psi$  is a solution of the wave equation  $\square_H \psi = 0$ . We shall also make use of the following standard integration by parts formulae<sup>21</sup>,

$$\int_{\Sigma_t} FN(G) = - \int_{\Sigma_t} \left( N(F) + (\text{tr}\theta + n^{-1}N(n))F \right) G, \quad (78)$$

where  $N$  is the unit normal to  $S_{t,u}$ .

If  $Y$  is a vectorfield in  $T\Sigma_t$  tangent to  $S_{t,u}$  then

$$\int_{\Sigma_t} F \text{div} Y = - \int_{\Sigma_t} \left( \nabla F + (b^{-1}\nabla b + n^{-1}\nabla n)F \right) \cdot Y. \quad (79)$$

It is also not difficult to verify that

$$\int_{[t_0, t] \times \mathbf{R}^3} FT(G) = - \int_{[t_0, t] \times \mathbf{R}^3} (T(F) + \text{Tr}k + \text{div}v)G + \int_{\Sigma_t} FG - \int_{\Sigma_{t_0}} FG \quad (80)$$

Writing  $\underline{L} = T - N$  we integrate by parts and express the integral  $\mathcal{B}^e$  in the form,

$$\mathcal{B}^e = -I_1 + I_2 + I_3 - I_4 \quad (81)$$

$$I_1 = \int_{[t_0, t] \times \mathbf{R}^3} \zeta nt(t-u)z(\underline{L}L\psi)\psi$$

$$I_2 = \int_{[t_0, t] \times \mathbf{R}^3} \left( -\underline{L}(\zeta nt(t-u)z) + (\text{tr}\theta + n^{-1}N(n) - \text{Tr}k - \text{div}v)\zeta nt(t-u)z \right) L\psi\psi$$

$$I_3 = \int_{\Sigma_t} \zeta nt(t-u)zL\psi\psi$$

$$I_4 = \int_{\Sigma_{t_0}} \zeta nt(t-u)zL\psi\psi$$

We first handle the boundary terms  $I_3, I_4$ . With the help of proposition 4.8( which we can apply in view of the estimates (57) for  $\Theta$  as well as the estimate (26) for  $\partial H$ .) we have

$$\|n(t-u)z\psi\|_{L^2(E_{xt_t})} \lesssim \lambda^{C\epsilon} \sup_{u \leq \frac{t}{2}} \|nz\|_{L^2(S_{t,u})}^{1-\epsilon/2} \mathcal{E}^{\frac{1}{2}}[\psi](t).$$

<sup>21</sup>These are simple adaptations of the formulae in lemma 6.2., [Kl-Ro].

Therefore,

$$\begin{aligned}
\int_{\Sigma_t} |\zeta n t(t-u) z L\psi\psi| &\lesssim \int_{Ext_t} |n(t-u) z t L\psi\psi| \\
&\lesssim \|t L\psi\|_{L^2(\Sigma_t)} \|n(t-u) z \psi\|_{L^2(Ext_t)} \\
&\lesssim \|n(t-u) z \psi\|_{L^2(Ext_t)} \mathcal{E}^{\frac{1}{2}}[\psi](t) \\
&\lesssim \lambda^{C\epsilon} \sup_{s \geq \frac{t}{2}} \|nz\|_{L^2(S_{t,u})}^{1-\epsilon/2} \mathcal{E}[\psi](t) \lesssim \lambda^{-\epsilon_0} \mathcal{E}[\psi](t).
\end{aligned}$$

The last inequality followed from the boundness of  $n$  and (75). Similar estimate holds for the second boundary term  $I_4$ .

To estimate  $I_2$  we observe that, as an immediate consequence of theorem 4.5, we have

$$|\underline{L}(t)|, |\underline{L}(t-u)| \lesssim 1, \quad |\underline{L}(\zeta)| \lesssim t^{-1}$$

Denoting

$$\Theta(t, x) = |\text{tr}\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta| + |\partial H|$$

we easily find,

$$|I_2| \lesssim \int_{t_0}^t \int_{Ext_\tau} \left( \tau^2 |\underline{L}(z)| + \tau |z| + \tau^2 \Theta |z| \right) |L\psi\psi| d\tau$$

To treat the term involving  $\underline{L}(z)$  we proceed as in the case of  $I_1$ ; We estimate  $\int_{Ext_\tau} \tau^2 |\underline{L}(z)| |L\psi\psi| d\tau$  by Cauchy-Schwartz followed by an application of proposition 4.8. The space integral of the other two terms can be estimated as follows:

$$\int_{Ext_\tau} (\tau |z| + \tau^2 \Theta |z|) |L\psi\psi| d\tau \leq (\|z\|_{L_x^\infty} + \tau \|\Theta\|_{L_x^\infty} \|z\|_{L_x^\infty}) \mathcal{E}[\psi](\tau).$$

Consequently, using the inequalities (75)-(76) for  $z$  ( as well as the weak estimate (60)) and the estimates for  $\Theta$  from the Asymptotics Theorem 4.5

$$\begin{aligned}
I_2 &\lesssim \int_{t_0}^t \left( \lambda^{C\epsilon} \sup_{u \leq \frac{\tau}{2}} \|\underline{L}(z)\|_{L^2(S_{t,u})}^{1-\epsilon/2} + \|z\|_{L^\infty(\Sigma_\tau)} + \tau \|\Theta\|_{L^\infty(\Sigma_\tau)} \|z\|_{L^\infty(\Sigma_\tau)} \right) \mathcal{E}[\psi](\tau) d\tau \\
&\lesssim \lambda^{C'\epsilon} \left( \|\sup_{u \leq \frac{\tau}{2}} \|\underline{L}(z)\|_{L^2(S_{t,u})}\|_{L_t^1}^{1-\epsilon/2} + \|z\|_{L_t^1 L_x^\infty} + \lambda \|\Theta\|_{L_t^2 L_x^\infty} \|z\|_{L_t^2 L_x^\infty} \right) \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \\
&\lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau)
\end{aligned}$$

as desired.

It remains therefore to consider  $I_1$ . We shall make use of the fact that  $\psi$  is a solution of the wave equation. This allows us to express the  $\underline{L}L(\psi)$  in terms of the angular laplacian<sup>22</sup>  $\Delta$  and lower order terms. Expressed relative to a null frame the wave operator  $\square_H \psi$  takes the form

$$\square_H \psi = H^{\alpha\beta} \psi_{,\alpha\beta} = -\psi_{,43} + \psi_{,AA},$$

<sup>22</sup>the Laplace-Beltrami operator on  $S_{t,u}$ .

where  $\psi_{;e_i e_j} = e_j(e_i(\psi)) - \mathbf{D}_{e_i} e_j(\psi)$ . We use the Ricci formulas:  $\mathbf{D}_3 e_4 = 2\eta_A e_A + \bar{k}_{NN} e_4$ , and  $\mathbf{D}_B e_A = \nabla_B e_A + \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\underline{\chi}_{AB} e_4$  to derive

$$\square_H \psi = -\underline{L}L\psi + \underline{\Delta}\psi + 2\eta_A \nabla_A \psi + \frac{1}{2}\text{tr}\chi \underline{L}\psi + \left(\frac{1}{2}\text{tr}\underline{\chi} + \bar{k}_{NN}\right)L\psi. \quad (82)$$

As a result of this calculation

$$\begin{aligned} I_1 &= \int_{[t_0, t] \times \mathbf{R}^3} \zeta n\tau(\tau - u)z \underline{L}L\psi \psi = \int_{[t_0, t] \times \mathbf{R}^3} \zeta n\tau(\tau - u)z \underline{\Delta}\psi \psi \\ &+ \frac{1}{2} \int_{[t_0, t] \times \mathbf{R}^3} \zeta n\tau(\tau - u)z \text{tr}\chi(\underline{L}\psi) \psi \\ &+ \int_{[t_0, t] \times \mathbf{R}^3} \zeta n\tau(\tau - u)z \left(2\eta_A \nabla_A \psi + \left(\frac{1}{2}\text{tr}\underline{\chi} + \bar{k}_{NN}\right)L\psi\right) \psi \\ &= I_{11} + I_{12} + I_{13}. \end{aligned} \quad (83)$$

Consider first  $I_{13}$ . Taking into account that  $t - u \geq \frac{t}{2}$

$$\begin{aligned} |I_{13}| &\lesssim \int_{t_0}^t \int_{Ext_\tau} \tau^2 |z| \left( \Theta \nabla \psi + \left(\frac{1}{\tau} + \Theta\right)L\psi \right) \psi \\ &\lesssim \int_{t_0}^t \left( \tau \|z\|_{L^\infty(\Sigma_\tau)} \|\Theta\|_{L^\infty(\Sigma_\tau)} + \|z\|_{L^\infty(\Sigma_\tau)} \right) \mathcal{E}[\psi](\tau) d\tau \\ &\lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \end{aligned} \quad (84)$$

as before.

To estimate  $I_{12}$  we need first to integrate once more by parts.

$$\begin{aligned} I_{12} &= \frac{1}{4} \int_{[t_0, t] \times \mathbf{R}^3} \left( -\underline{L}(\zeta n\tau(\tau - u)z \text{tr}\chi) \right. \\ &\quad \left. + (\text{tr}\theta + n^{-1}N(n) - \text{Tr}k - \text{div}v) \zeta n\tau(\tau - u)z \text{tr}\chi \right) \psi^2 \\ &\quad + \frac{1}{4} \int_{\Sigma_t} \zeta n\tau(\tau - u)z \text{tr}\chi(\psi)^2 - \frac{1}{4} \int_{\Sigma_{t_0}} \zeta n\tau(\tau - u)z \text{tr}\chi(\psi)^2 \end{aligned}$$

All terms can be treated as above. Take, for example, the worst term involving  $\underline{L}(\text{tr}\chi)$ . Recall that

$$\underline{L}(\text{tr}\chi) = \underline{L}\left(\text{tr}\chi - \frac{2}{r}\right) + \underline{L}\left(\frac{2}{r}\right) \lesssim \underline{L}\left(\text{tr}\chi - \frac{2}{r}\right) + \frac{2}{r^2} + \frac{1}{r}\Theta$$

Thus

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} |\zeta n\tau(t - u)z \underline{L}(\text{tr}\chi)(\psi)^2| &\lesssim \int_{t_0}^t \int_{Ext_\tau} \tau^2 |z| \left( \left| \underline{L}\left(\text{tr}\chi - \frac{2}{r}\right) \right| + \frac{1}{\tau^2} + \frac{1}{\tau}\Theta \right) (\psi)^2 \\ &\lesssim \int_{t_0}^t \int_{Ext_\tau} \tau^2 |z| \left| \underline{L}\left(\text{tr}\chi - \frac{2}{r}\right) \right| \psi^2 \\ &\quad + \int_{t_0}^t \left( \|z\|_{L^\infty(\Sigma_\tau)} + \tau \|z\|_{L^\infty(\Sigma_\tau)} \|\Theta\|_{L^\infty(\Sigma_\tau)} \right) \mathcal{E}[\psi](\tau) d\tau \end{aligned}$$

The second term has already been treated above, see (83). To estimate the first we apply first Cauchy-Schwartz and then make use of proposition 4.8,

$$\begin{aligned} \int_{t_0}^t \int_{E_{xt_\tau}} \tau^2 |z| |\underline{L}(\text{tr}\chi - \frac{2}{r})| \psi^2 &\lesssim \int_{t_0}^t \|\tau^2 |z| |\underline{L}(\text{tr}\chi - \frac{2}{r})| \psi\|_{L^2(E_{xt_\tau})} \mathcal{E}^{\frac{1}{2}}[\psi](\tau) d\tau \\ &\lesssim \int_{t_0}^t \lambda^{C\epsilon} \sup_{u \leq \frac{\tau}{2}} \|\tau |z| |\underline{L}(\text{tr}\chi - \frac{2}{r})|\|_{L^2(S_{t,u})}^{1-\epsilon/2} \mathcal{E}[\psi](\tau) d\tau \end{aligned}$$

Taking into account the estimates in (75)–(76) and the Remark 5.2 we deduce,

$$\begin{aligned} \lambda^{C\epsilon} \int_{t_0}^t \sup_{u \leq \frac{\tau}{2}} \|\tau |z| |\underline{L}(\text{tr}\chi - \frac{2}{r})|\|_{L^2(S_{t,u})}^{1-\epsilon/2} &\lesssim \lambda^{C'\epsilon} \left( t \|z\|_{L_t^2 L_x^\infty} \sup_{u \leq \frac{t}{2}} \|\underline{L}(\text{tr}\chi - \frac{2}{r})\|_{L^2(S_{t,u})} \|L_t^2\| \right)^{1-\epsilon/2} \\ &\lesssim \lambda^{-\epsilon_0} \end{aligned}$$

Therefore,

$$|I_{12}| \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (85)$$

Finally we estimate  $I_{11} = \int_{[t_0, t] \times \mathbf{R}^3} \zeta nt(t-u)z \nabla \psi \psi$  by integrating once more by parts as follows:

$$\begin{aligned} I_{11} &= - \int_{[t_0, t] \times \mathbf{R}^3} \zeta nt(t-u)z |\nabla \psi|^2 \\ &\quad - \int_{[t_0, t] \times \mathbf{R}^3} n^{-1} b^{-1} \nabla_A (bn \zeta nt(t-u)z) \nabla_A \psi \psi. \end{aligned}$$

The first integral on the right can be easily estimated

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} \zeta nt(t-u)z |\nabla \psi|^2 &\lesssim \int_{t_0}^t \|z\|_{L_x^\infty} \mathcal{E}[\psi](\tau) d\tau \\ &\lesssim \|z\|_{L_t^1 L_x^\infty} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \\ &\lesssim \lambda^{-3\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (86) \end{aligned}$$

To estimate the second we write schematically

$$\nabla (bn^2 \zeta t(t-u)z) \approx t(t-u)(\nabla b)z + t(t-u)\nabla z + t(t-u)z\Theta = t(t-u)\nabla z + t(t-u)z\Theta$$

since  $\nabla_A b = b(\eta_A - k_{AN})$ . Thus with the help of proposition 4.8 (using also the weak estimate (60)),

$$\begin{aligned} \int_{[t_0, t] \times \mathbf{R}^3} |n^{-1} b^{-1} \nabla_A (bn^2 \zeta \tau(\tau-u)z) \nabla_A \psi \psi| &\lesssim \int_{t_0}^t \int_{E_{xt_\tau}} (\tau |\nabla z| + \tau |z| |\Theta|) |\tau \nabla_A \psi| |\psi| \\ &\lesssim \int_{t_0}^t \left( \lambda^{C\epsilon} \sup_{u \leq \frac{\tau}{2}} \|\nabla z\|_{L^2(S_{t,u})}^{1-\epsilon/2} + \tau \|z\|_{L^\infty(\Sigma_\tau)} \|\Theta\|_{L^\infty(\Sigma_\tau)} \right) \mathcal{E}[\psi](\tau) d\tau. \end{aligned}$$

Using (76) once more we have,

$$\begin{aligned} & \int_{t_0}^t \left( \lambda^{C\epsilon} \sup_{u \leq \frac{\tau}{2}} \|\nabla z\|_{L^2(S_{t,u})}^{1-\epsilon/2} + \tau \|z\|_{L^\infty(\Sigma_\tau)} \|\Theta\|_{L^\infty(\Sigma_\tau)} \right) d\tau \\ & \lesssim \lambda^{C'\epsilon} \|\sup_{u \leq \frac{t}{2}} \|\nabla z\|_{L^2(S_{t,u})}\|_{L_t^1}^{1-\epsilon/2} + t \|z\|_{L_t^2 L_x^\infty} \|\Theta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\epsilon_0} \end{aligned}$$

Therefore, combining with (86) we infer that,

$$I_{11} \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (87)$$

Recalling also (85) and (84) we conclude that

$$I_1 \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (88)$$

Since  $I_2, I_3, I_4$  and  $\mathcal{B}^i$  have already been estimated we finally derive,

$$\mathcal{B} \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (89)$$

as desired. This combined with (73) yields,

$$\mathcal{J} \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (90)$$

Going back to the identity (72) we still have to estimate  $\mathcal{Y}$ . For this we only need to observe that  $\square_H t$  depends only on the first derivatives of  $H$ . Thus also

$$\mathcal{Y} \lesssim \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \quad (91)$$

Therefore,

$$\sup_{[t_0, t]} \mathcal{Q}[\psi](\tau) \leq \mathcal{Q}[\psi](t_0) + \lambda^{-\epsilon_0} \sup_{[t_0, t]} \mathcal{Q}[\psi](\tau)$$

which implies the boundedness theorem.

## 6. PROOF OF THE COMPARISON THEOREM

We proceed precisely as in [Kl-Ro], section 6.1. Define  $S$  and  $\underline{S}$ ,

$$S = \frac{1}{2}(\underline{u}L + u\underline{L}), \quad \underline{S} = \frac{1}{2}(\underline{u}L - u\underline{L}). \quad (92)$$

Since  $\underline{u} = -u + 2t$ ,  $\underline{L} = T - N$ ,  $L = T + N$

$$\begin{aligned} tT &= \frac{1}{4}(u + \underline{u})(L + \underline{L}) = S - \frac{1}{4}(u - \underline{u})(L - \underline{L}) = S - (t - u)N, \\ tT &= \frac{1}{2}t(\underline{L} + L) = \frac{t}{t - u}\underline{S} - \frac{t^2}{t - u}N. \end{aligned}$$

Therefore, with the help of the identities (78), and  $N(t) = 0$ ,  $N(u) = -b^{-1}$

$$\begin{aligned} 2 \int_{\Sigma_t} \psi t T(\psi) &= 2 \int_{\Sigma_t} (\psi(S\psi) - \frac{1}{2}(t-u)N(\psi^2)) \\ &= 2 \int_{\Sigma_t} \psi(S\psi) + \int_{\Sigma_t} \left( b^{-1} + (t-u)(\text{tr}\theta + n^{-1}N(n)) \right) \psi^2, \\ 2 \int_{\Sigma_t} \psi t T(\psi) &= 2 \int_{\Sigma_t} (\psi \frac{t}{t-u}(\underline{S}\psi) - \frac{1}{2} \frac{t^2}{t-u} N(\psi^2)) \\ &= 2 \int_{\Sigma_t} \psi \frac{t}{t-u}(\underline{S}\psi) + \int_{\Sigma_t} \frac{t^2}{(t-u)^2} \left( -b^{-1} + (t-u)(\text{tr}\theta + n^{-1}N(n)) \right) \psi^2. \end{aligned}$$

Recall that  $\theta_{AB} = \chi_{AB} + k_{AB}$ . Recall also that  $\Theta$  was defined in (62).

$$\Theta(t, x) = |\text{tr}\chi - \frac{2}{r}| + |\text{tr}\chi - \frac{2}{n(t-u)}| + |\hat{\chi}| + |\eta| + |\partial H|$$

Thus,

$$\begin{aligned} 2 \int_{\Sigma_t} \psi t T(\psi) &= 2 \int_{\Sigma_t} (\psi(S\psi) + \int_{\Sigma_t} \left( b^{-1} + \frac{2}{n} + (t-u)\Theta \right) \psi^2, \\ 2 \int_{\Sigma_t} \psi t T(\psi) &= 2 \int_{\Sigma_t} (\psi \frac{t}{t-u}(\underline{S}\psi) + \int_{\Sigma_t} \frac{t^2}{(t-u)^2} (-b^{-1} + \frac{2}{n} + (t-u)\Theta) \psi^2. \end{aligned}$$

Recall, from the Asymptotics Theorem 4.5,

$$|b - n| \lesssim \lambda^{-4\epsilon_0}$$

Also, since  $n$  is bounded away from zero so is  $b$ . Therefore,

$$\begin{aligned} 2 \int_{\Sigma_t} \psi t T(\psi) &= 2 \int_{\Sigma_t} (\psi(S\psi) + \int_{\Sigma_t} \left( \frac{3}{n} + (t-u)\Theta + \lambda^{-4\epsilon_0} \right) \psi^2, \\ 2 \int_{\Sigma_t} \psi t T(\psi) &= 2 \int_{\Sigma_t} (\psi \frac{t}{t-u}(\underline{S}\psi) + \int_{\Sigma_t} \frac{t^2}{(t-u)^2} (\frac{1}{n} + (t-u)\Theta + \lambda^{-4\epsilon_0}) \psi^2. \end{aligned}$$

Since

$$\bar{Q}(K, T)[\psi] = \frac{n}{4} (\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (\underline{u}^2 + u^2)|\nabla\psi|^2) + 2t\psi T\psi - n^{-1}\psi^2,$$

and

$$\frac{1}{4} (\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2) = \frac{1}{2} ((S\psi)^2 + (\underline{S}\psi)^2)$$

we can introduce positive constants  $A, B$  :  $A + B = 2$  such that

$$\begin{aligned} \mathcal{Q}[\psi](t) &= \frac{1}{2} \int_{\Sigma_t} \left( n(S\psi)^2 + 2A\psi(S\psi) + \left( \frac{3}{n}A - \frac{2}{n} \right) \psi^2 + ((t-u)\Theta + \lambda^{-4\epsilon_0}) \psi^2 \right) \\ &+ \frac{1}{2} \int_{\Sigma_t} \left( n(\underline{S}\psi)^2 + 2B\psi \frac{t}{t-u}(\underline{S}\psi) + \left( \frac{1}{n} \frac{t^2}{(t-u)^2} B \psi^2 + \frac{t^2}{(t-u)^2} ((t-u)\Theta + \lambda^{-4\epsilon_0}) \psi^2 \right) \right) \\ &+ \frac{1}{2} \int_{\Sigma_t} n(\underline{u}^2 + u^2)|\nabla\psi|^2. \end{aligned}$$

For any values of  $A, B$  such that  $1 < A < 2$  and  $0 < B < 1$  it is possible to find positive constants  $c_1, c_2$  such that

$$\begin{aligned} n(S\psi)^2 + 2A\psi(S\psi) + \frac{1}{n}(3A-2)\psi^2 &\geq c_1((S\psi)^2 + \psi^2), \\ n(\underline{S}\psi)^2 + 2B\psi\frac{t}{t-u}(\underline{S}\psi) + \frac{1}{n}B\frac{t^2}{(t-u)^2}\psi^2 &\geq c_2((\underline{S}\psi)^2 + \frac{t^2}{(t-u)^2}\psi^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{Q}[\psi](t) &\gtrsim \int_{\Sigma_t} (\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (u^2 + \underline{u}^2)|\nabla\psi|^2 + (1 + \frac{t^2}{(t-u)^2})\psi^2 \\ &\quad - \int_{\Sigma_t} (1 + \frac{t^2}{(t-u)^2}) \left( (t-u)\Theta + \lambda^{-4\epsilon_0} \right) \psi^2 \end{aligned}$$

$$\begin{aligned} \mathcal{Q}[\psi](t) &\gtrsim \int_{\Sigma_t} (\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (u^2 + \underline{u}^2)|\nabla\psi|^2 + (1 + \frac{t^2}{(t-u)^2})\psi^2 \\ &\quad - \int_{\Sigma_t} (1 + \frac{t^2}{(t-u)^2})(t-u)\Theta\psi^2 \end{aligned}$$

Therefore it suffices to show that

$$\int_{\Sigma_t} (1 + \frac{t^2}{(t-u)^2})(t-u)\Theta\psi^2 \leq \lambda^{-\epsilon_0} \int_{\Sigma_t} t^2|\nabla\psi|^2 + (1 + \frac{t^2}{(t-u)^2})\psi^2 \quad (93)$$

Consider the worst term

$$\int_{\Sigma_t} \frac{t^2}{(t-u)}\Theta\psi^2 \lesssim \left( \int_{\Sigma_t} t^2\Theta^2\psi^2 \right)^{\frac{1}{2}} \left( \int_{\Sigma_t} \frac{t^2}{(t-u)^2}\psi^2 \right)^{\frac{1}{2}} \quad (94)$$

According to the estimate (68) of proposition 4.8, applied to exponent  $p$  such  $2p' = q$ ,

$$\int_{\Sigma_t} t^2\Theta^2\psi^2 \lesssim t^{2-\frac{4}{q}} \sup_u \|\Theta\|_{L^q(S_{t,u})}^2 t^2 \int_{\Sigma_t} (|\nabla\psi|^2 + r^{-2}|\psi|^2).$$

Or, since according to (61),  $c^{-1} \leq \frac{r}{(t-u)} \leq c$ , and with the help of the estimate (57) for  $\Theta$  with  $q > 2$  sufficiently close to 2,

$$\int_{\Sigma_t} t^2\Theta^2\psi^2 \lesssim \lambda^{-5\epsilon_0} \int_{\Sigma_t} \left( t^2|\nabla\psi|^2 + \frac{t^2}{(t-u)^2}|\psi|^2 \right).$$

Thus, back to (94)

$$\int_{\Sigma_t} \frac{t^2}{(t-u)}\Theta\psi^2 \lesssim \lambda^{-2\epsilon_0} \int_{\Sigma_t} \left( t^2|\nabla\psi|^2 + \frac{t^2}{(t-u)^2}|\psi|^2 \right) \quad (95)$$

as desired in the proof of (93). The remaining term on the left hand side of (93) is easier to treat.

7. PROOF OF THE  $L^2 - L^\infty$  DECAY ESTIMATE; THEOREM 2.1

In this section we rely on the Boundedness Theorem 3.3 to prove the crucial theorem 2.1.

Recall that  $\mathcal{E}[\psi] = \mathcal{E}^i[\psi] + \mathcal{E}^e[\psi]$ , where

$$\begin{aligned}\mathcal{E}^i[\psi](t) &= \int_{\Sigma_t} \left( t^2 |\partial\psi|^2 + |\psi|^2 \right) (1 - \zeta), \\ \mathcal{E}^e[\psi](t) &= \int_{\Sigma_t} \left( \underline{u}^2 |L\psi|^2 + \underline{u}^2 |\nabla\psi|^2 + u^2 |\underline{L}\psi|^2 + |\psi|^2 \right) \zeta\end{aligned}$$

with a cut-off function  $\zeta$  equal to 1 in the region  $u \leq \frac{t}{2}$ .

**Estimate for  $(1 - \zeta)P\psi$ :**

Observe that since the projector  $P$  is an averaging operator on the scale of size 1 and  $(1 - \zeta)$  is a cut-off function with the scale of size  $t \geq 1$ , we can essentially write that  $(1 - \zeta)P\psi \approx P(\psi(1 - \zeta))$ . Thus the Bernstein inequality, followed by the fact  $\|(1 - \zeta)\nabla\psi\|_{L^2(\Sigma_t)} \leq t^{-1}\mathcal{E}^{\frac{1}{2}}[\psi](t)$  and  $|\nabla\zeta| \lesssim t^{-1}$ , implies that

$$\|P(\psi(t))(1 - \zeta)\|_{L^\infty_x} \lesssim \|\nabla(\psi(1 - \zeta))\|_{L^2(\Sigma_t)} \leq t^{-1}\mathcal{E}^{\frac{1}{2}}[\psi](t) \quad (96)$$

as desired.

**Estimate for  $\zeta P\psi$ :** It clearly suffices to establish the estimate for  $P\psi(t, x)$  at any point  $(t, x)$  with  $0 \leq u \leq \frac{t}{4}$ . According to the Sobolev inequality (64), with  $p = 4$ , of proposition 4.7 we have for any positive  $\delta < 1$ ,

$$\sup_{S_{t,u}} |P\psi|^2 \lesssim t^{\frac{4\delta}{4+\delta}} \left( \int_{S_{t,u}} (|\nabla P\psi|^2 + \frac{1}{t^2}|P\psi|^2) \right)^{1-\frac{4\delta}{4+\delta}} \left[ \int_{S_{t,u}} (|\nabla P\psi|^4 + \frac{1}{t^4}|P\psi|^4) \right]^{\frac{2\delta}{4+\delta}}.$$

Using the isoperimetric inequality (63) applied to  $(P\psi)^2$  and  $|\nabla P\psi|^2$ ,

$$\begin{aligned}\left( \int_{S_{t,u}} |P\psi|^4 \right)^{\frac{1}{2}} &\lesssim \left( \int_{S_{t,u}} |\nabla P\psi|^2 \right)^{\frac{1}{2}} \left( \int_{S_{t,u}} |P\psi|^2 \right)^{\frac{1}{2}} + \frac{1}{t} \int_{S_{t,u}} |P\psi|^2, \\ \left( \int_{S_{t,u}} |\nabla P\psi|^4 \right)^{\frac{1}{2}} &\lesssim \left( \int_{S_{t,u}} |\nabla^2 P\psi|^2 \right)^{\frac{1}{2}} \left( \int_{S_{t,u}} |\nabla P\psi|^2 \right)^{\frac{1}{2}} + \frac{1}{t} \int_{S_{t,u}} |\nabla P\psi|^2.\end{aligned}$$

In addition, making use of the trace inequality (65),<sup>23</sup>

$$\int_{S_{t,u}} |f|^2 \lesssim \left( \int_{\text{Ext}_t} |N(f)|^2 \right)^{\frac{1}{2}} \left( \int_{\text{Ext}_t} |f|^2 \right)^{\frac{1}{2}} + \frac{1}{t} \int_{\text{Ext}_t} |f|^2.$$

Here,  $\text{Ext}_t = \Sigma_t \cap \{0 \leq u \leq \frac{t}{2}\}$  and  $N$  is the vectorfield of the unit normals to  $S_{t,t-\rho}$ .

<sup>23</sup>The tensor version of the estimate requires the covariant  $\nabla_N$  derivative. Recall that  $\nabla_N$  denotes the projection on  $S_{t,u}$  of the covariant derivative  $\nabla_N$ .



Thus, setting  $\varepsilon = \frac{4\delta}{4+\delta}$ , using the fact that  $t \geq 1$ , and applying the Hölder inequality, we obtain

$$\sup_{S_{t,u}} |P\psi|^2 \lesssim t^\varepsilon \left( \int_{\text{Ext}_t} |\nabla_N \nabla P\psi|^2 + |\nabla P\psi|^2 + \frac{1}{t^2} (|N(P\psi)|^2 + |P\psi|^2) \right)^{1-\varepsilon} \cdot \mathcal{I}^\varepsilon \quad (97)$$

$$\mathcal{I} = \int_{\Sigma_t} |\nabla_N \nabla^2 P\psi|^2 + |\nabla^2 P\psi|^2 + |\nabla_N \nabla P\psi|^2 + |\nabla P\psi|^2 + \frac{1}{t^4} (|N(P\psi)|^2 + |P\psi|^2).$$

Note that we can always replace the outside  $N$  derivative with a generic derivative  $\partial$ . More precisely,  $|N(f)|^2 \lesssim \sum_i |\partial_i f|^2$ .

We make the following three observations:

1) The derivatives in the second factor  $\mathcal{I}$  can be ignored in view of the presence of the projection  $P$ . Thus we can crudely bound it by  $\mathcal{I} \lesssim \int_{\Sigma_t} |\psi|^2 \leq \mathcal{E}[\psi](t)$ .

2) The terms  $\frac{1}{t^2} \int_{\text{Ext}_t} (|N(P\psi)|^2 + |P\psi|^2)$  are easily estimated by  $t^{-2} \mathcal{E}[\psi](t)$ .

3) It remains to handle the terms

$$\int_{\text{Ext}_t} |\nabla_N \nabla P\psi|^2 + |\nabla P\psi|^2$$

Consider first the integral  $\int_{\text{Ext}_t} |\nabla P\psi|^2$ . Let  $\zeta$  be a cut-off function of the exterior region  $\text{Ext}_t$  such that  $\zeta|_{\text{Ext}_t} = 1$  and  $|\nabla\zeta| \lesssim t^{-1}$ . We introduce the angular vectorfields  $A_i = \zeta(\partial_i - \langle \partial_i, N \rangle N)$ . Clearly, for any scalar function  $f$ ,  $|\nabla f|^2 \approx \sum_{i=1}^3 |A_i f|^2$  in the exterior region  $\text{Ext}_t$ . Now write,

Thus,

$$\begin{aligned} \int_{\text{Ext}_t} |\nabla P\psi|^2 &\approx \sum_{i=1}^3 \int_{\text{Ext}_t} |A_i P\psi|^2 \\ &\lesssim \sum_{i=1}^3 \int_{\text{Ext}_t} |P A_i \psi|^2 + \sum_{i=1}^3 \int_{\text{Ext}_t} |[P, A_i] \psi|^2 \\ &\lesssim \sum_{i=1}^3 \int_{\Sigma_t} |P A_i \psi|^2 + \text{error} \\ &\lesssim \int_{\Sigma_t} |\nabla \psi|^2 + \text{Error} \end{aligned}$$

We estimate the error term  $\int_{\text{Ext}_t} \sum_i |[P, A_i] \psi|^2$  with the help of the following

**Lemma 7.1.** *Consider a vectorfield  $X = \sum_i X^i \partial_i$  vanishing on the complement of the exterior region  $\text{Ext}_t$  of  $\Sigma_t$  and  $P$  the standard Littlewood-Paley projection on*

frequencies of size 1. Then, for arbitrary scalar functions  $f$  we have the inequality<sup>24</sup>:

$$\|[P, X]f\|_{L^2(\text{Ext}_t)} \lesssim \sup_{i,j} \|\partial_i X^j\|_{L^\infty(\text{Ext}_t)} \|f\|_{L^2(\Sigma_t)}$$

**Proof** We postpone the proof until the end of section 8, see lemma 8.38.  $\blacksquare$

We apply the above lemma to the vectorfields  $A_k = \zeta(\delta_k^j - N_k N^j)\partial_j$ . Observe that the components  $A_k^j$  are bounded and  $|\nabla\zeta| \lesssim t^{-1}$ . Thus

$$\text{Error} \lesssim \left( t^{-2} + \|\nabla N\|_{L^\infty(\text{Ext}_t)}^2 \right) \|\psi\|_{L^2(\Sigma_t)}^2$$

Recall the expression, see (62),  $\Theta = |\text{tr}\chi - \frac{2}{r}| + |\hat{\chi}| + |\eta| + |\partial H|$  and the inequality (52)  $|\nabla N| \lesssim \frac{1}{r} + \Theta$ . Observe also that in the exterior region  $\text{Ext}_t$ ,  $\frac{1}{r} \leq \frac{2}{t}$ . Therefore,

$$\text{Error} \lesssim (t^{-1} + \|\Theta\|_{L^\infty(\text{Ext}_t)})^2 \|\psi\|_{L^2(\Sigma_t)}^2$$

We can finally conclude that

$$\begin{aligned} \int_{\text{Ext}_t} |\nabla P \psi|^2 &\lesssim \int_{\Sigma_t} |\nabla \psi|^2 + (t^{-1} + \|\Theta\|_{L^\infty(\text{Ext}_t)})^2 \int_{\Sigma_t} |\psi|^2 \\ &\lesssim (t^{-2} + \|\Theta\|_{L^\infty(\text{Ext}_t)}^2) \mathcal{E}[\psi](t) \end{aligned} \quad (98)$$

We now consider  $\int_{\text{Ext}_t} |\nabla_N \nabla P \psi|^2$ . In view of the simple commutation estimates (53) we can write:

$$\begin{aligned} \int_{\text{Ext}_t} |\nabla_N \nabla P \psi|^2 &\lesssim \int_{\text{Ext}_t} |\nabla(NP\psi)|^2 + \int_{\text{Ext}_t} (r^{-1} + \Theta)^2 |\nabla P \psi|^2 \\ &\approx \sum_{i=1}^3 \int_{\text{Ext}_t} |A_i(NP\psi)|^2 + \int_{\text{Ext}_t} (r^{-1} + \Theta)^2 |\nabla P \psi|^2 \end{aligned}$$

Observe that

$$\begin{aligned} A_i(NP\psi) &= A_i N^j \partial_j (P\psi) = N^j A_i \partial_j (P\psi) + [A_i, N^j] \partial_j (P\psi) \\ &= NP(A_i\psi) + N^j [A_i, \partial_j P] \psi + [A_i, N^j] \partial_j (P\psi) \end{aligned}$$

Therefore, using the lemma 7.1, with  $P$  replaced by  $\nabla P$ , as well as the estimates (52)

$$\int_{\text{Ext}_t} |A_i(NP\psi)|^2 \lesssim \int_{\text{Ext}_t} |A_i\psi|^2 + (t^{-1} + \|\Theta\|_{L^\infty(\text{Ext}_t)})^2 \int_{\Sigma_t} |\psi|^2$$

and finally,

$$\int_{\text{Ext}_t} |\nabla_N \nabla P \psi|^2 \lesssim (t^{-2} + \|\Theta\|_{L^\infty(\text{Ext}_t)}^2) \mathcal{E}[\psi](t) \quad (99)$$

<sup>24</sup>In fact the exterior region on the right hand side of the inequality should be somewhat enlarged( by size one ). Since this enlargement does not affect our arguments we prefer to ignore it.

Substituting (98)-(99) back into (97) we infer that in the exterior region

$$\begin{aligned} \sup_{S_{t,u}} |P\psi|^2 &\lesssim t^\varepsilon (t^{-2} + \|\Theta\|_{L^\infty(\text{Ext}_t)}^2)^{1-\varepsilon} \mathcal{E}^{1-\varepsilon}[\psi](t) \cdot \mathcal{I}^\varepsilon \\ &\lesssim t^\varepsilon (t^{-2} + \|\Theta\|_{L^\infty(\text{Ext}_t)}^2)^{1-\varepsilon} \mathcal{E}[\psi](t) \end{aligned}$$

Finally, together with the interior estimates (96) this implies that

$$\|P\psi(t)\|_{L_x^\infty} \lesssim \left( \frac{1}{(1+t)^{1-2\varepsilon}} + t^\varepsilon \|\Theta\|_{L^\infty(\text{Ext}_t)}^{1-\varepsilon} \right) \mathcal{E}^{\frac{1}{2}}[\psi](t). \quad (100)$$

Observe that according to (57) of the Asymptotics Theorem  $\Theta$  obeys the following estimate in the exterior region:

$$\|\Theta(t)\|_{L^\infty(\text{Ext}_t)} \lesssim t^{-1} \lambda^{-\varepsilon_0} + \lambda^\varepsilon \|\partial H(t)\|_{L_x^\infty}.$$

Define

$$d(t) = t^\varepsilon (\lambda^\varepsilon \|\partial H(t)\|_{L_x^\infty})^{1-\varepsilon}$$

Therefore,

$$\|P\psi(t)\|_{L_x^\infty} \lesssim \left( \frac{1}{(1+t)^{1-2\varepsilon}} + d(t) \right) \mathcal{E}^{\frac{1}{2}}[\psi](t).$$

To prove the desired  $L^2 - L^\infty$  decay estimate it remains to check that for some <sup>25</sup>  $q > 2$ ,

$$t_*^{\frac{1}{q}} \|d\|_{L_{[0,t_*]}^q} \lesssim 1$$

Since  $t_* \leq \lambda^{1-4\varepsilon_0}$  it clearly suffices to show that  $\|d\|_{L_{[0,t_*]}^q} \lesssim \lambda^{-\frac{1}{2}}$ . In view of the estimates, see proposition 2.4,

$$\|\partial H\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}-4\varepsilon_0}, \quad \|\partial H\|_{L_t^\infty L_x^\infty} \lesssim \lambda^{-\frac{1}{2}+\varepsilon_0},$$

we infer that

$$\|d\|_{L_{[0,t_*]}^q} \lesssim t_*^\varepsilon \lambda^\varepsilon \|\partial H\|_{L_t^{q(1-\varepsilon)} L_x^\infty}^{1-\varepsilon} \lesssim t_*^\varepsilon \lambda^\varepsilon \|\partial H\|_{L_t^\infty L_x^\infty}^{1-\frac{2}{q}-\varepsilon} \cdot \|\partial H\|_{L_t^2 L_x^\infty}^{\frac{2}{q}} \lesssim \lambda^{-\frac{1}{2}},$$

as desired.

## 8. PROOF OF THE REDUCTION STEPS

In this section we give precise statements and proofs for the reduction steps discussed in section 2. Recall the equation (3), written in the form (6),

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \phi = N(\phi, \partial\phi) \quad (101)$$

where  $\phi = (\mathbf{g}_{\mu\nu})$ ,  $N = N_{\mu\nu}$  and  $\mathbf{g}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}(\phi)$ . In fact  $(\mathbf{g}^{\alpha\beta}) = \phi^{-1}$ . We consider solutions  $\phi$  of (101) such that the components of both  $\phi$  and  $\phi^{-1}$  are uniformly bounded. Moreover  $\mathbf{g}_{\mu\nu}$  approach the Minkowski metric  $\mathbf{m}_{\mu\nu}$  at infinity according to (5). To avoid repeating this statement in what follows we introduce the following notation:

<sup>25</sup>We can assume that  $\frac{2}{1-\varepsilon} < q < 2 + 10^{-1}\varepsilon_0$ .

**Definition 8.1.** We say that  $f \in \mathbf{H}^s = \mathbf{H}^s(\mathbb{R}^3)$  if  $\nabla f \in H^{s-1}$ ,  $f$  is continuous and tends to zero as  $|x| \rightarrow \infty$ . Observe that  $\mathbf{H}^s$ , with  $s > \frac{3}{2}$ , is the closure of  $C_0^\infty$  in the norm  $\|\nabla f\|_{H^{s-1}}$ . Given a solution  $\phi$  of (101) we say that  $\phi = (\mathbf{g}_{\mu\nu}) \in C([0, T]; \mathbf{m} + \mathbf{H}^s)$  if, for every  $t \in [0, T]$ ,  $(\mathbf{g}_{\mu\nu}(t) - \mathbf{m}_{\mu\nu}) \in \mathbf{H}^s(\Sigma_t)$  and  $\partial_t \phi \in H^{s-1}(\Sigma_t)$ .

Throughout the section we shall use the following notation:

**Definition 8.2.** For any function  $f$  on  $\Sigma_t = \mathbb{R}^3$ ,  $P_\lambda f = \mathcal{F}^{-1}(\chi(\lambda^{-1}\xi)\hat{f}(\xi))$  with  $\chi$  supported in the unit dyadic region  $\frac{1}{2} \leq |\xi| \leq 2$ . Also  $f = \sum_\lambda P_\lambda f$ . We shall denote by  $f_{\leq \lambda} = P_{\leq \lambda} f = \sum_{\mu \leq \lambda} f^\mu$ . We shall also use the notation  $f_{< \lambda} = P_{< \lambda} f = \sum_{\mu < 2^{-M_0 \lambda}} f^\mu$ , for a sufficiently, fixed, large constant  $M_0$ , such as 100.

*Remark 8.3.* Observe that if  $f$  is continuous, approaches a constant  $c$  at infinity, i.e  $\sup_{|x|=r} |f(x) - c| \rightarrow 0$  as  $r \rightarrow \infty$ , and  $\nabla f \in H^{s-1}$ ,  $s > \frac{3}{2}$ , then<sup>26</sup>  $P_\lambda f \in H^s$ .

**8.4. Energy estimates.** We start with the following well known statement:

**Proposition 8.5** (Energy estimate). *Let  $\phi \in C([0, T]; \mathbf{m} + \mathbf{H}^s)$  be a solution of (101) on the time interval  $[0, T]$  for some  $s > \frac{3}{2}$  such that  $\|\phi, \phi^{-1}\|_{L_{[0, T]}^\infty L_x^\infty} \leq \Lambda_0$ . Then  $\phi$  verifies the following energy estimate.*

$$\|\partial \phi\|_{L_{[0, T]}^\infty \dot{H}^{s-1}} \leq C(\|\partial \phi\|_{L_{[0, T]}^1 L_x^\infty}, \Lambda_0) \|\partial \phi(0)\|_{\dot{H}^{s-1}}. \quad (102)$$

*Remark 8.6.* Throughout this section we shall often ignore the dependence on  $\Lambda_0$  and the constant  $M_0$  involved in the definition of  $P_{< \lambda}$ .

**Proof:** The proof of proposition 8.5 can be easily reduced to the following lemma.

**Lemma 8.7.** *Let  $\phi$  satisfy the conditions of proposition 8.5. Then for each dyadic  $\lambda \in 2^{\mathbb{Z}}$ ,  $\phi^\lambda = P_\lambda \phi$  verifies the equation*

$$-\partial_t^2 \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{0i})_{< \lambda}(\phi) \partial_i \partial_i \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{ij})_{< \lambda}(\phi) \partial_i \partial_j \phi^\lambda = R_\lambda, \quad (103)$$

where for any  $s > 1$  and  $t \in [0, T]$  the right hand-side  $R_\lambda$  has Fourier support in  $\{\xi : \frac{1}{4}\lambda \leq |\xi| \leq 4\lambda\}$  and obeys the estimate

$$\left( \sum_\lambda \|R_\lambda(t)\|_{\dot{H}^{s-1}}^2 \right)^{\frac{1}{2}} \leq C \|\partial \phi(t)\|_{L_x^\infty} \cdot \|\partial \phi(t)\|_{\dot{H}^{s-1}}. \quad (104)$$

with  $C$  a constant depending only on  $\Lambda_0$ . Moreover  $\phi^\lambda$  also satisfies the equation

$$\mathbf{g}_{< \lambda}^{\alpha\beta} \partial_\alpha \partial_\beta \phi^\lambda = R_\lambda \quad (105)$$

with a different  $R_\lambda$  which verifies the same estimate (104) and the frequency property.

<sup>26</sup>This can be easily proved by a density argument.

**Proof of lemma 8.7**

The proof of the lemma is based on the technique of the paradifferential calculus and is standard<sup>27</sup>. For the sake of completeness we provide an outline of the arguments. For a more detailed treatment see for example [Ba-Ch1] or [Kl-Ro].

Recall that  $P_\lambda$  denotes the projection on the frequencies of size  $\lambda$ , so that  $\phi^\lambda = P_\lambda \phi$ . We write the equation  $\mathbf{g}^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\phi = N$  in the form  $-\partial_t^2\phi + \mathbf{n}^2\mathbf{g}^{0i}(\phi)\partial_t\partial_i\phi + \mathbf{n}^2\mathbf{g}^{ij}(\phi)\partial_i\partial_j\phi = \mathbf{n}^2N$ . Then

$$-\partial_t^2\phi^\lambda + P_\lambda(\mathbf{n}^2\mathbf{g}^{0i}(\phi)\partial_t\partial_i\phi) + P_\lambda(\mathbf{n}^2\mathbf{g}^{ij}(\phi)\partial_i\partial_j\phi) = P_\lambda(\mathbf{n}^2N).$$

For convenience we introduce

$$G \cdot \partial^2\phi = \mathbf{n}^2\mathbf{g}^{0i}(\phi)\partial_t\partial_i\phi + \mathbf{n}^2\mathbf{g}^{ij}(\phi)\partial_i\partial_j\phi \quad (106)$$

and note that at least one of the derivatives on the right hand-side is a spatial derivative. Then

$$\begin{aligned} P_\lambda(G \cdot \partial^2\phi) &= P_\lambda \sum_{\mu,\nu} G^\mu \cdot \partial^2\phi^\nu = P_\lambda \sum_{\mu < \frac{1}{2}\nu, \nu} G^\mu \cdot \partial^2\phi^\nu + \\ P_\lambda \sum_{\nu < \frac{1}{2}\mu, \mu} G^\mu \cdot \partial^2\phi^\nu &+ P_\lambda \sum_{2^{-M_0}\nu \leq \mu \leq 2^{M_0}\nu, \nu} G^\mu \cdot \partial^2\phi^\nu = E_1(\lambda) + E_2(\lambda) + E_3(\lambda). \end{aligned}$$

It is clear that in the case when one frequencies  $\mu$  or  $\nu$  dominate, the projection  $P_\lambda$  on the frequencies of size  $\lambda$  forces the dominant frequency to be of the same size. We say that  $\mu \sim \lambda$  if  $\frac{1}{4}\lambda \leq \mu \leq 4\lambda$ .

**Treatment of  $E_1$** 

$$E_1 = \sum_{\mu < \frac{1}{2}\lambda} G^\mu \cdot \partial^2\phi^\lambda + \sum_{\nu \sim \lambda} [P_\lambda, G_{<\frac{1}{2}\nu}] \partial^2\phi^\nu.$$

The first term is precisely the term to keep<sup>28</sup> on the left hand side of the equation. To estimate the second term we need to make use of the standard commutator estimate, which implies that

$$\|[P_\lambda, G_{<\frac{1}{2}\nu}] \partial^2\phi^\nu\|_{L_x^2} \leq \lambda^{-1} \|\nabla G_{<\frac{1}{2}\nu}\|_{L_x^\infty} \|\partial^2\phi^\nu\|_{L_x^2} \leq \lambda^{-1} C(\Lambda_0) \|\nabla\phi\|_{L_x^\infty} \|\partial^2\phi^\nu\|_{L_x^2}.$$

Then, since the expression  $\partial^2\phi^\nu$  contains at least one spatial derivative, we obtain

$$\begin{aligned} \left\| \sum_{\nu \sim \lambda} [P_\lambda, G_{<\frac{1}{2}\nu}] \partial^2\phi^\nu \right\|_{\dot{H}^{s-1}} &\approx \lambda^{s-1} \left\| \sum_{\nu \sim \lambda} [P_\lambda, G_{\frac{1}{2}\nu}] \partial^2\phi^\nu \right\|_{L_x^2} \\ &\lesssim \lambda^{s-1} \sum_{\nu \sim \lambda} \|\partial\phi\|_{L_x^\infty} \|\partial\phi^\nu\|_{L_x^2} \\ &\lesssim \sum_{\nu \sim \lambda} \|\partial\phi\|_{L_x^\infty} \|\partial\phi^\nu\|_{\dot{H}^{s-1}}. \end{aligned}$$

<sup>27</sup>The equations discussed in the literature are somewhat different from the one treated here because of the non triviality of the components  $\mathbf{g}^{00}$  and  $\mathbf{g}^{0i}$  of the metric. This adds only minor technical complications.

<sup>28</sup>Observe that  $\sum_{\mu < \frac{1}{2}\lambda} G^\mu \cdot \nabla^2\phi^\lambda = (n^2 g^{0i})_{<\lambda} \partial_t\partial_i\phi^\lambda + (n^2 g^{ij})_{<\lambda} \partial_i\partial_j\phi - \sum_{\mu=2^{-M_0}\lambda}^{2^{-M_0-1}\lambda} G^\mu \cdot \partial^2\phi^\lambda$  and the second term is of the type  $E_3$

Squaring and summing over  $\lambda$  we obtain the bound

$$\left( \sum_{\lambda} \left\| \sum_{\nu \sim \lambda} [P_{\lambda}, G_{\leq \frac{1}{2}\nu}] \partial^2 \phi^{\nu} \right\|_{\dot{H}^{s-1}}^2 \right)^{\frac{1}{2}} \lesssim \|\partial \phi\|_{L^{\infty}} \|\partial \phi\|_{\dot{H}^{s-1}}.$$

as desired.

### Treatment of $E_2$

$$E_2(\lambda) = P_{\lambda} \sum_{\mu \sim \lambda} G^{\mu} \cdot \partial^2 \phi_{< \frac{1}{2}\mu}$$

We make use of the presence of a spatial derivative in  $\partial^2 \phi_{< \frac{1}{2}\mu}$  by estimating<sup>29</sup>,

$$\begin{aligned} \|E_2(\lambda)\|_{\dot{H}^{s-1}} &\leq \lambda^{s-1} \sum_{\mu \sim \lambda} \|G^{\mu}\|_{L_x^2} \|\partial^2 \phi_{< \frac{1}{2}\mu}\|_{L_x^{\infty}} \\ &\leq \sum_{\mu \sim \lambda} \lambda^{s-1} \mu \|G^{\mu}\|_{L_x^2} \|\partial \phi_{< \frac{1}{2}\mu}\|_{L_x^{\infty}} \lesssim \sum_{\mu \sim \lambda} \|\nabla G^{\mu}\|_{\dot{H}^{s-1}} \|\partial \phi\|_{L_x^{\infty}}. \end{aligned}$$

Thus, squaring and summing over  $\lambda$  we obtain

$$\left( \sum_{\lambda} \|E_2(\lambda)\|_{\dot{H}^{s-1}}^2 \right)^{\frac{1}{2}} \lesssim \|\nabla G\|_{\dot{H}^{s-1}} \|\partial \phi\|_{L_x^{\infty}}.$$

Clearly, in view of our assumptions,  $G(\phi) = \phi^{-1}$  is a smooth function of  $\phi$ . By a standard result on the composition properties of Sobolev spaces,

$$\|\nabla G(\phi)\|_{\dot{H}^{s-1}} \leq C(\Lambda_0) \|\nabla \phi\|_{\dot{H}^{s-1}} \quad (107)$$

Thus,

$$\left( \sum_{\lambda} \|E_2(\lambda)\|_{\dot{H}^{s-1}}^2 \right)^{\frac{1}{2}} \lesssim \|\nabla \phi\|_{\dot{H}^{s-1}} \|\partial \phi\|_{L_x^{\infty}}.$$

### Treatment of $E_3$

$$E_3(\lambda) = P_{\lambda} \sum_{2^{-M_0}\nu \leq \mu \leq 2^{M_0}\nu, \nu \geq 2^{-M_0}\lambda} G^{\mu} \cdot \partial^2 \phi^{\nu}.$$

Hence,

$$\begin{aligned} \|E_3\|_{\dot{H}^{s-1}} &\leq \lambda^{s-1} \sum_{2^{-M_0}\nu \leq \mu \leq 2^{M_0}\nu, \nu \geq 2^{-M_0}\lambda} \|G^{\mu}\|_{L_x^{\infty}} \|\partial^2 \phi^{\nu}\|_{L_x^2} \\ &\leq \sum_{2^{-M_0}\mu \leq \nu \leq 2^{M_0}\mu, \mu \geq 2^{-M_0}\lambda} \left(\frac{\lambda}{\nu}\right)^{s-1} \|\nabla G^{\mu}\|_{L_x^{\infty}} \cdot \|\partial \phi^{\nu}\|_{\dot{H}^{s-1}} \lesssim \sum_{\nu > 2^{-M_0}\lambda} \left(\frac{\lambda}{\nu}\right)^{s-1} \|\nabla \phi\|_{L_x^{\infty}} \cdot \|\partial \phi^{\nu}\|_{\dot{H}^{s-1}}. \end{aligned}$$

To check that the multiplicative type convolution with  $\nu^{(1-s)}$  maps  $l^2 \rightarrow l^2$  observe that  $\sum_{\nu > \frac{1}{4}} \nu^{1-s} < \infty$ , for  $s > 1$ . Thus,

$$\left( \sum_{\lambda} \|E_3(\lambda)\|_{\dot{H}^{s-1}}^2 \right)^{\frac{1}{2}} \lesssim \|\nabla \phi\|_{L_x^{\infty}} \cdot \|\partial \phi\|_{\dot{H}^{s-1}}.$$

<sup>29</sup>Observe that, in view of the remark 8.3  $\|G^{\mu}\|_{L_x^2}$  are finite.

It remains to treat the term  $\mathbf{n}^2 N(\phi, \partial\phi)$  which depends quadratically on  $\partial\phi$ . This is standard, it can be done in the same way as above. This ends the proof of the estimate (104). It remains to prove (105). We multiply the equation (103),  $(\mathbf{n}^2 \mathbf{g}^{\alpha\beta})_{<\lambda} \partial_\alpha \partial_\beta \phi^\lambda = R_\lambda$ , by  $\mathbf{n}_{<\lambda}^{-2}$ ,

$$-(\mathbf{n}^{-2})_{<\lambda} \partial_t^2 \phi^\lambda + (\mathbf{n}^{-2})_{<\lambda} (\mathbf{n}^2 \mathbf{g}^{0i})_{<\lambda} \partial_t \partial_i \phi^\lambda + (\mathbf{n}^{-2})_{<\lambda} (\mathbf{n}^2 \mathbf{g}^{ij})_{<\lambda} \partial_i \partial_j \phi^\lambda = (\mathbf{n}^{-2})_{<\lambda} R_\lambda.$$

It is easy to verify that the new right hand-side has the same properties as  $R_\lambda$ . Observe also that for arbitrary smooth functions  $f, g$

$$(fg)_{<\lambda} = f_{<\lambda} g_{<\lambda} + P_{<2\lambda}([P_{<\lambda}, f]g) + P_{<\lambda} \sum_{2^{-M_0}\lambda \leq \mu \leq 2^{-M_0+1}\lambda} f^\mu g_{<\lambda}.$$

Applying this to  $f = \mathbf{n}^{-2}$  and  $g = \mathbf{n}^2 \mathbf{g}^{\alpha\beta}$  with  $a = 0, \dots, 3$ ,  $\beta = 1, \dots, 3$ , we obtain

$$\begin{aligned} \mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta \phi^\lambda &= (\mathbf{n}^{-2})_{<\lambda} R_\lambda + P_{<2\lambda} \left( [P_{<\lambda}, \mathbf{n}^{-2}] \mathbf{n}^2 \mathbf{g}^{\alpha\beta} \right) \partial_\alpha \partial_\beta \phi^\lambda \\ &+ \sum_{\alpha=0, \dots, 3, \beta=1, \dots, 3} \sum_{2^{-M_0}\lambda \leq \mu \leq 2^{-M_0+1}\lambda} P_{<\lambda} \left( (\mathbf{n}^{-2})^\mu (\mathbf{n}^2 \mathbf{g}^{\alpha\beta})_{<\lambda} \right) \partial_\alpha \partial_\beta \phi^\lambda \end{aligned}$$

The commutator term on the right hand-side of the expression above is precisely of the type  $E_1(\lambda)$  and can be handled similarly. The metric component  $\mathbf{n}^{-2}$  appearing in the second term contains only frequencies  $\mu \geq 2^{-M_0}\lambda$ . This allows us to move one spatial derivative from  $\partial_\alpha \partial_\beta \phi^\lambda$ . Hence, the new right hand side  $R_\lambda$  possesses the same properties as the old  $R_\lambda$ .  $\square$

*Remark 8.8.* In the subsequent paper we shall also need the following more general result concerning other dyadic projections of our equation.

**Lemma 8.9.** *Under the assumptions of lemma 8.7 we have*

$$\mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta \phi_{<\lambda} = F_\lambda.$$

The function  $F_\lambda$  obeys the estimates

$$\|F_\lambda\|_{L_t^1 L_x^2} \leq C \|\partial\phi\|_{L_t^1 L_x^\infty} \|\partial\phi\|_{L_t^\infty L_x^2}, \quad \|F_\lambda\|_{L_t^1 \dot{H}^1} \leq C \|\partial\phi\|_{L_t^1 L_x^\infty} \|\partial\phi\|_{L_t^\infty \dot{H}_x^1}$$

In addition, for any dyadic  $\mu \geq 1$

$$\mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta P_{\lambda\mu} \phi = F_{\lambda,\mu},$$

where  $F_{\lambda,\mu}$  verifies

$$\begin{aligned} \|F_{\lambda,\mu}\|_{L_t^1 L_x^2} &\leq C (\lambda\mu)^{-\gamma} \lambda^{-1} \|\partial\phi\|_{L_t^1 L_x^\infty} \|\partial\phi\|_{L_t^\infty \dot{H}^{1+\gamma}}, \\ \|F_{\lambda,\mu}\|_{L_t^1 \dot{H}^1} &\leq C \lambda^{-\gamma} \mu^{1-\gamma} \|\partial\phi\|_{L_t^1 L_x^\infty} \|\partial\phi\|_{L_t^\infty \dot{H}^{1+\gamma}} \end{aligned}$$

The function  $\mathbf{g} = \phi^{-1}$  satisfies similar equations.

The proof of lemma 8.9 proceeds in the same manner as the proof of lemma 8.7 after applying the respective projections  $P_{<\lambda}$  and  $P_{\lambda\mu}$ .

To finish the proof of the proposition 8.5 we choose a large parameter  $\Lambda$  in such a way that for any  $\lambda \geq \Lambda$  the metric  $(\mathbf{n}^2 \mathbf{g}^{ij})_{<\lambda}$  is uniformly elliptic. This is always possible since  $P_{<\lambda}$  is an approximation of the identity and the original metric  $(\mathbf{n}^2 \mathbf{g}^{ij})$  is uniformly elliptic in  $[0, T]$ .

For the values of the dyadic parameter  $\lambda \leq \Lambda$  rewrite the equation for  $\phi^\lambda$  in the form

$$-\partial_t^2 \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{0i})_{<\lambda} \partial_t \partial_i \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{ij})_{<\lambda} \partial_i \partial_j \phi^\lambda = R'_\lambda$$

noting that the change of the metric introduces the error term of the type  $E_2$ .

For  $\lambda \geq \Lambda$  we keep the form of the equation as in lemma 8.7

$$-\partial_t^2 \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{0i})_{<\lambda} \partial_t \partial_i \phi^\lambda + (\mathbf{n}^2 \mathbf{g}^{ij})_{<\lambda} \partial_i \partial_j \phi^\lambda = R_\lambda$$

In either case, the standard  $H^1$  energy estimate for the wave equation yields

$$\|\partial \phi^\lambda\|_{L^\infty_{[0,T]} L^2} \leq C(\Lambda_0) (\|\partial \phi^\lambda(0)\|_{L^2} + \|R_\lambda\|_{L^1_{[0,T]} L^2_x}).$$

Using lemma 8.7 and the Gronwall inequality we immediately obtain for  $s > 1$

$$\|\partial \phi\|_{L^\infty_{[0,T]} \dot{H}^{s-1}} \lesssim \exp(\|\partial \phi\|_{L^1_{[0,T]} L^\infty_x}) \|\partial \phi(0)\|_{\dot{H}^{s-1}}.$$

The estimate for  $s = 1$  follows by standard energy estimates without the paradifferential decomposition.

**8.10. Reduction to the Strichartz type estimates.** As discussed in section 2 we need to prove the Strichartz type inequality (16). This is achieved by the following

**Theorem 8.11 (A1).** *Let  $\phi \in C([0, T]; \mathbf{m} + \mathbf{H}^{1+\gamma})$  be a solution of (101) on the time interval  $[0, T]$ ,  $T \leq 1$ . Assume that*

$$\|\partial \phi\|_{L^\infty_{[0,T]} H^{1+\gamma}} + \|\partial \phi\|_{L^2_{[0,T]} L^\infty_x} \leq B_0, \quad (108)$$

*There exists a small positive exponent  $\delta = \delta(B_0)$  such that  $\phi$  satisfies the following local in time Strichartz type estimate,*

$$\|\partial \phi\|_{L^2_{[0,T]} L^\infty_x} \leq C(B_0) T^\delta \quad (109)$$

*Remark 8.12.* In view of the remark 2.2 and definition 2.3 we shall treat  $B_0$  as a universal constant in what follows and hide the dependence on it in the notation  $\lesssim$ .

**8.13. The dyadic version of the Strichartz type estimate.** Fix a large frequency parameter  $\Lambda$ . It easily follows from the triangle inequality that for  $p \in [1, \infty]$ ,

$$\|\partial \phi\|_{L^p_x} \leq \|\partial \phi_{l \leq \Lambda}\|_{L^p_x} + \sum_{\lambda > \Lambda} \|\partial \phi^\lambda\|_{L^p_x}.$$

Thus, theorem 8.11 follows from the following dyadic version of the Strichartz type estimates for  $\phi^\lambda = P_\lambda \phi$ .

**Theorem 8.14 (A2).** *Let  $\phi$  be as in theorem 8.11. There exists a small positive exponent  $\delta = \delta(B_0)$  such that for each  $\lambda \geq \Lambda$ , the function  $\phi^\lambda$  satisfies the Strichartz type estimate*

$$\|\partial \phi^\lambda\|_{L^2_{[0,T]} L^\infty_x} \lesssim c_\lambda T^\delta \quad (110)$$



with constants  $c_\lambda$  such that  $\sum_\lambda c_\lambda \leq 1$ .

*Remark 8.15.* The corresponding estimate for small frequencies, i.e. for  $\phi_{<\lambda}$ , follows trivially from the Sobolev inequality,

$$\|\partial\phi_{<\lambda}\|_{L^2_{[0,T]}L^\infty_x} \lesssim T^{\frac{1}{2}}\|\partial\phi_{<\lambda}\|_{L^\infty_{[0,T]}H^{\frac{3}{2}+\gamma}} \lesssim \Lambda^{\frac{1}{2}}T^{\frac{1}{2}}\|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} \lesssim \Lambda^{\frac{1}{2}}T^{\frac{1}{2}}.$$

Since  $\Lambda$  is a fixed large parameter, which could depend only upon  $B_0$ , we have the desired bound for the low frequency part of  $\phi$ .

*Remark 8.16.* We shall need the following version of the estimate (104) for  $R_\lambda$  and any  $s < 2 + \gamma$ :

$$\|R_\lambda(t)\|_{\dot{H}^{s-1}} \lesssim c_\lambda \|\partial\phi\|_{L^\infty_x} \|\partial\phi\|_{H^{1+\gamma}} \quad (111)$$

with constants  $c_\lambda$ :  $\sum_\lambda c_\lambda \leq 1$ . The estimate (111) can be easily obtained from (104) by making use of the fact that the Fourier support of  $R_\lambda$  is localized on the set  $\{\xi : \lambda \leq |\xi| \leq 4\lambda\}$ . As a consequence, using the bootstrap assumption (108), we also have the estimate

$$\|R_\lambda(t)\|_{L^1_{[0,T]}\dot{H}^{s-1}} \lesssim c_\lambda T^{\frac{1}{2}}\|\partial\phi\|_{L^2_{[0,T]}L^\infty_x} \|\partial\phi\|_{L^\infty_{[0,T]}H^{1+\gamma}} \lesssim c_\lambda \quad (112)$$

**8.17. Dyadic linearization and time restriction.** This step reduces theorem 8.14 to a Strichartz type estimate for the linearized equation  $\mathbf{g}_{<\lambda}^{\alpha\beta}\partial_\alpha\partial_\beta\psi = 0$  on smaller subintervals of  $[0, T]$ . We partition  $[0, T]$  by the intervals  $I_k = [t_k, t_{k+1}]$ ,  $k = 0, \dots, \lambda^{8\epsilon_0}$  with the properties  $|I_k| \leq T\lambda^{-8\epsilon_0}$  and  $\|\partial\phi\|_{L^2_{I_k}L^\infty_x} \leq \lambda^{-4\epsilon_0}B_0$ . The existence of such partition is insured by the bootstrap condition (108).

**Theorem 8.18 (A3).** *Fix  $\lambda \geq \Lambda$  and  $k \in \mathbb{Z} \cap [0, \lambda^{8\epsilon_0}]$  and let  $\psi$  be a solution of the linear wave equation*

$$\mathbf{g}_{<\lambda}^{\alpha\beta}\partial_\alpha\partial_\beta\psi = 0$$

on the interval  $I_k = [t_k, t_{k+1}]$ , verifying,

$$(2^{-10}\lambda)^m \|\partial\psi(t_k)\|_{L^2_x} \leq \|\nabla^m\partial\psi(t_k)\|_{L^2_x} \leq (2^{10}\lambda)^m \|\partial\psi(t_k)\|_{L^2_x} \quad (113)$$

for every  $m \geq 0$ . Then there exists a sufficiently small exponent  $\delta > 0$  such that:

$$\|P_\lambda\partial\psi\|_{L^2_{I_k}L^\infty_x} \lesssim |I_k|^\delta \|\partial\psi(t_k)\|_{\dot{H}^{1+\delta}} \quad (114)$$

The size of  $\delta$  depends only on  $\epsilon_0, B_0$ . In particular, for any  $\epsilon_0 > 0$ , we can chose  $\delta$  such that,  $\delta < 10^{-1}\gamma$ .

*Remark 8.19.* The condition (113) implies that, modulo a negligible ‘‘tail’’, the Fourier support of  $\partial\psi(t_k)$  belongs to the set  $\{\xi : 2^{-10}\lambda \leq |\xi| \leq 2^{10}\lambda\}$ . In general, we shall say that function  $f$  obeys the property (115)<sub>M</sub> if

$$(2^{-M}\lambda)^m \|f\|_{L^2_x} \leq \|\nabla^m f\|_{L^2_x} \leq (2^M\lambda)^m \|f\|_{L^2_x} \quad (115)$$

**Lemma 8.20.**

1. Assume  $f$  in  $\mathbb{R}^3$  is a function whose frequency is localized to the region  $|\xi| \leq 2^{-M_0}\lambda$  and  $c \leq f \leq c^{-1}$  for some positive number  $c$ . Then  $u = f^{-1}$  verifies,

$$\|\nabla^m u\|_{L^\infty} \lesssim (2^{-M_0}\lambda)^m. \quad (116)$$

2. Assume<sup>30</sup> that  $u$  verifies (116) and  $c \leq u \leq c^{-1}$ . Let  $v$  be another function verifying the condition (115)<sub>5</sub>. Then<sup>31</sup>  $u \cdot v$  verifies (115)<sub>10</sub>.

**Proof** The proof of 1. is based on the trivial identity  $f \cdot f^{-1} = 1$ . Differentiating it and applying the Leibnitz rule we conclude that, although the Fourier support of  $f^{-1}$  does not belong to the set  $\{\xi : |\xi| \leq 2^{-M_0}\lambda\}$ , we still have the property,

$$\|\nabla^m(f^{-1})\|_{L_x^\infty} \lesssim (2^{-M_0}\lambda)^m.$$

The proof of 2. is once again an exercise in Leibnitz rule. In particular, for  $m = 1$  we have

$$\begin{aligned} \|\nabla(u \cdot v)\|_{L_x^2} &\lesssim \|\nabla u\|_{L_x^\infty} \|v\|_{L_x^2} + \|u\|_{L_x^\infty} \|\nabla v\|_{L_x^2} \\ &\lesssim 2^{-M_0}\lambda \|v\|_{L_x^2} + 2^5\lambda \|v\|_{L_x^2} \lesssim 2^{10}\lambda \|u \cdot v\|_{L_x^2} \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla(u \cdot v)\|_{L_x^2} &\gtrsim \|u\|_{L_x^\infty} \|\nabla v\|_{L_x^2} - \|\nabla u\|_{L_x^\infty} \|v\|_{L_x^2} \\ &\gtrsim 2^{-5}\lambda \|v\|_{L_x^2} - 2^{-M_0}\lambda \|v\|_{L_x^2} \gtrsim 2^{-10}\lambda \|u \cdot v\|_{L_x^2} \end{aligned}$$

■

**Proof of the implication Theorem (A3)  $\rightarrow$  Theorem (A2):** We shall first prove an inhomogeneous version of the Strichartz estimate (114) for solutions of the equation  $\mathbf{g}_{<\lambda}^{\alpha\beta}\psi = F$ , with the right hand side  $F$  verifying (115)<sub>5</sub>. Recall that  $\mathbf{g}_{<\lambda}^{\alpha\beta} = P_{<2^{-M_0}\lambda}\mathbf{g}^{\alpha\beta}$ . The Duhamel formula on the interval  $I_k$  for the inhomogeneous equation  $\mathbf{g}_{<\lambda}^{\alpha\beta}\partial_\alpha\partial_\beta\psi = F$  takes the form

$$\psi(t) = [W(t, 0)]\psi[t_k] + \int_0^t W(t, s) \left( (\mathbf{g}_{<\lambda}^{00})^{-1} F(s) \right) ds. \quad (117)$$

with  $\psi[t]$  denoting the vector  $(\psi(t), \partial_t\psi(t))$ . Here  $[W(t, s)]$  is the solution operator of the homogeneous equation acting on the pair of initial data  $(w_0, w_1)$  at time  $s$ , and  $W(t, s)$  is a solution operator corresponding to the special type of the initial data  $(0, w_1)$ . We need to check that  $(\mathbf{g}_{<\lambda}^{00})^{-1}F(s)$  verifies the same conditions (115) as  $F$ .

Recall  $-\mathbf{g}^{00} = \mathbf{n}^{-2}$ . Since  $F$  verifies (115)<sub>5</sub>, using 1. and 2. of lemma 8.20, we conclude that  $[(\mathbf{n}^{-2})_{<\lambda}]^{-1}F$  verifies (115)<sub>10</sub>.

We now apply theorem 8.18 to (117), assuming also that the initial data  $\partial\psi(t_k)$  verify the assumption (115)<sub>10</sub>,

$$\|P_\lambda \partial\psi\|_{L_{I_k}^2 L_x^\infty} \lesssim |I_k|^\delta \left( \|\partial\psi(t_k)\|_{\dot{H}^{1+\delta}} + \|F\|_{L_{I_k}^1 \dot{H}^{1+\delta}} \right). \quad (118)$$

Fix a sufficiently small  $\epsilon_0$  such that  $5\epsilon_0 + \delta < \gamma$ . Consider the  $\lambda$ -dyadic piece  $\phi^\lambda$  of  $\phi$ , solution of the equation (101), as in Theorem (A2). We know that  $\phi^\lambda$  verifies the equation  $\mathbf{g}_{<\lambda}^{\alpha\beta}\partial_\alpha\partial_\beta\phi^\lambda = R_\lambda$  on  $[0, T]$  and the Fourier support of  $R_\lambda$  belongs to

<sup>30</sup>Recall that  $M_0$  is a large positive constant

<sup>31</sup>This property is analogous to the standard paraproduct rule concerning the multiplication of functions  $u, v$  where the frequency of  $v$  dominates.

the set  $\{\xi : \frac{1}{4}\lambda \leq |\xi| \leq 4\lambda\}$ , thus automatically satisfying property (115)<sub>5</sub>. We can therefore apply (118) to  $\phi^\lambda$  on each  $I_k$  to obtain:

$$\begin{aligned} \|\partial\phi^\lambda\|_{L^2_{[0,T]}L^\infty_x} &= \left( \sum_{k=0}^{\lambda^{8\epsilon_0}-1} \|\partial\phi^\lambda\|_{L^2_{I_k}L^\infty_x}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left( \sum_{k=0}^{\lambda^{8\epsilon_0}-1} |I_k|^{2\delta} \left( \|\partial\phi^\lambda(t_k)\|_{\dot{H}^{1+\delta}} + \|R_\lambda\|_{L^1_{[0,T]}\dot{H}^{1+\delta}} \right)^2 \right)^{\frac{1}{2}} \\ &\lesssim |T|^\delta \lambda^{4\epsilon_0} \left( \|\partial\phi^\lambda\|_{L^\infty_{[0,T]}\dot{H}^{1+\delta}} + \|R_\lambda\|_{L^1_{[0,T]}\dot{H}^{1+\delta}} \right) \\ &\lesssim |T|^\delta \left( \|\partial\phi^\lambda\|_{L^\infty_{[0,T]}\dot{H}^{1+\gamma}} + \|R_\lambda\|_{L^1_{[0,T]}\dot{H}^{1+4\epsilon_0+\delta}} \right) \\ &\lesssim |T|^\delta c_\lambda \end{aligned}$$

The last two inequalities follow from the inequality  $\delta + 5\epsilon_0 < \gamma$  and the estimate (112).

**8.21. Properties of the metric  $\mathbf{g}_{<\lambda}$ .** Recall that  $\mathbf{g}_{<\lambda}^{\mu\nu} = P_{<2^{-M_0\lambda}}(\mathbf{g}^{\mu\nu})$  where  $\mathbf{g}^{\mu\nu}$  is the inverse of the Lorentz metric  $\mathbf{g}_{\mu\nu} = \phi$ . We shall use the notation  $\mathbf{g}_{<\lambda}$  to denote the inverse of  $\mathbf{g}_{<\lambda}^{\mu\nu}$ . Observe that, in view of our assumption  $\lambda \geq \Lambda$ ,  $\mathbf{g}_{<\lambda}$  defines a Lorentz metric in our spacetime region  $[0, T] \times \mathbb{R}^3$ . It clearly depends on the solution  $\phi$  of the quasilinear problem (101). In the next proposition we state the properties of the family  $\mathbf{g}_{<\lambda}$  which follow from the bootstrap condition (108) on  $\phi$ . We denote by  $\mathbf{R}_{\alpha\beta}(\mathbf{g}_{<\lambda})$  the components of Ricci curvature of the metric  $\mathbf{g}_{<\lambda}$ .

**Proposition 8.22.** *Let  $\phi \in C([0, T]; \mathbf{m} + \mathbf{H}^{1+\gamma})$  be a solution of (101) on  $[0, T]$ ,  $T \leq 1$ . Assume that  $\phi$  verifies the assumption (108) of theorem 8.11. Then the family of metrics  $\mathbf{g}_{<\lambda}$  obeys the following conditions on each interval  $I_k$  such that  $|I_k| \leq T\lambda^{-8\epsilon_0}$ , and  $\|\partial\phi\|_{L^2_{I_k}L^\infty_x} \leq \lambda^{-4\epsilon_0}$  :*

$$\|\partial^{1+m}\mathbf{g}_{<\lambda}\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0+m}, \quad (119)$$

$$\|\partial^{1+m}\mathbf{g}_{<\lambda}\|_{L^2_{I_k}L^\infty_x} \lesssim \lambda^{-4\epsilon_0+m}, \quad (120)$$

$$\|\partial^{1+m}\mathbf{g}_{<\lambda}\|_{L^\infty_{I_k}L^\infty_x} \lesssim \lambda^{\frac{1}{2}-4\epsilon_0+m}, \quad (121)$$

$$\|\nabla^{\frac{1}{2}+m}(\partial g g_{<\lambda})\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{\frac{1}{2}+m} \quad \text{for } 0 \leq m \leq \frac{1}{2} + 4\epsilon_0 \quad (122)$$

$$\|\nabla^{\frac{1}{2}+m}(\partial^2\mathbf{g}_{<\lambda})\|_{L^\infty_{[0,t_*]}L^2_x} \lesssim \lambda^{\frac{1}{2}+m-4\epsilon_0} \quad \text{for } -\frac{1}{2} + 4\epsilon_0 \leq m \quad (123)$$

$$\|\nabla^m \mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}_{<\lambda}\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0+m}, \quad (124)$$

$$\|\nabla^m (\nabla^{\frac{1}{2}} \mathbf{R}_{\alpha\beta}(\mathbf{g}_{<\lambda}))\|_{L^\infty_{I_k}L^2_x} \lesssim \lambda^m, \quad (125)$$

$$\|\nabla^m \mathbf{R}_{\alpha\beta}(\mathbf{g}_{<\lambda})\|_{L^1_{I_k}L^\infty_x} \lesssim \lambda^{-8\epsilon_0+m}. \quad (126)$$

*Remark 8.23.* It suffices to prove the above estimates for the inverse metric  $\mathbf{g}_{<\lambda}^{\mu\nu} = P_{<\lambda}(\mathbf{g}^{\mu\nu})$ . This can be easily seen by Leibnitz rule and the non degeneracy of  $\mathbf{g}_{<\lambda}$ .

On the other hand, due to the explicit presence of  $P_\lambda$ , the estimates for  $\mathbf{g}_{<\lambda}^{\mu\nu}$  can be immediately reduced to  $m = 0$ .

To be precise, the argument above works only for the spatial derivatives  $\nabla$ , since  $P_{<\lambda}$  truncates the frequencies of  $\mathbf{g}^{\mu\nu}$  only with respect to the space variable  $x$ . However, using the fact that  $\mathbf{g}_{\mu\nu} = \phi$  is a solution of the wave equation, one can recover the corresponding estimates for the time derivatives. Let us illustrate this by proving the estimate<sup>32</sup> (119) with  $m = 1$ . We assume that we have already proved (119)-(124) for  $m = 0$ . Then, clearly the derivatives  $\nabla^2 \mathbf{g}_{<\lambda}$  and  $\nabla \partial_t \mathbf{g}_{<\lambda}$  can be estimated with an additional factor of  $\lambda$ . It remains to address the derivative  $\partial_t^2 \mathbf{g}_{<\lambda}$ . Observe that

$$\mathbf{g}_{<\lambda}^{00} \partial_t^2 = \mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta + \sum_{\alpha=0,\dots,3, \beta=1,\dots,3} \mathbf{g}_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta.$$

The desired estimate follows from the condition (124) with  $m = 0$  and the fact that the second term in the previous formula contains at least one spatial derivative.

In view of the above remark we shall make no distinction between  $\mathbf{g}_{<\lambda}$  and  $\mathbf{g}_{<\lambda}^{-1}$  in what follows.

**Proof of (119)-(126) for  $m = 0$ :** The proof of inequality (120) follows immediately from the definition of  $I_k$ , since

$$\|\partial \mathbf{g}_{<\lambda}\|_{L_{I_k}^2 L_x^\infty} \lesssim \|\partial \phi\|_{L_{I_k}^2 L_x^\infty} \lesssim \lambda^{-4\epsilon_0}$$

Moreover, we have an even stronger estimate,

$$\|\partial \mathbf{g}\|_{L_{I_k}^2 L_x^\infty} \lesssim \|\partial \phi\|_{L_{I_k}^2 L_x^\infty} \lesssim \lambda^{-4\epsilon_0} \quad (127)$$

The Hölder inequality yields (119) from (120).

The estimates (121), (122), and (123) follow by a simple application of the Sobolev inequality, the composition properties of Sobolev spaces and the condition  $\gamma > 4\epsilon_0$ .

$$\begin{aligned} \|\partial(P_{<\lambda} \mathbf{g}(\phi))\|_{L_{I_k}^\infty L_x^\infty} &\lesssim \|\partial(P_{<\lambda} \mathbf{g}(\phi))\|_{L_{I_k}^\infty H^{\frac{3}{2}+\epsilon}} \\ &\lesssim \lambda^{\frac{1}{2}-4\epsilon_0} \|\partial \phi\|_{L_{I_k}^\infty H^{1+\gamma}} \lesssim \lambda^{\frac{1}{2}-4\epsilon_0}. \end{aligned} \quad (128)$$

The most interesting part of the proposition are the estimates (124), (126). Recall that the original metric  $\mathbf{g}$  satisfied the Einstein equation,  $\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0$ . In addition, since  $(\mathbf{g}^{\mu\nu}) = \phi^{-1}$  and  $\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \phi = N$ , each component of  $\mathbf{g}^{\mu\nu}$  satisfies the equation which can be written schematically as  $\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}^{\mu\nu} = |\partial \phi|^2$ . Thus,

$$\|\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}\|_{L_{I_k}^1 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}. \quad (129)$$

<sup>32</sup>This is one of the few estimates with  $m \neq 0$  which we shall actually use.

On the other hand we recall the expression for  $\mathbf{R}_{\alpha\beta}(\mathbf{g})$  relative to arbitrary coordinates,

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = \frac{1}{2} \mathbf{g}^{\mu\nu} (\partial_{\mu\beta}^2 \mathbf{g}_{\alpha\nu} + \partial_{\alpha\nu}^2 \mathbf{g}_{\mu\beta} - \partial_{\alpha\beta}^2 \mathbf{g}_{\mu\nu} - \partial_{\mu\nu}^2 \mathbf{g}_{\alpha\beta}) + \mathbf{g}_{\gamma\delta} (\Gamma_{\mu\beta}^{\gamma} \Gamma_{\alpha\nu}^{\delta} - \Gamma_{\mu\nu}^{\gamma} \Gamma_{\alpha\beta}^{\delta}).$$

Here  $\Gamma_{\mu\beta}^{\gamma}$  are the Christoffel symbols of the metric  $\mathbf{g}$ . It is then easy to see that the equation  $\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0$  also implies that

$$\|\mathbf{g}^{\mu\nu} (\partial_{\mu\beta}^2 \mathbf{g}_{\alpha\nu} + \partial_{\alpha\nu}^2 \mathbf{g}_{\mu\beta} - \partial_{\alpha\beta}^2 \mathbf{g}_{\mu\nu} - \partial_{\mu\nu}^2 \mathbf{g}_{\alpha\beta})\|_{L_{I_k}^1 L_x^\infty} \lesssim \|\partial\mathbf{g}\|_{L_{I_k}^2 L_x^\infty}^2 \lesssim \lambda^{-8\epsilon_0}. \quad (130)$$

and

$$\|\mathbf{g}^{\mu\nu} (\partial_{\mu\beta}^2 \mathbf{g}_{\alpha\nu} + \partial_{\alpha\nu}^2 \mathbf{g}_{\mu\beta} - \partial_{\alpha\beta}^2 \mathbf{g}_{\mu\nu} - \partial_{\mu\nu}^2 \mathbf{g}_{\alpha\beta})\|_{L_{I_k}^\infty \dot{H}^{\frac{1}{2}}} \lesssim \|\partial\mathbf{g} \cdot \partial\mathbf{g}\|_{L_{I_k}^\infty \dot{H}^{\frac{1}{2}}} \lesssim 1. \quad (131)$$

The last inequality follows from the generalized Leibnitz rule and the fact that  $\partial\mathbf{g} \in H^{1+\gamma}$ .

To derive the desired estimates (124)-(126) we simply<sup>33</sup> need to apply the following lemma to the estimates (129) and (130).

**Lemma 8.24.** *Let  $\mathbf{A} = (A_{\gamma\delta}^{\alpha\beta\mu\nu})$  be a fixed constant tensor. Denote  $\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g} = \mathbf{g}^{\gamma\delta} A_{\gamma\delta}^{\alpha\beta\mu\nu} \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu}$ . Assume that the linear combination  $\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}$  of the second derivatives of the metric  $\mathbf{g}$  satisfies the estimate  $\|\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}\|_{L_{I_k}^1 L_x^\infty} \leq c(B_0) \lambda^{-8\epsilon_0}$ . Then the same estimate holds for the linear combination associated with the metric  $\mathbf{g}_{<\lambda}$ :*

$$\|\mathbf{g}_{<\lambda} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}_{<\lambda}\|_{L_{I_k}^1 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}, \quad \|\mathbf{g}_{<\lambda} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}_{<\lambda}\|_{L_{I_k}^\infty \dot{H}^{\frac{1}{2}}} \lesssim 1 \quad (132)$$

**Proof** Recall that  $\mathbf{g}_{<\lambda} = P_{<\lambda} \mathbf{g}$ . Clearly,

$$\|\mathbf{g}_{<\lambda} - \mathbf{g}\|_{L_{I_k}^2 L_x^\infty} \lesssim \lambda^{-1} \|\nabla \mathbf{g}\|_{L_{I_k}^2 L_x^\infty} \lesssim \lambda^{-1-4\epsilon_0}. \quad (133)$$

Then

$$\begin{aligned} \|(\mathbf{g}_{<\lambda} - \mathbf{g}) \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}_{<\lambda}\|_{L_{I_k}^1 L_x^\infty} &\leq \|\mathbf{g}_{<\lambda} - \mathbf{g}\|_{L_{I_k}^2 L_x^\infty} \|\partial^2 \mathbf{g}_{<\lambda}\|_{L_{I_k}^2 L_x^\infty} \\ &\lesssim \lambda^{-1-4\epsilon_0} \lambda \|\partial \mathbf{g}_{<\lambda}\|_{L_{I_k}^2 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}. \end{aligned} \quad (134)$$

We can now consider the term  $\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g}_{<\lambda}$ . We have

$$\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 P_{<\lambda} \mathbf{g} = \mathbf{g} P_{<\lambda} \partial \cdot \mathbf{A} \cdot \partial \mathbf{g} = [\mathbf{g}, P_{<\lambda} \partial] \cdot \mathbf{A} \cdot \partial \mathbf{g} + P_{<\lambda} (\mathbf{g} \cdot \mathbf{A} \cdot \partial^2 \mathbf{g} + \partial \mathbf{g} \cdot \mathbf{A} \cdot \partial \mathbf{g}).$$

The commutator term can be estimated

$$\|([\mathbf{g}, P_{<\lambda} \partial]) f\|_{L_{I_k}^2 L_x^\infty} \lesssim \|\partial \mathbf{g}\|_{L_{I_k}^2 L_x^\infty} \|f\|_{L_x^\infty} \lesssim \lambda^{-4\epsilon_0} \|f\|_{L_x^\infty}.$$

It then follows that

$$\|([\mathbf{g}, P_{<\lambda} \partial]) \cdot \mathbf{A} \cdot \partial \mathbf{g}\|_{L_{I_k}^1 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}.$$

<sup>33</sup>The estimates (125) and (126) also require the following obvious estimates,

$$\|\partial \mathbf{g}_{<\lambda}\|_{L_{I_k}^2 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}, \quad \|\partial \mathbf{g}_{<\lambda} \cdot \partial \mathbf{g}_{<\lambda}\|_{L_{I_k}^\infty \dot{H}^{\frac{1}{2}}} \lesssim 1.$$

The remaining term satisfies the desired estimate by the assumptions of the lemma. The proof of the  $\dot{H}^{\frac{1}{2}}$  estimate in (132) is similar.  $\blacksquare$

**8.25. Rescaling.** According to theorem 8.18 we need to prove a Strichartz estimate for any solution of the problem  $g_{<\lambda}^{\alpha\beta} \partial_\alpha \partial_\beta \psi = 0$  on the interval  $I_k = [t_k, t_{k+1}]$ , with initial data  $\psi[t_k] = (\psi(t_k), \partial_t \psi(t_k))$  obeying condition (113), uniformly in  $\lambda, k$ .

It is convenient to replace the above problem by its rescaled version, so that the initial data satisfies condition (113) with  $\lambda = 1$  and the rescaled time interval  $I$  has length  $\leq \lambda^{1-8\epsilon_0}$ .

Introduce the family of the rescaled metrics<sup>34</sup>

$$H_{(\lambda)}(t, x) = g_{<\lambda}(\lambda^{-1}(t - t_k), \lambda^{-1}x) \quad (135)$$

We decompose the Lorentz metric  $H = H_{(\lambda)}$  relative to our spacetime coordinates;

$$-n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt) \quad (136)$$

where  $n$  and  $v$  are related to  $\mathbf{n}, \mathbf{v}$  according to the rule (135). In view of our choice of  $\lambda \geq \Lambda$  and (8) it easily follows that  $H$  is indeed a Lorentz metric and

$$c|\xi|^2 \leq h_{ij}\xi^i\xi^j \leq c^{-1}|\xi|^2, \quad n^2 - |v|_h^2 \geq c > 0, \quad |n|, |v| \leq c^{-1} \quad (137)$$

Proposition 8.22 implies that  $H = H_{(\lambda)}$  obeys the following estimates on the time interval  $I = [0, t_*]$  with  $t_* \leq \lambda^{1-8\epsilon_0}$ :

*Background Estimates (see proposition 2.4):*

$$\|\partial^{1+m} H\|_{L^1_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-8\epsilon_0}, \quad (138)$$

$$\|\partial^{1+m} H\|_{L^2_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (139)$$

$$\|\partial^{1+m} H\|_{L^\infty_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \quad (140)$$

$$\|\nabla^{\frac{1}{2}+m}(\partial H)\|_{L^\infty_{[0, t_*]} L^2_x} \lesssim \lambda^{-m} \quad \text{for} \quad -\frac{1}{2} \leq m \leq \frac{1}{2} + 4\epsilon_0 \quad (141)$$

$$\|\nabla^{\frac{1}{2}+m}(\partial^2 H)\|_{L^\infty_{[0, t_*]} L^2_x} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0} \quad \text{for} \quad -\frac{1}{2} + 4\epsilon_0 \leq m \quad (142)$$

$$\|\partial^m(H^{\alpha\beta} \partial_\alpha \partial_\beta H)\|_{L^1_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}, \quad (143)$$

$$\|\nabla^m(\nabla^{\frac{1}{2}} \mathbf{Ric}(H))\|_{L^\infty_{[0, t_*]} L^2_x} \lesssim \lambda^{-1}, \quad (144)$$

$$\|\partial^m \mathbf{R}_{\alpha\beta}(H)\|_{L^1_{[0, t_*]} L^\infty_x} \lesssim \lambda^{-1-8\epsilon_0}. \quad (145)$$

We now formulate the rescaled version of the desired Strichartz estimate.

**Theorem 8.26 (A4).** *Let  $\psi$  be a solution of the linear wave equation*

$$H^{\alpha\beta} \partial_\alpha \partial_\beta \psi = 0, \quad (146)$$

*on the time interval  $[0, t_*]$  with  $t_* \leq \lambda^{1-8\epsilon_0}$ . Assume that the parameter  $\lambda \geq \Lambda$  for a sufficiently large constant  $\Lambda$  and that the metric  $H$  verifies (138)-(145) with*

<sup>34</sup>Just as for  $g_{<\lambda}$  we make no distinction between  $H_{(\lambda)}$ , as Lorentz metric and its inverse.

a sufficiently small  $\epsilon_0 > 0$ . Let  $P$  be the operator of projection on the set  $\{\xi : 1 \leq |\xi| \leq 2\}$  in Fourier space. Then there exists a small constant  $\delta = \delta(\epsilon_0) > 0$  such that

$$\|P \partial \psi\|_{L^2_{[0,t_*]} L^\infty_x} \lesssim |t_*|^\delta \|\partial \psi(0)\|_{L^2_x} \quad (147)$$

**Remark:** Note that Theorem (A4) does not contain any assumptions on the Fourier support of the initial data  $\psi[0]$ .

**8.27. Decay estimates.** A variation of the standard  $TT^*$  type argument, see [K11], allows us to reduce the Strichartz estimate (147) to a corresponding dispersive inequality, see (148). In the process we replace<sup>35</sup> the equation  $H^{\alpha\beta} \partial_\alpha \partial_\beta \psi = 0$  by the *geometric* wave equation  $\square_H \psi = \frac{1}{\sqrt{|H|}} \partial_\alpha (H^{\alpha\beta} \sqrt{|H|} \partial_\beta \psi) = 0$ .

**Theorem 8.28 (A5).** Let  $\psi$  be a solution of the linear wave equation

$$\begin{aligned} \square_H \psi &= 0, \\ \psi|_{t_0} &= \psi_0, \quad \partial_t \psi|_{t_0} = \psi_1 \end{aligned} \quad (148)$$

on the time interval  $[0, t_*]$  with  $t_* \leq \lambda^{1-4\epsilon_0}$  and with initial data  $\psi[t_0] = (\psi(t_0), \partial_t \psi(t_0))$ . We consider only large values of the parameter  $\lambda \geq \Lambda$ . Assume that the metric  $H$  verifies (138)-(145). Then there exists a function  $d(t)$  obeying the condition

$$t_*^{\frac{1}{q}} \|d\|_{L^q_{[0,t_*]}} \leq 1, \quad \text{for some } q > 2 \text{ sufficiently close to } 2, \quad (149)$$

such that for all  $t_0 \leq t \leq t_*$ , a fixed arbitrary small  $\epsilon > 0$ , and a sufficiently large integer  $m$ ,

$$\|P \partial \psi(t)\|_{L^\infty_x} \lesssim \left( \frac{1}{(1 + |t - t_0|)^{1-\epsilon}} + d(t) \right) \sum_{k=0}^m \|\nabla^k \psi[t_0]\|_{L^1_x}. \quad (150)$$

We make the final reduction by decomposing the initial data  $\psi[t_0]$  in the physical space into a sum of functions with essentially disjoint supports contained in balls of radius  $\frac{1}{2}$ . Using the additivity of the  $L^1$  norm and the standard Sobolev inequality we can reduce the dispersive inequality (150) to an  $L^2 - L^\infty$  decay estimate.

**Theorem 8.29 ( $L^2 - L^\infty$  decay).** Let  $\psi$  be a solution of the linear wave equation (148) on the time interval  $[0, t_*]$  with  $t_* \leq \lambda_0^\epsilon$  and with initial data  $\psi[t_0]$  supported in the ball  $B_{\frac{1}{2}}(0)$  of radius  $\frac{1}{2}$  centered at the origin in the physical space. We fix a big constant  $\Lambda$  and consider only large values of the parameter  $\lambda \geq \Lambda$ . Assume that the metric  $H$  verifies (138)-(145). Then there exists a function  $d(t)$  obeying the condition (149) such that for all  $t_0 \leq t \leq t_*$ , an arbitrary small  $\epsilon > 0$ , and a sufficiently large integer  $m > 0$ ,

$$\|P \partial \psi(t)\|_{L^\infty_x} \lesssim \left( \frac{1}{(1 + |t - t_0|)^{1-\epsilon}} + d(t) \right) \sum_{k=0}^m \|\nabla^k \psi[t_0]\|_{L^2_x}. \quad (151)$$

<sup>35</sup>The two wave operators differ only by lower order terms in so far as the Strichartz estimates are concerned.

**8.30. Proof of the implication Theorem (A5)  $\rightarrow$  Theorem (A4); Decay  $\rightarrow$  Strichartz.** On this step of the reduction we assume that the family of metrics  $H = H_{(\lambda)}$  satisfies conditions (138)-(145) and that any solution of the geometric wave equation  $\square_H \psi = 0$  obeys the decay estimate

$$\|P \partial \psi(t)\|_{L_x^\infty} \lesssim \left( \frac{1}{(1 + |t - t_0|)^{1-\epsilon}} + d(t) \right) \sum_{k=0}^m \|\nabla^k \psi[t_0]\|_{L_x^1}.$$

We need to show that under these assumptions any solution<sup>36</sup> of the wave equation  $H^{\alpha\beta} \partial_\alpha \partial_\beta \phi = 0$  satisfies the Strichartz estimate  $\|P \partial \phi\|_{L_{[0, t_*]}^2 L_x^\infty} \lesssim |t_*|^\delta \|\psi[0]\|_{L_x^2}$ .

First, observe that it suffices to prove the following estimate:

$$\|P \partial \phi\|_{L_{[0, t_*]}^q L_x^\infty} \lesssim \|\phi[0]\|_{L_x^2} \quad (152)$$

with  $\delta = 1 - \frac{2}{q} > 0$  arbitrarily small. Observe also that the solutions of either the geometric wave equation  $\square_H \psi = F$  or the equation  $H_\lambda^{\alpha\beta} \partial_\alpha \partial_\beta \psi = F$  obey the following energy inequality for any  $t, t_0 \in [t_0, t_*]$ :

$$\begin{aligned} \|\partial \psi(t)\|_{L_x^2} &\leq \exp(C \|\partial H\|_{L_{[0, t_*]}^1 L_x^\infty}) \left( \|\partial \psi(t_0)\|_{L_x^2} + \|F\|_{L_{[0, t_*]}^1 L_x^2} \right) \\ &\leq 2 \left( \|\partial \psi(t_0)\|_{L_x^2} + \|F\|_{L_{[0, t_*]}^1 L_x^2} \right), \end{aligned} \quad (153)$$

where the last inequality follows<sup>37</sup> from the condition (138) on the metric  $H$ .

Furthermore, since

$$\square_H = H^{\alpha\beta} \partial_\alpha \partial_\beta + \frac{1}{\sqrt{|H|}} \partial_\alpha (\sqrt{|H|} H^{\alpha\beta}) \partial_\beta,$$

it is easy to show<sup>38</sup> that it suffices to establish (152) for a solution of the geometric wave equation. We shall now prove a stronger result.

**Proposition 8.31.** *Let  $\phi$  verifies the wave equation  $\square_H \phi = 0$ . Assume that the metric  $H$  is Lorentzian<sup>39</sup> and satisfies the condition*

$$C \|\partial H\|_{L_{[0, t_*]}^1 L_x^\infty} \leq \frac{1}{2} \quad (154)$$

for some sufficiently large positive constant  $C$ . We also assume that the conclusions of Theorem (A5) hold true. Then, for any  $q > 2$ ,

$$\|P \partial \phi\|_{L_{[0, t_*]}^q L_x^\infty} \lesssim \|\partial \phi(0)\|_{L_x^2}, \quad (155)$$

<sup>36</sup>Remark that we don't require any assumptions on the initial data. This is due to the presence of the projection  $P$  in the estimate.

<sup>37</sup>Recall that we consider  $\lambda \geq \Lambda$  for a sufficiently large constant  $\Lambda$ .

<sup>38</sup>By the Duhamel Principle we would obtain

$$\|P \partial \phi\|_{L_{[0, t_*]}^q L_x^\infty} \leq M (\|\phi[0]\|_{L_x^2} + \|\partial H\|_{L_{[0, t_*]}^1 L_x^\infty} \|\partial \phi\|_{L_{[0, t_*]}^\infty L_x^2})$$

and the condition (138) together with the energy inequality for  $\phi$  would imply (152).

<sup>39</sup>for simplicity we can assume that the ellipticity constant of the restrictions of the metric  $H$  to the time slices  $\Sigma_t$  is 2



**Proof** As in [Kl1], [Kl-Ro] we start by observing that our desired estimate

$$\|P \partial \phi\|_{L^q_{[0,t_*]} L^\infty_x} \leq M \|\partial \phi(0)\|_{L^2_x}, \quad (156)$$

is trivially true with a constant  $M > 0$  which may depend on  $\lambda$ . Thus we only need to prove that the constant  $M$  is in fact independent of  $\lambda$ .

*Remark 8.32.* We shall first prove the estimate (155) for  $P \partial_t \phi$ .

**Definition 8.33.** Setting  $(w_0, w_1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$ ,  $w = (w_0, w_1)$  we denote by  $\Phi(t, s; w)$  the vector  $(\phi, \partial_t \phi)$ , where  $\phi(t, s; w)$  is the solution at time  $t$  of the homogeneous equation  $\square_H \phi = 0$  subject to the initial data at time  $s$ ,  $\phi(s, s; w) = w_0, \partial_t \phi(s, s; w) = w_1$ .

By a standard uniqueness argument <sup>40</sup> we can easily prove the following:

$$\Phi\left(t, s; \Phi(s, t_0; w)\right) = \Phi(t, t_0; w) \quad (157)$$

**Definition 8.34.** Denote by  $\mathcal{H}$  the set of vector functions  $w = (w_0, w_1)$  with  $(w_0, w_1) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$ . The scalar product in  $\mathcal{H}$  is defined by

$$\langle w, v \rangle = \int_{\Sigma_0} \left( -H^{00} w_1 \cdot v_1 + H^{ij} \partial_i w_0 \cdot \partial_j v_0 \right)$$

*Remark 8.35.* Observe that the above scalar product is positive definite. Indeed  $H^{00}$  is strictly negative and  $H^{ij}$  is positive definite. To see the last assertion let  $h_{ij}$  denote the metric induced by  $H$  on  $\Sigma_t$ . In fact the metric  $H$  is given by  $-n^2 dt^2 + h_{ij}(dx^i + v^i dt) \otimes (dx^j + v^j dt)$ . Thus  $H^{ij} = h^{ij} - n^{-2} v^i v^j$ . Observe first <sup>41</sup> that  $H^{ij} v_i v_j > c|v|_h^2$ . This follows easily from  $n^2 - |v|_h^2 > 0$ , see (137). On the other hand, denoting by  $T_v = \{\omega / h_{ij} \omega^i \omega^j = 0\}$  the orthogonal complement to  $v$ , we easily check that  $H^{ij} \omega_i \omega_j > c|\omega|^2$ . This follows from the positivity of  $h$ , see (137). Finally  $H^{ij} \omega_i v_j = 0$ .

Let  $X = L^q_{[0,t_*]} L^\infty_x$  and its dual  $X' = L^{q'}_{[0,t_*]} L^1_x$ . Let  $\mathcal{T}$  be the operator from  $\mathcal{H}$  to  $X$  defined by:

$$\mathcal{T}(w) = -P \partial_t \phi(t, 0; w) \quad (158)$$

with  $\phi$  defined according to definition 8.33.

The adjoint  $\mathcal{T}^*$  is defined from  $X'$  to  $\mathcal{H}$ . To prove the estimate (155) it suffices to check that  $\mathcal{T} \cdot \mathcal{T}^*$  is a bounded operator from  $X'$  to  $X$ . In view of (156) we have <sup>42</sup>  $\|\mathcal{T}\|_{\mathcal{H} \rightarrow X} = M$  where  $\|\mathcal{T}\|_{\mathcal{H} \rightarrow X}$  denotes the operator norm of  $\mathcal{T}$ . Thus,

$$\|\mathcal{T} \cdot \mathcal{T}^*\|_{X' \rightarrow X} = M^2.$$

To calculate  $\mathcal{T}^*$  we write,

$$\langle \mathcal{T}^* f, w \rangle := \langle f, \mathcal{T}(w) \rangle = - \int_{[0,t_*] \times \mathbf{R}^3} \partial_t \phi P f dt dx = \int_{[0,t_*] \times \mathbf{R}^3} \partial_t \phi \bar{\square}_H \psi,$$

<sup>40</sup>which follows from the energy estimate (153), which still holds under assumption (154) on the metric  $H$

<sup>41</sup>Here  $v_i = h_{ij} v^j$ .

<sup>42</sup>We may assume that  $M$  is the smallest constant for which (156) holds true.

where  $\psi$  is the unique solution to the equation

$$\begin{aligned}\bar{\square}_H \psi &= \partial_\alpha (H^{\alpha\beta} \partial_\beta \psi) = -P f, \\ \phi(t_*) &= \partial_t \phi(t_*) = 0.\end{aligned}\tag{159}$$

Consequently, integrating by parts, we obtain

$$\langle \mathcal{T}^* f, w \rangle = - \int_{\Sigma_0} \left( \partial_t \phi H^{0\beta} \partial_\beta \psi - H^{0\beta} \partial_\beta \partial_t \phi \psi \right) + \int_{[0, t_*] \times \mathbf{R}^3} \bar{\square}_H \partial_t \phi \psi.$$

Observe that

$$\bar{\square}_H \partial_t \phi = \partial_t \bar{\square}_H \phi - \partial_\alpha (\partial_t (H^{\alpha\beta}) \partial_\beta \phi).$$

Therefore, integrating by parts once more, we have

$$\begin{aligned}\langle \mathcal{T}^* f, w \rangle &= - \int_{\Sigma_0} \left( \partial_t \phi H^{0\beta} \partial_\beta \psi - H^{0\beta} \partial_\beta \partial_t \phi \psi \right. \\ &\quad \left. + \bar{\square}_H \phi \psi - \partial_t (H^{0\beta}) \partial_\beta \phi \psi \right) \\ &\quad - \int_{[0, t_*] \times \mathbf{R}^3} \left( \bar{\square}_H \phi \partial_t \psi - \partial_t (H^{\alpha\beta}) \partial_\beta \phi \partial_\alpha \psi \right).\end{aligned}$$

Further note that  $-H^{0\beta} \partial_\beta \partial_t \phi + \bar{\square}_H \phi - \partial_t (H^{0\beta}) \partial_\beta \phi = \partial_i (H^{i\beta} \partial_\beta \phi)$ , and therefore,

$$\begin{aligned}\int_{\Sigma_0} \left( \partial_t \phi H^{0\beta} \partial_\beta \psi - H^{0\beta} \partial_\beta \partial_t \phi \psi + \bar{\square}_H \phi \psi - \partial_t (H^{0\beta}) \partial_\beta \phi \psi \right) \\ = \int_{\Sigma_0} \left( \partial_t \phi H^{0\beta} \partial_\beta \psi - H^{i\beta} \partial_\beta \phi \partial_i \psi \right) \\ = \int_{\Sigma_0} \left( H^{00} \partial_t \phi \partial_t \psi - H^{ij} \partial_i \phi \partial_j \psi \right).\end{aligned}$$

Thus, since

$$\square_H = \bar{\square}_H + H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta$$

and  $\square_H \phi = 0$

$$\begin{aligned}\langle \mathcal{T}^* f, w \rangle &= \int_{\Sigma_0} \left( -H^{00} \partial_t \phi \partial_t \psi + H^{ij} \partial_i \phi \partial_j \psi \right) \\ &\quad - \int_{[0, t_*] \times \mathbf{R}^3} \left( \bar{\square}_H \phi \partial_t \psi - \partial_t (H^{\alpha\beta}) \partial_\beta \phi \partial_\alpha \psi \right) \\ &= \int_{\Sigma_0} \left( -H^{00} \partial_t \phi \partial_t \psi + H^{ij} \partial_i \phi \partial_j \psi \right) \\ &\quad + \int_{[0, t_*] \times \mathbf{R}^3} \left( H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \phi \partial_t \psi + \partial_t (H^{\alpha\beta}) \partial_\beta \phi \partial_\alpha \psi \right)\end{aligned}$$

Thus, since  $\phi[0] = w$  and recalling the definition of  $\langle, \rangle_{\mathcal{H}}$

$$\langle \mathcal{T}^* f, w \rangle = \langle \psi[0], w \rangle + \langle R(f), w \rangle$$

with  $R(f)$  the linear operator from  $X'$  to  $\mathcal{H}$  defined by the formula,

$$\langle R(f), w \rangle = \int_{[0, t_*] \times \mathbf{R}^3} \left( H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \phi \partial_t \psi + \partial_t (H^{\alpha\beta}) \partial_\beta \phi \partial_\alpha \psi \right) \quad (160)$$

Therefore,

$$\mathcal{T}^* f = \psi[0] + R(f) \quad (161)$$

with  $\psi[0] = (\psi(0), \partial_t \psi(0))$ .

Henceforth,

$$\mathcal{T}\mathcal{T}^* f = \mathcal{T}\psi[0] + \mathcal{T}R(f) \quad (162)$$

Observe that  $\square_H \psi = -Pf + e$  with  $e = H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \psi$ . Thus we can write  $\psi = -\psi_1 + \psi_2$  with,

$$\begin{aligned} \square_H \psi_1 &= Pf, \\ \square_H \psi_2 &= e \end{aligned}$$

with both  $\psi_1, \psi_2$  verifying zero initial conditions at  $t = t_*$  as in (159). Now  $\mathcal{T}\psi[0] = -\mathcal{T}\psi_1[0] + \mathcal{T}\psi_2[0]$  and, recalling the definition 8.33,  $\mathcal{T}\psi_1[0] = -P\partial_t \phi(t, 0; \psi_1[0])$ . According to the Duhamel principle, as in (117) we have, with  $\psi[t] = (\psi(t), \partial_t \psi(t))$ ,

$$\psi_1[t] = \int_{t_*}^t \Phi(t, s; F(s)) ds$$

with  $F(s) = (0, (H^{00})^{-1} Pf(s)) = (0, -n^{-2} Pf(s))$  and therefore,

$$\psi_1[0] = - \int_0^{t_*} \Phi(0, s; F(s)) ds$$

and, in view of (157),

$$\mathcal{T}\psi_1[0] = P\partial_t \phi \left( t, 0; \int_0^{t_*} \Phi(0, s; F(s)) ds \right) = P \int_0^{t_*} \partial_t \phi(t, s; F(s)) ds.$$

We are now in a position to apply the dispersive inequality of Theorem (A6).

$$\|P\partial_t \phi(t, s; F(s))\|_{L^\infty} \leq C \left( (1 + |t - s|)^{-1+\epsilon} + d(t) \right) \sum_{k=0}^m \|\nabla^k (n^{-2} Pf(s))\|_{L^1}.$$

In view of (137) and (140), we have  $\|\nabla^k n^{-2}\|_{L^\infty} \lesssim 1$ . Thus, since  $P$  is the projection on the frequencies of size 1, we infer that

$$\|P\partial_t \phi(t, s; F(s))\|_{L^\infty} \leq C \left( (1 + |t - s|)^{-1+\epsilon} + d(t) \right) \|f(s)\|_{L^1}.$$

Therefore, by the Hardy-Littlewood-Sobolev inequality,

$$\|\mathcal{T}\psi_1[0]\|_{L_{[0, t_*]}^q L_x^\infty} \leq C \|f\|_{L_{[0, t_*]}^{q'} L_x^1} + \left\| \int_0^{t_*} d(t) \|f(s)\|_{L^1} ds \right\|_{L_{[0, t_*]}^q}$$

We can now make use of the assumption (149) of Theorem (A5) and infer that,

$$\left\| \int_0^{t_*} d(t) \|Pf(s)\|_{L^1} ds \right\|_{L_{[0, t_*]}^q} \leq C t_*^{\frac{1}{q}} \|d\|_{L_{[0, t_*]}^q} \|f\|_{L_{[0, t_*]}^{q'} L_x^1} \leq C \|f\|_{L_{[0, t_*]}^{q'} L_x^1}$$

Thus

$$\|\mathcal{T}\psi_1[0]\|_{L^q_{[0,t_*]}L^\infty_x} \leq C\|f\|_{L^{q'}_{[0,t_*]}L^1_x} \quad (163)$$

with  $C$  a constant, independent of  $\lambda$ .

To estimate  $\mathcal{T}\psi_2[0]$  we apply the Strichartz inequality with a bound  $M$ , see(156),

$$\|\mathcal{T}\psi_2[0]\|_{L^q_{[0,t_*]}L^\infty_x} \leq M\|\psi_2[0]\|_{\mathcal{H}}$$

where,

$$\|\psi_2[0]\|_{\mathcal{H}} = \sup_{\|w\|_{\mathcal{H}} \leq 1} \langle w, \psi_2[0] \rangle_{\mathcal{H}} \leq C\|\partial\psi_2(0)\|_{L^2}.$$

We shall now make use of the energy estimate (153) for  $\psi_2$  verifying the equation  $\square_H\psi_2 = e$ , subject to the initial conditions  $\psi_2(t_*) = \partial_t\psi_2(t_*) = 0$ ,

$$\|\partial\psi_2(0)\|_{L^2_x} \leq C\|e\|_{L^1_{[0,t_*]}L^2_x} \leq C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x} \|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}$$

Therefore, with the help of the condition (154), we have

$$\|\mathcal{T}\psi_2[0]\|_{L^q_{[0,t_*]}L^\infty_x} \leq \frac{1}{4}M\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x} \quad (164)$$

We shall now estimate the other error term  $\mathcal{T}Rf$ . Since the operator norm of  $\mathcal{T}$  is bounded by  $M$ ,

$$\|\mathcal{T}R(f)\|_{L^q_{[0,t_*]}L^\infty_x} \leq M\|R(f)\|_{\mathcal{H}}.$$

On the other hand,

$$\begin{aligned} \|R(f)\|_{\mathcal{H}} &= \sup_{\|w\|_{\mathcal{H}} \leq 1} \langle w, R(f) \rangle_{\mathcal{H}} \\ &= - \sup_{\|w\|_{\mathcal{H}} \leq 1} \int_{[0,t_*] \times \mathbf{R}^3} \left( H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \phi \partial_t \psi + \partial_t (H^{\alpha\beta}) \partial_\beta \phi \partial_\alpha \psi \right) \end{aligned}$$

Estimating in a straightforward manner we derive,

$$\|R(f)\|_{\mathcal{H}} \leq C\|\partial H\|_{L^1_{[0,t_*]}L^\infty_x} \|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x} \|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}.$$

We use the energy inequality (153) to estimate  $\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x}$ . Since the initial data  $\|w\|_{\mathcal{H}} \leq 1$  we infer that,  $\|\partial\phi\|_{L^\infty_{[0,t_*]}L^2_x} \leq C$ . Therefore, with the help of (154), we have

$$\|\mathcal{T}R(f)\|_{L^q_{[0,t_*]}L^\infty_x} \leq \frac{1}{4}M\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}. \quad (165)$$

To estimate  $\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x}$  we rely on the following:

**Lemma 8.36.** *The solution  $\psi$  of the equation  $\bar{\square}_H\psi = -Pf$ ,  $\psi(t_*) = \partial_t\psi(t_*) = 0$  verifies the estimate,*

$$\|\partial\psi\|_{L^\infty_{[0,t_*]}L^2_x} \leq 2M\|f\|_{L^{q'}_{[0,t_*]}L^1_x} \quad (166)$$

Gathering together (163),(164),(165) and (166) we infer that,

$$\|\mathcal{T}\mathcal{T}^*f\|_X = \|\mathcal{T}(\psi_1[0] + \psi_2[0] + R(f))\|_{L^q_{[0,t_*]}L^\infty_x} \leq (C + \frac{1}{2}M^2)\|f\|_{L^{q'}_{[0,t_*]}L^1_x}$$

Therefore, in view of (162),

$$M^2 = \|\mathcal{T}\mathcal{T}^*\|_{X' \rightarrow X} \leq (C + \frac{1}{2}M^2).$$

Thus we infer that  $M$  is a universal constant, as desired.

It only remains to prove the lemma 8.36. We proceed as follows. Let  $t$  be fixed in the interval  $[0, t_*]$ . We rewrite the equation  $\square_H \phi = 0$  in the form,

$$\bar{\square}_H \phi = F = -H^{\alpha\beta} \frac{\partial_\alpha \sqrt{|H|}}{\sqrt{|H|}} \partial_\beta \phi \quad (167)$$

with initial data  $\phi(t) = w_0, \partial_t \phi(t) = w_1$ , and  $(w_0, w_1) = w \in \mathcal{H}_t$ ,  $\|w\|_{\mathcal{H}_t} \leq 1$ . Here, the space  $\mathcal{H}_t$  is defined by the scalar product  $\langle w, v \rangle_{\mathcal{H}_t} = \int_{\Sigma_t} -H^{00} w_1 v_1 + H^{ij} \partial_i w_0 \partial_j v_0$ . We also recall that, see (159),

$$\bar{\square}_H \psi = -Pf \quad (168)$$

with initial data  $\psi_1(t_*) = \partial_t \psi_1(t_*) = 0$ . As in [Kl1] and [Kl-Ro] we multiply (167) by  $\partial_t \psi$  and (168) by  $\partial_t \phi$  after which we sum and integrate on our spacetime slab  $[t, t_*] \times \mathbb{R}^3$ . Observe that,

$$\begin{aligned} \partial_\alpha (H^{\alpha\beta} \partial_\beta \psi) &= (\partial_\alpha H^{\alpha\beta}) \partial_\beta \psi + H^{\alpha\beta} \partial_\alpha \partial_\beta \psi \\ H^{\alpha\beta} \partial_\alpha \partial_\beta \psi \partial_t \phi + H^{\alpha\beta} \partial_\alpha \partial_\beta \phi \partial_t \psi &= H^{\alpha\beta} \partial_\alpha (\partial_t \phi \partial_\beta \psi) + H^{\alpha\beta} \partial_\beta (\partial_t \psi \partial_\alpha \phi) \\ &\quad - H^{\alpha\beta} (\partial_\alpha \partial_t \phi \partial_\beta \psi) - H^{\alpha\beta} (\partial_\beta \partial_t \psi \partial_\alpha \phi) \\ &= H^{\alpha\beta} \partial_\alpha (\partial_t \phi \partial_\beta \psi) + H^{\alpha\beta} \partial_\beta (\partial_t \psi \partial_\alpha \phi) \\ &\quad - H^{\alpha\beta} \partial_t (\partial_\alpha \phi \partial_\beta \psi) \end{aligned}$$

Thus

$$\begin{aligned} \partial_\alpha (H^{\alpha\beta} \partial_\beta \psi) \partial_t \phi + \partial_\alpha (H^{\alpha\beta} \partial_\beta \phi) \partial_t \psi &= \partial_\alpha (H^{\alpha\beta} \partial_t \phi \partial_\beta \psi + \partial_t \psi \partial_\beta \phi) \\ &\quad - \partial_t (H^{\alpha\beta} \partial_\alpha \phi \partial_\beta \psi) + (\partial_t H^{\alpha\beta}) \partial_\alpha \phi \partial_\beta \psi \\ &= \partial_t (H^{i\beta} \partial_t \phi \partial_\beta \psi + \partial_t \psi \partial_\beta \phi) \\ &\quad + \partial_t (-H^{00} \partial_t \phi \partial_t \psi + H^{ij} \partial_i \phi \partial_j \psi) \\ &\quad + (\partial_t H^{\alpha\beta}) \partial_\alpha \phi \partial_\beta \psi \end{aligned}$$

Integrating in the region  $[t, t_*] \times \mathbf{R}^n$  we derive the identity,

$$\int_{\Sigma_t} \left( -H^{00} \partial_t \phi \partial_t \psi + H^{ij} \partial_i \phi \partial_j \psi \right) = - \int_t^{t_*} \int_{\Sigma_\tau} \left( -\partial_t \phi Pf + \partial_t \psi F + \partial_t (H^{\alpha\beta}) \partial_\alpha \phi \partial_\beta \psi \right).$$

Therefore,

$$\|\partial \psi(t)\|_{L^2} \leq \|Pf\|_{L^q_{[0,t_*]}L^\infty_x} \|\phi\|_{L^{q'}_{[0,t_*]}L^1_x} + C \|\partial H\|_{L^1_{[0,t_*]}L^\infty_x} \|\partial \phi\|_{L^\infty_{[0,t_*]}L^2_x} \|\partial \psi\|_{L^\infty_{[0,t_*]}L^2_x}$$

We recall that according to our assumption  $\|P \partial_t \phi\|_{L_{[0,t_*]}^q L_x^\infty} \leq M \|w\|_{\mathcal{H}_t} \leq M$ . Also according to the energy estimate,  $\|\partial \phi\|_{L_{[0,t_*]}^\infty L_x^2} \leq 2 \|w\|_{\mathcal{H}_t} \leq 2$ . Therefore,

$$\|\partial \psi\|_{L_{[0,t_*]}^\infty L_x^2} \leq M \|f\|_{L_{[0,t_*]}^{q'} L_x^1} + C \|\partial H\|_{L_{[0,t_*]}^1 L_x^\infty} \|\partial \psi\|_{L_{[0,t_*]}^\infty L_x^2}$$

and therefore, since  $C \|\partial H\|_{L_{[0,t_*]}^1 L_x^\infty} \leq \frac{1}{2}$ , we conclude that,

$$\|\partial \psi\|_{L_{[0,t_*]}^\infty L_x^2} \leq 2M \|f\|_{L_{[0,t_*]}^{q'} L_x^1}$$

as desired.

To prove the Strichartz estimate for the spatial derivatives we rely on the proof, given above, for  $P \partial_t \phi$ . We thus assume that the estimate (8.36) holds true for  $P \partial_t \phi$  with a universal constant  $M$ .

To estimate  $\|P \partial_k \phi\|_{L_{[0,t_*]}^q L_x^\infty}$  it suffices to estimate the integral,  $\mathcal{I} = \int_{[0,t_*] \times \mathbf{R}^3} P \partial_k \phi f dt dx$  for functions  $f$  with  $\|f\|_{L_{[0,t_*]}^{q'} L_x^\infty} \leq 1$ . Let  $\psi$  verify the equation  $\bar{\square}_H \psi = P f$  with  $\psi(t_*) = \partial_t \psi(t_*) = 0$ . Integrating by parts as before we infer that

$$\begin{aligned} \mathcal{I} &= \int_{[0,t_*] \times \mathbf{R}^3} \partial_k \phi \bar{\square}_H \psi = \int_{\Sigma_0} H^{0\beta} \left( \partial_k \phi \partial_\beta \psi + \partial_k \psi \partial_\beta \phi \right) \\ &\quad - \int_{[0,t_*] \times \mathbf{R}^3} \left( \bar{\square}_H \phi \partial_k \psi - (\partial_k H^{\alpha\beta}) \partial_\alpha \phi \partial_\beta \psi \right) \end{aligned}$$

Once again

$$\left| \int_{[0,t_*] \times \mathbf{R}^3} \bar{\square}_H \phi \partial_k \psi \right| \leq C \|\partial H\|_{L_{[0,t_*]}^1 L_x^\infty} \|\partial \phi\|_{L_{[0,t_*]}^\infty L_x^2} \|\partial \psi\|_{L_{[0,t_*]}^\infty L_x^2}$$

Also,

$$\int_{\Sigma_0} H^{0\beta} \left( \partial_k \phi \partial_\beta \psi + \partial_k \psi \partial_\beta \phi \right) \leq \|\partial \phi(0)\|_{L^2} \|\partial \psi\|_{L_{[0,t_*]}^\infty L_x^2}.$$

The energy estimate (153) gives  $\|\partial \phi\|_{L_{[0,t_*]}^\infty L_x^2} \leq 2 \|\partial \phi(0)\|_{L^2}$ . According to the lemma 8.36 we have,

$$\|\partial \psi\|_{L_{[0,t_*]}^\infty L_x^2} \leq 2M \|f\|_{L_{[0,t_*]}^{q'} L_x^1}.$$

Observe that the  $M$  in lemma 8.36 depends only on the Strichartz estimate (155) for  $P \partial_t \phi$  which we have already proved. Therefore,

$$|\mathcal{I}| \leq CM \|\partial \phi(0)\|_{L^2} (1 + \|\partial H\|_{L_{[0,t_*]}^1 L_x^\infty}) \|f\|_{L_{[0,t_*]}^{q'} L_x^1} \leq CM \|\partial \phi(0)\|_{L^2}$$

which implies,  $\|P \partial_a \phi\|_{L_{[0,t_*]}^q L_x^\infty} \leq CM \|\partial \phi(0)\|_{L^2}$  as desired.  $\blacksquare$

**8.37. Commutator lemma.** We conclude this section by presenting the proof of lemma 7.1 from section 2.1. Recall that the definition of the exterior region  $\text{Ext}_t = \{u \leq t/2\}$ .

**Lemma 8.38.** *Consider a vectorfield  $X = \sum_i X^i \partial_i$  vanishing on the complement of the exterior region  $\text{Ext}_t$  of  $\Sigma_t$  and  $P$  the standard Littlewood-Paley projection on frequencies of size 1. Then, for arbitrary scalar functions  $f$  we have the inequality:*

$$\|[P, X]f\|_{L^2(\text{Ext}_t)} \lesssim \sup_{i,j} \|\partial_i X^j\|_{L^\infty(\text{Ext}_t)} \|f\|_{L^2(\Sigma_t)} \quad (169)$$

**Proof** First observe, by expanding  $X = X^j \partial_j$  relative to our system of our coordinates on  $\Sigma_t$ , that  $[P, X] = [P \partial_j, X^j] - P(\partial_j X^j)$ . We shall denote  $P_j = P \partial_j$ , the modified cut-off of the unit frequencies. In what follows, the roles of  $P$  and  $P_j$  are identical. The convolution kernels of  $P, P_j$  are represented by the smooth functions  $P(x), P_j(x)$  verifying the condition that  $|P(x)|, |P_j(x)| \lesssim |x|^{-k}$  for any  $k > 0$  and  $|x| \geq 1$ . In particular, for any functions  $w, v$

$$\begin{aligned} v &= \int_{\Sigma_t} P(x-y)(w(y) - w(x))v(y) dy \\ &= - \int_0^1 \int_{\Sigma_t} P(x-y)(x-y)^i \partial_i w(\tau x + (1-\tau)y)v(y) dy d\tau \end{aligned}$$

As a consequence,

$$\|[P, w]v\|_{L^2(\Sigma_t)} \lesssim \|\nabla w\|_{L^\infty(\text{Ext}_t)} \|v\|_{L^2(\Sigma_t)} \quad (170)$$

Similar inequality also holds for  $P_j$ .

We shall show that

$$\|[P_j, X^j]f\|_{L^2(\Sigma_t)} + \|P((\partial_j X^j)f)\|_{L^2(\Sigma_t)} \lesssim \sup_{i,j} \|\partial_i X^j\|_{L^\infty(\text{Ext}_t)} \|f\|_{L^2(\Sigma_t)}$$

Since all  $X^j$  vanish outside of  $\text{Ext}_t$  and  $P$  is a bounded operator on  $L^2(\Sigma_t)$ , we can easily estimate the second term,

$$\|P((\partial_j X^j)f)\|_{L^2(\Sigma_t)} \lesssim \sup_{i,j} \|\partial_i X^j\|_{L^\infty(\text{Ext}_t)} \|f\|_{L^2(\Sigma_t)}$$

According to (170) we also have

$$\|[P_j, X^j]f\|_{L^2(\Sigma_t)} \lesssim \sup_{i,j} \|\partial_i X^j\|_{L^\infty(\text{Ext}_t)} \|f\|_{L^2(\Sigma_t)}$$

■

## REFERENCES

- [Ba-Ch1] H. Bahouri and J. Y. Chemin. *Équations d'ondes quasilineaires et estimation de Strichartz*. Amer. J. Math., vol. 121; (1999), pp. 1337–1777
- [Ba-Ch2] H. Bahouri and J. Y. Chemin. *Équations d'ondes quasilineaires et effet dispersif*. IMRN, vol. 21; (1999), pp. 1141–1178
- [Br] Y. Choquet Bruhat *Theoreme d'Existence pour certains systemes d'equations aux derivees partielles nonlineaires.*, Acta Math. **88** (1952), 141-225.

- [Ch-Kl] D. Christodoulou and S. Klainerman. *The Global Nonlinear Stability of the Minkowski Space*. Princeton Mathematical Series, 41. Princeton University Press, 1993
- [H-K-M] Hughes, T. Kato and J. Marsden *Well posed quasilinear second order hyperbolic systems* Arch. Rat. Mech. Anal. **63**(1976) no 3, 273–294.
- [Kl1] S. Klainerman. *A commuting vectorfield approach to Strichartz type inequalities and applications to quasilinear wave equations*. IMRN, 2001, No 5, 221–274.
- [Kl2] S. Klainerman. *PDE as a unified subject* Special Volume GAFA 2000, 279–315
- [Kl-Ro] S. Klainerman and I. Rodnianski, *Improved local well posedness for quasilinear wave equations in dimension three*. submitted to Duke Math. Journ.
- [Kl-Ro2] S. Klainerman and I. Rodnianski, *The causal structure of microlocalized, rough, Einstein metrics*
- [Kl-Ro3] S. Klainerman and I. Rodnianski, *Ricci defects of microlocalized, rough, Einstein metrics*
- [Li] H. Linblad. *Counterexamples to local existence for semilinear wave equations*. AJM, vol. 118; (1996), pp. 1–16
- [Po-Si] G. Ponce and T. Sideris. *Local regularity of non linear wave equations in three space dimensions*. CPDE, vol. 18; (1993), pp. 169–177
- [Sm] H. Smith. *A parametrix construction for wave equations with  $C^{1,1}$  coefficients*. Annales de L'Institut Fourier, vol. 48; (1998), pp. 797–835
- [Sm-So] H. Smith and C. Sogge. *On Strichartz and eigenfunction estimates for low regularity metrics*. Math. Res. Lett., vol. 1; (1994), pp. 729–737
- [Sm-Ta] H. Smith and D. Tataru. *Sharp counterexamples for Strichartz estimates for low regularity metrics*. Preprint
- [Ta2] D. Tataru. *Strichartz estimates for second order hyperbolic operators with non smooth coefficients*. Preprint
- [Ta1] D. Tataru. *Strichartz estimates for operators with non smooth coefficients and the nonlinear wave equation*. Amer. J. Math., vol. 122; (2000), pp. 349–376

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544

*E-mail address:* `seri@math.princeton.edu`

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544

*E-mail address:* `irod@math.princeton.edu`