

Excluding sums of Kuratowski graphs

Neil Robertson
Ohio State University, Columbus, OH 43210

Paul Seymour¹
Princeton University, Princeton, NJ 08544

April 7, 2024; revised April 28, 2024

¹Supported by AFOSR grant FA9550-22-1-0234, and NSF grant DDMS-2154169.

Abstract

We prove that a graph does not contain as a minor a graph formed by 0-, 1-, 2- or 3-summing k copies of K_5 or $K_{3,3}$, if and only if it has bounded genus.

1 Introduction

In this paper, graphs are finite and have no loops or parallel edges. By a *surface* we mean a non-null compact 2-manifold without boundary. (In a companion paper [3], surfaces could have boundary, but here we will not need that, so we might as well define it away.) The *genus* of an orientable connected surface is the number of handles we add to a 2-sphere to make the surface, and the genus of a non-orientable connected surface is the number of crosscaps we add to a 2-sphere to make it. The genus of a general (disconnected) surface is the sum of the genera of its components. A graph is a *k-Kuratowski graph* if it has exactly k components, each isomorphic to K_5 or to $K_{3,3}$. It is known [3] that:

1.1 *If H is a k -Kuratowski graph and Σ is a surface, then H can be drawn in Σ if and only if the genus of Σ is at least k .*

The main theorem of [3] says:

1.2 *For all integers $k \geq 1$, there exists $n \geq 0$ such that if a graph G contains no k -Kuratowski graph as a minor, then there exists $X \subseteq V(G)$, with $|X| \leq n$, such that $G \setminus X$ can be drawn on a surface of genus less than k (and hence in which no k -Kuratowski graph can be drawn).*

But that does not mean that G itself has bounded genus, and in this paper we look at the question of what else must be excluded to guarantee that G has bounded genus. For instance, if H is made from a k -Kuratowski graph by choosing one vertex from each component and identifying these vertices, then H has large genus, and so it is necessary to exclude all such graphs H if we want to have small genus. (Let us call such a graph H a $(k, 1)$ -Kuratowski graph). Similarly, if we start with a k -Kuratowski graph, choose two distinct vertices u_i, v_i from the i th component for $1 \leq i \leq k$, and make the identifications $u_1 = \dots = u_k$ and $v_1 = \dots = v_k$, forming some graph H (deleting any edge between these two vertices), again it is necessary to exclude all such graphs H . (We call such a graph H a $(k, 2)$ -Kuratowski graph). And finally, it is necessary to exclude $K_{k,3}$. (Let us call $K_{k,3}$ a $(k, 3)$ -Kuratowski graph, and call a k -Kuratowski graph a $(k, 0)$ -Kuratowski graph, for convenience.) It turns out that this is all that is necessary. We will prove:

1.3 *For all $k \geq 0$, there exists $n \geq 0$ such that if for $0 \leq i \leq 3$, G contains no (k, i) -Kuratowski graph as a minor, then G has genus at most n .*

We first proved this and the result of [3] in the early 1990's, but did not write them up at that time, because our proof was very long and complicated; and now, unfortunately (or perhaps fortunately) that proof is forgotten. The proof given in this paper for 1.4 is certainly different, because we use a result of [1] which was not known at that time.

(k, i) -Kuratowski graphs are in general a mix of K_5 's and $K_{3,3}$'s, but if we have a (k', i) -Kuratowski graph with $k' \geq 3k$, it contains a (k', i) -Kuratowski graph made purely from K_5 or purely from $K_{3,3}$. When $i = 2$, there are two ways to piece $K_{3,3}$'s together, on an edge or on a nonedge, and we could sort them out as well. So we could state the result just in terms of these "purified" (k, i) -Kuratowski graphs, which are shown in figure 1. We call them *basic*.

In fact, we will prove a stronger result that we explain next. Let w be a vertex of a graph G , and let us partition the set of neighbours of w into two sets P, Q . Delete w and add two new vertices u, v , where u is adjacent to the vertex in P , and v is adjacent to those in Q (and u, v have no other

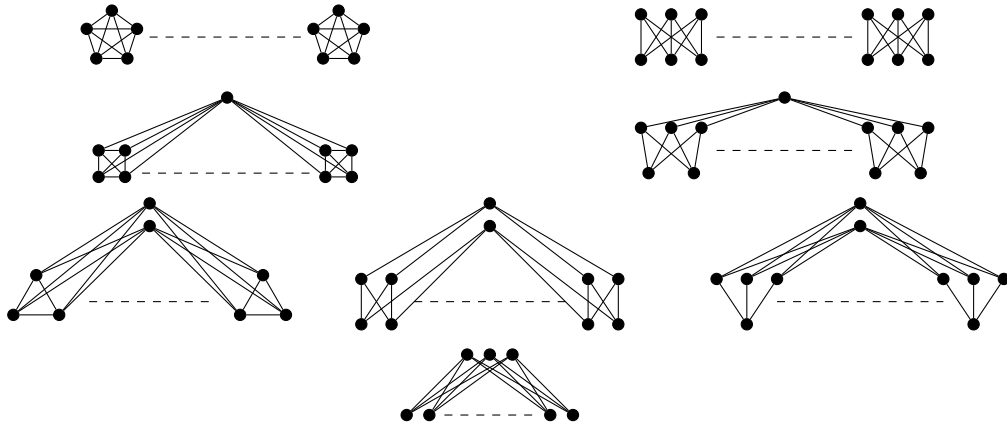


Figure 1: Basic (k, i) -Kuratowski graphs for $i = 0, 1, 2, 3$

neighbours). Let H be the graph we produce. We say that H is obtained from G by *splitting* w , and the process is a *split*. Doing one split might increase or decrease the genus. It might well increase it dramatically, but it does not decrease it by more than one. To see this, observe that if G' is obtained from G by one split, then G can be obtained from G' by adding an edge (which increases genus by at most one) and contracting the new edge (which does not increase genus at all).

In order to prove 1.3, we can first apply 1.1, and find a set $X \subseteq V(G)$, of bounded size, such that $G \setminus X$ can be drawn on a surface Σ of genus less than k . We need to show that G itself has bounded genus, and to prove this, we will prove a stronger statement, that by a bounded number of splits, the resultant graph G' can be drawn in the same surface Σ . (This is a stronger statement, because doing a bounded number of splits can only reduce genus by up to the same number, as we saw.)

Since the genus of the surface Σ is controlled just by the largest k such that G contains a k -Kuratowski graph as a minor, we take advantage of this to get the strongest result we can. We will prove:

1.4 *For all k, ℓ , there exists $n \geq 0$ such that if no minor of G is an ℓ -Kuratowski graph, and for $0 \leq i \leq 3$ no minor of G is a (k, i) -Kuratowski graph, then there is a graph G' that can be obtained from G by at most n splits and can be drawn in a surface of genus less than ℓ .*

As we saw, in order to prove this, it suffices (because of 1.2) to prove that the result holds for graphs G such that deleting a bounded set of vertices X makes a graph that can be drawn in Σ . When Σ is a 2-sphere, it turns out that something stronger is true: we don't need to do any split of the vertices that are already drawn in Σ , we only need to split the vertices in X . More exactly:

1.5 *If Σ is a 2-sphere, then for all integers $k, \xi \geq 1$, there exists $n \geq 0$ such that, if for $0 \leq i \leq 3$ no minor of G is a (k, i) -Kuratowski graph, and there exists $X \subseteq V(G)$ with $|X| \leq \xi$ such that $G \setminus X$ can be drawn in Σ , then by at most n splits of the vertices in X , the resultant graph can be drawn in Σ .*

We have not been able to decide whether 1.5 is true when Σ is not a 2-sphere. We discuss this further at the end of section 5.

2 Equivalent formulations

Saying that a graph does not contain a k -Kuratowski graph as a minor, is evidently equivalent to saying that it does not contain k nonplanar subgraphs, pairwise vertex-disjoint; and that is also equivalent to saying that it does not contain k pairwise vertex-disjoint subgraphs, each a subdivision of K_5 or $K_{3,3}$.

This suggests a different formulation of our main result, which we will explain next. A *rooted graph* is a pair (H, R) where H is a graph and $R \subseteq V(H)$. (We will only need to consider rooted graphs where $|R| \leq 3$.) A rooted graph (H, R) is *flat* if H can be drawn in a closed disc in such a way that the vertices in R are drawn in the boundary of the disc.

For $k \geq 1$ and for $0 \leq i \leq 3$, a (k, i) -*junction* is a graph J such that for some $R \subseteq V(J)$ with $|R| = i$, there are k subgraphs H_1, \dots, H_k with union J , such that $R \subseteq V(H_i)$ for each i , and $V(H_i \cap H_j) = R$ for $1 \leq i < j \leq k$, and

- if $i \in \{0, 1\}$ then H_i is nonplanar;
- if $i = 2$ then (H_i, R) is not flat; and
- if $i = 3$ then H_i is connected, and every vertex in R has a neighbour in $V(H_i) \setminus R$.

Let us say a graph G is k -*subgraph-restricted* if for $i = 0, \dots, 3$, no subgraph is a (k, i) -junction; and G is k -*restricted* if for $0 \leq i \leq 3$, no minor of G is a (k, i) -junction. Here are three possible ways to restrict a graph G :

- For $0 \leq i \leq 3$, no minor of G is a (k, i) -Kuratowski graph;
- G is k -subgraph-restricted;
- G is k -restricted.

Evidently, the third implies the other two. We claim that

2.1 *If the first bullet holds, then the third holds with k replaced by $8k$.*

Proof. Suppose that G is not $8k$ -restricted; then it has a minor G' that is an $(8k, i)$ -junction for some $i \in \{1, \dots, 3\}$. Let R and $(H_1, R), \dots, (H_{8k}, R)$ be as in the definition of $(8k, i)$ -junction. If $i = 0$ or 1 then H_i is not planar, and so it has a subgraph J_i that is a subdivision of K_5 or $K_{3,3}$. If $k = 0$, these are vertex-disjoint and so G' has a minor which is a $(k, 0)$ -Kuratowski graph. If $i = 1$, and k of the J_i 's do not contain the vertex of R , then the same holds; and if k of them do contain the vertex in R , then G' has a minor which is a $(k, 1)$ -Kuratowski graph. Next suppose that $i = 2$. For $1 \leq j \leq 4k$, since (H_{2j-1}, R) and (H_{2j}, R) are not flat, it follows that $H_{2j-1} \cup H_{2j}$ is not planar, and so contains a subdivision J_j of K_5 or $K_{3,3}$. At least k of the J_j 's have equal intersections with R and the result follows easily. Finally, if $i = 3$ then G' evidently contains $K_{k,3}$ as a minor. This proves 2.1. ■

So, in order to prove 1.4, it suffices to prove the next result:

2.2 *For every surface Σ , and all integers $k, \xi \geq 1$, there exists $n \geq 0$ such that, if G is k -restricted and there exists $X \subseteq V(G)$ with $|X| \leq \xi$ such that $G \setminus X$ can be drawn in Σ , then by at most n splits of vertices of G , the resultant graph can be drawn in Σ .*

Returning to the three bullets above: what is much less obvious, is that if the second bullet holds, then so do the first and third (with larger values of k , but independent of G). We will prove this in section 6. In this sense, all three bullets above are equivalent. Consequently 2.2 implies the following apparently much stronger statement:

2.3 *For every surface Σ , and all integers $k, \xi \geq 1$, there exists $n \geq 0$ such that, if G is k -subgraph-restricted and there exists $X \subseteq V(G)$ with $|X| \leq \xi$ such that $G \setminus X$ can be drawn in Σ , then by at most n splits of the vertices of G , the resultant graph can be drawn in Σ .*

3 Two-connected apical pairs

Our main goal is to prove 2.2; and for that we may evidently assume that X is stable, because we can handle any edges between vertices in X with a bounded number of extra splits of vertices in X . We will first prove it when Σ is a 2-sphere, and in that case, we will show the stronger result that only need split vertices in X . Then an application of a result of [4] will prove it in general. Thus, the objective of most of the paper is to prove:

3.1 *For all integers $k, \xi \geq 1$, there exists $n \geq 0$ such that, if G is k -restricted and there exists a stable set $X \subseteq V(G)$ with $|X| \leq \xi$ such that $G \setminus X$ is planar, then by at most n splits of vertices in X , we can convert G to a planar graph.*

If $X \subseteq V(G)$ is a stable set and $G \setminus X$ is planar, we call the pair (G, X) an *apical pair*, and if $X = \{x\}$, we write (G, x) for $(G, \{x\})$. Let us say the *nonplanarity* of an apical pair (G, X) is the minimum n such that G can be converted into a planar graph by splitting the vertices in X into a set of n vertices. We frequently look at a subgraph A of $G \setminus X$, and want to consider the graph consisting of A, X and the edges between X and $V(A)$, which we will call $A + X$; and again, if $X = \{x\}$ we write $A + x$ for $A + \{x\}$. In the proofs, it often happens that we are trying to bound the nonplanarity of (G, X) , and $G \setminus X$ is the union of some subgraphs A_1, \dots, A_t say; and we know that $(A_i + X, X)$ has bounded nonplanarity, for each i . A natural first step is to split each $x \in X$ into t vertices, where the i th vertex is adjacent only to the neighbours of x in $V(A_i)$ (and not necessarily to all of them, if the A_i 's overlap). When we do this for each x , we obtain $t|X|$ vertices, which we can partition into t subsets X_1, \dots, X_t , each containing exactly one vertex that was made by splitting x , for each $x \in X$. Briefly, we call this “splitting X into sibling-free sets”. (After splitting, the graph we obtain might not be k -restricted any more, so it has to be treated with caution.)

The proof of 3.1 breaks into cases depending on the connectivity of $G \setminus X$. If the latter is 3-connected, the result is easy, because of a theorem of Böhme and Mohar below. At the other extreme, when $G \setminus X$ is not connected, the result can easily be deduced from the connected case. To reduce the connected case to the 2-connected case, we use a more complicated inductive argument with weightings, but that is also reasonably straightforward. In going from 2-connected to 3-connected, however, there is an issue that causes some headaches. If (G, X) is an apical pair, and $G \setminus X$ is 2-connected, and (A, B) is a 2-separation of $G \setminus X$, and we know that $(A + X, X)$ and $(B + X, X)$ have bounded nonplanarity, how can we get a bound on the nonplanarity of (G, X) ? To do so, we need to bound the nonplanarity of $(A' + X, X)$ and $(B' + X, X)$, where A', B' are obtained from A, B by adding an edge joining the two vertices in $A \cap B$. In this section we prove a variety of lemmas related to this problem. Let us say an apical pair (G, X) is *2-connected* if $G \setminus X$ is 2-connected.

In this paper we do not permit loops or parallel edges, so we must make adjustments for that. When we contract a set of edges, this may make loops or parallel edges, and if so, we delete all such loops and all but one from each set of parallel edges. When we speak of “adding an edge between u, v ”, again we only add such an edge if u, v are not already adjacent.

We say a region of a drawing *covers* a vertex of the drawing if the region is incident with the vertex, and *covers* an edge if the edge is incident with the region. We use a theorem of Böhme and Mohar [1], that says:

3.2 *For all integers $k \geq 1$ there is an integer $k' \geq 1$ with the following property. Let G be a 3-connected graph drawn in the plane, and let $U \subseteq V(G)$. Then either:*

- *there is a set of at most k' regions of the drawing, together covering every vertex in U ; or*
- *there are two vertex-disjoint connected subgraphs A, B of G , and k distinct vertices $u_1, \dots, u_k \in U$ and not in $V(A \cup B)$, such that for $1 \leq i \leq k$, u_i has a neighbour in $V(A)$ and a neighbour in $V(B)$.*

If U is the set of neighbours of an extra vertex x , then the first outcome implies that $(G + x, x)$ has nonplanarity at most k , and the second implies that $G + x$ contains $K_{k,3}$ as a minor. Our next task is to prove something similar, with the single vertex x replaced by a set X of vertices with bounded size, and with 3-connectivity replaced by 2-connectivity. But that needs a good deal of preparation. First, we need the following lemma.

3.3 *Let (G, X) be a 2-connected apical pair, such that G is planar. Let $u, v \in V(G \setminus X)$ be distinct, and suppose that there is a planar drawing of $G \setminus X$ such that u, v are incident with the same region. Then by at most $3|X| + 64|X|^2$ splits of the vertices in X , we can obtain from G a graph that can be drawn in the plane such that both u, v are incident with the same region.*

Proof. Let $\xi = |X|$, and let $H = G \setminus X$. Since H is 2-connected, and admits a drawing with u, v incident with the same region, there is a cycle C of H containing u, v such that there is a drawing of H in a plane with C the boundary of the infinite region.

Fix a drawing of G in a 2-sphere Σ . This gives us a (different, in general) planar drawing of H . By an *H-bridge* we mean a subgraph of H which is either

- a subgraph of H consisting of one edge e and its ends, such that both ends of e are in $V(C)$ and $e \notin E(C)$, or
- a subgraph of H that is the union of a component D of $H \setminus V(C)$, all vertices in C that have a neighbour in $V(D)$, and all edges between $V(C)$ and $V(D)$.

If D is an *H-bridge*, each vertex of C with a neighbour in $V(D)$ is an *attachment* of D . Let P_1, P_2 be the two paths of C between u, v . We may assume that both P_1, P_2 have at least two edges, because otherwise the result is true since G is planar. Now C bounds a disc Δ in the given drawing of G ; and each *H-bridge* is either inside C or outside, in the natural sense, and we call them *inner* and *outer* respectively. Let us say an *H-bridge* D is *major* if it has an attachment not in $V(P_1)$ and an attachment not in $V(P_2)$, and *minor* otherwise. Now we have four kinds of *H-bridges*, outer or inner, and major or minor. Minor *H-bridges* fall into two types: those with all their attachments in $V(P_1)$, and those with all attachments in $V(P_2)$. Let us call them *P₁-type* and *P₂-type* respectively.

If $x \in X$, then it might have neighbours in P_1 or P_2 , or in major H -bridges, or in minor H -bridges of either type. But all neighbours of x lie on the same region of H , which is bounded by a cycle since H is 2-connected. And there are at most two major H -bridges in which x has neighbours. Consequently, we can split x into a set Y_x of at most four vertices, such that:

- for each $y \in Y_x$, either all neighbours of y lie in a major H -bridge, or for some $i \in \{1, 2\}$ they all lie in the union of P_i and minor H -bridges of P_i -type;
- let G' be the graph produced from G by carrying out this split for all $x \in X$; then the drawing of H can be extended to a drawing of G' .

This was a total of at most 3ξ splits so far. To simplify notation, let us replace X by the set of vertices just obtained by splitting. We lose the hypothesis that $|X| = \xi$, but instead we have:

(1) $|X| \leq 4\xi$; and for each $x \in X$, either all neighbours of x lie in a major H -bridge, or for some $i \in \{1, 2\}$ they all lie in the union of P_i and minor H -bridges of P_i -type.

So far we have been working with H -bridges, but now we need G -bridges, which are defined similarly: either

- a subgraph of G consisting of one edge e and its ends, such that both ends of e are in $V(C)$ and $e \notin E(C)$, or
- a subgraph of G that is the union of a component D of $G \setminus V(C)$, all vertices in C that have a neighbour in $V(D)$, and all edges between $V(C)$ and $V(D)$.

For G -bridges, we define major and minor, outer and inner, P_1 -type and P_2 -type, in the same way as before. Let D_1, D_2 be G -bridges. Every major G -bridge D' includes a unique major H -bridge D , and has the same attachments as D .

Let us first see when the theorem is satisfied with no further splitting of the vertices in X . To obtain a drawing of G with u, v both on the same region (say the infinite region), we need to redraw all major outer bridges inside C . Minor bridges can stay where they are, or be redrawn inside C , or be redrawn outside C , as necessary. We say two G -bridges D_1, D_2 *conflict* if $C \cup D_1 \cup D_2$ cannot be drawn such that C bounds a region, that is, D_1, D_2 cannot both be redrawn inside C without crossings. It is easy to see (although we will not use this fact) that D_1, D_2 conflict if and only if either:

- there are distinct vertices c_1, c_2, c_3, c_4 of C , in order in C , such that c_1, c_3 are attachments of D_1 and c_2, c_4 are attachments of D_2 ; or
- the sets of attachments of D_1, D_2 are equal, and of size three.

We define conflict for H -bridges in the same way, but no two H -bridges conflict, since H can be drawn in the plane such that C bounds a region.

For $i = 1, 2$, if $a, a' \in V(P_i)$, we denote the path of P_i between them by $P_i(a, a')$. If $Z \subseteq V(P_i) \neq \emptyset$, the u -most member of Z is the vertex in Z closest (in P_i) to u , and the v -most is defined similarly. Let H' be the union of C and all major outer H -bridges. A region of H' outside C is *important* if it is incident with a vertex of $V(C) \setminus V(P_1)$ and incident with a vertex in $V(C) \setminus V(P_2)$. For each

$x \in X$ drawn outside C , there is a region of H' in which x is drawn, say $r(x)$. If $r(x)$ is important, and is incident with at least two vertices of P_1 , then the u -most and v -most vertices of P_1 incident with $r(x)$ are called *important* for x , and similarly in P_2 . Let us say a vertex is *important* if either it is important for some x drawn outside C , or it equals u or v . For each $x \in X$ outside of C there are at most two important vertices for x ; so there are at most $8\xi + 2$ important vertices in P_1 and the same in P_2 .

Let the important vertices in P_1 be $u = a_1, \dots, a_s = v$, numbered in order, and let $u = b_1, \dots, b_t = v$ be the important vertices in P_2 in order. Thus $s, t \leq 8\xi + 2$.

Let us say an H -bridge or G -bridge is *local* if there exist $i \in \{1, \dots, s-1\}$ and $j \in \{1, \dots, t-1\}$ such that all attachments of D belong to $V(P_1(a_i, a_{i+1}) \cup P_2(b_j, b_{j+1}))$.

(3) *If every G -bridge is local, then G can be drawn in the plane with u, v both incident with the same region.*

Let K be the ‘‘conflict graph’’ between G -bridges: its vertex set is the set of all G -bridges, and two of them are adjacent in K if they conflict. Since no two inner G -bridges conflict, and the same for outer G -bridges, it follows that K is bipartite; and it suffices to show that no component of K contains a major outer G -bridge and a major inner G -bridge. Suppose then that there is a sequence A_1, \dots, A_n of G -bridges, with n even, such that A_1 is outer, A_n is inner, and for $1 \leq i < n$, A_i conflicts with A_{i+1} . So $n > 2$ since each major G -bridge have the same attachment sets as some major H -bridge, and there is no conflict between H -bridges. By choosing n minimal we may assume that A_2, \dots, A_{n-1} are minor; and since each conflicts with the next, it follows that they are all of P_1 -type or all of P_2 -type, and so we may assume they are all of P_1 -type. Moreover, A_1, \dots, A_n all have the same i -value, since each conflicts with the next. Let i be this common i -value.

Since A_n is a major inner G -bridge, and A_{n-1} conflicts with it, and no H -bridge conflicts with a major G -bridge, it follows that there is a vertex in X that belongs to $V(A_{n-1})$, say x . We claim that $r(x)$ is important. Let a, b be the u -most and v -most attachment of A_{n-1} in P_1 . If $r(x)$ is not important, then since $r(x)$ is incident with at least two vertices of P_1 (a and b), $r(x)$ is not incident with any vertex in P_2 ; and so there is a major outer H -bridge D with attachments in both $P_1(u, a)$ and in $P_1(b, v)$. But then D conflicts with A_n , a contradiction, since no two major G -bridges conflict. This proves that $r(x)$ is important. It follows that the u -most and v -most vertices in P_1 incident with $r(x)$ are important vertices. But they both belong to $P_1(a_i, a_{i+1})$; and so a_i is the u -most vertex of P_1 incident with $r(x)$, and a_{i+1} is the v -most.

Since A_1 is a major outer G -bridge, it has the same attachment set as a major outer H -bridge A'_1 say; and A'_1 has two distinct attachments in $P_1(a_i, a_{i+1})$, since it conflicts with A_2 . But that contradicts that a_i, a_{i+1} are both incident with $r(x)$, and hence proves (3).

It need not be true that every G -bridge is local; but next we will show that we can make it true with a few more splits.

(4) *By at most $64\xi^2$ splits of vertices in X , making a graph G' , we can arrange that every G' -bridge is local (without changing the set of vertices of C that are important).*

Each outer G -bridge is already local, but we need to worry about inner G -bridges. Every inner H -bridge is local, since there is no conflict between H -bridges; and if D is a major inner G -bridge,

then it has the same attachment set as a major inner H -bridge and so is local. For each minor inner H -bridge D of type P_1 , D has at least two attachments in P_1 since H is 2-connected, and so, since D is local, there is a unique i as in the definition of local; let us call it the i -value of D . (The same holds for P_2 .)

Let D be a minor inner G -bridge, say of P_1 -type. Then D is the union of some H -bridges of P_1 -type, which are local, and some vertices in X that serve to connect up the H -bridges. For each $x \in X$ drawn inside C , we can split x into at most $s - 1$ vertices, such that for each of them, say y , all the H -bridges in which y has neighbours have the same i -value. We can follow the same procedure for minor inner G -bridges of P_2 -type. This requires $\max(s - 2, t - 2)$ splits for each vertex in X , and since $s, t \leq 8\xi + 2$ and $|X| \leq 4\xi$, this step needs at most $64\xi^2$ splits. The meaning of “local” depends on which vertices of C are important, but that only depends on H and vertices of X drawn outside C ; and here we only split vertices of X inside C , so the meaning of “local” is not affected. After these splits, the statement of (4) holds, so this proves (4).

By (3), applied to G' , it follows that G' can be drawn in the plane with u, v both incident with the same region. In total, starting with a set X , we did at most 3ξ splits in the first round, and then another $64\xi^2$ to arrange that (4) holds. This proves 3.3. \blacksquare

An *interval* of the set of integers means a subset $\{p, p + 1, \dots, q\}$ for some two integers p, q with $p \leq q$, or the null set.

3.4 *Let (G, X) be an apical pair, such that G is k -restricted. Suppose that there exist $t \geq 1$ and a sequence a_0, \dots, a_t of distinct vertices of $G \setminus X$, and connected subgraphs A_1, \dots, A_t of $G \setminus X$ with union $G \setminus X$, such that $a_0 \in V(A_1), a_t \in V(A_t)$, and for $1 \leq i < j \leq t$, if $j \geq i + 2$ then $V(A_i \cap A_j) = \emptyset$, and if $j = i + 1$ then $V(A_i \cap A_j) = \{a_i\}$. For each interval $I \subseteq \{1, \dots, t\}$, we denote $\bigcup_{i \in I} A_i$ by $A(I)$. Then:*

- *There exists $M_1 \subseteq \{1, \dots, t\}$ with $|M_1| \leq 4k|X|$ such that for all $x \in X$ and for every interval $I = \{p, p + 1, \dots, q\}$ of $\{1, \dots, t\} \setminus M_1$, the graph obtained from $A(I) + x$ by adding the edges xa_{p-1}, xa_q is planar.*
- *There exists $M_2 \subseteq \{1, \dots, t\}$ with $|M_2| \leq 4k \binom{|X|}{2}$ such that for all distinct $x, y \in X$ and for every interval $I = \{p, p + 1, \dots, q\}$ of $\{1, \dots, t\} \setminus M_2$, either one of x, y has no neighbour in $V(A(I))$, or the graph obtained from $A(I) + \{x, y\}$ by adding the edges $xa_{p-1}, xa_q, ya_{p-1}, ya_q, xy$ is planar.*
- *There exists $M_3 \subseteq \{1, \dots, t\}$ with $|M_3| \leq 2k \binom{|X|}{3}$ such that for every interval I of $\{1, \dots, t\} \setminus M_3$, at most two members of X have a neighbour in $A(I)$.*

Proof. We prove these in order. For the first statement, let $x \in X$. We say an interval $I = \{p, \dots, q\}$ of $\{1, \dots, t\}$ is $\{x\}$ -good if the graph obtained from $A(I)$ by adding the edges xa_{p-1}, xa_q is planar. Suppose there are $4k$ pairwise disjoint intervals that are not $\{x\}$ -good, say I_1, \dots, I_{4k} , numbered in order. For $1 \leq j \leq k$ let H_j be the interval $\{a, a + 1, \dots, b\}$ where a is the smallest member of I_{4j-3} and b is the largest member of I_{4j-1} , and let $C_j = A(H_j)$. Thus all the graphs C_j are pairwise vertex-disjoint. We claim that for $1 \leq j \leq k$, $C_j + x$ is not planar. To see this, let I_{4j-2} be the interval $I = \{p, p + 1, \dots, q\}$; then, since I_{4j-2} is not $\{x, y\}$ -good, the graph D_j obtained from

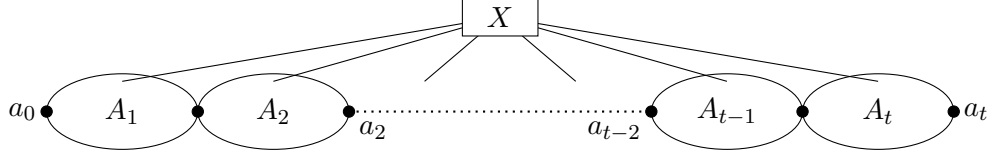


Figure 2: The subgraphs for 3.4.

$A(I) + x$ by adding the edges xa_{p-1}, xa_q is nonplanar. Moreover, x has a neighbour in $V(A(I_{4j-3}))$ (because I_{4j-3} is not $\{x\}$ -good) and x has a neighbour in $V(A(I_{4j-1}))$ similarly, and consequently $C_j + x$ contains D_j as a minor and hence is not planar. It follows that G contains a $(k, 1)$ -junction, a contradiction. Thus there do not exist $4k$ pairwise disjoint intervals that are not $\{x\}$ -good. By the Helly property of intervals, there exists $I_x \subseteq \{1, \dots, t\}$ with $|I_{x,y}| \leq 4k$ such that every interval containing no member of I_x is $\{x\}$ -good. Let M_1 be the union of I_x over all $x \in X$; then it satisfies the first statement of the theorem.

For the second statement, let $x, y \in X$ be distinct. We say an interval $I = \{p, \dots, q\}$ of $\{1, \dots, t\}$ is $\{x, y\}$ -good if either one of x, y has no neighbour in $A(I)$, or the graph obtained from $A(I) + \{x, y\}$ by adding the edges $xa_{p-1}, xa_q, ya_{p-1}, ya_q, xy$ is planar. Suppose there are $4k$ pairwise disjoint intervals that are not $\{x, y\}$ -good, say I_1, \dots, I_{4k} , numbered in order. For $1 \leq j \leq k$ let H_j be the interval $\{a, a+1, \dots, b\}$ where a is the smallest member of I_{4j-3} and b is the largest member of I_{4j-1} , and let $C_j = A(H_j)$. Thus all the graphs C_j are pairwise vertex-disjoint. We claim that for $1 \leq j \leq k$, $(C_j + x, y, \{x, y\})$ is not flat. To see this, let I_{4j-2} be the interval $I = \{p, p+1, \dots, q\}$; then, since I_{4j-2} is not $\{x, y\}$ -good, the graph D_j obtained from $A(I)$ by adding the edges $xa_{p-1}, xa_q, ya_{p-1}, ya_q, xy$ is nonplanar. Moreover, x, y both have a neighbour in $V(A(I_{4j-3}))$ (because I_{4j-3} is not $\{x, y\}$ -good) and x, y both have a neighbour in $V(A(I_{4j-1}))$ similarly, and consequently if we add the edge xy to $C_j + \{x, y\}$, the graph we obtain contains D_j as a minor and hence is not planar. This proves that $(C_j + \{x, y\}, \{x, y\})$ is not flat, for $1 \leq j \leq k$. It follows that G contains a $(k, 2)$ -junction, a contradiction. Thus there do not exist $4k$ pairwise disjoint intervals that are not $\{x, y\}$ -good. By the Helly property of intervals, there exists $I_{x,y} \subseteq \{1, \dots, t\}$ with $|I_{x,y}| \leq 4k$ such that every interval containing no member of $I_{x,y}$ is $\{x, y\}$ -good. Let M_2 be the union of $I_{x,y}$ over all distinct x, y ; then it satisfies the second statement.

For the third statement, we say an interval $J = \{i, i+1, \dots, j\}$ is *tripled* if there are at least three members of X with a neighbour in $V(A(J))$. Suppose there are $2k \binom{|X|}{3}$ pairwise disjoint tripled intervals. For each of them, say I , some triple of vertices in X all have neighbours in $V(A(I))$ it, and so there are $2k$ of them that have the same triple, say $\{x, y, z\}$. Hence there are k such intervals, say I_1, \dots, I_k , such that the subgraphs $A(I_1), A(I_2), \dots, A(I_k)$ are pairwise vertex-disjoint. But then G contains a $(k, 3)$ -junction, a contradiction. So there do not exist $2k|X|^3$ pairwise disjoint tripled intervals, and hence by the Helly property of intervals, there exists $M_3 \subseteq \{1, \dots, t\}$ with $|M_3| \leq 2k|X|^3$ such that every tripled interval contains a member of M_3 . This proves the third statement and so proves 3.4. \blacksquare

From 3.3 and 3.4 we deduce:

3.5 *Let (G, X) be a 2-connected apical pair such that G is k -restricted; and let $e = uv$ be an edge of $G \setminus x$. Suppose that for every 2-connected subgraph A of $G \setminus e$, $(A + x, x)$ has nonplanarity at most K . Then (G, X) has nonplanarity at most $300kK^2|X|^4$.*

Proof. Let $G' = (G \setminus X) \setminus e$. Since $G \setminus X$ is 2-connected, it follows that G' is connected, and for each of its 2-separations (A, B) with $V(A), V(B) \neq V(G')$, $V(A \setminus V(B))$ contains one of u, v , and $V(B) \setminus V(A)$ contains the other. Consequently there is a sequence $u = a_0, \dots, a_t = v$ of distinct vertices of G' , and subgraphs A_1, \dots, A_t of G' with union G' , as in 3.4. Let M_1, M_2, M_3 be as in 3.4, and let $M = M_1 \cup M_2 \cup M_3$. Thus

$$|M| \leq 2k \binom{|X|}{3} + 4k \binom{|X|}{2} + 4k|X| \leq 4k|X|^3.$$

Let I_1, \dots, I_s be the maximal intervals included in $\{1, \dots, t\} \setminus M$. There are at most $|M| + 1$ of them, and they are pairwise disjoint. For each one, say $I = \{p, \dots, q\}$, we have by (1), (2), and (3) that at most two members of X have a neighbour in $V(A_i)$; if two members x, y both have neighbours in $V(A_i)$ then the graph obtained from $A(I) + \{x, y\}$ by adding the edges $xa_{p-1}, xa_q, ya_{p-1}, ya_q, xy$ is planar; and if only one member $x \in X$ has a neighbour in $V(A(I))$ then the graph obtained from $A(I) + x$ by adding the edges xa_{p-1}, xa_q is planar.

Suppose that x, y both have neighbours in $V(A(I))$, and take a planar drawing of the graph obtained from $A(I) + \{x, y\}$ by adding the edges $xa_{p-1}, xa_q, ya_{p-1}, ya_q, xy$. It follows that the graph obtained from $A(I) + \{x, y\}$ by adding the edges $xa_{p-1}, xa_q, ya_{p-1}, ya_q$ can be drawn in the plane such that the cycle $x-a_{p-1}-y-a_q-x$ bounds a region, and so $A(I) + \{x, y\}$ admits a planar drawing with a_{p-1}, a_q on the same region.

If $x \in X$ is the only member of X with a neighbour in $A(I)$, then the graph obtained from $A(I) + x$ by adding the edges xa_{p-1}, xa_q is planar, and so a graph obtained from $A(I) + x$ by one split of x is planar, and can be drawn with a_{p-1}, a_q on the same region.

For each $i \in M$, by hypothesis $(A_i + X, X)$ has nonplanarity at most K , and A_i is 2-connected or trivial, so by 3.3, there is a planar graph obtained from $A_i + X$ by at most $3K + 64K^2$ splits that can be drawn with a_{i-1}, a_i on the same region. For each vertex in X , let us split it into several vertices, one for each $i \in M$, say y_i and one for each interval I_j , say z_j . Make y_i adjacent to the neighbours of x in $V(A_i)$, and z_j adjacent to the neighbours of x in $A(I_j)$. It follows that by at most $3K + 64K^2$ splits of each y_i , and at most one split of each z_i , (G', X) can be converted to a planar graph that can be drawn with a_0, a_t on the same region. This is a total of at most

$$(3K + 64K^2)|M| \cdot |X| + t(|M| + 1)|X|$$

splits. Since $|M| \leq 4k|X|^3$, it follows that (G, X) has nonplanarity at most

$$|X| + (3K + 64K^2)(4k|X|^3)|X| + (4k|X|^3 + 1)|X| \leq 300kK^2|X|^4.$$

This proves 3.5. ■

3.6 *Let $k \geq 1$, and let (G, X) be a 2-connected apical pair such that G is k -restricted. Suppose that $u, v \in V(G)$ are distinct and adjacent, joined by an edge e , and there are subgraphs A_1, \dots, A_t of $G \setminus X$, with union $G \setminus X$, each containing the edge e , such that each $(A_i + X, X)$ has nonplanarity at most K , and such that $V(A_i \cap A_j) = \{u, v\}$ for $1 \leq i < j \leq k$. Then (G, X) has nonplanarity at most $kK|X|$.*

Proof. Let $I \subseteq \{1, \dots, t\}$ be the set of i such that some vertex in X has a neighbour in $V(A_i) \setminus \{u, v\}$. Since G does not contain $K_{k,3}$ as a minor, it follows that $|I| \leq (k-1)|X|$. For each $i \in I$, $(A_i + X, X)$ has nonplanarity at most K ; and for each $i \in \{1, \dots, t\} \setminus I$, $(A_i + X, X)$ has nonplanarity one. Consequently, (G, X) has nonplanarity at most $(k-1)K|X| + 1 \leq kK|X|$. This proves 3.6. ■

We will be looking at different graphs with the same vertex sets, and it is sometimes convenient to say “ G -adjacent” to mean “adjacent in G ”, and so on.

3.7 *For all integers $k, \xi \geq 1$ there exists an integer $\ell \geq 1$ such that for all integers $K \geq 1$, the following holds. Let (G, X) be a 2-connected apical pair, such that G is k -restricted and $|X| \leq \xi$. Suppose that there are separations (A_i, B_i) ($i \in I$) of G satisfying:*

- *Each (A_i, B_i) has order two, and $(A_i + X, X)$ has nonplanarity at most K ;*
- *$A_i \subseteq B_j$ for all distinct $i, j \in I$;*
- *Let $B = \bigcap_{i \in I} B_i$, and let H be the graph obtained from B by adding an edge between the two vertices in $A_i \cap B_i$ for each $i \in I$. Then either H is 3-connected or has at most three vertices.*

Then (G, X) has nonplanarity at most ℓK^2 .

Let k' be as in 3.2, with k replaced by $4k\xi$, and let $\ell = 1204k^4(k')^2\xi^8$. We will show that ℓ satisfies the theorem.

Let F be the set of edges uv of H such that for some $i \in I$, $V(A_i \cap B_i) = \{u, v\}$. For each $e = uv \in F$, let I_e be the set of $i \in I$ such that $V(A_i \cap B_i) = \{u, v\}$. Let $A_e = \bigcup_{i \in I_e} A_i$ and $B_e = \bigcap_{i \in I_e} B_i$, and let C_e be obtained from A_e by adding the edge e . For each $i \in I_e$, $(A_i + X, X)$ has planarity at most K , by hypothesis, and so by 3.5, $(A'_i + X, X)$ has nonplanarity at most $300kK^2|X|^4$, where A'_i is obtained from A_i by adding the edge $e = uv$. Thus $(C_e + x, x)$ has nonplanarity at most $k(300kK^2|X|^4)|X| \leq 300k^2K^2|X|^5$ by 3.6.

Let G' be the graph obtained from H by adding the vertices of X , such that for each $x \in X$,

- x is G' -adjacent to each vertex in B that is G -adjacent to x ; and
- x is G' -adjacent to both ends of each edge $e = uv \in E(F)$ such that for some $i \in I$, $V(A_i \cap B_i) = \{u, v\}$ and x is G -adjacent to a vertex in $V(A_i) \setminus V(B_i)$.

(1) *There is a set of at most $k(k')^2|X|$ regions of H together covering all vertices of H G -adjacent to a vertex in X , and covering all edges in F .*

Let us apply 3.2 to G' , with k replaced by $4k|X|$, and with U the set of vertices of H adjacent in G to a vertex in X . Suppose first that there are two vertex-disjoint connected subgraphs P, Q of H , and $4k|X|$ distinct vertices $y_1, \dots, y_{4k|X|}$ of $V(H) \setminus V(P \cap Q)$, all in U , and each with a neighbour in $V(P)$, and a neighbour in $V(Q)$. Since each y_i has a neighbour in X , some $4k$ of them have the same neighbour; say y_1, \dots, y_{4k} are all adjacent in G' to $x \in X$. Since H is planar, the four-colour theorem implies that we may choose k of y_1, \dots, y_{4k} pairwise nonadjacent, say y_1, \dots, y_k . In particular, no two of them are ends of the same edge in F . For $1 \leq i \leq k$, y_i is G' -adjacent to x , and hence either y_i is G -adjacent to x , or there is an edge $e \in F$ incident with y_i such that x has a G -neighbour in $V(A_e) \setminus V(B_e)$. But then G has a $K_{k,3}$ minor (because no two different vertices y_i use the same $e \in F$) and the theorem holds.

Thus, by 3.2, we may assume that there is a set of at most k' regions of H , say R_1, \dots, R_t where $t \leq k'$, together covering U . In particular, for every $e \in F$, both ends of e are incident with one of these regions. Let F_1 be the set of edges $e = uv$ in F , such that none of R_1, \dots, R_t is incident with e . Since H is 3-connected, it follows that none of R_1, \dots, R_t is incident with both u, v . Hence we

may assume that there is a set $F_2 \subseteq F_1$ with $|F_2| \geq |F_1|/t^2$, such that for each $e \in F_2$, one end of e is incident with R_1 , the other end is incident with R_2 , and e is incident with neither of these regions. For each $e \in F_2$, there exists $x \in X$ with a neighbour in $V(A_e) \setminus V(B_e)$, so there exists $F_3 \subseteq F_2$ with $|F_3| \geq |F_2|/|X|$ and $x \in X$ such that x has a neighbour in $V(A_e) \setminus V(B_e)$ for each $e \in F_3$. Let C_1, C_2 be the cycles of $G \setminus X$ forming the boundaries of R_1, R_2 respectively. They might intersect, but if their intersection is non-null then it is either just one vertex or an edge; so in each case $C_1 \setminus V(C_2)$ and $C_2 \setminus V(C_1)$ are connected and vertex-disjoint. Since for each $e \in F_3$, there is an edge between $V(A_e) \setminus V(B_e)$ and each of $\{x\}, C_i \setminus V(C_j)$ and $C_j \setminus V(C_i)$, we may assume that $|F_3| < k$, since otherwise G has a $K_{k,3}$ minor. Hence $|F_1| \leq t^2(k-1)|X|$; and so the regions R_1, \dots, R_t , with an additional $t^2(k-1)|X|$ regions, will suffice to cover all neighbours of x in G . Since $t \leq k'$ and so $k' + t^2(k-1)|X| \leq k(k')^2|X|$, this proves (1).

Let R_1, \dots, R_s be regions as in (1), where $s \leq k(k')^2|X|$. Let us split X into s sibling-free subsets X_1, \dots, X_s , such that for $1 \leq r \leq s$, each vertex in X_r is adjacent only to vertices in $V(H)$ that are incident with R_r , and to vertices in $V(A_e)$ for some $e \in F$ incident with R_r . Moreover, we can arrange this splitting such that for each $e \in F$, at most one of X_1, \dots, X_r contains vertices with a neighbour in $V(A_e)$. We need to do some further splitting of the vertices in X_r to insert them into the drawing. (Not quite: we will also have to change the drawing of some of the individual subgraphs A_e to accommodate the extra vertices.)

Fix r with $1 \leq r \leq s$, and let C be the cycle of H that forms the boundary of the region R_r . For each edge $e \in E(C)$, if $e \notin F$ let A_e be the subgraph of G consisting of e and its ends. (A_e is already defined if $e \in F$.) For each path P of C , let $A(P)$ denote $\bigcup_{e \in E(P)} A_e$.

Since H is 3-connected, $H \setminus V(C)$ is connected and every vertex of C is adjacent in H to a vertex in $V(H \setminus V(C))$. Not all edges of H are edges of G ; but for every edge $e = uv$ of H that is not in $E(G)$, there is a path of A_e between u, v . Consequently, letting $W = G \setminus (V(C) \cup X)$, we deduce that W is connected. and every vertex in $V(C)$ is G -adjacent to a vertex in $V(W)$.

(2) *There exists $M_1 \subseteq E(C)$ with $|M_1| \leq 2k \binom{|X|}{2}$ such that for each path P of $C \setminus M_1$, at most one member of X has a neighbour in $A(P)$.*

Suppose there are $2k \binom{|X|}{2}$ pairwise vertex-disjoint paths of C such that for each of them, say P , there are two members of X with a neighbour in $A(P)$. Then for some distinct $x, y \in X$, there are $2k$ vertex-disjoint subpaths of C such paths P such that x, y both have a neighbour in $A(P)$; and for some k of these paths P , the subgraphs $A(P)$ are pairwise vertex-disjoint. But then, by contracting W to a single vertex, we obtain a $K_{k,3}$ minor, a contradiction. So there do not exist $2k \binom{|X|}{2}$ such subgraphs of C . Subpaths of a cycle do not have the Helly property, but they do with an additive error of one; so (2) follows.

(3) *There exists $M_2 \subseteq E(C)$ with $|M_2| \leq 4k|X|$ such that for each path P of $C \setminus M_2$ and each $x \in X$, $(A(P) + x, \{u, v, x\})$ is flat, where u, v are the ends of P .*

Let $x \in X$. We say a subpath of P of C is x -bad if $(A(P) + x, \{u, v, x\})$ is not flat, where u, v are the ends of P . It follows that x has a neighbour in $V(A(P))$ for each x -bad path P . Let Q be a path of C that includes two vertex-disjoint x -bad paths P_1, P_2 , and let J be obtained from $A(Q) + x$ by adding a vertex w adjacent to each vertex of Q . Let u, v be the ends of P_1 . Since u, v, x all have

a neighbour in the connected graph $A(P_2) + w$, and since $(A(P_1) + x, \{u, v, x\})$ is not flat, it follows that J is not planar. Consequently, if there are $4k$ vertex-disjoint x -bad paths, there are $2k$ x -bad paths P such that the graphs $A(P)$ are pairwise vertex-disjoint; and by pairing them in consecutive pairs, and contracting W to a single vertex w , and applying the observation just made, we obtain a minor of G that contains a $(k, 2)$ -junction, a contradiction. From the “near-Helly” property of subpaths of a cycle, this proves (3).

Choose M_1, M_2 as in (2), (3), not both empty, and let $M = M_1 \cup M_2$. Thus $|M| \leq 2k \binom{|X|}{2} + 4k|X| \leq 4k|X|^2$. As we saw earlier, $(C_e + X, X)$ has nonplanarity at most $300k^2K^2|X|^5$ for each $e \in M$, and hence the same is true for $(C_e + X_r, X_r)$. For each path P of $C \setminus M$, there is at most one vertex $x \in X_r$ with a neighbour in $A(P)$, and if so then $(A(P) + x, \{u, v, x\})$ is flat, where u, v are the ends of P . Let us split each $x \in X_r$ into at most $8k|X|^2$ vertices (one assigned for each $e \in M$, and one for each component of $C \setminus M$), and then splitting further the vertices assigned to each $e \in M$ into a set of at most $300k^2K^2|X|^5$ vertices, and repeating for $1 \leq r \leq s$, we convert G into a planar graph. Since $s \leq k(k')^2|X|$, this shows that (G, X) has nonplanarity at most

$$(k(k')^2|X|)(4k|X|^2(300k^2K^2|X|^5) + 4k|X|^2) \leq 1204k^4(k')^2K^2|X|^8 \leq \ell K^2.$$

This proves 3.7. ■

We deduce:

3.8 *Let $k, \xi \geq 1$, and let $\ell \geq 900k\xi^4$ be as in 3.7; then for all integers $K \geq 1$, the following holds. Let (G, X) be a 2-connected apical pair such that G is k -restricted and $|X| \leq \xi$. Suppose that for every 2-separation (A, B) of $G \setminus X$, one of $(A + X, X), (B + X, X)$ has nonplanarity at most K . Then (G, X) has nonplanarity at most ℓK^2 .*

Proof. We may assume that (G, X) has nonplanarity more than $900kK^2|X|^4$, since $\ell > 900k\xi^4$. Let us say a 2-separation (A, B) of $G \setminus X$ is *oriented* if $(A + X, X)$ has nonplanarity at most K and $V(B) \neq V(G) \setminus \{X\}$; and (A, B) is *extreme* if there is no oriented 2-separation (A', B') with $A \subseteq A'$ and $B' \subseteq B$ such that $(A, B) \neq (A', B')$. Let $\{(A_i, B_i) : i \in I\}$ be the set of all extreme 2-separations.

(1) *If $i, j \in I$ are distinct, then $A_i \subseteq B_j$ and $A_j \subseteq B_i$.*

We take $i = 1, j = 2$ for simplicity. Suppose that the separation $(A_1 \cup A_2, B_1 \cap B_2)$ has order at least three. Then $(A_1 \cap A_2, B_1 \cup B_2)$ has order at most one, and so $V(B_1 \cup B_2) = V(G) \setminus X$. Since $(A_1 \cap A_2, B_1 \cup B_2)$ has order at most one, we may assume from the symmetry between (A_1, B_1) and (A_2, B_2) that $A_1 \cap B_1 \subseteq B_2$. If also $A_2 \cap B_2 \subseteq B_1$, then $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$ as required, so we assume that $V(A_2 \cap B_2) \setminus V(B_1)$ is nonempty, and consequently has exactly one vertex, and $A_1 \cap B_1 \cap A_2 \cap B_2$ is null. Since $(A_1 \cap B_2, B_1 \cup A_2)$ has order at most one, it follows that $V(B_2) = V(G) \setminus X$, a contradiction.

Thus $(A_1 \cup A_2, B_1 \cap B_2)$ has order at most two. We claim that $((B_1 \cap B_2) + X, X)$ has nonplanarity at most K . If $(A_1 \cup A_2, B_1 \cap B_2)$ has order at most one, then $B_1 \cap B_2$ has at most one vertex and the claim is trivial; and if $(A_1 \cup A_2, B_1 \cap B_2)$ has order exactly two, then since (A_1, B_1) is extreme, it follows that $((A_1 \cup A_2) + X, X)$ does not have nonplanarity at most K , and so $((B_1 \cap B_2) + X, X)$ has nonplanarity at most K from the hypothesis.

From the symmetry between (A_1, B_1) and (A_2, B_2) , and since the sum of the orders of $(A_1 \cup B_2, A_2 \cap B_1)$ and $(A_1 \cap B_2, A_2 \cup B_1)$ is four, we may assume that $(A_1 \cup B_2, A_2 \cap B_1)$ has order at most two. Let us define $C_1 = A_1, C_2 = B_1 \cap A_2, C_3 = B_1 \cap B_2$. Then $(C_i + X, X)$ has nonplanarity at most K for $i = 1, 2, 3$; and $C_1 + X, C_2 + X, C_3 + X$ have union G , and each of the separations $(C_1, C_2 \cup C_3), (C_2, C_1 \cup C_3), (C_3, C_1 \cup C_2)$ has order two. Let W be the set of vertices in at least two of C_1, C_2, C_3 . Every vertex in W belongs to at least two of $V(C_1 \cap (C_2 \cup C_3)), V(C_2 \cap (C_1 \cup C_3)), V(C_3 \cap (C_1 \cup C_2))$, and so $|W| \leq 3$.

If $|W| = 3$, we may number $X = \{w_1, w_2, w_3\}$ such that $w_1 \in V(C_2 \cap C_3), w_2 \in V(C_1 \cap C_3)$ and $w_3 \in V(C_1 \cap C_2)$. Let D_1 be the graph obtained from C_1 by adding the edge $w_2 w_3$, and define D_2, D_3 similarly. By 3.5, $(D_i + X, X)$ has nonplanarity at most $300kK^2|X|^4$ for $i = 1, 2, 3$; and so by splitting X into three sibling-free sets, one for each of D_1, D_2, D_3 , we deduce that (G, X) has nonplanarity at most $900kK^2|X|^4$, a contradiction.

If $|W| = 2$, let $W = \{u, v\}$; then for $i = 1, 2, 3$, either $u, v \in V(C_i)$, or $|V(C_i)| \leq 1$. If $u, v \in V(C_i)$ let D_i be obtained from C_i by adding the edge uv , and otherwise $D_i = C_i$. Again, by 3.5, $(D_i + X, X)$ has nonplanarity at most $300kK^2|X|^4$ for $i = 1, 2, 3$; and so (G, X) has nonplanarity at most $900kK^2|X|^4$, a contradiction. This proves (1).

Let $B = \bigcap_{i \in I} B_i$, and let H be the graph obtained from B by adding an edge between the two vertices in $A_i \cap B_i$ for each $i \in I$.

(3) *Either H is 3-connected or $|H| \leq 3$.*

If not then there is a separation (P, Q) of H of order at most two, with $V(P), V(Q) \neq V(H)$. Let I_1 be the set of $i \in I$ such that $uv \in E(P)$ where $V(A_i \cap B_i) = \{u, v\}$, and let P' be the union of $P \cap G$ and A_i for all $i \in I_1$, Define Q' similarly. Then (P', Q') is a 2-separation of $G \setminus X$, and so we may assume that (P', Q') is oriented. Hence there exists $i \in I$ such that $P' \subseteq A_i$ and $B_i \subseteq Q$ from the definition of I . From the definition of H , it follows that $V(H) \subseteq V(B_i)$, and so $V(P) \subseteq V(B_i) \subseteq V(Q')$. But then $V(P) \subseteq V(Q)$ and so $V(Q) = V(H)$, a contradiction. This proves (3).

But then the result follows from 3.7. This proves 3.8. ■

4 Weightings of apical pairs

If $X \subseteq V(G)$, we denote by $\sigma(X, G)$ the maximum t such that there are subgraphs H_1, \dots, H_t of G , such that

- every vertex in more than one of these subgraphs is in X ;
- for each i , $|V(H_i) \cap X| \leq 3$; and
- for each i , if $|V(H_i) \cap X| \leq 2$ then H_i is nonplanar, and if $|V(H_i) \cap X| = 3$ then H_i and $H_i \setminus X$ are both connected.

If $v \in V(G) \setminus X$, $\sigma(X, v, G)$ denotes the maximum t such that there are t nonplanar subgraphs H_1, \dots, H_t of G , such that

- $v \in V(H_i)$ for $1 \leq i \leq t$;
- every vertex in more than one of these subgraphs is in $X \cup \{v\}$; and
- for each i , H_i is nonplanar and $|V(H_i) \cap X| \leq 2$.

Our present goal is to prove 2.2 when Σ is a 2-sphere, but for inductive purposes we need to introduce weights. Let $k \geq 1$ be an integer. A k -weighting for an apical pair (G, X) is a pair (α, β) where $\alpha \geq 0$ is an integer, and β is a map from $V(G) \setminus X$ to the set of nonnegative integers, such that

- G is k -restricted;
- $\sigma(X, G \setminus Z) < k|X|^3 - \alpha - |Z|$ for every $Z \subseteq V(G) \setminus X$ such that $\beta(v) > 0$ for each $v \in Z$ (and consequently $\alpha < k|X|^3$, and there are fewer than $k|X|^3$ vertices v with $\beta(v) > 0$);
- $\sigma(X, v, G) < k|X|^2 - \beta(v)$ for each $v \in V(G) \setminus X$ (and consequently $\beta(v) < k|X|^2$).

The *cost* of a weighting (α, β) is

$$k^3|X|^6 - k^2|X|^5\alpha - \sum_{v \in V(G) \setminus X} \beta(v).$$

Every weighting has cost between 1 and $k^3|X|^6$.

4.1 *Let (G, X) be an apical pair that admits a weighting (α, β) of cost c . Let A be a subgraph of $G \setminus X$, such that either*

- *there is a connected subgraph H of $G \setminus X$, vertex-disjoint from A , such that at least three member of X have a neighbour in $V(H)$; or*
- *there is a nonplanar subgraph H of G with at most one vertex in $V(A)$ and at most two vertices in X ; or*
- *there exists $w \in V(G) \setminus (V(A) \cup X)$ such that $\beta(w) > 0$.*

Then the apical pair (A, X) admits a weighting with cost less than c .

Proof. Suppose first that there exists A' vertex-disjoint from A , satisfying the first or second bullet, or there exists w satisfying the third bullet. Define $\alpha' = \alpha + 1$ and $\beta'(v) = \beta(v)$ for each $v \in V(A)$. We claim that (α', β') is a weighting for (A, X) . To show this, we must check the three conditions in the definition of a weighting. The first and third are clear. For the second, we must check that for every $Z \subseteq V(A) \setminus X$ such that $\beta(v) > 0$ for each $v \in Z$, $\sigma(X, A \setminus Z) < k|X|^3 - \alpha' - |Z|$. If w exists, then, setting $Z' = Z \cup \{w\}$,

$$\sigma(X, A \setminus Z) \leq \sigma(X, G \setminus Z') < k|X|^3 - \alpha - |Z'| = k|X|^3 - \alpha' - |Z|$$

as required. If H exists, then

$$\sigma(x, A \setminus Z) + 1 \leq \sigma(x, G \setminus Z) < k|X|^3 - \alpha - |Z|$$

as required. So (α', β') is a weighting for (A, x) . We claim its cost is less than that of (α, β) , and this is true since

$$\sum_{v \in V(G) \setminus V(A)} \beta(v) < k^2 |X|^5 = k^2 |X|^5 (\alpha' - \alpha).$$

Next suppose that $\beta(v) = 0$ for all $v \in V(G) \setminus (V(A) \cup \{x\})$, and there is no H as in the case above. Consequently there is a nonplanar subgraph H of G with at most two vertices in X and exactly one in $V(A)$, say a . Define $\beta'(a) = \beta(a) + 1$, and $\beta'(v) = \beta(v)$ for each $v \in V(A) \setminus \{a\}$. We claim that (α, β') is a weighting for (A, X) . To show this, we must check the three conditions in the definition of a weighting. The first is clear. For the second, we must check that $\sigma(X, A \setminus Z) < k|X|^3 - \alpha - |Z|$ for every $Z \subseteq V(A)$ such that $\beta'(v) > 0$ for each $v \in Z$. This is true if $a \notin Z$, so we assume that $a \in Z$. But then, since H exists and contains at most two vertices of X , we have

$$\sigma(X, A \setminus Z) + 1 \leq \sigma(X, G \setminus (Z \setminus \{a\})) < k|X|^3 - \alpha - (|Z| - 1)$$

as required. For the third condition, we must check that $\sigma(X, v, A) < k|X|^2 - \beta'(v)$ for each $v \in V(A)$. This is true if $v \neq a$ since then $\beta'(v) = \beta(v)$, so we assume that $v = a$. But then, since H contains a and at most two vertices of X , we have

$$\sigma(X, a, A) + 1 \leq \sigma(X, a, G) < k|X|^2 - \alpha - (|Z| - 1)$$

as required. This proves that (α, β') is a weighting for (A, X) . But its cost is one less than the cost of (α, β) , since $\beta(v) = 0$ for all $v \in V(G) \setminus (V(A) \cup X)$. This proves 4.1. \blacksquare

Now we are ready to prove the main theorem of this section. We want to prove that if (G, X) is a 2-connected apical pair and G is k -restricted then (G, X) has bounded nonplanarity, but for inductive purposes, we prove a stronger statement:

4.2 *Let $k, \xi \geq 1$ be integers. Then for all $c \in \{0, \dots, k^3 \xi^6\}$, there exists $f(c)$ (with $f(c) \geq f(c-1)$ if $c \geq 1$) with the following property. If (G, X) is a 2-connected apical pair, with $|X| \leq \xi$, that admits a weighting with cost c , then (G, X) has nonplanarity at most $f(c)$.*

Proof. Let $\ell \geq 900k\xi^4$ be as in 3.7. We proceed by induction on c (with k, ξ fixed). We take $f(0) = 0$, which works since no weighting of an apical pair has cost zero; so we may assume that $c > 0$ and $f(c-1)$ exists. Let $f(c) = \max(\ell f(c-1)^2, 90000k^5 \ell \xi^{11})$. We will prove that $f(c)$ satisfies the theorem.

If for every 2-separation (A, B) of G , one of $(A + X, X)$, $(B + X, X)$ has nonplanarity at most $f(c-1)$, then the claim follows from 3.8. So we may assume that there is a 2-separation (B_1, B_2) of $G \setminus X$ such that $(B_1 + X, X)$, $(B_2 + X, X)$ both have nonplanarity more than $f(c-1)$. Let $V(B_1 \cap B_2) = \{u, v\}$. From 4.1, for $i = 1, 2$ and for every connected subgraph A of $B_i \setminus \{u, v\}$, at most two vertices in X have a neighbour in $V(A)$; and for $i = 1, 2$, every nonplanar subgraph of $B_i + X$ with at most two vertices in X contains both u and v . Choose $t \geq 2$ maximum such that there are t subgraphs A_1, \dots, A_t of G , each with at least three vertices, and each containing both u, v and otherwise vertex-disjoint, with union $G \setminus X$. It follows that $A_i \setminus \{u, v\}$ is connected for each i , from the maximality of t . For $i = 1, \dots, t$, let C_i be obtained from A_i by adding the edge $e = uv$.

(1) *For $1 \leq i \leq t$, there is no 2-separation (P, Q) of C_i such that either*

- there is a nonplanar subgraph P' of $P + X$ with at most two vertices in X , or
- there is a connected subgraph P'' of P vertex-disjoint from Q , such that at least three vertices in X have a neighbour in P'' ;

and the same for Q .

Suppose that such a separation exists. We may assume that $e \in E(Q)$. If P'' as above exists, then P'' is a connected subgraph of one of $B_1 \setminus \{u, v\}, B_2 \setminus \{u, v\}$, which is impossible from the properties of B_1, B_2 . Thus P' exists, and is a subgraph of $A_i + X$, and hence of one of $B_1 + X, B_2 + X$. Since P' is nonplanar and $V(P' \cap X) \leq 2$, it follows that $u, v \in V(P')$, from the properties of B_1, B_2 ; and so $u, v \in V(P)$. But $u, v \in V(Q)$, and so $V(P \cap Q) = \{u, v\}$, contradicting that $C_i \setminus \{u, v\}$ is connected. This proves (1).

(2) For $1 \leq i \leq t$, and every 2-separation (P, Q) of C_i , let P', Q' be obtained from P, Q respectively by adding the edge pq , where $V(P \cap Q) = \{p, q\}$. Then one of $(P' + X, X), (Q' + X, X)$ has nonplanarity at most $300k^2|X|^5$.

From (1), we may assume that there is no nonplanar subgraph of $P + X$ with at most two vertices in X , and there is no connected subgraph of $P \setminus \{p, q\}$ vertex-disjoint from Q , and containing neighbours of at least three members of X . Let L_1, \dots, L_s be the components of $P \setminus \{p, q\}$, and for $1 \leq r \leq s$ let M_r be the subgraph of P induced on $L_i \cup \{p, q\}$. It follows that $M_r + X$ is planar for $1 \leq r \leq s$; and so from 3.5 with $K = 1$, if M'_r denotes the graph obtained from M_r by adding the edge pq , then $(M'_r + X, X)$ has nonplanarity at most $300k|X|^4$. Hence from 3.6, $(P' + X, X)$ has nonplanarity at most $300k^2|X|^5$. This proves (2).

for $1 \leq i \leq t$, From (2) and 3.8 with $K = 300k^2|X|^5$, $(C_i + X, X)$ has nonplanarity at most $\ell(300k^2|X|^5)^2$, for $1 \leq i \leq t$. By 3.6 with $K = \ell(300k^2|X|^5)^2$, (G, X) has nonplanarity at most $k(\ell(300(k^2|X|^5)^2)|X| \leq 90000k^5\ell|X|^{11} \leq f(c)$. This proves 4.2. \blacksquare

We deduce:

4.3 Let $k, \xi \geq 1$ be integers. Then there exists τ_2 such that if (G, X) is a 2-connected apical pair, and $|X| \leq \xi$ and G is k -restricted, then (G, X) has nonplanarity at most τ_2 .

Proof. Let $\tau_2 = f(k^3\xi^6)$ where f is as in 4.2. If (G, X) is a 2-connected apical pair, let $\alpha = 0$ and $\beta(v) = 0$ for each $v \in V(G)$. We claim that (α, β) is a weighting. To see this, we must check that $\sigma(X, G \setminus Z) < k|X|^3$, which is true since G is k -restricted; and that for each $v \in V(G) \setminus \{x\}$, $\sigma(X, v, G) < k|X|^2$, which is also true for the same reason. Since this weighting has cost $k^3|X|^6$, the result follows from 4.2. This proves 4.3. \blacksquare

Next we will replace “2-connected” by “connected”, by proving the following:

4.4 Let $k, \xi \geq 1$ be integers. Then for all $c \in \{0, \dots, k^3\xi^6\}$, there exists $f(c)$ (with $f(c) \geq f(c-1)$ if $c \geq 1$) with the following property. If (G, X) is an apical pair such that $G \setminus X$ connected and $|X| \leq \xi$, that admits a weighting with cost c , then (G, X) has nonplanarity at most $f(c)$. Consequently there exists τ_1 such that, if (G, X) is an apical pair, and $|X| \leq \xi$ and $G \setminus X$ is connected and G is k -restricted, then (G, X) has nonplanarity at most τ_1 .

Proof. Again, the proof is by induction on c , and we may assume that $c > 0$, and $f(c-1)$ exists. Let τ_2 be as in 4.3, and let $f(c) = \max(2f(c-1), \tau_2 + k\xi^2)$. We may assume that (G, X) has nonplanarity more than $2f(c-1)$, and so:

(1) For every 1-separation (A, B) of $G \setminus X$, one of $A + X, B + X$ is planar and the other is not.

If both $A + X, B + X$ are nonplanar, then by 4.1, both $(A + X, X)$ and $(B + X, X)$ admits weightings with cost less than c , and so both have nonplanarity at most $f(c-1)$. Hence (G, X) has nonplanarity at most $2f(c-1)$, a contradiction. If both $A + X, B + X$ are planar, then (G, X) has nonplanarity at most two, a contradiction. This proves (1).

Say a 1-separation (A, B) of $G \setminus X$ is *oriented* if $(A + X, X)$ is planar and $V(B) \neq V(G) \setminus \{X\}$, and $A \setminus V(B)$ is connected; and (A, B) is *extreme* if there is no oriented 1-separation (A', B') with $A \subseteq A'$ and $B' \subseteq B$ such that $(A, B) \neq (A', B')$. Let $\{(A_i, B_i) : i \in I\}$ be the set of all extreme 1-separations, and let $V(A_i \cap B_i) = \{v_i\}$ for each $i \in I$.

(2) $A_i \subseteq B_j$ for all distinct $i, j \in I$.

If $v_i = v_j$ then the claim is clear, so we assume that $v_i \neq v_j$. Suppose that $v_j \in V(A_i)$. Since B_i is connected and does not contain v_j , it is a subgraph of one of A_j, B_j , and not A_j since B_i is nonplanar. So $B_i \subseteq B_j$, and hence $A_j \subseteq A_i$, contrary to the extremeness of (A_j, B_j) . So $v_j \in V(B_i)$ and similarly $v_i \in V(B_j)$. Thus $A_j \subseteq B_i$ and $A_i \subseteq B_j$. This proves (2).

Let B be the intersection of all the graphs B_i ($i \in I$).

(3) B is either 2-connected or has at most two vertices.

If there is a separation (P, Q) of B with order at most one with $V(P), V(Q) \neq V(B)$, let P' be the union of P and all A_i with $i \in I$ such that $v_i \in V(P)$, and let Q' be the union of Q with all A_i such that $v_i \notin V(P)$. It follows that (P, Q) is a separation of G of order at most one, with $V(P), V(Q) \neq V(G)$; and so it has order 1. By (1), we may assume that Q is nonplanar and P is planar, and by replacing P by a subgraph, we may therefore assume that (P, Q) is oriented. Hence there exists $i \in I$ with $P \subseteq A_i$; but $P \subseteq B \subseteq B_i$, which is impossible. This proves (3).

Thus G is the union of B and all the graphs A_i ($i \in I$). Let J_1 be the set of all $j \in I$ such that for some $x \in X$, the rooted graph $(A_j + x, \{x, v_j\})$ is not flat; that is, the graph obtained from A_j by adding an edge xv_j is not planar. It follows that $|J_1| < k|X|$, because otherwise the graph obtained from G by contracting B to a single vertex contains a $(k, 2)$ -junction.

Let J_2 be the set of all $j \in I$ such that at least two vertices in X have a neighbour in $V(A_v) \setminus \{v\}$. It follows that $|J_2| \leq k \binom{|X|}{2}$, since otherwise by contracting each subgraph $A_v \setminus \{v\}$ to a vertex, and contracting B to a vertex, we obtain a graph that has a $K_{k,3}$ minor.

If $j \in I$ and no vertex in X has a neighbour in $V(A_v) \setminus \{v\}$, then we may delete $A_i \setminus \{v_i\}$; because if the theorem is true for the graph with this subgraph deleted, then it is true for G . Similarly, if $j \in I$ and there is exactly one vertex $x \in X$ with a neighbour in $V(A_v) \setminus \{v\}$, then $(A_i + x, \{v_i, x\})$ is flat, and so we may contract A_i to an edge xv_i ; again, because if the theorem is true for the

smaller graph then it is true for the original. Thus we may assume that $I = J_1 \cup J_2$, and hence $|I| \leq k|X| + k\binom{|X|}{2} \leq k|X|^2$.

Let us split X into $|I| + 1$ sibling-free sets of vertices X_i ($i \in I$) and Y say, where for each $i \in I$, vertices in X_i only have neighbours in $V(A_i) \setminus \{v_i\}$, and vertices in Y only have neighbours in $V(B)$. For each $i \in I$, $A_i + X_i$ is planar, since $A_i + X$ is planar. But $(B + X, X)$ has planarity at most τ_2 , by 4.3. Hence, adding, we deduce that (G, X) has nonplanarity at most $|I| + \tau_2$. Since $|I| \leq k\xi^2$, this proves 4.4. \blacksquare

Finally, we can prove 3.1, which we restate:

4.5 *Let $k, \xi \geq 1$ be integers. There exists τ_0 such that if (G, X) is an apical pair and G is k -restricted, then (G, X) has nonplanarity at most τ_0 .*

Proof. Let τ_1 be as in 4.4, and let $\tau_0 = k\xi^3\tau_1$. Let A_i ($i \in I$) be the components of $G \setminus x$. Let J_1 be the set of $i \in I$ such that $A_i + x$ is nonplanar for some $x \in X$. Then as usual, $|J_1| \leq k|X|$. Let J_2 be the set of $j \in I$ such that exactly two vertices $x, y \in X$ have a neighbour in $V(A_j)$, and $(A_j + \{x, y\}, \{x, y\})$ is not flat. Again, as usual $|J_2| \leq k\binom{|X|}{2}$. Let J_3 be the set of $i \in I$ such that at least three vertices in X have a neighbour in $V(A_i)$. Thus, $|J_3| \leq k\binom{|X|}{3}$.

We claim that if $i \in I \setminus (J_1 \cup J_2 \cup J_3)$ then A_i may be deleted. This is clear if no vertex in X has a neighbour in $V(A_i)$; and we know that at most two vertices in X have such a neighbour. If there is only one, say x , then $A_i + x$ is planar, so deleting A_i does not change the nonplanarity of (G, X) . Similarly if $x, y \in X$ both have a neighbour in $V(A_i)$ then $(A_i + \{x, y\}, \{x, y\})$ is flat, so again deleting A_i does not change the nonplanarity of (G, X) .

So we may assume that $I = J_1 \cup J_2 \cup J_3$, and hence $|I| \leq k\xi^3$. Let us split X into $|I|$ sibling-free subsets X_I ($i \in I$), where vertices in X_i only have neighbours in $V(A_i)$. Since each $(A_i + X_i, X_i)$ has nonplanarity at most τ_1 , it follows that (G, X) has nonplanarity at most $|I|\tau_1 \leq k\xi^3\tau_1$. This proves 4.5. \blacksquare

5 General surfaces

Now let us deduce 2.2 from 3.1. This will be via an application of a theorem from [4], and needs some definitions. Let G be a graph drawn in a connected surface Σ . We denote by $U(G)$ the set of points of Σ that belong to the drawing (that is, either belong to $V(G)$ or belong to some edge of G). A subset of Σ homeomorphic to a circle is called an *O-arc*, and a subset F of Σ is *G-normal* if $F \cap U(G) \subseteq V(G)$. If Σ is not a 2-sphere, there is an O-arc that is not null-homotopic in Σ , and it can be chosen to be *G-normal*. We say G is *θ -representative*, where $\theta \geq 1$ is an integer, if $|F \cap V(G)| \geq \theta$ for every *G-normal* O-arc F . If G is θ -representative, then for every *G-normal* O-arc F with $|F \cap V(G)| < \theta$, there is a closed disc in Σ with boundary F (unique, since Σ is not a sphere), and we denote it by $\text{ins}(F)$.

It is proved in theorem (4.1) of [5], that in these circumstances, there is associated a “ θ -respectful tangle” of order θ in G ; and consequently, by the discussion in section 9 of that paper, there is a distance function d defined on the set of “atoms” of G , where an *atom* is either a vertex, an edge or a region. We call d the *θ -restraint distance function* of G . Roughly, the distance between two atoms a, b is the minimum $t < \theta$ such that some (possibly self-intersecting) *G-normal* closed curve

“surrounds” both a, b and passes through at most t vertices, if there is such a curve, and θ otherwise. “Surrounds an atom a ” means either the curve passes through a , or includes an O-arc F with a in $\text{ins}(F)$. See [5] for the exact definition.

We need:

5.1 *Let Σ be a connected surface, not a 2-sphere, and let $k \geq 1$ be an integer. Then there exists $\theta \geq 1$ with the following property. Let G be a ϕ -representative graph drawn in Σ , for some $\phi \geq \theta$, and let $U \subseteq V(G)$ with $|U| = k$. Let d be the ϕ -restraint distance function of G , and suppose that $d(u, v) \geq \theta$ for all distinct $u, v \in U$. Then for each $u \in U$ there is a path P_u of G with ends u and p_u say, pairwise vertex-disjoint, and there are two vertex-disjoint connected subgraphs A, B of G , vertex-disjoint from each of the paths P_u , such that for each $u \in U$, p_u has a neighbour in $V(A)$ and a neighbour in $V(B)$.*

Proof. Choose $\theta \geq 7$ some large number (we will say how large later). Let G be a ϕ -representative graph drawn in Σ , for some $\phi \geq \theta$. Let d be the ϕ -restraint distance function of G . Since $\phi \geq 1$, the drawing is 2-cell (that is, each region is homeomorphic to an open disc), and G is connected.

(1) *For each $u \in U$, there is a G -normal O-arc F with $|F \cap V(G)| \in \{2, 3\}$, such that:*

- $u \in \text{ins}(F)$;
- if $|F \cap V(G)| = 2$ then there is a connected subgraph A of $G \cap F$ with $u \in V(A)$ such that $F \cap V(A) = \emptyset$, and both vertices in F have a neighbour in $V(A)$;
- if $|F \cap V(G)| = 3$, there is a connected subgraph A of $G \cap F$ with $u \in V(A)$ such that $|F \cap V(A)| = 1$, and both vertices in $F \setminus V(A)$ have a neighbour in $V(A)$;
- there is no G -normal O-arc F' with $\text{inf}(F) \subseteq \text{ins}(F')$ and $|F' \cap V(G)| < |F \cap V(G)|$.

There is an O-arc that passes through u and no other vertex; so we may choose a G -normal O-arc F_1 with $x \in \text{ins}(F_1)$ and $|F_1 \cap V(G)| \leq 1$, with $G \cap \text{ins}(F_1)$ maximal. It follows that $G \cap \text{inf}(F_1)$ is connected. From the maximality of $G \cap \text{ins}(F_1)$, there is an edge of G with one end in $\text{ins}(F_1)$ and the other not in $\text{ins}(F_1)$; and such edges come in a natural linear order, from the drawing. Let $e = ab$ be the first such edge, where $a \in F_1$. There is a G -normal O-arc that passes through both a, b and no other vertex, and bounds a disc that includes $\text{ins}(F_1) \cup e$. (We recall that e is a subset of Σ .) Let F_2 be a G -normal O-arc with $\text{ins}(F_1) \cup e \subseteq \text{ins}(F_2)$ and $|F_2 \cap V(G)| \leq 2$, with $G \cap \text{ins}(F_2)$ maximal. The maximality of $G \cap \text{ins}(F_1)$ implies that $|F_2 \cap V(G)| = 2$. There are two vertex-disjoint paths between $\{a, b\}$ and F , from the maximality of $G \cap \text{ins}(F_1)$ and Menger’s theorem. If $a \notin F_2$ then (1) is satisfied, so we assume that $a \in F_2$.

From the maximality of $G \cap \text{ins}(F_2)$, there is an edge ac with $c \notin \text{ins}(F_2)$; and by taking the first such edge, we deduce that there is a G -normal O-arc F_3 with $u \in F_3$ and $\text{ins}(F_2) \subseteq \text{ins}(F_3)$ and $|F_3 \cap V(G)| = 3$, such that the third bullet of (1) is satisfied. From the maximality of $G \cap \text{ins}(F_2)$, the fourth bullet is also satisfied. This proves (1).

For each $u \in U$, let F_u be an O-arc as in (1). Since $\theta \geq 7$, and $d(u, x) \leq 3$ for all $x \in \text{ins}(F_u)$, it follows that the closed discs $\text{ins}(F(u))$ ($u \in U$) are pairwise disjoint. For each $u \in U$, if $|F_u \cap V(G)| = 2$ let $F_u \cap V(G) = \{p_u, q_u\}$, and if $|F_u \cap V(G)| = 3$, let A be as in the third bullet of (1) and let p_u, q_u

be the two vertices in $F \setminus V(A)$. By theorem (5.9) of [4], applied to the manifold (with boundary) Σ' obtained by deleting the interiors of the discs $\text{ins}(F'_u)$ ($u \in U$), if θ is sufficiently large in terms of Σ, k (independent of G), there are two vertex-disjoint trees A, B of $G \cap \Sigma'$, where for each $u \in U$, $p_u \in V(A)$ and $q_u \in V(B)$, and neither of A, B has any other vertex in F . This proves 5.1. \blacksquare

We deduce:

5.2 *Let Σ be a connected surface, not a 2-sphere, and let $k, \xi \geq 1$ be integers. Then there exist $n, \phi \geq 0$ with the following property. Let G be a k -restricted graph, and let $X \subseteq V(G)$ be stable, with $|X| \leq \xi$. Suppose that $G \setminus X$ is drawn in Σ and is ϕ -representative. Then, by at most n splits of the vertices in X , we can convert G to a graph that can be drawn in Σ .*

Proof. Let τ_0 be as in 3.1. Let θ be as in 5.1 with k replaced by $k\xi$. Define $t_{k\xi} = \theta$, and inductively $t_i = 2t_{i+1} + 10$ for $i = k\xi - 1, k\xi - 2, \dots, 0$. Define $\phi = t_0$. Now let $n = k\xi\tau_0$. Now let G, X be as in the theorem. Let W be the set of vertices in $G \setminus X$ with a neighbour in X . Let d be the ϕ -restraint distance function defined by $G \setminus X$. Suppose first that there exists $U \subseteq W$ with $|U| = k\xi$, pairwise with distance at least θ . Then there is a subset $U' \subseteq U$ with $|U'| = k$, all with a common neighbour in $|X|$; and by 5.1, G contains a $K_{k,3}$ minor, a contradiction. So there is no such U .

Let us say that $U \subseteq W$ is *pivotal* if $|U| \leq k\xi$ and the vertices in U are pairwise at distance at least $t_{|U|}$. Certainly \emptyset is pivotal; choose a maximal pivotal set U . Thus $|U| < k\xi$, as we saw above. From the maximality of U , for every vertex $w \in W \setminus U$, there exists $u \in U$ such that $d(u, w) < t_{|U|+1}$. For each $u \in U$, let W_u be the set of $w \in W$ such that $d(u, w) < t_{|U|+1}$. From the maximality of U , it follows that $\bigcup_{u \in U} W_u = W$. Moreover, for all distinct $u, u' \in U$, since $d(u, u') \geq t_{|U|} \geq 2t_{|U|+1} + 10$, the sets W_u ($u \in U$) are pairwise disjoint, and if $w \in W_u$ and $w' \in W_{u'}$ then $d(w, w') \geq 10$.

Let $u \in U$. Since $2 \leq \kappa \leq \phi - 3$, by theorem (9.2) of [6], there is a cycle C of $G \setminus X$ that bounds a closed disc Δ in Σ , such that:

- $d(a, u) \leq \kappa + 2$ for every atom with $a \subseteq \Delta$;
- $d(a, u) \geq \kappa + 1$ for every atom with $a \not\subseteq \Delta$.

Similarly, since $2 \leq \kappa - 3 \leq \phi - 3$, there is a cycle C' bounding a disc Δ' , such that $d(a, u) \leq \kappa - 1$ for every atom with $a \subseteq \Delta'$, and $d(a, u) \geq \kappa - 2$ for every atom with $a \not\subseteq \Delta'$. Consequently $W_u \subseteq \Delta'$, since $t_{|U|+1} - 1 < \kappa - 2$. Moreover, every vertex w in Δ with a neighbour in X satisfies $d(u, w) \leq \kappa + 2$, and so $d(u', w) \geq t_{|U|} - (\kappa - 2) \geq t_{|U|+1}$ for each $u' \in U \setminus \{u\}$. Consequently every such vertex belong to W_u .

If a is a vertex or edge of C , then $d(a, a') \leq 2$ for some region a' not included in Δ ; and since $d(u, a') \geq \kappa + 1$, it follows that $d(u, a) \geq \kappa - 1$. Consequently C, C' are vertex-disjoint.

Let Z be the component of $G \cap \Delta_1$ that contains C and hence contains u , and let $H = G[V(C \cup Z)]$. Thus H is planar, and C bounds a region in *every* planar drawing of H , since Z is connected. But $(H + X, X)$ has planarity at most τ_0 , by 3.1; and so by splitting the vertices in X into a set X' of at most τ_0 vertices, we can convert $H + X$ into a planar graph H' . All vertices in $V(H)$ that are G -adjacent to a vertex in X belong to $V(Z)$; and so there is a planar drawing of H' such that C bounds a region of it. Consequently, by splitting X into at most τ_0 vertices, we can convert $(G \cap \Delta) + X$ into a planar graph that can be drawn such that C bounds a region.

Let us write $\Delta_u = \Delta$ and $C_u = C$. Let us split X into $|U|$ sibling-free subsets X_u ($u \in U$), where for each $u \in U$, vertices in X_u only have neighbours in Δ_u . Then by splitting each X_u into at most τ_0 vertices, we can convert G to a graph that can be drawn in Σ . This proves 5.2. \blacksquare

Finally, we can prove 2.2, which we restate:

5.3 *For every surface Σ , and all integers $k, \xi \geq 1$, there exists $n(\Sigma, k, \xi) \geq 0$ such that, if G is k -restricted and there is a stable subset $X \subseteq V(G)$ with $|X| \leq \xi$ such that $G \setminus X$ can be drawn in Σ , then by splitting the vertices in X into at most $n(\Sigma, k, \xi)$ vertices total, the resultant graph can be drawn in Σ .*

Proof. We proceed by induction on the maximum genus of components of Σ , and with this maximum genus fixed, by induction on the number of components of Σ . Thus, let Σ be a surface. Suppose first that Σ is not connected, and let $\Sigma_1, \dots, \Sigma_t$ be its components. From the second inductive hypothesis, $n(\Sigma_i, k, \xi)$ exists for $1 \leq i \leq t$. Define $n(\Sigma, k, \xi) = \sum_{1 \leq i \leq t} n(\Sigma_i, k, \xi)$. Let G, X be as in the theorem. Let us split the vertices of X into t sibling-free sets X_1, \dots, X_t , where for $1 \leq i \leq t$, all neighbours of each vertex in X_i belong to Σ_i . By splitting each set X_i into $n(\Sigma_i, k, \xi)$ vertices, we deduce that the result holds for Σ, k, ξ .

Thus we may assume that Σ is connected. By 3.1 we may assume that Σ is not a 2-sphere. Let n, ϕ be as in 5.2. Now let G, X be as in the theorem. Let n' be the maximum of $n(\Sigma', k, \xi + \phi)$ over all surfaces Σ' with at most two components, both of genus strictly less than that of Σ . Define $n(\Sigma, k, \xi) = \max(n, n')$. If $G \setminus X$ is ϕ -representative, the result holds by 5.2. Thus we assume there is a non-null-homotopic $(G \setminus x)$ -normal O -arc F in Σ with $|F \cap V(G)| < \phi$. By cutting the surface Σ along F , we obtain a manifold with boundary, with one or two components, and its boundary also has one or two components. Let us paste a disc onto each component of the boundary: we obtain a surface Σ' with at most two components, and each of its components has genus strictly less than the genus of Σ . Let $X' = F \cap V(G)$. Then $G \setminus (X \cup X')$ is drawn in Σ' ; and so by splitting $X \cup X'$ into at most n' vertices, the graph we obtain can be drawn in Σ' and hence in Σ . This proves 5.3. ■

It is annoying that we were not able to extend 3.1 to the nonspherical case. Let us sketch two approaches to this:

- The proof of 5.3 is neat, via 5.2, but the approach only works when the drawing has large representativity. If it has small representativity, one naturally cuts along the corresponding curve and tries to use induction on genus somehow. In the proof above, we removed the vertices in F from the drawing, and this is how vertices in the surface end up being split. We could instead cut Σ along F , splitting each vertex in F into two vertices in the natural way, so that they stay in the manifold; apply induction to the simpler surface(s) we obtain, and then sew the original surface back together again. This works well if cutting along F disconnects the surface, but it might not; and in that case the new graph might not be k -restricted, and not even k' -restricted for any bounded value of k' . We don't see how to get past that.
- Here is a second approach, which might well work but which will need a lot more writing. One could abandon hope of using 5.3, and try a method like that of 3.1. That all seems to work, and one can reduce the problem to the case when $G \setminus X$ is 3-connected, and the drawing of $G \setminus X$ is 1-representative. (Not 3-representative, unfortunately.) To finish it off, we need the analogue of 3.2 for such drawings, but this has not been proved.

6 Kuratowski subdivisions

In this section we prove a result promised earlier:

6.1 For all $k \geq 1$ there exists an integer $k' \geq 1$ such that if G is k -subgraph-restricted, then G contains no (k', j) -Kuratowski graph as a minor for $j \in \{0, \dots, 3\}$.

We need the following lemma (logarithms are to base two):

6.2 Let T_1, \dots, T_t be trees, each with maximum degree less than d , where $d \geq 2$ is an integer, and for $1 \leq i \leq n$ and $1 \leq s \leq t$ let $v_i^s \in V(T_s)$, all different. Suppose that $n \geq 3d^{t \log(2k)}$. Then for $1 \leq s \leq t$ there are k pairwise vertex-disjoint subtrees S_i^s ($1 \leq i \leq k$) of T_s , and pairwise disjoint subsets X_1, \dots, X_k of $\{1, \dots, n\}$, each of cardinality three, such that $v_i^s \in V(S_i^s)$ for $1 \leq i \leq k$ and $1 \leq s \leq t$.

Proof. We begin with:

(1) If T is a tree with $|T| \geq 2$ and with maximum degree less than d , and $P \subseteq V(T)$, then there is an edge e such that both components T' of $T \setminus e$ satisfy $|P \cap V(T')| \geq (|P| - 1)/(d - 1)$.

For each vertex v of G , there is a component C of $T \setminus v$ that contains at least $(|P| - 1)/(d - 1)$ vertices of P ; let e be the edge between v and this component, and call (v, e) a *good pair*. Thus there are $|T|$ good pairs, and so some edge belongs to two of them, and hence satisfies the claim. This proves (1).

If A, B are subtrees of a tree T , vertex-disjoint and with $V(T) = V(A) \cup V(B)$, we call them *complementary subtrees*.

(2) Suppose that $n \geq d^t$. Then for $1 \leq s \leq t$, there are complementary subtrees A_s, B_s of T_s and two disjoint subsets X, Y of $\{1, \dots, n\}$, such that $|X|, |Y| \geq n/d^t$, and for $1 \leq s \leq t$, $t_i \in V(A_s)$ for each $i \in X$, and $t_i \in V(B_s)$ for each $i \in Y$.

For $1 \leq s \leq t$, we construct a complementary pair A_s, B_s of subtrees of T_s , such that there are disjoint subsets X_s, Y_s of $\{1, \dots, n\}$, such that $|X_s|, |Y_s| \geq n/d^s$, and for $1 \leq r \leq s$, $t_i \in V(A_r)$ for each $i \in X_s$, and $t_i \in V(B_r)$ for each $i \in Y_s$. By (1), the statement is true for $s = 1$, since $(n - 1)/(d - 1) \geq n/d$. We proceed by induction on s . Thus we assume that $s \geq 2$ and the result holds for $s - 1$. By (1) applied to T_s , there are complementary subtrees A_s, B_s of T_s and a partition (C, D) of X_{s-1} such that $|C|, |D| \geq (|X_{s-1}| - 1)/(d - 1) \geq |X_{s-1}|/d$, and $t_i \in V(A_s)$ for each $i \in C$, and $t_i \in V(B_s)$ for each $i \in D$. One of $V(A_s), V(B_s)$ contains at least half the vertices $\{v_i^s : i \in Y_{s-1}\}$, and so by exchanging A_s, B_s if necessary we may assume that there exists $Y_s \subseteq Y_{s-1}$ with $|Y_s| \geq |Y_{s-1}|/2$ such that $v_i^s \in B_s$ for each $i \in Y_s$. Thus for $1 \leq r \leq s$, $t_i \in V(B_r)$ for each $i \in Y_s$. Moreover, let $X_s = C$; then for $1 \leq r \leq s$, $t_i \in V(A_r)$ for each $i \in X_s$. Finally, we observe that $|X_s| = |C| \geq |X_{s-1}|/d \geq n/d^s$, and $|Y_s| \geq |Y_{s-1}|/2 \geq n/(2d^{s-1}) \geq n/d^s$, and so the inductive definition is complete. This proves (2).

(3) Suppose that $c \geq 1$ is an integer and $n \geq d^{ct}$. Then for $1 \leq s \leq t$, there are 2^c pairwise vertex-disjoint subtrees $S_s(1), \dots, S_s(2^c)$ of T_s , and 2^c pairwise disjoint subsets $X(1), \dots, X(2^c)$ of $\{1, \dots, n\}$, all of size at least n/d^{ct} , such that $v_i^s \in S_s(i)$ for $1 \leq s \leq t$ and $1 \leq j \leq 2^c$.

The proof is by induction on c , and it is true by (2) when $c = 1$. Suppose that $c > 1$ and the

result holds for $c - 1$. Let X, Y and A_s, B_s for $1 \leq i \leq t$ be as in (2). The result follows from the inductive hypothesis, applied to the trees A_1, \dots, A_t and the set X , and applied to B_1, \dots, B_t and the set Y . This proves (3).

Let c be the smallest integer with $2^c \geq k$; so $2^c < 2k$ and therefore $d^c < d^{\log(2k)}$. By (3), if $n \geq 3(2k)^{t \log d}$, the theorem holds. This proves 6.2. \blacksquare

6.3 *For all integers $k \geq 1$ there exists an integer $k' \geq 1$, such that if G is k -retracted then G does not contain $K_{k',3}$ as a minor.*

Proof. Define $f(k, 0) = k$, and for $t = 1, 2, 3$ inductively define $f(k, t) = \lceil 3f(k, t-1)^{1+t \log(2k)} \rceil$. We will show that taking $k' = f(k, 3)$ satisfies the theorem. Since G contains $K_{k',3}$ as a minor, there are three subtrees T_1, T_2, T_3 of G , and k' connected subgraphs $C_1, \dots, C_{k'}$ of G , such that $T_1, T_2, T_3, C_1, \dots, C_{k'}$ are pairwise vertex-disjoint, and for $1 \leq i \leq k'$ the result holds. Let t be the number of T_1, \dots, T_3 that have more than one vertex; we will prove, by induction on t , that if $k' \geq f(k, t)$, then for some $j \in \{3-t, \dots, 3\}$, G contains a (k, j) -junction as a subgraph.

If $t = 0$ the result is true, since $f(k, 0) = k$. Thus we may assume that $1 \leq t \leq 3$, and T_1, \dots, T_t have more than one vertex, and T_{t+1}, \dots, T_3 have only one.

For $1 \leq i \leq k'$ and $1 \leq s \leq t$ let $v_i^s \in V(T_s)$ have a neighbour in $V(C_i)$. If some $f(k, t-1)$ of the vertices t_i^s are equal (where $1 \leq i \leq k'$ and $1 \leq s \leq t$), then the result follows from the inductive hypothesis; so we may assume that there exists $|I| \subseteq \{1, \dots, k'\}$ with $|I| \geq k'/f(k, t-1)$ such that the vertices t_i^s are all distinct, for $i \in I$ and $1 \leq s \leq t$. We may assume that for $1 \leq s \leq 3$, every leaf of T_s equals v_i^s for some $i \in \{1, \dots, k'\}$, because otherwise this leaf can be deleted. If some vertex v of T_1 has degree at least $f(k, t-1)$, then the result follows from the induction on t , replacing T_1 by the one-vertex tree with vertex v , and choosing a subset $I' \subseteq I$ with cardinality at least $f(k, t-1)$, such that the vertices t_i^1 ($i \in I'$) all belong to different components of $T_1 \setminus v$. Thus we may assume that T_1 , and similarly T_1, \dots, T_t all have maximum degree at most $f(k, t-1) - 1$. But $|I| \geq f(k, t)/f(k, t-1) \geq 3(2k)^{t \log f(k, t-1)}$, and so by 6.2, for $1 \leq s \leq t$ there are k pairwise vertex-disjoint subtrees S_i^s ($1 \leq i \leq k$) of T_s , and pairwise disjoint subsets X_1, \dots, X_k of $\{1, \dots, n\}$, each of cardinality three, such that $v_i^s \in V(S_i^s)$ for $1 \leq i \leq k$ and $1 \leq s \leq t$. Consequently G contains a $(k, 3-t)$ -junction as a subgraph. This proves 6.3. \blacksquare

For the next case we need the following lemma:

6.4 *For all integers $a, b, c, d \geq 1$, let $n \geq abcd$. If T_1, \dots, T_n are subtrees of a tree T , each with maximum degree at most d , then either there are a of T_1, \dots, T_n that are pairwise vertex-disjoint, or there is an edge of T that belongs to b of T_1, \dots, T_n , or there is a vertex $v \in V(T)$ and c of T_1, \dots, T_n that pairwise intersect exactly in $\{v\}$.*

Proof. We may assume that there do not exist a of T_1, \dots, T_n that are pairwise vertex-disjoint, and so there is a subset $X \subseteq V(T)$ with $|X| < a$ such that each T_i contains a vertex in X . Hence there exist $v \in X$ and $I \subseteq \{1, \dots, n\}$ with $|I| \geq n/(a-1) \geq bcd$ such that $v \in V(T_i)$ for each $i \in I$. We may assume that no edge of T belongs to b of T_1, \dots, T_n . Since each T_i has maximum degree at most d , it contains at most d edges of T incident with v , and so shares more than one vertex with at most $(b-1)d$ other trees T_j ($j \in I$). Hence there exists $J \subseteq I$ with $|J| \geq |I|/((b-1)d+1)$ such that the trees in J pairwise intersect in exactly v . Since $|J|/((b-1)d+1) \geq c$, this proves 6.4. \blacksquare

A *near-K graph rooted at $\{u, v\}$* is a graph H with $u, v \in V(H)$, distinct and nonadjacent, such that either H is a subdivision of K_5 or $K_{3,3}$, or adding the edge uv to H makes a subdivision of K_5 or $K_{3,3}$.

6.5 For all integers $k \geq 1$ there exists an integer $k' \geq 1$, such that if G is k -subgraph-restricted, then G contains no (k', m) -Kuratowski graph for $m = 0, 1, 2, 3$ as a minor.

Proof. Let G be k -subgraph-restricted, and suppose that G contains a (k', m) -Kuratowski graph as a minor, where k_m is some large function of m, k to be decided later. If $m = 0$ then taking $k_m = k$ works, and if $m = 3$ then the result is true by 6.3, so we may assume that $m \in \{1, 2\}$. Hence, there are vertex-disjoint trees $T_j (1 \leq j \leq m)$ in G , with union T_0 say, and for $1 \leq i \leq k'$ there is a connected subgraph D_i including T_0 , such that $D_i \cap D_{i'} = T_0$ for $1 \leq i < i' \leq k'$; and for $1 \leq i \leq k'$, if we start with D_i and contract each tree T_j to a single vertex t_j , we obtain a near-K graph rooted at $\{t_1, t_2\}$ if $m = 2$, or a subdivision of K_5 or $K_{3,3}$ if $m = 1$. We need to show that if k_m is a large enough function of m, k then this is impossible.

For $1 \leq i \leq k_m$ and $1 \leq j \leq m$, if $m = 1$ there are two, three or four edges of D_i between $V(C_i)$ and $V(T_j)$, and if $m = 2$ there are one, two or three such edges. We call these edges (i, j) -connectors. There is a subset $I \subseteq \{1, \dots, k_m\}$ with $|I| \geq k_m/9$ such that for all $j \in \{1, \dots, m\}$, there exists d_i such that there are exactly d_i (i, j) -connector for all $i \in I$. We need to replace the trees T_j that have more than one vertex with something else, but T_j 's with only one vertex are good for us. We may assume that T_1 has more than one vertex; so let $m' = 1$ if $m = 1$, or if $|T_2| = 1$, and otherwise let $m' = 2$. If $m' = 2$ let us number such that $d_1 \leq d_2$.

Thus the possibilities for the sequence $d_1, \dots, d_{m'}$ are

$$(1), (1, 1), (2), (1, 2), (3), (1, 3), (2, 2), (4), (2, 3), (3, 3)$$

and we will prove the result for each such sequence, in order from left to right. For $1 \leq j \leq m'$ and each $i \in I$ let $T_{i,j}$ be the minimal subtree of T_j containing an end of each (i, j) -connector. There are too many cases, all very similar, to write them all out carefully, so we will just sketch the arguments.

For (1), each $T_{i,1}$ has only one vertex t_i say, and if many of them are equal, we can replace T_1 by a one-vertex tree and win. So we may choose a large $I' \subseteq I$ such that the vertices $t_i (i \in I')$ are all different, and then we win by applying 6.2.

For (1, 1) the same approach either replaces T_i by a one-vertex tree, or reduces it to the (1) case with an application of 6.2.

For (2), we apply 6.4 to T_1 and the paths $T_{i,1}$. If many of them share the same edge, then we reduce the problem to the (1, 1) case by deleting the edge. If many pairwise intersect in the same vertex, we reduce T_1 to a one-vertex tree.

So we may assume that many of them (say $T_{i,1}$ for $i \in I'$) are pairwise vertex-disjoint. If $m = 1$, we win, so we assume $m = 2$ and $|T_2| = 1$. Let \mathcal{A} be the set of all subtrees of T_1 that include two of the trees $T_{i,1} (i \in I')$. If many members of \mathcal{A} are pairwise disjoint, we win, and otherwise, there is a vertex v of T_1 such that many components of $T_1 \setminus v$ contain some $T_{i,1}$ with $i \in I'$. In that case we can replace T_1 by a one-vertex tree with vertex v , and add to C_i a minimal subtree of T_1 including $T_{i,1}$ and v .

For (1, 2), let $T_{i,1}$ have one vertex $t_{i,1}$. If many of the $T_{i,1}$'s are equal we win, so we may assume they are all different; but then applying 6.2 to T_1 works as before.

For (3), we apply 6.4 and win in each case immediately.

For (1, 3) we apply 6.2 to T_1 as before. (In this case we don't need to partition into triples; a partition into pairs would be good enough, but triples also works.)

For (2, 2) and (2, 3), we follow the same procedure as in the (2) case.

For (4), it follows that $m = 1$. We apply 6.4. If many of the trees share an edge we reduce to the (2, 2) or (1, 3) case, and in the other two outcomes of 6.4 we win immediately.

For (3, 3), we use the same method as for (2). We may assume that the trees $T_{i,1}(i \in I')$ are pairwise vertex-disjoint, by 6.2, where $I' \subseteq I$ is as large as we want. Let \mathcal{A} be the set of all subtrees of T_1 that include two of the trees $T_{i,1}(i \in I')$. If many members of \mathcal{A} are pairwise disjoint, we win by reducing to the (3) case, and otherwise, there is a vertex v of T_1 such that many components of $T \setminus v$ contain some $T_{i,1}$ with $i \in I'$. In that case, as before, we can replace T_1 by a one-vertex tree with vertex v , and add to C_i a minimal subtree of T_1 including $T_{i,1}$ and v . This proves 6.5. ■

References

- [1] T. Böhme and B. Mohar, “Labeled $K_{2,t}$ -minors in plane graphs”, *J. Combinatorial Theory, Ser. B*, bf 84 (2002), 291–300.
- [2] K. Kuratowski, “Sur le problème des courbes gauches en topologie”, *Fund. Math.* 15 (1930), 271–283.
- [3] N. Robertson and P. Seymour, “Excluding disjoint Kuratowski graphs”, in preparation.
- [4] N. Robertson and P. Seymour, “Graph minors. VII. Disjoint paths on a surface”, *J. Combinatorial Theory, Ser. B*, 45 (1988), 212–254.
- [5] N. Robertson and P. Seymour, “Graph minors. XI. Circuits on a surface”, *J. Combinatorial Theory, Ser. B*, 60 (1994), 72–106.
- [6] N. Robertson and P. Seymour, “Graph minors. XIV. Extending an embedding”, *J. Combinatorial Theory, Ser. B*, 65 (1995), 23–50.