A BOUND ON THE EXCLUDED MINORS FOR A SURFACE

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Abstract

For any connected surface Σ, we find explicitly an upper bound on the number of vertices in any graph that cannot be embedded in Σ and is minor-minimal with this property.

1. INTRODUCTION

In this paper, by a *surface* we mean a compact, connected 2-manifold without boundary. An *excluded minor* for a surface Σ is a graph G that cannot be embedded in Σ, without isolated vertices, such that for every edge e of G, both $G\backslash e$ and G/e can be embedded in Σ. ($G\backslash e$ and G/e are the graphs obtained by deleting e and contracting e respectively.) Kuratowski's theorem implies that the excluded minors for the sphere are precisely K_5 and $K_{3,3}$ (up to isomorphism - we shall omit this henceforth), and the excluded minors for the projective plane are given by Archdeacon's result [1] - there are 35 of them. For much more complicated surfaces, one would not expect to find all the excluded minors explicitly, because there are too many, and an easier task is to find an upper bound on their size (that is, number of vertices). Archdeacon and Huneke [2] did this for every non-orientable surface, but for the orientable surfaces no bound has yet been found. That there *is* a bound, that is, that the number of excluded minors is finite, was shown by Robertson and the author in [6]; indeed, we show in [10] that for any property of graphs that can be characterized by excluded minors (such as having an embedding in Σ) the list of excluded minors is finite.

The main result of this paper is such a bound, for every surface. By the *complexity* of a surface Σ we mean twice its orientable genus if it is orientable (that is, twice the number of handles we must add to a sphere to obtain it), and its non-orientable genus if it is non-orientable (that is, the number of crosscaps we must add to a sphere to obtain it). We shall show the following.

(1.1) *Let Σ be a surface of complexity g. Then every excluded minor for Σ has at most 2^{2^k} vertices, where $k = (3g+9)^9$.*

To show (1.1), we first show that every excluded minor for Σ has "tree-width" $\leq$

$14(g+3)^3$ (we define tree-width in section 3), and then use Thomas' theorem [12] to bound its size. Obtaining the tree-width bound occupies sections 2 and 3, and in section 4 we convert it to the size bound.

M. Fellows told me in 1989 that finding an explicit bound on the tree-width of the excluded minors for a surface Σ would yield an algorithm to find the excluded minors (see [4]). That motivated the research reported here, and I would like to express my thanks to Fellows for this idea.

2. REDRAWING A GRAPH

If Σ is a compact 2-manifold, an *O-arc* in Σ is a subset homeomorphic to a circle, and a *line* is a subset homeomorphic to the closed interval [0, 1]. If $X \subseteq \Sigma$, its closure is denoted by $\bar{X}$. A *closed disc* in Σ is a subset homeomorphic to $\{(x,y) : x^2+y^2 \leq 1\}$, and an *open disc* is defined similarly.

A *drawing* in a compact 2-manifold Σ is a pair (U,V), where $U \subseteq \Sigma$ is closed, $V \subseteq U$ is finite, $U - V$ has only finitely many arc-wise connected components, called *edges*, and for each edge e, either $|\bar{e} - e| = 1$ and $\bar{e}$ is an O-arc, or $|\bar{e} - e| = 2$ and $\bar{e}$ is a line with ends the members of $\bar{e} - e$. If $\Gamma = (U,V)$ is a drawing, we write $U(\Gamma) = U, V(\Gamma) = V$, and the members of V are the *vertices* of Γ. The arc-wise connected components of $\Sigma - U(\Gamma)$ are the *regions* of Γ. If every region is an open disc, Γ is 2-*cell* (in Σ). If Γ is 2-cell in Σ, it follows that $bd(\Sigma) \subseteq U(\Gamma)$ where $bd(\Sigma)$ denotes the boundary of Σ. If Γ is a drawing in Σ and $\Delta \subseteq \Sigma$ is such that $\bar{e} \subseteq \Delta$ or $e \cap \Delta = \emptyset$ for every edge e of Γ, then $(U(\Gamma) \cap \Delta, V(\Gamma) \cap \Delta)$ is a drawing which we denote by $\Gamma \cap \Delta$. A drawing is obviously a graph with vertices and edges as given, and we use graph-theoretic terminology for drawings in the natural way.

Graphs in this paper are finite, and may have loops or parallel edges. If G is a graph, we write $H \subseteq G$ to denote that H is a subgraph of G. $V(G)$ and $E(G)$ denote the vertex- and edge-sets of a graph G. A *circuit* of G is a non-null connected subgraph in which every vertex has valency 2 (for instance, a loop forms a 1-edge circuit). A *path* is a non-null tree in G in which every vertex has valency ≤ 2. In particular, paths and circuits have no "repeated" vertices or edges. We denote by $G\backslash X$ the graph obtained from G by deleting X (here X may be a vertex or an edge, or a set of vertices or edges). If H is a subgraph of G, a *bridge* of H in G is a connected subgraph B of G with $|E(B)| \neq \emptyset$ and $E(B \cap H) = \emptyset$, such that either

(i) $|E(B)| = 1$ and the edge of B has both ends in $V(H)$; such bridges are called *trivial* bridges

(ii) $|E(B)| > 1$ and B consists of a component C of $G\backslash V(H)$ together with all edges of G between $V(C)$ and $V(H)$ and their ends; these are called *non-trivial* bridges.

An *embedding* (α, Γ) of a graph G in a compact 2-manifold Σ is an isomorphism α between G and a drawing Γ in Σ; and if (α, Γ) is an embedding of G in Σ we say that Γ is a *drawing of* G in Σ.

Let $k \geq 2$ be an integer, and let G be a graph. A *k-nest* in G consists of a $(k+1)$-tuple $(A, C_1, ..., C_k)$, where

(i) $A \subseteq G$

(ii) $C_1, ..., C_k$ are mutually disjoint circuits of A

(iii) every edge of G with an end in $V(A) - V(C_1)$ belongs to $E(A)$

(iv) A is planar, and there is an embedding (α, Γ) of A in a closed disc Δ_1, so that for $1 \leq i \leq k$ the O-arc $U(\alpha(C_i))$ bounds a closed disc $\Delta_i \subseteq \Delta_1$ with $\Delta_k \subseteq \Delta_{k-1} \subseteq ... \subseteq \Delta_1$, and $\Delta_k - bd(\Delta_k)$ is a region of Γ.

A circuit C of G is *k-nested* if there is a k-nest $(A, C_1, ..., C_k)$ in G with $C_k = C$. The main result of this section is the following.

(2.1) *Let Σ be a surface of complexity g, and let G be a graph that has an embedding in Σ. Let C be a $(g+2)$-nested circuit of G. Then G has an embedding (α, Γ) in Σ so that $U(\alpha(C))$ bounds a region of Γ that is an open disc.*

Proof. We proceed by induction on g, and for fixed g by induction on $|V(G)| + |E(G)|$. Let $(A, C_1, ..., C_{g+2})$ be a $(g+2)$-nest in G with $C_{g+2} = C$. From the hypothesis, A has an embedding in a disc Δ_1, and to simplify notation we may therefore assume that A itself is such a drawing; that is, A is a drawing in a disc $\Delta_1, U(C_1) = bd(\Delta_1)$, and for $1 \leq i \leq g+2$ Δ_2 is the closed disc bounded by $U(C_i)$, and $\Delta_1 \supseteq \Delta_2 \supseteq ... \supseteq \Delta_{g+2}$ and $\Delta_{g+2} - U(C)$ is a region of A. We may assume that

(1) *There is no circuit $C' \neq C$ of A with $U(C') \subseteq \Delta_{g+1} - bd(\Delta_{g+1})$ bounding a closed disc Δ' in Δ_{g+1} with $\Delta_{g+2} \subseteq \Delta'$.*

Subproof. If there is, then C' is a $(g+2)$-nested circuit in G', where G' is obtained from G by deleting all vertices and edges of A in $\Delta' - bd(\Delta')$. Since G' is a proper subgraph of G, from the second inductive hypothesis G' can be drawn in Σ so that C' bounds an open disc region; but then the remainder of G can be drawn inside this region and the result holds. Thus we may assume (1).

We may assume that

(2) *There is a path of A between $V(C)$ and $V(C_1)$.*

Subproof. Suppose not, and let the component of A containing C be G'. Since $V(G' \cap C_1) = \emptyset$ it follows that G' is a component of G; let $G \backslash V(G') = G''$. Now G' is a subgraph of A, and hence can be drawn in a closed disc so that C bounds an open disc region. But G'' can be drawn in Σ; take such a drawing, choose a closed disc in Σ disjoint from the drawing of G'', and draw G' in it in the way just described. This gives a drawing of G in Σ satisfying the theorem. Thus we may assume (2).

Now since $g+2 \geq 2$, there is a bridge B_1 of C containing C_1. Therefore B_1 contains all $C_1, C_2, ..., C_{g+1}$ since by (2) and the planarity of A, these circuits all belong to the same bridge. We may assume

(3) *There is no bridge $B \neq B_1$ of C in G with $V(B \cap A) \neq \emptyset$.*

Subproof. Suppose there is such a bridge. Then B is a subgraph of A since $B \cap C_1$ is null; and indeed, $U(B) \subseteq \Delta_{g+1} - U(C_{g+1})$. It follows that $|V(B \cap C)| \leq 1$, for if $|V(B \cap C)| \geq 2$ there would be a circuit C' as in (1). Let G' be obtained from G by deleting all vertices and edges of B not in C. Then C is $(g+2)$-nested in G', and G' is a proper subgraph of G, and so from the inductive hypothesis G' can be drawn in Σ so that C bounds an open disc region. But since B is planar (because $U(B) \subseteq \Delta_{g+1}$) we can augment this drawing to a drawing of G with the desired property. Thus we may assume (3).

Let G' be obtained from G by deleting all edges and vertices of A in $\Delta_{g+1} - U(C_{g+1})$. We may assume

(4) *G' cannot be drawn in Σ so that some non-null-homotopic O-arc F is disjoint from the drawing.*

Subproof. Suppose there is such a drawing. By cutting Σ along F we obtain a 2-manifold with boundary, with one or two components. Its boundary is either one O-arc, or two disjoint O-arcs, and by pasting discs onto these O-arcs, we obtain either

(i) a surface Σ' of complexity $g-1$ or $g-2$, or

(ii) two disjoint surfaces Σ_1, Σ_2 of complexity g_1 and g_2, where $g_1, g_2 > 0$ and $g_1 + g_2 = g$.

In the first case G' can be drawn in Σ', and since C_{g+1} is $(g+1)$-nested in G', there is an embedding of G' in Σ' so that C_{g+1} bounds an open disc region. But then G'' can be drawn in this region to obtain an embedding of G in Σ' so that C bounds an open disc region r; and to convert this to an embedding in a surface homeomorphic to Σ, we choose a region $r' \neq r_1$ and add a handle or crosscap within it appropriately. Thus in this case the result holds.

In the second case, let G_1, G_2 be the subgraphs of G' drawn in Σ_1 and Σ_2 respectively, where $C_{g+1} \subseteq G_1$. By (2), all of $C_1, ..., C_g$ belong to G_1 and so C_{g+1} is $(g+1)$-nested in G_1. Hence G_1 can be embedded in Σ_1 so that C_{g+1} bounds an open disc region. But then, as in the first case, we may draw the remainder of G within this region, and so G can be embedded in $\Sigma_1 \cup \Sigma_2$ so that C bounds an open disc region r. By choosing an O-arc $F_i \subseteq \Sigma_i$ disjoint from the drawing ($i = 1, 2$), with $F_1 \cap r = \emptyset$, and adding to $\Sigma_1 \cup \Sigma_2$ a cylinder with boundary $F_1 \cup F_2$, we obtain an embedding of G in Σ so that C bounds an open disc region. Thus we may assume (4).

Take an embedding (Γ, α) of G in Σ. By (4), $U(\alpha(C))$ is null-homotopic in Σ, and so by [3, theorem (1.7)], there is a disc $\Delta \subseteq \Sigma$ bounded by $U(\alpha(C))$. If $\Delta - bd(\Delta)$ is a region of Γ we are done, and so we suppose not. If $U(\alpha(B_1))$ meets $\Delta - bd(\Delta)$ then since B_1 is a bridge it follows that $U(\alpha(B_1)) \subseteq \Delta$. In that case it follows that $B_1 \cup C$ is planar, and can be drawn in a sphere so that C bounds a region; and by (3), $B_1 \cup C$ is a component of G.

Take an embedding of $G\backslash V(B_1 \cup C)$ in Σ, choose a closed disc $\Delta' \subseteq \Sigma$ disjoint from the drawing, and embed $B_1 \cup C$ in Δ' so that C bounds a region. This gives an embedding of G in Σ satisfying the theorem. We may therefore assume that $U(\alpha(B_1)) \cap \Delta \subseteq bd(\Delta)$. By (3), any bridge B of C in G with $U(\alpha(B)) \cap \Delta \not\subseteq bd(\Delta)$ satisfies $V(B \cap C) = \emptyset$, and hence is a planar component of G (because $U(\alpha(B)) \subseteq \Delta$); and we may therefore change the embedding (Γ, α) so that B is not drawn inside Δ. The result follows. ■

We deduce

(2.2) *Let Σ, Σ' be surfaces, where Σ has complexity g. Let G be a drawing in Σ', let $e \in E(G)$, and suppose there are $g+2$ disjoint circuits of $G\backslash e$, all bounding discs in Σ' which include e. If $G\backslash e$ can be drawn in Σ then so can G.*

Proof. Let $C_1, ..., C_{g+2}$ be the disjoint circuits of $G\backslash e$, bounding discs $\Delta_1, ..., \Delta_{g+2}$ in Σ' respectively, where $\Delta_{g+2} \subseteq \Delta_{g+1} \subseteq ... \subseteq \Delta_1$. Let G' be obtained from G by deleting all vertices and edges in $\Delta_{g+2} - bd(\Delta_{g+2})$ (and in particular, deleting e). Let $A = G' \cap \Delta_1$; then $(A, C_1, ..., C_{g+2})$ is a $(g+2)$-nest in G'. If $G\backslash e$ can be embedded in Σ, then so can G', and by (2.1) there is an embedding of G' in Σ so that C_{g+2} bounds an open disc region. Then the remainder of G can be drawn in this region, and so G can be embedded in Σ, as required. ■

3. TANGLES AND DISTANCE

A *separation* in a graph G is a pair of subgraphs (A, B) with $A \cup B = G$ and $E(A \cap B) = \emptyset$; its *order* is $|V(A \cap B)|$. Let $\theta \geq 1$ be an integer. A *tangle of order* θ in a graph G is a set $\mathcal{T}$ of separations of G, all of order $< \theta$, such that

(i) $\mathcal{T}$ contains one of $(A, B), (B, A)$, for every separation (A, B) of G of order $< \theta$

(ii) if $(A_i, B_i) \in \mathcal{T} (i = 1, 2, 3)$ then $A_1 \cup A_2 \cup A_3 \neq G$

(iii) if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Intuitively, a tangle in G describes a piece of G that is in some sense θ-connected. For instance, there is a 1-1 correspondence between the tangles of order 1 and the components of G, between the tangles of order 2 and the loopless blocks of G, and between the tangles of order 3 and the non-trivial constituents of Tutte's decomposition of G into 3-connected pieces. Tangles were introduced and studied in [7].

Let Γ be a drawing in a surface Σ. We say $X \subseteq \Sigma$ is Γ-*normal* if $X \cap U(\Gamma) \subseteq V(\Gamma)$. Let $\mathcal{T}$ be a tangle of order $\theta \geq 1$ in Γ. We say that $\mathcal{T}$ is *respectful* if for every Γ-normal O-arc F in Σ with $|F \cap V(\Gamma)| < \theta$, there is a closed disc $\Delta \subseteq \Sigma$ bounded by F such that

$$(\Gamma \cap \Delta, \Gamma \cap \overline{\Sigma - \Delta}) \in \mathcal{T}.$$

If there is such a disc Δ, it is necessarily unique, and we denote it by $ins(F)$. A *curve* in Σ is a continuous function $\phi : S^1 \to \Sigma$, where S^1 is the unit circle. We denote $\{\phi(x) : x \in S^1\}$ by $\bar{\phi}$. It is Γ-*normal* if $\bar{\phi}$ is Γ-normal, and its length (with respect to Γ) is the cardinality of the set $\{x \in S^1 : \phi(x) \in U(\Gamma)\}$. If $\mathcal{T}$ is a respectful tangle of order θ in Γ, and ϕ is a Γ-normal curve with length $< \theta$, we define $ins(\phi)$ to be the union of $\bar{\phi}$ and $ins(F)$ taken over all O-arcs $F \subseteq \bar{\phi}$. The *atoms* of Γ are the regions of Γ, the edges of Γ, and the sets $\{v\}$ $(v \in V(\Gamma))$, and the set of atoms is denoted by $A(\Gamma)$. Let $\mathcal{T}$ be a respectful tangle in Γ, and let $a, b \in A(\Gamma)$. If $a = b$ we define $d(a, b) = 0$. If $a \neq b$ and there is a Γ-normal curve ϕ of length $< \theta$ with $a \cap ins(\phi) \neq \emptyset \neq b \cap ins(\phi)$, we define $d(a, b)$ to be the minimum length of such a curve. If $a \neq b$ and there is no curve ϕ as in (ii), we define $d(a, b) = \theta$. It is shown in [8, theorem (9.1)] that d is a metric if Γ is 2-cell. We call d the *metric* of $\mathcal{T}$.

We need the following, from [9, theorem (9.2)].

(3.1) *Let Γ be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle of order θ in Γ, with metric d. Let $z \in A(\Gamma)$, and let κ be an integer with $2 \leq \kappa \leq \theta - 3$. Then there is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) \subseteq U(\Gamma)$, such that*

(i) $d(z, x) \leq \kappa + 2$ *for every* $x \in A(\Gamma)$ *with* $x \subseteq \Delta$

(ii) $d(z, x) \geq \kappa$ *for every* $x \in A(\Gamma)$ *with* $x \not\subseteq \Delta - bd(\Delta)$.

We use (3.1) to prove the following.

(3.2) *Let Σ be a surface of complexity g, and let $t \geq 0$ and $k \geq 2$ be integers. Let Γ be a drawing in Σ, and let $\mathcal{R}$ be a set of regions of Γ with $|\mathcal{R}| \leq t$. If Γ has a tangle of order $\geq 2(g+1)(t+g+1)(3k+4)$ then there is an edge e of Γ and k disjoint circuits of $\Gamma \backslash e$, all bounding closed discs in Σ including e and including no member of $\mathcal{R}$.*

Proof. We proceed by induction on $|V(\Gamma)| + g$. Now by [11, theorem (2.11)], some component Γ' of Γ has a tangle of order $\geq 2(g+1)(t+g+1)(3k+4)$, and if the result holds for Γ' then it holds for Γ. We may therefore assume that Γ is connected. Let $\theta = 2(t+1)(3k+4)$.

Suppose first that there is a Γ-normal O-arc F with $|F \cap V(\Gamma)| < \theta$ which is non-null-homotopic. Let $F \cap V(\Gamma) = Z$; then by [7, theorem (8.5)], $\Gamma \backslash Z$ has a tangle of order

$$\geq 2(g+1)(t+g+1)(3k+4) - \theta = 2g\,((t+2)+(g-1)+1)(3k+4).$$

Let Σ' be the 2-manifold obtained from Σ by cutting along F. By pasting discs on the (one or two) components of $bd(\Sigma')$, we obtain either

(i) a surface Σ_1 of complexity $< g$, or

(ii) two disjoint surfaces Σ_1, Σ_2 with complexity g_1 and g_2, such that $g_1, g_2 < g$ and $g_1 + g_2 = g$.

In the first case, $\Gamma \backslash Z$ is a drawing in Σ_1; let $\mathcal{R}_1$ be the set of regions of $\Gamma \backslash Z$ in Σ_1 that either are in $\mathcal{R}$ or are not regions of Γ in Σ (there are at most two of the latter, at most one for each of the discs pasted onto Σ'). Hence $|\mathcal{R}_1| \leq t+2$. From the inductive hypothesis the result holds for $\Gamma \backslash Z$ and Σ_1; let $\Delta_1, ..., \Delta_k$ be the corresponding closed discs in Σ_1. Since each Δ_i includes no member of $\mathcal{R}_1$ it follows that each Δ_i is a subset of Σ, and so the result holds for Γ and Σ, as required.

In the second case, let $\Gamma_i = (\Gamma \backslash Z) \cap \Sigma_i$ $(i = 1, 2)$. Now (Γ_1, Γ_2) is a separation of $\Gamma \backslash Z$ of order 0. By [11, theorem (2.11)], one of Γ_1, Γ_2 has a tangle of order $\geq 2g((t+2)+(g-1)+1)(3k+4)$, say Γ_1. Let $\mathcal{R}_1$ be the set of regions of Γ_1 in Σ_1 that are either in $\mathcal{R}$ or not regions of Γ in Σ; then $|\mathcal{R}_1| \leq t+1$, and the result follows as in the first case.

We may therefore assume that every Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < \theta$ is null-homotopic. In particular, since $\theta \geq 1$ every O-arc F with $F \cap U(\Gamma) = \emptyset$ is null-homotopic, and since Γ is connected and non-null it follows that Γ is 2-cell.

We claim that Γ has a respectful tangle of order θ. If Σ is a sphere this is true, since $g = 0$ and therefore Γ has a tangle of order $\geq 2(t+1)(3k+4) = \theta$ and hence has one of order θ (take all members of the first tangle which have order $< \theta$), and since every tangle in a drawing in a sphere is respectful. If Σ is not a sphere, then the claim follows from [8, theorem (4.1)] since Γ is 2-cell and every Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < \theta$ is null-homotopic. This proves our claim that Γ has a respectful tangle, $\mathcal{T}$ say, of order θ. Let d be its metric.

Choose $e_0 \in E(\Gamma)$. By [8, theorem (8.12)] there is an edge e_θ of Γ with $d(e_0, e_\theta) = \theta$. Since G is connected, there is a path P of Γ with first edge e_0 and last edge e_θ. For $0 \leq i \leq \theta$, let e_i be the last edge e of P with $d(e_0, e) \leq i$. If f is the next edge of P, then

$d(e_0, f) \geq i+1$ and $d(e_i, f) \leq 4$, and so by the triangle inequality, $d(e_0, e_i) \geq i-3$. (If there is no such f, then $e_i = e_\theta$ and hence $d(e_0, e_i) = \theta \geq i-3$.)

Let $r \in \mathcal{R}$. We claim there are at most $6k+8$ values of i with $0 \leq i \leq \theta$ and with $d(r, e_i) \leq 3k+2$. For suppose there are $\geq 6k+9$. Then there are two, say i and j, such that $j-i \geq 6k+8$. But then

$$\begin{aligned} j-3 &\leq d(e_0, e_j) \leq d(e_0, e_i) + d(e_i, e_j) \leq i + d(e_i, e_j) \\ &\leq i + d(r, e_i) + d(r, e_j) \leq i + 2(3k+2), \end{aligned}$$

a contradiction. Thus there are at most $6k+8$ such values of i. Since $|\mathcal{R}| \leq t$ and there are $\theta + 1 > t(6k+8)$ values of i altogether, it follows that for some i with $0 \leq i \leq \theta, d(r, e_i) \geq 3k+2$ for all $r \in \mathcal{R}$. Let $e = e_i$.

By (3.1), taking $z = e$ and $\kappa = 3, 6, ..., 3k$, there are closed discs $\Delta_1, ..., \Delta_k \subseteq \Sigma$ such that for $1 \leq i \leq k$

(i) $bd(\Delta_i) \subseteq U(\Gamma)$; let C_i be the circuit $\Gamma \cap bd(\Delta_i)$

(ii) $d(e, x) \leq 3i+2$ for every $x \in A(\Gamma)$ with $x \subseteq \Delta_i$

(iii) $d(e, x) \geq 3i$ for every $x \in A(\Gamma)$ with $x \not\subseteq \Delta_i - bd(\Delta_i)$.

From (iii) it follows that $e \subseteq \Delta_i - bd(\Delta_i)$ for each i. Moreover, $C_1, ..., C_k$ are mutually disjoint; for if $1 \leq i < j \leq k$ and $v \in V(C_i \cap C_j)$ say, then $d(e, \{v\}) \leq 3i+2$ since $\{v\} \subseteq \Delta_i$, and $d(e, \{v\}) \geq 3j$ since $\{v\} \not\subseteq \Delta_j - bd(\Delta_j)$, a contradiction. Finally, since $d(e, r) \geq 3k+3$ for each $r \in \mathcal{R}$, it follows from (ii) that each Δ_i includes no $r \in \mathcal{R}$. ■

A *tree-decomposition* of a graph G is a pair $(T, (X_t : t \in V(T)))$ where T is a tree and each X_t is a subset of $V(G)$, such that

(i) $\bigcup(X_t : t \in V(T)) = V(G)$, and for every edge e of G there exists $t \in V(T)$ so that X_t contains both ends of e

(ii) if $t, t', t'' \in V(T)$ and t' lies on the path of T between t and t'' then $X_t \cap X_{t''} \subseteq X_{t'}$.

It has *width* $\leq w$ if $|X_t| \leq w + 1$ for all $t \in V(T)$; and the *tree-width* of G is the minimum width of a tree-decomposition. From (3.2) we deduce

(3.3) *Let Σ be a surface of complexity g. Let G be a graph that cannot be embedded in Σ, such that $G \backslash e$ can be embedded in Σ for every edge e. Then G has tree-width $\leq 3(g+3)^2(3g+16) - 3$.*

Proof. Choose $f \in E(G)$; then $G \backslash f$ can be embedded in Σ, and so G has a drawing Γ in a surface Σ' of complexity $g' \leq g + 2$ (add a handle to Σ appropriately and draw the edge f running along the handle). By (2.2), for each edge e of G, there do not exist $g + 2$ disjoint circuits of $G \backslash e$ all bounding discs in Σ' including e. By (3.2) (with $\Sigma, g, t, k, \Gamma, \mathcal{R}$ replaced by $\Sigma', g', 0, g' + 2, \Gamma, \emptyset$), G has no tangle of order $\geq 2(g+3)^2(3g+16)$. By [7, theorem (5.2)], G has tree-width at most

$$\frac{3}{2}(2(g+3)^2(3g+16) - 1) - 1$$

as required. ■

4. FROM TREE-WIDTH TO SIZE

Now we come to the second half of the proof. Our objective here is to prove the following.

(4.1) *Let Σ be a surface of complexity g, and let G be an excluded minor for Σ, with tree-width $< w$. Then $|V(G)| \leq (2w + 2g)^p$ where $p = \prod_{1 \leq h \leq w} (12(g + h - 1))!$*

From (4.1) and (3.3), our main result (1.1) follows after some arithmetic, which we leave to the reader.

To prove (4.1), first we need the following. For $X \subseteq V(G)$, a *bridge* of X in G means a bridge of H in G, where H is the subgraph of G with $V(H) = X$ and $E(H) = \emptyset$.

(4.2) *Let Σ be a surface of complexity g, let G be a drawing in Σ, and let $X \subseteq V(G)$. Let $\mathcal{B}$ be the set of non-trivial bridges of X in G, and suppose that for each $B \in \mathcal{B}$ with $|X \cap V(B)| \leq 2, B$ cannot be drawn in a disc with $X \cap V(B)$ drawn in the boundary. Then $|\mathcal{B}| \leq 2|X| + 2g - 1$ unless $|V(G)| = g = 0$.*

Proof. We proceed by induction on $|V(G)| + |E(G)| + g$. We may assume that

(1) *$X \cap V(C) \neq \emptyset$ for every circuit C of G with $U(C)$ non-null-homotopic.*

Subproof. Suppose that C is a circuit of $G \backslash X$ and $U(C)$ is non-null-homotopic. Let D be the component of $G \backslash X$ containing C. Then $G \backslash V(D)$ is a drawing in the 2-manifold Σ' obtained by cutting Σ along $U(C)$. By pasting discs on the components of $bd(\Sigma')$, we obtain either

(i) a surface Σ_1 of complexity of $g_1 < g$, or

(ii) two disjoint surfaces Σ_1, Σ_2 of complexity of g_1 and g_2, such that $0 < g_1, g_2 < g$ and $g_1 + g_2 = g$.

Suppose that (i) holds. Since $G \backslash V(D)$ is a drawing in Σ_1, it follows from the inductive hypothesis that X has at most

$$max(2|X| + 2g_1 - 1, 0) \leq 2|X| + 2g - 2$$

bridges in $G\backslash V(D)$, and hence at most $2|X|+2g-1$ bridges in G, and so the result holds.

Now suppose that (ii) occurs. For $i=1,2$, let $G_i=(G\backslash V(D))\cap\Sigma_i$, let $X_i=X\cap\Sigma_i$ and let $\mathcal{B}_i$ be the set of bridges of X_i in G_i. From the inductive hypothesis, since $g_1,g_2\neq 0$, $|\mathcal{B}_i|\leq 2|X_i|+2g_i-1$ for $i=1,2$. Since $|\mathcal{B}|=|\mathcal{B}_1|+|\mathcal{B}_2|+1$ and $g=g_1+g_2$, we deduce that $|\mathcal{B}|\leq 2|X|+2g-1$, as required. This proves (1).

We may assume that

(2) *G is simple, and no edge has both ends in X, and $|V(B)\cap X|\leq 3$ for each $B\in\mathcal{B}$, and for $B\in\mathcal{B}$, if $|V(B)\cap X|=3$ then $|V(B)-X|=1$.*

Subproof. If there is an edge e which is a loop, or which is parallel to another edge, or with both ends in X, or with one end in X and with $e\in E(B)$ for some $B\in\mathcal{B}$ with $|V(B)\cap X|\geq 4$, then the result follows from the inductive hypothesis applied to $G\backslash e$. Also, if there exists $B\in\mathcal{B}$ with $|V(B)\cap X|=3$ and $|V(B)-X|\geq 2$, let e be an edge of B with both ends in $V(B)-X$; then the result follows from the inductive hypothesis applied to G/e. This proves (2).

A *Kuratowski subgraph* of G is a subgraph that is a subdivision of K_5 or $K_{3,3}$. From (1) it follows that

(3) *If K is a Kuratowski subgraph of G then $|V(K)\cap X|\geq 2$.*

Subproof. Suppose that $|V(K)\cap X|\leq 1$. Since K is non-planar there is a circuit C of K such that $U(C)$ is non-null-homotopic, and so $|V(K)\cap X|=1$, $V(K)\cap X=\{x\}$ say. By (1), there is no circuit C of $K\backslash x$ such that $U(C)$ is non-null-homotopic, and so by [5, theorem (11.10)] there is a closed disc $\Delta\subseteq\Sigma$ with $U(K\backslash x)\subseteq\Delta$. Let C be a circuit of $K\backslash x$, chosen so that the disc Δ' in Δ bounded by $U(C)$ is maximal. Now there are four cases, depending whether K is a subdivision of K_5 or of $K_{3,3}$, and whether x has valency 2 or > 2 in K. In each case it is easy to check that however $K\backslash x$ is drawn in Δ, some neighbour of x belongs to $\Delta' - U(C)$. Hence $x \in \Delta' - U(C)$, and so $U(K) \subseteq \Delta'$, which is impossible since K is non-planar. This proves (3).

Let $B \in \mathcal{B}$ with $|V(C) \cap X| \leq 2$. If $|V(B) \cap X| \leq 1$ then from (3) and Kuratowski's theorem, B is planar contrary to hypothesis. Thus $|V(B) \cap X| = 2$, $V(B) \cap X = \{x_1, x_2\}$ say. Let B^+ be obtained from B by adding a new edge f to B with ends x_1, x_2. By hypothesis, B^+ is non-planar, and so by Kuratowski's theorem, B^+ has a Kuratowski subgraph K. By (3), $\{x_1, x_2\} \subseteq V(K)$, and we may choose K so that $K\backslash\{x_1, x_2\}$ is connected. If $E(B) \not\subseteq E(K)$, the result follows from the inductive hypothesis applied to the graph obtained from G by deleting all vertices and edges of B not in K. We may assume therefore that $E(B) \subseteq E(K)$, and so $E(K) = E(B)$ or $E(B) \cup \{f\}$. If some vertex v of K has valency 2 in K, let $e \in E(B)$ be incident with v; then the result follows from the inductive hypothesis applied to G/e. We may therefore assume that K is isomorphic to K_5 or to $K_{3,3}$. If K is isomorphic to K_5, then x_1, x_2 are adjacent in K, and so $f \in E(K)$ since x_1, x_2 are not adjacent in B. In this case, B has three vertices different from x_1, x_2, and they are mutually adjacent and are all adjacent to both x_1 and x_2; and we say B is of *type* 1. In this case, define $l(e)$ for each edge e of B by: $l(e) = 2$ if e has an end in X, and $l(e) = 4$ if e has no end in X.

The second possibility is that K is isomorphic to $K_{3,3}$, and then possibly $f \in E(K)$ and possibly $f \notin E(K)$. If $f \in E(K)$, we say B is of *type* 2; B has four vertices a, b, c, d different from x_1, x_2, and eight edges $ab, bc, cd, da, ax_1, cx_1, bx_2, dx_2$. For $e \in E(B)$, we define $l(e) = 1$ if e has an end in X, and $l(e) = 3$ if e has no end in X. If $f \notin E(K)$, we say B is of *type* 3; B has four vertices a, b, c, d different from x_1, x_2, and nine edges $ax_1, bx_1, cx_1, ax_2, bx_2, cx_2$, ad, bd, cd. For $e \in E(B)$, we define $l(e) = 2$.

Thus, there are altogether three types of bridges B with $|V(B) \cap X| = 2$. If $B \in \mathcal{B}$

with $|V(B) \cap X| \neq 2$, then by (2) $|V(B) \cap X| = 3$, $V(B) \cap X = \{x_1, x_2, x_3\}$ say, and B has only one vertex $a \neq x_1, x_2, x_3$, and has just three edges ax_1, ax_2, ax_3. In this case we say B has *type* 4, and we define $l(e) = 2$ for each edge of B.

Since every edge of G belongs to $E(B)$ for some (unique) $B \in \mathcal{B}$ by (2), we have defined $l(e)$ for each $e \in E(B)$. Let there be q_i members of $\mathcal{B}$ of type i for $i = 1, 2, 3, 4$. Then

$$\begin{aligned} |V(G)| &= |X| + 3q_1 + 4q_2 + 4q_3 + q_4 \\ |E(G)| &= 9q_1 + 8q_2 + 9q_3 + 3q_4. \end{aligned}$$

But we may assume that G is non-null, and so by Euler's formula, if G has p regions, then

$$|V(G)| - |E(G)| + p \geq 2 - g.$$

(Note that equality need not hold here since G may not be 2-cell.) On substitution, we obtain

(4) $p \geq 2 - g - |X| + 6q_1 + 4q_2 + 5q_3 + 2q_4$.

In particular, if $p \leq 1$ then by (4),

$$|\mathcal{B}| = q_1 + q_2 + q_3 + q_4 \leq 6q_1 + 4q_2 + 5q_3 + 2q_4 \leq |X| + g - 1 \leq 2|X| + 2g - 1$$

as required, and so we may assume that $|\mathcal{R}| = p \geq 2$.

Now let $B \in \mathcal{B}$ of type 1 or 2, and let C_B be the circuit $B \backslash (V(B) \cap X)$. By (1), $U(C_B)$ is null-homotopic, and so by [3, theorem (1.7)] there is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = U(C_B)$. Let $G' = G \backslash V(C_B)$, $G_1 = G' \cap (\Sigma - \Delta)$, $X_1 = X \cap (\Sigma - \Delta)$, $G_2 = G' \cap \Delta$ and $X_2 = X \cap \Delta$, and let $\mathcal{B}_i$ be the set of bridges of X_i in G_i ($i = 1, 2$). Then $|\mathcal{B} = |\mathcal{B}_1| + |\mathcal{B}_2| + 1$. Since B has type 1 or 2, not both vertices in $X \cap V(B)$ are drawn within Δ, and so $V(G_1) \neq \emptyset$. If also $V(G_2) \neq \emptyset$, then from the inductive hypothesis, $|\mathcal{B}| \leq 2|X_1| + 2g - 1$ and $|\mathcal{B}_2| \leq 2|X_2 - 1$, and so $|\mathcal{B}| \leq 2|X| + 2g - 1$ as required. We may therefore assume that $V(G_2) = \emptyset$, and so $U(C_B)$ bounds a region of G.

For each region r of G, let $l(r)$ denote the sum of $l(e)$, taken over all edges e of G incident with r and counting twice those edges e incident with no other region. Now every circuit C of G satisfies $\sum_{e \in E(C)} l(e) \geq 8$, from the definition of l. Moreover, Σ is connected and $p \geq 2$, and so for every region r there is a circuit C of G such that every edge of C is incident with r. Consequently $l(r) \geq 8$ for every region r of G, and $l(r) \geq 12$ for at least $q_1 + q_2$ regions of G, namely those bounded by $U(C_B)$ where $B \in \mathcal{B}$ has type 1 or 2. Thus, denoting the set of all regions by $\mathcal{R}$,

$$\sum_{r \in \mathcal{R}} l(r) \geq 8R + 4(q_1 + q_2) \geq 8(2 - g - |X|) + 52q_1 + 36q_2 + 40q_3 + 16q_4$$

by (4). Now for each $B \in \mathcal{B}$, $\sum_{e \in E(B)} l(e) = 24, 16, 18$ or 6 depending whether B has type 1, 2, 3 or 4; and so

$$\sum_{e \in E(G)} l(e) = 24q_1 + 16q_2 + 18q_3 + 6q_4.$$

But $\sum_{r \in \mathcal{R}} l(r) = 2 \sum_{e \in E(G)} l(e)$; and on substitution we deduce that

$$8(2 - g - |X|) + 4q_1 + 4q_2 + 4q_3 + 4q_4 \leq 0,$$

that is, $|\mathcal{B}| \leq 2|X| + 2g - 4$, as required. ■

(4.3) *Let Σ be a surface of complexity g, let G be an excluded minor for Σ, and let $X \subseteq V(G)$. If B is a non-trivial bridge of X in G with $|V(B) \cap X| \leq 2$ then B cannot be drawn in a closed disc with $V(B) \cap X$ on the boundary.*

Proof. Let $A = G \backslash (V(B) - X)$, and let $A^+ = A$ if $|X \cap V(B)| \leq 1$, and let A^+ be obtained from A by adding an edge joining the two vertices in $X \cap V(B)$ if there are two such vertices. Now A^+ is isomorphic to a proper minor of G, and so can be drawn in Σ. If B can be drawn in a disc with $V(B) \cap X$ on the boundary, we may convert the drawing of A^+ to one of G, a contradiction. The result follows. ■

(4.4) *Let L be a drawing in a surface Σ of complexity g. Suppose that $|E(L)| \geq 2$ and L is simple. Then*

$$|E(L)| \leq 3(|V(L)| + g - 2).$$

Proof. Let L have p regions. Since L is simple and $|E(L)| \geq 2$, every region is incident with ≥ 3 edges (counting an edge twice if the region is incident with it on both sides). Hence $3p \leq 2|E(L)|$. But by Euler's formula,

$$|V(L)| - |E(L)| + p \geq 2 - g$$

(equality need not hold since L may not be 2-cell), and so

$$|V(L)| - \frac{1}{3}|E(L)| \geq 2 - g.$$

The result follows. ■

We need to look at several different kinds of drawings in a surface Σ, but in each case we say two such drawings are *equivalent* if there is a homeomorphism of Σ to itself taking one to the other.

(4.5) *Let Σ be a surface; and let $n \geq 1, m \geq 0$ be integers. There are at most*

$$2^{1-n} \cdot \frac{(2m)!}{(m-n+1)!}$$

equivalence classes of pairs (Γ, π) where Γ is a 2-cell drawing in Σ with n vertices and m edges, and π is a linear order of $V(\Gamma)$.

Proof. The number ν_1 of equivalence classes of pairs (Γ, π) as in the theorem is at most the number ν_2 of equivalence classes of triples (Γ, π, T) where Γ, π are as before and T is a spanning tree of Γ, because every 2-cell drawing is connected.

Given some (Γ, π, T) as above, we may regard π as a bijection from $V(G)$ into $\{1, ..., n\}$. Let $v_0 \in V(G)$ with $\pi(v_0) = n$. For every edge e of T, define $\pi(e) = \pi(v)$ where v is the end of e in the component of $T \backslash e$ not containing v_0. Thus π yields a bijection from $E(T)$ to $\{1, ..., n-1\}$.

Given some (Γ, π, T) and v_0 as above, we define its *signature* as follows. Choose a closed disc $\Delta \subseteq \Sigma$ with $U(T) \subseteq \Delta - bd(\Delta)$ so that $e \not\subseteq \Delta$ and $|e \cap bd(\Delta)| = 2$ for every $e \in E(\Gamma) - E(T)$. Let H be the drawing with $U(H) = U(\Gamma) \cap \Delta$ and $V(H) = V(T) \cup (U(\Gamma) \cap bd(\Delta))$; then H is a tree drawn in Δ. Choose a closed walk W of H following the boundary of the unique region of H in Σ, starting at v_0; thus, every edge of H occurs twice in W. Let S_1 be the sequence of edges in W. Every edge of H that is not an edge of T occurs in two consecutive positions in S_1. Let S_2 be obtained from S_1 by deleting the first occurrence of each edge of H not in $E(T)$ and replacing its second occurrence by the edge of Γ including it. Then every edge of Γ occurs exactly twice in S_2. Take a bijection $\mu : E(\Gamma) - E(T) \to \{n, ..., m\}$. Let S_2 be $e_1, ..., e_{2m}$. For $1 \leq i \leq 2m$, if $e_i \in E(T)$ let $\alpha_i = \pi(e_i)$. If $e_i \notin E(T)$ let $\alpha_i = \mu(e_i)$ if the unique O-arc in $U(T) \cup \{e_i\}$ is orientation-preserving, and $\alpha_i = -\mu(e_i)$ if it is orientation-reversing. The sequence $\alpha_1, ..., \alpha_{2m}$ is a *signature* of (Γ, T, λ).

Since there are $(m-n+1)!$ choices for μ, it follows that (Γ, π, T) has at least $(m-n+1)!$ signatures. But from a knowledge of a signature (and n) one can reconstruct (Γ, π, T) up to equivalence, as we can see as follows. By taking the subsequence of the signature consisting of those terms which are non-negative and at most $n-1$, we obtain the sequence of edges of W, where the edges are named by their values under π. From this we can reconstruct (T, π) up to equivalence. But from the signature we can also reconstruct the cyclic order around v of the edges of Γ incident with each vertex v, corresponding to some fixed orientation of Δ. Since we know which edges e in $E(\Gamma) - E(T)$ give rise to orientation - reversing circuits in $T + e$, we may reconstruct the entire "rotation scheme" of Γ and hence reconstruct Γ, since it is 2-cell.

We deduce that $(m-n+1)!\nu_2 \leq \nu_3$, where ν_3 is the number of sequences of length $2m$ in which every term is equal to exactly one other term, and every term lies in

$$\{1, 2, ..., m\} \cup \{-m, 1-m, ..., -n\},$$

and for $i \geq n$ not both i and $-i$ occur in the sequence. There are $2^{1-n}(2m)!$ such sequences, and so $\nu_3 \leq 2^{1-n}(2m)!$. Hence

$$\nu_1 \leq \nu_2 \leq \frac{\nu_3}{(m-n+1)!} \leq \frac{2^{1-n}(2m)!}{(m-n+1)!}$$

as required. ■

A *template* in Σ is a triple (Γ, π, R), where Γ is a drawing in Σ(not necessarily 2-cell), π is a linear order of $V(\Gamma)$, and R is a subset of the set of regions of Γ, and every edge of Γ is incident with two distinct regions.

(4.6) *Let Σ be a surface of complexity g, and let $n \geq 1, p \geq 1$ be integers. There are at most $(2n + 2p + 2g)! - 2$ equivalence classes of templates (Γ, π, R) such that $|V(\Gamma)| = n$ and Γ has $\leq p$ regions.*

Proof. Let Λ_i be the set of all templates (Γ, π, R) such that $|V(\Gamma)| = n$ and Γ has exactly i regions, and let Λ_i be the union of λ_i equivalence classes. Let $(\Gamma, \pi, R) \in \Lambda_i$. By adding edges to Γ we may obtain a 2-cell drawing Γ' in Σ with i regions; and the edges of Γ' not in Γ are precisely the edges of Γ' incident with only one region. Since for each Γ, π there are 2^i choices for R, it follows that $\lambda_i \leq 2^i \mu_i$ where μ_i is the number of equivalence classes of pairs (Γ, π), where Γ is a 2-cell drawing in Σ with n vertices and i regions, and π is a linear order of $V(\Gamma)$. By Euler's formula, $|E(\Gamma)| = n + g + i - 2$. By (4.5),

$$\mu_i \leq 2^{1-n} \frac{(2n + 2g + 2i - 4)!}{(g + i - 1)!}$$

and so

$$\lambda_i \leq 2^{1+i-n} \frac{(2n + 2g + 2i - 4)!}{(g + i - 1)!}.$$

It follows easily that

$$\sum_{1 \leq i \leq p} \lambda_i \leq (2n + 2p + 2g)! - 2$$

as required. (The -2 is for later convenience.) ■

Let Γ be a drawing in a surface Σ, and let $X \subseteq V(\Gamma)$. A drawing H *separates* Γ at X if

(i) $V(H) = X$ and $U(H) \cap U(\Gamma) = X$

(ii) every region of H intersects exactly one bridge of X in Γ, and

(iii) every edge of H is incident with two distinct regions of H.

(4.7) *Let Γ be a 2-cell drawing in Σ with $E(\Gamma) \neq \emptyset$, and let $X \subseteq V(\Gamma)$. Then there is a drawing H that separates Γ at X.*

Proof. Certainly there is a drawing K in Σ with $V(K) = X$ and $U(K) \cap U(\Gamma) = X$, such that every region of $\Gamma \cup K$ is incident with an edge of Γ; for taking $U(K) = V(K) = X$ is one such drawing, since $E(\Gamma) \neq \emptyset$. For any such K the drawing $\Gamma \cup K$ has at most $2|E(\Gamma)|$ regions, since each is incident with an edge of Γ and every edge of Γ is incident with ≤ 2 such regions. Consequently, by Euler's formula, $|E(K)|$ is bounded above by a function of Γ and Σ. We may therefore choose K maximal with the properties specified above.

We claim that for every region s of K, $s \cap U(B) \neq \emptyset$ for exactly one bridge B of X in Γ. If s is a region of $\Gamma \cup$K it is not incident with any edge of Γ, a contradiction. Thus s is not a region of $\Gamma \cup K$, and so s includes a vertex or edge of Γ. Hence $s \cap U(B) \neq \emptyset$ for at least one bridge B of X in Γ. Suppose that there is more than one such bridge. It follows that there is a region r of $\Gamma \cup K$ with $r \subseteq s$, and two edges e, f of Γ both incident with r, belonging to different bridges of X in Γ. Now Γ is 2-cell and hence so is $\Gamma \cup K$, and therefore r is an open disc; let

$$v_0, e_1, v_1, e_2, ..., e_k, v_k = v_0$$

be a closed walk of $\Gamma \cup K$ following the perimeter of r. We may assume that $e_i = e$ and $e_k = f$, say, where $1 \leq i < k$. Consequently one, p say, of $v_0, v_1, ..., v_{i-1}$ belongs to X and so does one, q say, of $v_i, v_{i+1}, ..., v_{k-1}$, since e and f belong to different bridges of X in Γ. Add to K an edge with ends p, q drawn in r, forming a drawing K' so that one of the two new regions of K' is incident with e and the other with f. This contradicts the maximality of K, and proves our claim that every region of K meets exactly one bridge of X in Γ.

Delete every edge of K that is incident with only one region, forming H; then H separates Γ at X, as required. ■

(4.8) *Let Σ be a surface of complexity g, and let G be an excluded minor for Σ. Let $(A_1, B_1), ..., (A_k, B_k)$ be a sequence of separations of G, all distinct and of the same order n say, such that*

(i) $A_1 \subseteq A_2 \subseteq ... \subseteq A_k$ *and* $B_k \subseteq B_{k-1} \subseteq ... \subseteq B_1$

(ii) *for $1 \leq i < k$ there are n disjoint paths of $B_i \cap A_{i+1}$ between $V(A_i \cap B_i)$ and $V(A_{i+1} \cap B_{i+1})$, and*

(iii) *for* $1 \le i \le k$*, no two vertices in* $V(A_i \cap B_i)$ *are adjacent in* B_i*.*

Then $k < (12(n+g))!$

Proof. First suppose that $n = 0$. By (4.2) and (4.3) with $X = \emptyset, G$ has $\le 2g - 1$ components. Hence $k \le 2g \le (12g)! - 1$ as required. Thus we may assume that $n \ge 1$.

For $1 \le i \le k$, let $V(A_i \cap B_i) = \{v_i^1, ..., v_i^n\}$, numbered so that for $1 \le i < k$ there are n disjoint paths of $B_i \cap A_{i+1}$ with ends v_i^j, v_{i+1}^j for $1 \le j \le n$. For $1 \le i \le k$, let $\mathcal{A}_i$ be the set of all templates (Γ, π, R) in Σ such that $|V(\Gamma)| = n$ and B_i can be drawn in Σ within $V(\Gamma) \cup \bigcup\{r : r \in R\}$ in such a way that for $1 \le j \le n$, the jth term of π represents v_i^j.

(1) *For* $1 \le i < k$, $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$.

Subproof. Let $(\Gamma, \pi, R) \in \mathcal{A}_i$, and take a drawing of B_i as above. Let $P_1, ..., P_n$ be disjoint paths of $B_i \cap A_{i+1}$ where P_j has ends v_i^j, v_{i+1}^j for $1 \le j \le n$. By contracting the edges of $P_1, ..., P_n$ and deleting all other vertices and edges of $B_i \cap A_{i+1}$, we obtain a drawing of B_{i+1} within $V(\Gamma) \cup \bigcup\{r : r \in R\}$ such that the jth term of π represents v_{i+1}^j for $1 \le j \le n$. Hence $(\Gamma, \pi, R) \in \mathcal{A}_{i+1}$.

(2) *For* $1 \le i < k$, $\mathcal{A}_i \ne \mathcal{A}_{i+1}$.

Subproof. Suppose that $\mathcal{A}_i = \mathcal{A}_{i+1}$. Let $P_1, ..., P_n$ be disjoint paths of $B_i \cap A_{i+1}$, where P_j has ends v_i^j, v_{i+1}^j for $1 \le j \le n$. Let G' be obtained from G by contracting all edges in $P_1, ..., P_n$ and deleting all other vertices and edges in $B_i \cap A_{i+1}$. Since $(A_i, B_i) \ne (A_{i+1}, B_{i+1})$ it follows that G' is not isomorphic to G, and so G' can be drawn in Σ. Let Γ be a drawing of G' in Σ. It follows that there is a separation (Γ_1, Γ_2) of Γ with $V(\Gamma_1 \cap \Gamma_2) = \{x_1, ..., x_n\}$ say, such that there is an isomorphism $\alpha : A_i \to \Gamma_1$

with $\alpha(v_i^j) = x_j$ $(1 \le j \le n)$, and an isomorphism $\beta : B_{i+1} \to \Gamma_2$ with $\beta(v_{i+1}^j) = x_j$ $(1 \le j \le n)$. Let $X = \{x_1, ..., x_n\}$. We claim that there are at most $5(n+g)$ bridges of X in Γ. For if C is a non-trivial bridge of X in Γ, then C is a non-trivial bridge of X in one of Γ_1, Γ_2, and therefore is isomorphic to either a non-trivial bridge of $V(A_i \cap B_i)$ in G or of $V(A_{i+1} \cap B_{i+1})$ in G. It follows from (4.3) that if $|V(C) \cap X| \le 2$ then C cannot be drawn in a closed disc with $V(C) \cap X$ on the boundary. By (4.2), there are at most $2n + 2g - 1$ non-trivial bridges of X in Γ. Let us bound the number of trivial bridges. Let L be the restriction of Γ to X. From hypothesis (iii) and the fact that G is simple, it follows that L is simple. From (4.4),

$$|E(L)| \le max(1, 3(n + g - 2)) \le 3n + 3g + 1.$$

Consequently there are at most $5(n+g)$ bridges of X in Γ, as claimed.

By (4.5) there is a drawing H that separates Γ at X. Since every region of H meets a bridge B of X in Γ, and hence includes $U(B) - X$, it follows that H has $\le 5(n+g)$ regions. Let π be the linear order $x_1, x_2, ..., x_n$ of $V(H)$, and let R be the set of regions of H that meet $U(\Gamma_2)$. Then (H, π, R) is a template, and $(H, \pi, R) \in \mathcal{A}_{i+1}$ by definition of $\mathcal{A}_i$ since Γ_2 is a drawing of B_{i+1}. Since $\mathcal{A}_i = \mathcal{A}_{i+1}$ it follows that $(H, \pi, R) \in \mathcal{A}_i$, and so there is a drawing Γ_2' of B_i in Σ and an isomorphism $\beta' : B_i \to \Gamma_2'$, such that $U(\Gamma_2') \subseteq X \cup \bigcup(r : r \in \mathcal{R})$ and $\beta'(v_i^j) = x_j$ $(1 \le j \le n)$. We claim that $U(\Gamma_1) \cap U(\Gamma_2') = X$; for if $y \in U(\Gamma_1) \cap U(\Gamma_2') - X$, then y belongs to a bridge of X in Γ_1, and hence $y \in r_0$ for some region r_0 of H which meets a bridge of X in Γ_1. Since r_0 meets only one bridge of X, it follows that $r_0 \notin R$, contradicting that $U(\Gamma_2') \subseteq X \cup \bigcup(r : r \in R)$. This proves that $U(\Gamma_1) \cap U(\Gamma_2') = X$, and so $\Gamma' = (U(\Gamma_1) \cup U(\Gamma_2'), V(\Gamma_1) \cup V(\Gamma_2'))$ is a drawing in Σ. But let $\alpha'(x) = \alpha(x)$ for x in A_i, and $\alpha'(x) = \beta'(x)$ for x in B_i; then α' is an isomorphism between G and Γ', a contradiction since G cannot be drawn in Σ. This proves (2).

Now for each $i, \mathcal{A}_i$ is a union of equivalence classes of templates (H, π, R) such that $|V(H)| = n$ and H has $\le 5(n+g)$ regions. There are, by (4.6), at most $(12(n+g))! - 2$ such equivalence classes; and from (1) and (2) we deduce that $k \leq (12(n+g))!-1$ as required. ■

(4.9) *Let $n \geq 0$, and let $x_1, ..., x_k$ be a sequence of integers with $0 \leq x_i \leq n$ for each i. For $1 \leq h \leq n$ let $k_h \geq 0$ be an integer, and suppose that for all h with $0 \leq h \leq n$ there do not exist j_1, j_2 with $1 \leq j_1 \leq j_2 \leq k$ such that*

(i) *$x_i \geq h$ for all i with $j_1 \leq i \leq j_2$, and*

(ii) *$x_i = h$ for at least k_h values of i with $j_1 \leq i \leq j_2$.*

Then $k < k_0 k_1 ... k_n$.

Proof. We proceed by induction on n. If $n = 0$ then $x_1 = ... = x_k = 0$ and so since (i), (ii) do not hold with $h = 0$, $j_1 = 1$ and $j_2 = k$, it follows that $k < k_0$ as required. We assume then that $n \geq 1$. From the inductive hypothesis applied to the subsequence of $x_1, ..., x_k$ consisting of all terms $< n$, we deduce that there are at most $t - 1$ such terms where $t = k_0 k_1 ... k_{n-1}$. By an *interval* we mean a set $\{j_1, j_1 + 1, j_1 + 2, ..., j_2\}$ with $1 \leq j_1 \leq j_2 \leq k$ such that

(i) $x_i = n$ for all i with $j_1 \leq i \leq j_2$

(ii) either $j_1 = 1$ or $x_{j_1 - 1} < n$, and

(iii) either $j_2 = n$ or $x_{j_2+1} < n$.

Between any two intervals there is a term with value $< n$, and so there are $\leq t$ intervals. Since (i), (ii) do not hold with $h = n$, it follows that $j_2 - j_1 \leq k_n - 1$ for every interval $\{j_1, ..., j_2\}$. But every term with value n belongs to an interval, and so there are at most $t(k_n - 1)$ such terms. Since there are $\leq t - 1$ with value $< n$ it follows that

$$k \leq t(k_n - 1) + t - 1 = k_0 k_1 ... k_n - 1$$

as required. ■

Proof of (4.1)

Now G has tree-width $< w$, and so it has a tree-decomposition $(T, (X_t : t \in V(T)))$ such that $|X_t| \leq w$ for all $T \in V(T)$. Choose $t_0 \in V(T)$. For every $e \in E(T)$, let T_e, T^e be the two components of $T \backslash e$ where $t_0 \in V(T_e)$. Let $X_e = \bigcup(X_t : t \in V(T_e))$ and define X^e similarly. Let A_e be the subgraph of G with vertex set X_e and edges all edges of G with both ends in X_e; and let B_e be the subgraph of G with vertex set X^e and edges all edges of G with an end in $X^e - X_e$. Then (A_e, B_e) is a separation of G with $V(A_e \cap B_e) = X_e \cap X^e$. Now by the theorem of [12], we may choose $(T, (X_t : t \in V(T)))$ and t_0 so that the following holds:

(1) (i) *$|X_t| \leq w$ for all $t \in V(T)$.*

(ii) *If $e, f \in E(T)$ and f lies on the path of T between t_0 and e, and $|V(A_e \cap B_e)| = |V(A_f \cap B_f)| = n$ say, then either there are n disjoint paths of $B_f \cap A_e$ between $V(A_e \cap B_e)$ and $V(A_f \cap B_f)$, or there is an edge g of T, in the path of T between e and f, so that $|V(A_g \cap B_g)| < n$.*

Choose $(T, (X_t : t \in V(T)))$ and t_0 satisfying (1) with $|V(T)|$ minimum. It follows that

(2) *$A_e \neq G$ for every $e \in E(T)$.*

Subproof. If $A_e = G$ then $(T_e, (X_t : t \in V(T_e)))$ is a tree-decomposition of G, still satisfying (1), with $|V(T_e)| < |V(T)|$, contrary to the choice of T.

(3) *If $e, f \in E(T)$ are distinct and f lies on the path of T between t_0 and e, then $(A_e, B_e) \neq (A_f, B_f)$.*

Subproof. Suppose that $(A_e, B_e) = (A_f, B_f)$. Construct T' from $T^e \cup T_f$ by adding an edge joining the end of e in $V(T^e)$ to the end of f in $V(T_f)$; then $(T', (X_t : t \in V(T')))$ still satisfies (1) and $|V(T')| < |V(T)|$, a contradiction. This proves (3).

(4) *For each $e \in E(T)$, (A_e, B_e) has order $< w$.*

Subproof. Let e have ends t_1, t_2, where t_1 is between t_0 and t_2. Then $V(A_e \cap B_e) = X_e \cap X^e = X_{t_1} \cap X_{t_2}$. But by (3), $X_{t_1} \neq X_{t_2}$ and so $|X_{t_1} \cap X_{t_2}| < w$, since $|X_{t_1}|, |X_{t_2}| \leq w$. This proves (4).

(5) *Every vertex of T has valency $\leq 2w + 2g$.*

Subproof. Let $t \in V(T)$. For each edge $e \in E(T)$ incident with t and not in the path between t_0 and $t, A_e \neq G$ by (2), and so $X_e \neq V(G)$. Consequently there is a non-trivial bridge C_e of X_t in G with $V(C_e) \cap X^e \not\subseteq X_t$. If $e, f \in E(T)$ are distinct and both incident with t and are not in the path between t_0 and t, then $C_e \neq C_f$, for otherwise $C_e \backslash (X_t \cap V(C_e))$ is a connected subgraph of G meeting both X^e and X^f and not meeting X_t which is impossible. Thus all the bridges C_e are distinct. By (4.2), t has valency $\leq 2|X| + 2g \leq 2w + 2g$. This proves (5).

(6) *Every path of T starting from t_0 has $< \prod_{1 \leq h \leq w} (12(g + h - 1))!$ edges.*

Subproof. Let P be a path of T starting from t_0, with k edges $e_1, ..., e_k$ in order. For $1 \leq i \leq k$, let $A_i = A_{e_i}, B_i = B_{e_i}$, and $x_i = |V(A_i \cap B_i)|$. For $0 \leq h \leq w - 1$, let $k_h = (12(g + h))!$. By (4.8) and (1)(ii) the sequence $x_1, ..., x_k$ satisfies the hypothesis of (4.9) (taking $n = w - 1$). It follows from (4.9) that $k < \prod_{0 \leq h \leq w-1} (12(g + h))!$. This proves (6).

Let $d = 2w + 2g$ and $p = \prod_{1\leq h\leq w}(12(g + h - 1))!$. Since G is non-planar and has tree-width $< w$, it follows that $w \geq 4$, and hence $d \geq 8$ and $p \geq 36!$ Now every vertex has valency $\leq d$ and every path starting from t_0 has $< p$ edges, and so

$$|V(T)| \leq 1 + d + d(d-1) + \ldots + d(d-1)^{p-2}.$$

Since $p \geq 4$ and $d \geq 3$, we deduce that

$$\begin{aligned} |V(T)| &\leq 1 + d((d-1)^{p-1} - 1)(d-2)^{-1} \leq d(d-1)^{p-1}(d-2)^{-1} \\ &\leq d^3(d-2)(d-1)^{p-4}(d-2)^{-1} \leq d^{p-1}. \end{aligned}$$

Consequently $|V(G)| \leq w|V(T)| \leq d^p$, as required. ■

References

[1] D. Archdeacon, *A Kuratowski Theorem for the Projective Plane*, Thesis, Ohio State University, 1980.

[2] D. Archdeacon and P. Huneke, "A Kuratowski theorem for non-orientable surfaces", *J. Combinatorial Theory, Ser. B,* 46 (1989), 173-231.

[3] D. B. A. Epstein, "Curves on 2-manifolds and isotopies", *Acta Math.* 115 (1966), 83-107.

[4] M. R. Fellows and M. A. Langston, "An analogue of the MyHill-Nerode theorem and its use in computing finite-basis characterizations", *30th Annual Symposium on Foundations of Computer Science* (1989), 520-525.

[5] N. Robertson and P. D. Seymour, "Graph Minors. VII. Disjoint paths on a surface", *J. Combinatorial Theory, Ser. B,* 45 (1988), 212-254.

[6] N. Robertson and P. D. Seymour, "Graph Minors. VIII. A Kuratowski theorem for general surfaces", *J. Combinatorial Theory, Ser. B,* 48 (1990), 255-288.

[7] N. Robertson and P. D. Seymour, "Graph Minors X. Obstructions to tree-decomposition", *J. Combinatorial Theory, Ser. B,* 52 (1991), 153-190.

[8] N. Robertson and P. D. Seymour, "Graph Minors. XI. Circuits on a surface", *J. Combinatorial Theory, Ser. B,* 60 (1994), 72-106.

[9] N. Robertson and P. D. Seymour, "Graph Minors. XIV. Extending an embedding", submitted for publication.

[10] N. Robertson and P. D. Seymour, "Graph Minors. XX. Wagner's conjecture", in preparation.

[11] N. Robertson, P. D. Seymour and R. Thomas, "Quickly excluding a planar graph", *J. Combinatorial Theory, Ser. B,* to appear.

[12] R. Thomas, "A Menger-like property of tree-width. The finite case", *J. Combinatorial Theory, Ser. B,* 48 (1990), 67-76.