FORMATION OF TRAPPED SURFACES II

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1. Introduction

2. Geometry of a null hypersurface

As in [?] we consider a region $\mathcal{D} = \mathcal{D}(u_*, \underline{u}_*)$ of a vacuum spacetime (M, g) spanned by a double null foliation generated by the optical functions $(u, \underline{u})$ increasing towards the future, $0 \le u \le u_*$ and $0 \le \underline{u} \le \underline{u}_*$. We denote by H_u the outgoing null hypersurfaces generated by the level surfaces of u and by $\underline{H}_{\underline{u}}$ the incoming null hypersurfaces generated level hypersurfaces of $\underline{u}$. We write $S_{u,\underline{u}} = H_u \cap \underline{H}_{\underline{u}}$ and denote by $H_u^{(\underline{u}_1,\underline{u}_2)}$, and $\underline{H}_{\underline{u}}^{(u_1,u_2)}$ the regions of these null hypersurfaces defined by $\underline{u}_1 \le \underline{u} \le \underline{u}_2$ and respectively $u_1 \le u \le u_2$. Let $L = -g^{\alpha\beta}\partial_\alpha u \partial_\beta$, $\underline{L} = -g^{\alpha\beta}\partial_\alpha \underline{u} \partial_\beta$, $\underline{L}$ be the geodesic vectorfields associated to the two foliations and define,

$$g(L, \underline{L}) := -2\Omega^{-2} = g^{\alpha\beta}\partial_\alpha u \partial_\beta \underline{u} \tag{1}$$

Observe that the flat value[1] of Ω is 1. As well known, our space-time slab $\mathcal{D}(u_*, \underline{u}_*)$ is completely determined (for small values of $u_*, \underline{u}_*$) by data along the null, characteristic, hypersurfaces H_0, $\underline{H}_0$ corresponding to $\underline{u} = 0$, respectively $u = 0$. Following [?] we assume that our data is trivial along $\underline{H}_0$, i.e. assume that H_0 extends for $\underline{u} < 0$ and the spacetime (M, g) is Minkowskian for $\underline{u} < 0$ and all values of $u \ge 0$. Moreover we can construct our double null foliation such that $\Omega = 1$ along H_0, i.e.,

$$\Omega(0, \underline{u}) = 1, \qquad 0 \le \underline{u} \le \underline{u}_*. \tag{2}$$

We denote by $r = r(u, \underline{u})$ the radius of the 2-surfaces $S = S(u, \underline{u})$, i.e. $|S(u, \underline{u})| = 4\pi r^2$. We denote by r_0 the value of r for $S(0,0)$, i.e. $r_0 = r(0,0)$. For simplicity we assume $r_0 = 1$.

Throughout this paper we work with the normalized null pair (e_3, e_4),

$$e_3 = \Omega\underline{L}, \quad e_4 = \Omega L, \qquad g(e_3, e_4) = -2.$$

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[1]Note that our normalization for Ω differ from that of [?]

Given a 2-surfaces $S(u,\underline{u})$ and $(e_a)_{a=1,2}$ an arbitrary frame tangent to it we define the Ricci coefficients,

$$\Gamma_{(\lambda)(\mu)(\nu)} = g(e_{(\lambda)}, D_{e_{(\nu)}} e_{(\mu)}), \quad \lambda,\mu,\nu = 1,2,3,4 \tag{3}$$

These coefficients are completely determined by the following components,

$$\begin{aligned} \chi_{ab} &= g(D_a e_4, e_b), \qquad \underline{\chi}_{ab} = g(D_a e_3, e_b),\\ \eta_a &= -\frac{1}{2} g(D_3 e_a, e_4), \quad \underline{\eta}_a = -\frac{1}{2} g(D_4 e_a, e_3)\\ \omega &= -\frac{1}{4} g(D_4 e_3, e_4), \qquad \underline{\omega} = -\frac{1}{4} g(D_3 e_4, e_3),\\ \zeta_a &= \frac{1}{2} g(D_a e_4, e_3) \end{aligned} \tag{4}$$

where $D_a = D_{e_{(a)}}$. We also introduce the null curvature components,

$$\begin{aligned} \alpha_{ab} &= R(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} = R(e_a, e_3, e_b, e_3),\\ \beta_a &= \frac{1}{2} R(e_a, e_4, e_3, e_4), \quad \underline{\beta}_a = \frac{1}{2} R(e_a, e_3, e_3, e_4),\\ \rho &= \frac{1}{4} R(Le_4, e_3, e_4, e_3), \quad \sigma = \frac{1}{4} {}^*R(e_4, e_3, e_4, e_3) \end{aligned} \tag{5}$$

Here ${}^*R$ denotes the Hodge dual of R. We denote by ∇ the induced covariant derivative operator on $S(u,\underline{u})$ and by ∇_3, ∇_4 the projections to $S(u,\underline{u})$ of the covariant derivatives D_3, D_4. Observe that,

$$\begin{aligned} \omega &= -\frac{1}{2} \nabla_4(\log\Omega), \qquad \underline{\omega} = -\frac{1}{2} \nabla_3(\log\Omega),\\ \eta_a &= \zeta_a + \nabla_a(\log\Omega), \quad \underline{\eta}_a = -\zeta_a + \nabla_a(\log\Omega) \end{aligned} \tag{6}$$

We recall the integral formulas[2] for a scalar function f in $\mathcal{D}$,

$$\begin{aligned} \frac{d}{d\underline{u}} \int_{S(u,\underline{u})} f &= \int_{S(u,\underline{u})} \big(\frac{df}{d\underline{u}} + \Omega \mathrm{tr}\chi f\big) = \int_{S(u,\underline{u})} \Omega\big(e_4(f) + \mathrm{tr}\chi f\big)\\ \frac{d}{du} \int_{S(u,\underline{u})} f &= \int_{S(u,\underline{u})} \big(\frac{df}{du} + \Omega \mathrm{tr}\underline{\chi} f\big) = \int_{S(u,\underline{u})} \Omega\big(e_3(f) + \mathrm{tr}\underline{\chi} f\big) \end{aligned} \tag{7}$$

In particular,

$$\frac{dr}{d\underline{u}} = \frac{1}{8\pi} \int_{S(u,\underline{u})} \Omega \mathrm{tr}\chi, \qquad \frac{dr}{du} = \frac{1}{8\pi} \int_{S(u,\underline{u})} \Omega \mathrm{tr}\underline{\chi} \tag{8}$$

We also recall the following commutation formulas: We record below commutation formulae between ∇ and ∇_4, ∇_3:

[2]see for example Lemma 3.1.3 in [**?**]

Lemma 2.1. *For a scalar function f:*

$$[\nabla_4, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})D_4 f - \chi \cdot \nabla f \tag{9}$$

$$[\nabla_3, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})D_3 f - \underline{\chi} \cdot \nabla f, \tag{10}$$

For a 1-form tangent to S:

$$\begin{aligned}
[\nabla_4, \nabla_a]U_b &= -\chi_{ac}\nabla_c U_b + \in_{ac} {}^*\beta_b U_c + \frac{1}{2}(\eta_a + \underline{\eta}_a)D_4 U_b \\
&- \chi_{ac}\underline{\eta}_b U_c + \chi_{ab}\underline{\eta} \cdot U \\
[\nabla_3, \nabla_a]U_b &= -\underline{\chi}_{ac}\nabla_c U_b + \in_{ac} *\underline{\beta}_b U_c + \frac{1}{2}(\eta_a + \underline{\eta}_a)D_3 U_b \\
&- \underline{\chi}_{ac}\eta_b U_c + \underline{\chi}_{ab}\eta \cdot U
\end{aligned}$$

In particular,

$$\begin{aligned}
[\nabla_4, div\,]U &= -\frac{1}{2}tr\chi\, div\, U - \hat{\chi} \cdot \nabla U - \beta \cdot U + \frac{1}{2}(\eta + \underline{\eta}) \cdot \nabla_4 U - \underline{\eta} \cdot \hat{\chi} \cdot U \\
[\nabla_3, div\,]U &= -\frac{1}{2}tr\underline{\chi}\, div\, U - \hat{\underline{\chi}} \cdot \nabla U + \underline{\beta} \cdot U + \frac{1}{2}(\eta + \underline{\eta}) \cdot \nabla_3 U - \eta \cdot \hat{\underline{\chi}} \cdot U
\end{aligned}$$

2.2. **Christodoulou's heuristic argument.** We recall here the assumptions needed in Christodoulou's heuristic argument for the formation of a trapped surface as described in [**?**]. As mentioned above we assume that our data is trivial along $\underline{H}_0$, i.e. assume that H_0 extends for $\underline{u} < 0$ and the spacetime (M, g) is Minkowskian for $\underline{u} < 0$ and all values of $u \geq 0$. We introduce a small parameter $\delta > 0$ and restrict the values of $\underline{u}$ to $0 \leq \underline{u} \leq \delta$, i.e. $\underline{u}_* = \delta$.

Main Assumptions. We assume that throughout $\mathcal{D} = \mathcal{D}(u_*, \underline{u}_*)$ we have the following estimates:

MA1. For small δ, Ω is comparable with its standard value in flat space, i.e.

$$\Omega = 1 + O(\delta^{1/2}).$$

MA2. The Ricci coefficients $\chi, \omega, \eta, \underline{\chi}, \underline{\omega}$ verify

$$|\hat{\chi}, \omega| = O(\delta^{-1/2}), \qquad |\mathrm{tr}\chi, \eta| = O(1), \qquad |\hat{\underline{\chi}}, \mathrm{tr}\underline{\chi} + \frac{2}{r}, \underline{\omega}| = O(\delta^{1/2}).$$

MA3. Also for some $c > 0$,

$$|\nabla\eta| = O(\delta^{-1/2+c}).$$

Note that in view of (8) we also have,

$$\frac{dr}{du} = -1 + O(r\delta^{1/2}), \qquad \frac{dr}{d\underline{u}} = O(r) \tag{11}$$

Thus, for δ sufficiently small, we infer that r is decreasing along the incoming null hypersurfaces and remains bounded, $0 \le r \le r_0 + 1 = 2$, in $\mathcal{D}$.

Christodoulou's argument for the formation of trapped surfaces in [**?**] rests on the equations,

$$\begin{aligned}
\nabla_4 \mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 &= -|\hat{\chi}|^2 - \frac{1}{2}(\mathrm{tr}\chi)^2 - 2\omega\mathrm{tr}\chi \\
\nabla_3 \hat{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} &= \nabla\widehat{\otimes}\eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\chi\underline{\hat{\chi}} + \eta\widehat{\otimes}\eta
\end{aligned}$$

In view of our Ricci coefficients assumptions we can rewrite,

$$\begin{aligned}
\nabla_4 \mathrm{tr}\chi &= -|\hat{\chi}|^2 + O(\delta^{-1/2}) \\
\nabla_3 \hat{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} &= O(\delta^{-1/2+c})
\end{aligned}$$

Multiplying the second equation by $\hat{\chi}$,

$$\nabla_4 |\hat{\chi}|^2 + \mathrm{tr}\underline{\chi}|\hat{\chi}|^2 = O(\delta^{-1+c})$$

Using also our assumptions for $u, \underline{u}, \Omega$ we deduce,

$$\frac{d}{d\underline{u}}\mathrm{tr}\chi = -|\hat{\chi}|^2 + O(\delta^{-1/2}) \tag{12}$$

$$\frac{d}{du}|\hat{\chi}|^2 + \mathrm{tr}\underline{\chi}|\hat{\chi}|^2 = O(\delta^{-1+c}) \tag{13}$$

Integrating (12) we obtain,

$$\mathrm{tr}\chi(u, \underline{u}) = \frac{2}{r(u,0)} - \int_0^{\underline{u}} |\hat{\chi}|(u, \underline{u}')^2 d\underline{u}' + O(\delta^{1/2}) \tag{14}$$

In view of our assumptions for $\mathrm{tr}\underline{\chi}$ and $\frac{dr}{du}$

$$\begin{aligned}
\frac{d}{du}(r^2|\hat{\chi}|^2) &= r^2\frac{d}{du}|\hat{\chi}|^2 + 2r\frac{dr}{du}|\hat{\chi}|^2 = r^2\big[-\mathrm{tr}\underline{\chi}|\hat{\chi}|^2 + O(\delta^{-1+c})\big] + 2r\big[-1 + O(r\delta^c)\big]|\hat{\chi}|^2 \\
&= r^2 O(\delta^{-1+c}).
\end{aligned}$$

Therefore,

$$r^2|\hat{\chi}|^2(u, \underline{u}) = r^2(0, \underline{u})|\hat{\chi}|^2(0, \underline{u}) + r^2 O(\delta^{-1+c})$$

As in [] we freely prescribe $\hat{\chi}$ along the initial hypersurface $H_0^{(0,\delta)}$, i.e.

$$\hat{\chi}(0, \underline{u}) = \hat{\chi}_0(\underline{u}) = O(\delta^{-1/2}) \tag{15}$$

for some traceless 2 tensor $\hat\chi_0$. We deduce, (need $0 < c \le \frac{1}{2}$),

$$|\hat\chi|^2(u,\underline{u}) = \frac{r^2(0,\underline{u})}{r^2(u,\underline{u})}|\hat\chi_0|^2(\underline{u}) + O(\delta^{-1+c})$$

or, since $|\underline{u}| \le \delta$ and $r(u,\underline{u}) = r_0 + \underline{u} - u + O(\delta^c)$,

$$|\hat\chi|^2(u,\underline{u}) = \frac{r_0^2}{r^2(u,0)}|\hat\chi_0|^2(\underline{u}) + O(\delta^{-1+c})$$

Thus, returning to (14),

$$\begin{aligned} \mathrm{tr}\chi(u,\delta) &= \frac{2}{r(u,0)} - \frac{r_0^2}{r^2(u,0)}\int_0^{u} |\hat\chi_0|^2(\underline{u}')d\underline{u}' + O(\delta^c) \\ &= \frac{2}{r(u,0)} - \frac{r_0^2}{r^2(u,0)}\int_0^{u} |\hat\chi_0|^2(\underline{u}')d\underline{u}' + O(\delta^c) \end{aligned}$$

We have thus proved the following.

Proposition 2.3. *Under the assumptions* **MA1- MA3** *we have, for sufficiently small $\delta > 0$ and fixed $c > 0$,*

$$tr\chi(u,\delta) = \frac{2}{r(u,0)} - \frac{r_0^2}{r^2(u,0)}\int_0^{\delta} |\hat\chi_0|^2(\underline{u}')d\underline{u}' + O(\delta^c) \tag{16}$$

Since $r(u,\underline{u}) = r_0 - u + \underline{u} + O(\delta^c)$ formula (16) can also be written in the form,

$$tr\chi(u,\delta) = \frac{2}{r(u,\delta)} - \frac{r_0^2}{r^2(u,\delta)}\int_0^{\delta} |\hat\chi_0|^2(\underline{u}')d\underline{u}' + O(\delta^c) \tag{17}$$

Corollary 2.4. *The necessary condition to have $tr\chi(u,\underline{u}=\delta) < 0$*

$$\frac{2r(u,0)}{r_0^2} < \int_0^{\delta} |\hat\chi_0|^2 + O(\delta^c) \tag{18}$$

for sufficiently small $\delta > 0$. Since $r(u,0) = r_0 - u + O(\delta^c)$, condition (18) can also be written in the form,

$$\frac{2(r_0-u)}{r_0^2} < \int_0^{\delta} |\hat\chi_0|^2 + O(\delta^c) \tag{19}$$

3. Change of foliation

3.1. **Main transformation formula.** To improve on (18) we plan to change the u foliation along $\underline{u} = \delta$ and compute the corresponding incoming expansion $\mathrm{tr}\chi'$. More precisely, given the foliation induced by u, we look for a new foliation $v = v(u,\omega)$ defined by the equations

$$\begin{aligned} \nabla_u v = e^f, &\qquad v|_{S_0} = u|_{S_0} = 0 \\ \nabla_u f = 0, &\qquad f|_{S_*} = f_0 \end{aligned} \tag{20}$$

with f_0 a function on $S_0 = S(0,\delta)$ to be carefully chosen later.

NOTE CHANGE: BEFORE WE HAD $\nabla_3 v = e^f$ WHICH LEADS TO THE UNDESIRED TERM $e^- f\nabla\log\Omega$ IN THE EQUATION FOR G.

We introduce the new null frame adapted to the v-foliation,

$$e_3' = e_3, \qquad e_a' = e_a - e^{-f}\Omega e_a(v)e_3, \qquad e_4' = e_4 - 2e^{-f}\Omega e_a(v)e_a + e^{-2f}\Omega^2|\nabla v|^2 e_3 \tag{21}$$

Indeed since $\nabla_u = \Omega\nabla_3$ we have, $e_a'(v) = e_a(v) - e^{-f}\Omega e_a(v)e_3(v) = e_a(v) - e^{-f}e_a(v)\nabla_u(v) = 0$. Also, ince e_3 is orthogonal to any vector tangent to $\underline{H}$ we easily check that

$$g(e_a', e_b') = g(e_a, e_b) = \delta_{ab}, \quad g(e_4', e_a') = g(e_4', e_4') = 0, \quad g(e_3', e_4') = -2.$$

We prove the following.

Lemma 3.2. *The new incoming expansion $tr\chi'$ verifies the transformation formula,*

$$tr\chi' = tr\chi - 2e^f\, div\,(e^{-f}F) - tr\underline{\chi}|F|^2 - 4\hat{\underline{\chi}}_{bc}F^bF^c - 2(\eta+\zeta)\cdot F \tag{22}$$

where $F_a = e^{-f}\Omega\nabla_a v$ and $tr\chi, \zeta, tr\underline{\chi}, \hat{\underline{\chi}}, \underline{\omega}$ are connection coefficients for the given double null foliation $(u, \underline{u})$.

Proof. We have,

$$\chi'(e_a', e_b') \;\; := \;\; g(D_a' e_4', e_b') = g(D_a e_4', e_b') - e^{-f}\Omega^{-1}e_a(v)g(D_3 e_4', e_b')$$

Now, writing $e_4' = e_4 - 2F + |F|^2 e_3$ with $F = F_c e_c$ and $e_b' = e_b - F_b e_3$,

$$\begin{aligned}
g(D_a e_4', e_b') &= g\big(D_a(e_4 - 2F + |F|^2 e_3)\,,\, e_b - F_b e_3\big)\\
&= \chi(e_a, e_b) - 2F_b\zeta_a - 2\nabla_a F_b + 2F_b g(D_a F, e_3) + |F|^2 g(D_a e_3, e_b - F_b e_3)\\
&= \chi_{ab} - 2\zeta_a F_b - 2\nabla_a F_b - 2F_b\,\underline{\chi}(F, e_a) + |F|^2\underline{\chi}_{ab}\\
&= \chi_{ab} - 2\zeta_a F_b - 2\nabla_a F_b - 2F_b\,F_c\,\underline{\chi}_{ac} + |F|^2\underline{\chi}_{ab}
\end{aligned}$$

Also,

$$\begin{aligned}
g(D_3 e_4', e_b') &= g\big(D_3(e_4 - 2F + |F|^2 e_3)\,,\, e_b - F_b e_3\big)\\
&= g(D_3 e_4, e_b) - F_b g(D_3 e_4, e_3) - 2\nabla_3 F_b\\
&= 2\eta_b + 4F_b\underline{\omega} - 2\nabla_3 F_b
\end{aligned}$$

Hence,

$$\begin{aligned}
\chi_{ab}' &= \chi_{ab} - 2\zeta_b F_a - 2\nabla_a F_b - 2F_b\,\underline{\chi}(F, e_b) + |F|^2\underline{\chi}_{ab} - F_a\big(2\eta_b + 4F_b\underline{\omega} - 2\nabla_3 F_b\big)\\
&= \chi_{ab} - 2\nabla_a F_b + 2F_a\nabla_3 F_b - 2\zeta_b F_a - 2F_a\eta_b + \big(|F|^2\underline{\chi}_{ab} - 2F_bF_c\underline{\chi}_{ac}\big) - 4\underline{\omega}F_aF_b
\end{aligned}$$

By symmetry in a, b we deduce the formula,

$$\begin{aligned}
\chi_{ab}' &= \chi_{ab} - (\nabla_a F_b + \nabla_b F_a) + \nabla_3(F_aF_b) - (\zeta_b + \eta_b)F_a + (\zeta_a + \eta_a)F_b\\
&+ \big(|F|^2\underline{\chi}_{ab} - F_bF_c\underline{\chi}_{ac} - F_aF_c\underline{\chi}_{bc}\big) - 4\underline{\omega}F_aF_b
\end{aligned} \tag{23}$$

and, taking the trace,

$$\begin{aligned} \mathrm{tr}\chi' &= \mathrm{tr}\chi - 2\mathrm{div}\ F + \nabla_3|F|^2 - 2(\eta+\zeta)\cdot F + (|F|^2\mathrm{tr}\underline{\chi} - 2\underline{\chi}_{bc}F^bF^c) - 4\underline{\omega}|F|^2 \\ &= \mathrm{tr}\chi - 2\mathrm{div}\ F + \nabla_3|F|^2 - 2(\eta+\zeta)\cdot F - 2\hat{\underline{\chi}}_{bc}F^bF^c - 4\underline{\omega}|F|^2 \end{aligned}$$

We next calculate $\nabla_3|F|^2$ using (20) and the commutation formula

$$[\nabla_3, \nabla]h = (\nabla \log\Omega)\nabla_3 h - \underline{\chi}\cdot\nabla h$$

or,

$$[\nabla_u, \nabla]h \ = \ -\Omega\underline{\chi}\cdot\nabla h$$

Since $\nabla_u f = 0$ and $F = \Omega^{-1}e^f\nabla v$ we deduce,

$$\begin{aligned} \nabla_u F_a &= \nabla_u(\Omega e^{-f}\nabla v) = \Omega e^{-f}\nabla_u\nabla v + \nabla_u\Omega e^{-f}\nabla v \\ &= \Omega e^{-f}\nabla\nabla_u v - \Omega e^{-f}\Omega\underline{\chi}\cdot\nabla v + \nabla_u\Omega e^{-f}\nabla v \\ &= \Omega\nabla f - \Omega^2 e^{-f}\underline{\chi}\cdot\nabla v + \nabla_u\Omega e^{-f}\nabla v \\ &= \Omega\nabla f - \Omega\underline{\chi}\cdot F - \Omega^{-1}\nabla_u\Omega F \end{aligned}$$

or,

$$\begin{aligned} \nabla_3 F &= \nabla F - \underline{\chi}\cdot F - \Omega^{-1}\nabla_3\Omega F \\ &= \nabla F - \underline{\chi}\cdot F + 2\underline{\omega}F \end{aligned}$$

i.e.,

$$\nabla_3 F + \frac{1}{2}\mathrm{tr}\underline{\chi}F = \nabla f - \hat{\underline{\chi}}\cdot F + 2\underline{\omega}F \tag{24}$$

from which,

$$\nabla_3|F|^2 \ = \ -\mathrm{tr}\underline{\chi}|F|^2 + 2F\cdot\nabla f - 2\hat{\underline{\chi}}_{bc}F^bF^c + 4\underline{\omega}|F|^2$$

Therefore,

$$\begin{aligned} \mathrm{tr}\chi' &= \mathrm{tr}\chi - 2\mathrm{div}\ F - 2(\eta+\zeta)\cdot F - 2\hat{\underline{\chi}}_{bc}F^bF^c - 4\underline{\omega}|F|^2 \\ &- \ \mathrm{tr}\underline{\chi}|F|^2 + 2F\cdot\nabla f - 2\hat{\underline{\chi}}_{bc}F^bF^c + 4\underline{\omega}|F|^2 \\ &= \mathrm{tr}\chi - 2\mathrm{div}\ F + 2F\cdot\nabla f - \mathrm{tr}\underline{\chi}|F|^2 - 4\hat{\underline{\chi}}_{bc}F^bF^c - 2(\eta+\zeta)\cdot F \\ &= \mathrm{tr}\chi - 2e^f\mathrm{div}\ (e^{-f}F) + 2F\cdot\nabla f - \mathrm{tr}\underline{\chi}|F|^2 - 4\hat{\underline{\chi}}_{bc}F^bF^c - 2(\eta+\zeta)\cdot F \end{aligned}$$

as desired.

□

Remark. Note that we can eliminate ζ from the formula (22) by writing the term $2(\eta+\zeta)\cdot F = 4\eta\cdot F - 2\Omega^{-1}\nabla\Omega\cdot F$ Thus,

$$\mathrm{tr}\chi' = \mathrm{tr}\chi - 2e^f\Omega\mathrm{div}\ (\Omega^{-1}e^{-f}F) - \mathrm{tr}\underline{\chi}|F|^2 - 4\hat{\underline{\chi}}_{bc}F^bF^c - 4\eta\cdot F \tag{25}$$

To understand how $\mathrm{tr}\chi'$ differs from $\mathrm{tr}\chi$ it only remains to derive a transport equation for div G with $G = e^{-f}F$.

3.3. Transport equation for div G.

In view of (24) and $e_3(f) = 0$ we have for $G := e^{-f}F$.

$$\nabla_3 G + \frac{1}{2}\mathrm{tr}\underline{\chi} G = e^{-f}\nabla f - \underline{\hat{\chi}} \cdot G + 2\underline{\omega} G \tag{26}$$

To derive a transport equation for div G we make us of the following

Lemma 3.4. *Assume that the S-tangent vectorfield V verifies an equation of the form,*

$$\nabla_3 V + \frac{1}{2} tr\underline{\chi} V = -\underline{\hat{\chi}} \cdot V + W$$

Then,

$$\begin{aligned}
\nabla_3(div\ V) + \frac{1}{2} tr\underline{\chi} div\ V &= div\ W + W \cdot \nabla(\log\Omega) - 2\underline{\hat{\chi}} \cdot \nabla V - \nabla tr\underline{\chi} \cdot V \\
&+ \big(tr\underline{\chi}\zeta - 2\underline{\hat{\chi}}\zeta - 2\underline{\hat{\chi}} \cdot \nabla(\log\Omega)\big) \cdot V
\end{aligned}$$

Proof.

$$\nabla_3(\mathrm{div}\ V) + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{div}\ V = \mathrm{div}\ \big(- \underline{\hat{\chi}} \cdot V + W) - \frac{1}{2}\nabla \mathrm{tr}\underline{\chi} \cdot V + [\nabla_3, \mathrm{div}\ ]V$$

We make use of the commutation formula, see lemma 2.1,

$$[\nabla_3, \mathrm{div}\ ]V = -\frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{div}\ V - \underline{\hat{\chi}} \cdot \nabla V + \big(\underline{\beta} - \eta \cdot \underline{\hat{\chi}}\big) \cdot V + \nabla(\log\Omega) \cdot \nabla_3 V$$

Therefore,

$$\begin{aligned}
\nabla_3(\mathrm{div}\ V) + \mathrm{tr}\underline{\chi}\mathrm{div}\ V &= \mathrm{div}\ \big(- \underline{\hat{\chi}} \cdot V + W) - \underline{\hat{\chi}} \cdot \nabla V + \big(\underline{\beta} - \frac{1}{2}\nabla\mathrm{tr}\underline{\chi} - \eta \cdot \underline{\hat{\chi}}\big) \cdot V \\
&+ \nabla(\log\Omega) \cdot \nabla_3 V \\
&= \mathrm{div}\ W - 2\underline{\hat{\chi}} \cdot \nabla V + \big(- \mathrm{div}\ \underline{\hat{\chi}} + \underline{\beta} - \frac{1}{2}\nabla\mathrm{tr}\underline{\chi} - \eta \cdot \underline{\hat{\chi}}\big) \cdot V \\
&+ \nabla(\log\Omega) \cdot \big(- \frac{1}{2}\mathrm{tr}\underline{\chi} V - \underline{\hat{\chi}} \cdot V + W\big) \\
&= \mathrm{div}\ W + W \cdot \nabla(\log\Omega)) - 2\underline{\hat{\chi}} \cdot \nabla V \\
&+ \big(- \mathrm{div}\ \underline{\hat{\chi}} - \frac{1}{2}\nabla\mathrm{tr}\underline{\chi} + \underline{\beta} - \eta \cdot \underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi}\nabla(\log\Omega) - \underline{\hat{\chi}} \cdot \nabla(\log\Omega)\big) \cdot V
\end{aligned}$$

Using the Codazzi equation, div $\underline{\hat{\chi}} = \frac{1}{2}\nabla \mathrm{tr}\underline{\chi} + \underline{\beta} + \zeta\cdot(\underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi})$ as well as $\eta = \zeta + \nabla(\log\Omega)$ we derive,

$$
\begin{aligned}
&-\mathrm{div}\ \underline{\hat{\chi}} - \frac{1}{2}\nabla \mathrm{tr}\underline{\chi} + \underline{\beta} - \eta\cdot\underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi}\nabla(\log\Omega) - \underline{\hat{\chi}}\cdot\nabla(\log\Omega))\\
&= -\nabla\mathrm{tr}\underline{\chi} - \zeta\cdot(\underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi} - \eta\cdot\underline{\hat{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi}\nabla(\log\Omega) - \underline{\hat{\chi}}\cdot\nabla(\log\Omega)\\
&= -\nabla\mathrm{tr}\underline{\chi} - \underline{\hat{\chi}}\cdot(\zeta + \eta + \nabla(\log\Omega)) + \frac{1}{2}\mathrm{tr}\underline{\chi}(\zeta + \nabla(\log\Omega))\\
&= -\nabla\mathrm{tr}\underline{\chi} - 2\underline{\hat{\chi}}\,(\zeta + \nabla(\log\Omega)) + \mathrm{tr}\underline{\chi}\eta
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\nabla_3(\mathrm{div}\ V) + \mathrm{tr}\underline{\chi}\mathrm{div}\ V &= \mathrm{div}\ W + W\cdot\nabla(\log\Omega) - 2\underline{\hat{\chi}}\cdot\nabla V - \nabla\mathrm{tr}\underline{\chi}\cdot V\\
&+ \big(\mathrm{tr}\underline{\chi}\eta - 2\underline{\hat{\chi}}\,\zeta - 2\underline{\hat{\chi}}\cdot\nabla(\log\Omega)\big)\cdot V
\end{aligned}
$$

as desired.

□

Applying the lemma to equation (26) we derive,

$$
\begin{aligned}
\nabla_3(\mathrm{div}\ G) + \mathrm{tr}\underline{\chi}\mathrm{div}\ G &= \mathrm{div}\ W + W\cdot\nabla(\log\Omega) - 2\underline{\hat{\chi}}\cdot\nabla G - \nabla\mathrm{tr}\underline{\chi}\cdot G\\
&+ \big(\mathrm{tr}\underline{\chi}\eta - 2\underline{\hat{\chi}}\,\zeta - 2\underline{\hat{\chi}}\cdot\nabla(\log\Omega)\big)\cdot G
\end{aligned}
$$

with $W = e^{-f}\nabla f + 2\underline{\omega}G$. Thus,

$$
\mathrm{div}\ W + W\cdot\nabla(\log\Omega) = \mathrm{div}\ (e^{-f}\nabla f) + e^{-f}\nabla(\log\Omega)\cdot\nabla f + 2\mathrm{div}\ (\underline{\omega}G) + 2\nabla(\log\Omega)\underline{\omega}G
$$

We deduce the following transport equation for div G,

$$
\nabla_3(\mathrm{div}\ G) + \mathrm{tr}\underline{\chi}\mathrm{div}\ G = \mathrm{div}\ (e^{-f}\nabla f) + 2\underline{\omega}\mathrm{div}\ G + \mathrm{Err}_1 \tag{27}
$$

with error term,

$$
\begin{aligned}
\mathrm{Err}_1 &= e^{-f}\nabla(\log\Omega)\cdot\nabla f - 2\underline{\hat{\chi}}\cdot\nabla G - \nabla\mathrm{tr}\underline{\chi}\cdot G\\
&+ \big(\mathrm{tr}\underline{\chi}\eta - 2\underline{\hat{\chi}}\,\zeta - 2\underline{\hat{\chi}}\cdot\nabla(\log\Omega) + 2\nabla\underline{\omega} + 2\underline{\omega}\nabla\log\Omega\big)\cdot G
\end{aligned}
$$

In the same manner we deduce a transport equation for the principal term div $(e^{-f}\nabla f)$ on the right hand side of (27). Indeed, since $\nabla_3 f = 0$ we derive,

$$
\nabla_3(\nabla f) + \frac{1}{2}\mathrm{tr}\underline{\chi}\nabla f = -\underline{\hat{\chi}}\,\nabla f
$$

Therefore, using lemma 3.4,

$$
\begin{aligned}
\nabla_3\mathrm{div}\ (e^{-f}\nabla f) + \mathrm{tr}\underline{\chi}\mathrm{div}\ (e^{-f}\nabla f) &= -2\underline{\hat{\chi}}\cdot\nabla(e^{-f}\nabla f) - \nabla\mathrm{tr}\underline{\chi}\cdot e^{-f}\nabla f\\
&+ \big(\mathrm{tr}\underline{\chi}\eta - 2\underline{\hat{\chi}}\,\zeta - 2\underline{\hat{\chi}}\cdot\nabla(\log\Omega)\big)\cdot e^{-f}\nabla f.
\end{aligned}
$$

We summarize the results of this subsection in the following proposition. We summarize the results of this subsection in the following proposition.

Proposition 3.5. *Let v, f be defined according to (20), $F = \Omega^{-1} e^{-f} \nabla v$ and $G = e^{-f} F$. The trace of the null second fundamental form χ', relative to the new frame (21), is given by the formula (22), i.e.,*

$$tr\chi' = tr\chi - 2e^f \, div \, G - tr\underline{\chi}|F|^2 - 4\hat{\underline{\chi}}_{bc} F^b F^c - 2(\eta + \zeta) \cdot F \tag{28}$$

F verifies the transport equation

$$\nabla_3 F + \frac{1}{2} tr\underline{\chi} F = \nabla f - \hat{\underline{\chi}} \cdot F + 2\underline{\omega} F \tag{29}$$

and $div \, G$ verifies,

$$\nabla_3 (div \, G) + tr\underline{\chi} div \, G = div \, (e^{-f} \nabla f) + Err_1 \tag{30}$$

where,

$$\begin{aligned} Err_1 &= e^{-f} \nabla (\log \Omega) \cdot \nabla f - 2\hat{\underline{\chi}} \cdot \nabla G \\ &- \nabla tr\underline{\chi} \cdot G + \big(tr\underline{\chi} \zeta - 2\hat{\underline{\chi}} \, \zeta - 2\hat{\underline{\chi}} \cdot \nabla(\log \Omega) + 2\nabla \underline{\omega} + 2\underline{\omega} \nabla \log \Omega \big) \cdot G \end{aligned}$$

Also,

$$\nabla_3 f = 0 \tag{31}$$

$$\nabla_3 (\nabla f) + \frac{1}{2} tr\underline{\chi} \nabla f = -\hat{\underline{\chi}} \, \nabla f \tag{32}$$

$$\nabla_3 [e^f \, div \, (e^{-f} \nabla f)] + tr\underline{\chi} [e^f \, div \, (e^{-f} \nabla f)] = Err_2 \tag{33}$$

with error term,

$$Err_2 = -2\hat{\underline{\chi}} \cdot \big(\nabla^2 f - \nabla f \nabla f \big) - \nabla tr\underline{\chi} \cdot \nabla f + \big(tr\underline{\chi} \eta - 2\hat{\underline{\chi}} \, \zeta - 2\hat{\underline{\chi}} \cdot \nabla(\log \Omega) \big) \cdot \nabla f$$

3.6. **Additional assumptions.** To proceed we need to make stronger assumptions than those of section 2.2. More precisely, we need, in addition **MA1** -**MA3** the following,

MA2-S. The Ricci coefficients $\eta, \underline{\eta}, \nabla \log \Omega$ verify the stronger assumptions,

$$|\eta|, |\underline{\eta}| = O(\delta^c)$$

MA3-S For a fixed $c > 0$,

$$|\nabla \eta|, |\nabla \underline{\eta}| = O(\delta^{-1/2+c}), \qquad |\nabla \underline{\chi}|, |\underline{\beta}| = O(\delta^c)$$

As a corollary of proposition 3.5 and these assumptions we deduce first,

$$\mathrm{tr}\chi' = \mathrm{tr}\chi - 2e^f \mathrm{div}\,(G) + \frac{2}{r}|F|^2 + |F|^2 O(\delta^c) \tag{34}$$

From the equation (32) $\nabla_3(\nabla f) + \frac{1}{2}\mathrm{tr}\underline{\chi}\nabla f = -\hat{\underline{\chi}}\,\nabla f$ we deduce,

$$\nabla_u(r|\nabla f|) = O(\delta^{1/2})\, r|\nabla f|$$

Therefore,

$$r|\nabla f| = r_0|\nabla f_0|\big(1 + O(\delta^{1/2})\big) \tag{35}$$

We can also deduce in the same manner an estimate for $r^2|\nabla^2 f|$. Indeed, differentiating (32) and commuting ∇ with ∇_3, according to lemma 2.1 we deduce,

$$\nabla_3(\nabla^2 f) + \mathrm{tr}\underline{\chi}\nabla^2 f = -\hat{\underline{\chi}}\,\nabla^2 f + O(\delta^c)(1 + \frac{1}{r})|\nabla f|$$

Note that the $\frac{1}{r}$ term is due to the contribution of the term $\mathrm{tr}\underline{\chi}\eta\cdot\nabla f$ which appear in the commutation lemma. Hence, since $r \le r_0 \le r$ we deduce using (35),

$$\begin{aligned}\nabla_u(r^2|\nabla^2 f)| &= O(\delta^{1/2})r^2|\nabla^2 f| + O(\delta^c)(1 + \frac{1}{r})r^2|\nabla f| \\ &= O(\delta^{1/2})r^2|\nabla^2 f| + O(r\delta^c)r_0|\nabla f_0|\end{aligned}$$

and we infer that,

$$r^2|\nabla^2 f| \lesssim C\big(r_0^2|\nabla^2 f_0| + O(\delta^c)r_0|\nabla f_0|\big) \tag{36}$$

Proceeding in the same manner with (33) we derive, for $H = e^f \mathrm{div}\,(e^{-f}\nabla f)$

$$\nabla_3 H + \mathrm{tr}\underline{\chi} H = O(\delta^{1/2})\big(|\nabla^2 f| + |\nabla f|^2\big) + O(\delta^c)(1 + \frac{1}{r})|\nabla f|$$

We deduce,

$$\begin{aligned}\nabla_u(r^2 H) &= O(\delta^{1/2})r^2|\nabla^2 f| + O(\delta^c)r|\nabla f| \\ &= O(\delta^c)\big[r_0^2|\nabla^2 f_0| + r_0|\nabla f_0|\big]\end{aligned}$$

Hence,

$$r^2 H = r_0^2 H_0 + O(\delta^c)\big[r_0^2|\nabla^2 f_0| + r_0|\nabla f_0|\big]$$

or,

$$r^2 \mathrm{div}\,(e^{-f})\nabla f = -r_0^2\Delta(e^{-f_0}) + O(\delta^c)\big[r_0^2|\nabla^2 f_0| + r_0|\nabla f_0|\big]e^{-f_0} \tag{37}$$

Now, from equation (38),

$$\nabla_3|F| + \frac{1}{2}\mathrm{tr}\underline{\chi}|F| = |\nabla f| + O(\delta^{1/2})F$$

we deduce,

$$\nabla_u(r|F|) = O(\delta^{1/2})r|F| + r|\nabla f| = O(\delta^{1/2})r|F| + r_0|\nabla f_0|\big(1 + O(\delta^c)\big)$$

and therefore, since $F_0 = e^{-f_0}|\nabla v_0| = 0$,

$$r|F| = ur_0|\nabla f_0|\big(1 + O(\delta^c)\big) \tag{38}$$

with $C > 0$ independent of δ or f_0.

We next calculate ∇F. Using the commutation lemma 2.1 we deduce,

$$\nabla_3|\nabla F| + \mathrm{tr}\underline{\chi}|\nabla F| \ \ \leq \ \ |\nabla^2 f| + O(\delta^{1/2})|\nabla F| + O(\delta^c)|F|$$

Thus, according to (36)

$$\begin{aligned} \nabla_u(r^2|\nabla F|) \ &\leq \ r^2|\nabla^2 f| + O(\delta^{1/2})r^2|\nabla F| + O(\delta^c)r^2|F| \\ &\leq \ O(\delta^{1/2})r^2|\nabla F| + C\big(r_0^2|\nabla^2 f_0| + O(\delta^c)r_0|\nabla f_0|\big) + O(\delta^c)rr_0|\nabla f_0| \end{aligned}$$

We deduce,

$$r^2|\nabla F| \ \ \leq \ \ C\big(r_0^2|\nabla^2 f_0| + O(\delta^c)r_0|\nabla f_0|\big) \tag{39}$$

Since $G = e^{-f}F$ we also deduce,

$$r^2|\nabla G| \ \ \leq \ \ C\big(r_0^2|\nabla^2 f_0| + O(\delta^c)r_0|\nabla f_0|\big)e^{-f_0} \tag{40}$$

Next we calculate div G from (30) which we write in the form,

$$\begin{aligned} \nabla_3(\mathrm{div}\ G) + \mathrm{tr}\underline{\chi}\mathrm{div}\ G \ &= \ \mathrm{div}\ (e^{-f}\nabla f) + O(\delta^c)I_0 e^{-f_0} \\ I_0 \ &:= \ \big(r_0^2|\nabla^2 f_0| + O(\delta^c)r_0|\nabla f_0|\big). \end{aligned}$$

Hence, making use of (37)

$$\begin{aligned} \nabla_u(r^2\mathrm{div}\ G) \ &= \ r^2\mathrm{div}\ (e^{-f}\nabla f) + O(\delta^c)r^2 I_0 e^{-f_0} \\ &= \ -r_0^2\Delta(e^{-f_0}) + O(\delta^c)I_0 + O(\delta^c)r^2 I_0 e^{-f_0} \end{aligned}$$

Therefore,

$$r^2\mathrm{div}\ G = -ur_0^2\Delta(e^{-f_0}) + O(\delta^c)I_0 \tag{41}$$

Finally, going back to (34), and formula (38) for $|F|$,

$$\begin{aligned} \mathrm{tr}\chi' \ &= \ \mathrm{tr}\chi - 2e^f\mathrm{div}\ (G) + \frac{2}{r}|F|^2 + |F|^2 O(\delta^c) \\ &= \ \mathrm{tr}\chi + 2ur^{-2}r_0^2 e^{f_0}\Delta(e^{-f_0}) + O(r^{-2}\delta^c)I_0 + \frac{2}{r^3}u^2 r_0^2|\nabla f_0|^2\big(1 + O(\delta^c)\big) \\ &= \ \mathrm{tr}\chi + 2ur^{-2}r_0^2\big(-\Delta f_0 + |\nabla f_0|^2\big) + \frac{2}{r^3}u^2 r_0^2|\nabla f_0|^2\big(1 + O(\delta^c)\big) + O(r^{-2}\delta^c)I_0 \\ &= \ \mathrm{tr}\chi + \frac{2ur_0^2}{r^2}\left(-\Delta f_0 + \big[1 + u/r\,\big(1 + O(\delta^c)\big]\,|\nabla f_0|^2\big]\right) + O(r^{-2}\delta^c)I_0 \end{aligned}$$

We summarize the result in the following proposition.

Proposition 3.7. *Assume that* **MA1-MA3** *and* **MA2-S**, **MA3-S** *are verified in the space-time region* $\mathcal{D}(u_*, \delta)$ *and* f, v *defined according to* (20) *The the expansion* $tr\chi'$ *of the* v *foliation verifies, for all* $0 \le u \le u_*$ *and* $0 \le \underline{u} \le \delta$, *with* $I_0 = \big(r_0^2|\nabla^2 f_0| + O(\delta^c) r_0 |\nabla f_0|\big)$ *verifies,*

$$tr\chi' \quad = \quad tr\chi + \frac{2ur_0^2}{r^2}\Bigg(- \Delta f_0 + \big[1 + u/r\,\big(1 + O(\delta^c)\big]\,|\nabla f_0|^2\big]\Bigg) + O(r^{-2}\delta^c) I_0 \tag{42}$$

In particular, if $\delta > 0$ *is sufficiently small,*

$$tr\chi' \quad \le \quad tr\chi + \frac{2ur_0^2}{r^2}\big[- \Delta f_0 + \big(1 + \frac{u}{r}\big)\big]\,|\nabla f_0|^2 + O(r^{-2}\delta^c) I_0 \tag{43}$$

3.8. **Main equation.** We now combine the results of propositions 2.3 and 3.7. For simplicity we shall also assume that $r_0 = 1$. According to proposition 2.3 we have,

$$\mathrm{tr}\chi(u, \delta) = \frac{2}{r(u, \delta)} - \frac{1}{r^2(u, \delta)} \int_0^\delta |\hat{\chi}_0|^2(\underline{u}') d\underline{u}' + O(\delta^c)$$

Thus, inserting in (42)

$$\mathrm{tr}\chi'(u, \delta) \quad \le \quad \frac{2}{r} + \frac{2u}{r^2}\Bigg(- \Delta f_0 + \big[1 + u/r\,\big(1 + O(\delta^c)\big]\,|\nabla f_0|^2\big]\Bigg) - \frac{1}{r^2} M_0 + O(r^{-2}\delta^c) I_0$$

where $r = r(u, \delta)$ and

$$M_0 = \int_0^\delta |\hat{\chi}_0|^2(\underline{u}') d\underline{u}'.$$

Now, along a level surface[3] $S_1 := \{v = 1\} \cap H_{\underline{u}}$ we can express both u and r as functions along S_0 which we denote by $U = U(f_0)$ and $R = R(f_0)$. In fact, since $v = ue^{f_0}$ we deduce $U = e^{-f_0}$. Moreover since according to (11) $\frac{dr}{du} = -1 + O(\delta^{1/2} r) = -1 + O(\delta^{1/2})$ we can write,

$$R = 1 - U + O(\delta^{1/2})\, U$$

To have $\mathrm{tr}\chi'$ non-positive along S_{v_0} we need,

$$\frac{2}{R} + \frac{2U}{R^2}\Bigg(- \Delta f_0 + \big[1 + U/R\,\big(1 + O(\delta^c)\big]\,|\nabla f_0|^2\big]\Bigg) \le \frac{1}{r^2}\big(M_0 - O(\delta^c) I_0\big)$$

We deduce the following.

Corollary 3.9. *A necessary condition for* S_1 *to be a trapped surface, is that,*

$$-\Delta f_0 + \big[1 + U/R\,\big(1 + O(\delta^c)\big]\,|\nabla f_0|^2\big] + \frac{R}{U} \le \frac{1}{2U}\big(M_0 - O(\delta^c) I_0\big) \tag{44}$$

where,

$$U \quad = \quad e^{-f_0}, \qquad R = 1 - e^{-f_0} + O(\delta^{1/2})\, e^{-f_0}$$

[3] with v the deformation function defined by (20)

Note that the inequality is only meaningful in the domain $\mathcal{D}$, i.e. for $U = e^{-f_0} \le u_*$, or for δ sufficiently small,

$$R > 1 - u_* \tag{45}$$

We now re-express (44) with respect to $R = R(f_0)$. We have,

$$\begin{aligned}\nabla R &= \frac{R}{df}(f_0)\nabla f_0\\ \Delta R &= \frac{R}{df}(f_0)\Delta f_0 + \frac{d^2 R}{d^2 f}(f_0)|\nabla f_0|^2\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{dR}{df}(f_0) &= \frac{dr}{du}\cdot\frac{dU}{df} = -\nabla_u r\cdot e^{-f_0}\\ \frac{d^2 R}{d^2 f}(f_0) &= \nabla_u^2 r\cdot e^{-f_0} + \nabla_u r e^{-f_0}\end{aligned}$$

In view of formula (7) and the equation,

$$\nabla_3 \mathrm{tr}\underline{\chi} + \frac{1}{2}(\mathrm{tr}\underline{\chi})^2 = -2\underline{\omega}\mathrm{tr}\underline{\chi} - |\hat{\underline{\chi}}\,|^2$$

we deduce,

$$\begin{aligned}\nabla_u(r\nabla_u r) &= \frac{1}{8\pi}\nabla_u\int_{S(u,\underline{u})}\Omega\mathrm{tr}\underline{\chi} = \frac{1}{8\pi}\int_{S(u,\underline{u})}\Omega\big(e_3(\Omega\mathrm{tr}\underline{\chi}) + \Omega\mathrm{tr}\underline{\chi}\mathrm{tr}\underline{\chi}\big)\\ &= \frac{1}{16\pi}\int_{S(u,\underline{u})}\Omega^2\mathrm{tr}\underline{\chi}^2 - \frac{1}{8\pi}\int_{S(u,\underline{u})}\Omega^2|\hat{\underline{\chi}}\,|^2 = 1 + rO(\delta^{1/2})\end{aligned}$$

Hence,

$$r\nabla_u^2 r + (\nabla_u r)^2 = \Omega^2 + rO(\delta^{1/2}) = 1 + O(\delta^{1/2})$$

from which we deduce,

$$r\nabla_u^2 r = O(\delta^{1/2}). \tag{46}$$

Hence,

$$\begin{aligned}|\nabla R|^2 &= |\nabla f_0|^2 e^{-2f_0}\,|\nabla_u r|^2 = |\nabla f_0|^2 e^{-2f_0}\left(1 + O(\delta^{1/2})\right)\\ \Delta R &= -\Delta f_0 e^{-f_0}\nabla_u r + \big(\nabla_u^2 r\cdot e^{-f_0} + \nabla_u r e^{-f_0}\big)|\nabla f_0|^2\\ &= e^{-f_0}\Delta f_0\big(1 + O(\delta^{1/2}\big) + (-1 + O(\delta^{1/2})e^{-f_0}|\nabla f_0|^2\\ &+ R^{-1}e^{-f_0}|\nabla f_0|^2 O(\delta^{1/2})\\ &= e^{-f_0}\big(\Delta f_0 - |\nabla f_0|^2\big) + O(\delta^{1/2})\big(\Delta f_0 + (1 + R^{-1})|\nabla f_0|^2\big)\end{aligned}$$

Thus,

$$\begin{aligned}|\nabla f_0|^2 &= e^{2f_0}|\nabla R|^2 + O(\delta^{1/2})|\nabla f_0|^2\\ \Delta f_0 &= e^{f_0}\Delta R + |\nabla f_0|^2 + O(\delta^{1/2})\big(\Delta f_0 + (1+R^{-1})|\nabla f_0|^2\big)\\ &= e^{f_0}\Delta R + e^{-2f_0}|\nabla R|^2 + O(\delta^{1/2})\big(\Delta f_0 + (1+R^{-1})|\nabla f_0|^2\big)\end{aligned}$$

Note that,

The left hand side of (44) becomes,

$$\begin{aligned}L &= -\Delta f_0 + \left[1 + U/R\left(1+O(\delta^c)\right)\right]|\nabla f_0|^2\big] + \frac{R}{U}\\ &= -e^{f_0}\Delta R - e^{-2f_0}|\nabla R|^2 + (1+e^{-f_0}R^{-1})e^{2f_0}|\nabla R|^2 + Re^{f_0}\\ &+ O(\delta^c)R^{-1}e^{f_0}|\nabla R|^2 + O(\delta^{1/2})\big(e^{f_0}\Delta R + e^{2f_0}|\nabla R|^2\big)\end{aligned}$$

Hence,

$$L = e^{f_0}\Big[-\Delta R + R^{-1}|\nabla R|^2\big(1+O(\delta^c)\big) + R + O(\delta^{1/2})\big(\Delta R + e^{f_0}|\nabla R|^2\big)\Big]$$

The inequality (44) becomes,

$$-\Delta R + R^{-1}|\nabla R|^2\big(1+O(\delta^c)\big) + R + O(\delta^{1/2})\big(\Delta R + e^{f_0}|\nabla R|^2 \le \frac{1}{2}(M_0 - O(\delta^c)I_0)$$

or,

$$-\Delta R + R^{-1}|\nabla R|^2\big(1+O(\delta^c)\big) + R \le 2^{-1}M_0 + O(\delta^c)J$$

where J is a fixed smooth function depending only on $\nabla^2 R$ and ∇R. We deduce the following.

Proposition 3.10. *A necessary condition such that S_1 is a trapped surface is that, for $\delta >$ sufficiently small there exists a smooth function R on $_0$ verifying $R > 1 - u_*$ and the differential inequality,*

$$-\Delta R + \big(1+O(\delta^c)\big)R^{-1}|\nabla R|^2 + R \le 2^{-1}M_0 + O(\delta^c)J \tag{47}$$

To proceed we need to make the assumption $R \ge \delta^{-c/2}$. Hence, it suffices to prove the inequality:

$$-\Delta R + R^{-1}|\nabla R|^2 + R \le 2^{-1}M_0 + O(\delta^{c/2})J$$

or, for δ sufficiently small,

$$-\Delta R + R^{-1}|\nabla R|^2 + R < 2^{-1}M_0. \tag{48}$$

In order that the initial surface, corresponding to $R = 1$, is not trapped we need $2^{-1}M_0 < 1$. We also note, by the maximum principle that,

$$\max R < 2^{-1}\max M_0$$

Thus,

$$1 - u_* < 2^{-1} \max_{S_0} M_0. \tag{49}$$

is a necessary condition for the formation of a trapped surface. Recall that u_* is the maximum of u for which our assumptions are satisfied in $\mathcal{D}(u, \delta)$. Is it sufficient ?

Performing the transformation $R = e^{-\phi}$ we derive,

$$\Delta\phi + 1 < 2^{-1} M_0 e^{\phi}$$

4. Solutions to the deformation equation on a fixed sphere (S, γ).

In this section we provide examples of solutions to our main deformation equation,

$$\Delta\phi + 1 < M e^{\phi} \tag{50}$$

on a smooth, compact, 2−dimensional Riemnannian manifold S, diffeomorphic to the standard sphere, with strictly positive Gaussian curvature K. We define $r = r(S)$ such that $|S| = 4\pi r^2$ and define,

$$k_m = \min_S r^2 K, \qquad k_M = \max_S r^2 K$$

We consider geodesic balls $B(p, \epsilon) = B(p, \epsilon)$ for sufficiently small $\epsilon > 0$. We start by proving the following lemma.

Lemma 4.1. *Given a ball $B(p, \epsilon) \subset S$, there exists a function w_ϵ, smooth outside the point p, such that*

$$\Delta w_\epsilon + K = 4\pi\delta_p \tag{51}$$

where δ_p is the Dirac measure at p. Moreover, if λ denotes the distance function from p,

$$w_\epsilon \;=\; \chi_\epsilon \log\lambda + v \tag{52}$$

with $v \in C^{1.1-}(S)$, $v(p) = 0$, smooth in $S \setminus \{p\}$ and χ_ϵ a smooth cutoff function,

$$\begin{cases} \chi_\epsilon = 1 & on \quad B(p, \epsilon) \\ \chi_\epsilon = 0 & on \quad B(p, 2\epsilon) \end{cases}$$

Assuming the lemma true we consider the cut-off function

$$\begin{cases} \varphi_\epsilon = 0 & \text{on} \quad B(p, \epsilon/2) \\ \varphi_\epsilon = 1 & \text{on} \quad B_\epsilon(p) \end{cases}$$

and define $w'_\epsilon = \varphi_\epsilon w_\epsilon$. Note that w'_ϵ verifies the following properties:

$$\begin{cases} w'_\epsilon = 0, & \text{on} \quad B(p,\epsilon/2) \\ w'_\epsilon = \log\epsilon + O(1), & \text{on} \quad S\setminus B(p,\epsilon/2) \\ \nabla^2 w'_\epsilon = O(\epsilon^{-2}\log\epsilon) & \text{on} \quad S\setminus B(p,\epsilon/2) \\ \Delta w'_\epsilon + K = 0 & \text{on} \quad S\setminus B(p,\epsilon) \end{cases} \tag{53}$$

Consider now the function $w_\epsilon = \Lambda w'_\epsilon$, for a fixed constant Λ and observe that, on $S\setminus B(p,\epsilon)$, we must have, $\Delta w_\epsilon + 1 = -\Lambda K + 1 < 0$ provided that $\Lambda > k_m^{-1}$. Thus, with a fixed choice of $\Lambda > k_m^{-1}$ we have,

$$\begin{cases} w_\epsilon = 0, & \text{on} \quad B(p,\epsilon/2) \\ w_\epsilon = \Lambda\log\epsilon + O(1), & \text{on} \quad S\setminus B(p,\epsilon/2) \\ \nabla^2 w_\epsilon = O(\epsilon^{-2}\log\epsilon) & \text{on} \quad S\setminus B(p,\epsilon/2) \\ \Delta w_\epsilon + 1 < 0 & \text{on} \quad S\setminus B(p,\epsilon) \end{cases} \tag{54}$$

It remains to check under what conditions for M, the function w_ϵ verifies (50). Clearly, on $S\setminus B(p,\epsilon)$, (50) is trivially verified in view of the fact that $M \ge 0$. Now let $M_\epsilon = \inf_{B(p,\epsilon)} M$. Thus, for some constant C,

$$\begin{aligned} Me^{w_\epsilon} &\ge M_\epsilon e^{w_\epsilon} \ge C\epsilon^\Lambda M_\epsilon, \\ \Delta w_\epsilon + 1 &= O(\epsilon^2\log\epsilon) \end{aligned}$$

Hence, to have (50) verified in $B(p,\epsilon)$ we need,

$$O(\epsilon^{-2}\log\epsilon) < M_\epsilon\epsilon^\Lambda$$

This proves the following.

Proposition 4.2. *Let $M_\epsilon = \min_{B(p,\epsilon)} M$ and let $\Lambda > (\min_S K)^{-1}$. Assume that, for some universal constant $C > 0$,*

$$M_\epsilon > C\epsilon^{-2-\Lambda}\log\epsilon \tag{55}$$

Then, for sufficiently small $\epsilon > 0$, there exists a function ϕ_ϵ verifying the inequality (50) and such that

$$\min\phi_\epsilon > \log\epsilon + O(1) \tag{56}$$

$$|\nabla\phi_\epsilon| = O(\epsilon^{-1}\log\epsilon), \quad |\nabla^2\phi_\epsilon| = O(\epsilon^{-2}\log\epsilon). \tag{57}$$

In remains to prove lemma 4.1. This is a standard argument, see for example chapter 2 in [**?**], which we sketch below.

Let λ be the geodesic distance function from p. In a neighborhood of p we can write,

$$ds^2 = d\lambda^2 + a^2(\lambda,\theta)d\theta^2, \qquad a(0) = 0, \frac{da}{d\lambda}(0) = 1.$$

Let h be the geodesic curvature of the level curves of λ, i.e. denoting, $e = \gamma(\partial_\theta, \partial_\theta)^{-1/2}\partial_\theta$ the unit tangent vector to these curves,

$$h = \gamma(\nabla_e \partial_\lambda, e) = a^{-1}\partial_\lambda a$$

h verifies the second variation formula,

$$\partial_\lambda h = -h^2 - K$$

or,

$$\partial_\lambda^2 a + Ka = 0. \tag{58}$$

Since $a(0) = 0, \frac{da}{d\lambda}(0) = 1$ we deduce from (58) that $\partial_\lambda^2 a(0) = 0$. Consequently,

$$a(\lambda) = \lambda + O(\lambda^3), \qquad \partial_\lambda a(\lambda) = 1 + O(\lambda^2), \quad h(\lambda) = \lambda^{-1} + O(\lambda)$$

Now,

$$\Delta \log \lambda = \lambda^{-1} h - \lambda^{-2} = O(1)$$

Thus, for $\delta < \epsilon$ converging to 0,

$$\int_{S \setminus B(p,\delta)} \Delta(\chi_\epsilon \log \lambda) dv_\gamma = -2\pi + O(\delta)^2$$

Hence, passing to the limit,

$$\int_S \Delta(\chi_\epsilon \log \lambda) dv_\gamma = -2\pi$$

Note also that,

$$\int_S \chi \Delta(\chi_\epsilon \log \lambda) dv_\gamma = -2\pi \chi(p) \tag{59}$$

for any smooth test function χ supported in $B(p, \epsilon)$.

We now solve the equation,

$$\Delta_S v = f. \tag{60}$$

where, f is the bounded function

$$\begin{cases} f = K + 2\Delta(\chi_\epsilon \log \lambda) & \text{on } S \setminus \{p\} \\ f = 0 & \text{at } p \end{cases}$$

Note that (60) admits a $C^{1,1}$ solution in view of the fact that $f \in L^\infty(S)$ and,

$$\int_S f dv_\gamma = \int_S K dv_\gamma + 2\int_S \Delta(\chi_\epsilon \log \lambda) dv_\gamma = 4\pi - 4\pi = 0.$$

We can also normalized v such $v(p) = 0$. We now define,

$$w_\epsilon = 2\chi_\epsilon \log \lambda - v$$

and note that,

$$\Delta w_\epsilon + K = 0, \qquad \text{on } S \setminus \{p\}$$

Moreover, in view of (59),

$$\Delta w_\epsilon + K = -4\pi\delta$$

as desired.

Remark 4.3. Note that the proof of the proposition only requires the existence of a smooth function w on S which verifies $\Delta w + 1 < 0$ in the complement of a closed domain D for which $\inf_D M : M_D > 0$. Indeed if such a function exists we can produce a solution to our inequality simply by taking $\phi = -\log s + w$ for a sufficiently small constant $s > 0$. Indeed, with such a choice (50) is automatically satisfied in the complement of D. In D, $\Delta\phi = \Delta w$, $e^\phi = s^{-1}e^w$ and therefore we need $\max_D \left[e^{-w}(\Delta w + 1)\right] < s^{-1}M_D$. Finally we note that such solutions can easily be constructed for balls $B(p, \delta)$ with $\delta < i_p$, the radius of injectivity of S at p.

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