Criticality for multicommodity flows

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Abstract

For $k \geq 1$, the *k-commodity flow problem* is, we are given k pairs of vertices in a graph G, and we ask whether there exist k flows in the graph, where

- the ith flow is between the ith pair of vertices, and has total value one; and
- for each edge e, the sum of absolute values of the flows along e is at most one.

We prove that for all k there exists $n(k)$ such that if G is connected, and contraction-minimal such that the k-commodity flow problem is infeasible (that is, minimal in the sense that contracting any edge makes the problem feasible) then $|V(G)| + |E(G)| \leq n(k)$.

For integers $k, p \geq 1$, the *(k,p)-commodity flow problem* is as above, with the extra requirement that the flow value of each flow along each edge is a multiple of $1/p$. We prove that if $p > 1$, and G is connected, and contraction-minimal such that the (k,p)-commodity flow problem is infeasible, then $|V(G)| + |E(G)| \leq n(k)$, with the same $n(k)$ as before, independent of p. In contrast, when $p = 1$ there are arbitrarily large contraction-minimal instances, even when $k = 2$.

We give some other corollaries of the same approach, including

- a proof that for all $k \geq 0$ there exists $p > 0$ such that every feasible k-commodity flow problem has a solution in which all flow values are multiples of $1/p$, and
- a very simple polynomial-time algorithm to solve the (k,p) multicommodity flow problem when $p > 1$.

1 Introduction

For integer $k \geq 0$, the *k-commodity flow problem* is, we are given k pairs of vertices in a graph G, and we ask whether there exist k flows in the graph, where

- the ith flow is between the ith pair of vertices, and has total value one; and
- for each edge e, the sum of the absolute values of the flows along e is at most one.

(All graphs in this paper are finite, and may have loops or parallel edges.) For convenience, let us direct the edges of G, arbitrarily; then, given G and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices, we are looking for k maps ϕ_i $(1 \leq i \leq k)$, where

- for $1 \leq i \leq k$, ϕ_i is a map from $E(G)$ to the set of real numbers $\mathbb{R}$
- for $1 \leq i \leq k$ and each vertex v of G,

$$\phi_i(\delta^+(v)) - \phi_i(\delta^-(v)) = \begin{cases} 0 & \text{if } v \neq s_i, t_i \\ 1 & \text{if } v = s_i \neq t_i \\ -1 & \text{if } v = t_i \neq s_i \\ 0 & \text{if } v = s_i = t_i, \end{cases}$$

  where $\delta^+(v), \delta^-(v)$ denote the sets of non-loop edges with tail v and head v respectively, and for $X \subseteq E(G)$, $\phi_i(X)$ denotes $\sum_{e \in X} \phi_i(e)$, and
- for every edge e, $\sum_{1 \leq i \leq k} |\phi_i(e)| \leq 1$.

(We allow $s_i = t_i$, for convenience when we contract edges, although then ϕ_i might as well be identically zero.) If there is a solution we say the k-commodity flow problem is *$\mathbb{R}$–feasible*. This is a linear programme of polynomial size, and so we can check $\mathbb{R}$–feasibility in polynomial time [4, 8], independent of k.

In this paper we are interested in restricting the values $\phi_i(e)$. Let $p \geq 1$ be an integer, and let $\mathbb{Z}/p$ denote the set of all rationals q/p where q is an integer. If $\phi_1, \ldots, \phi_k$ can be chosen as above so that in addition

- $\phi_i(e) \in \mathbb{Z}/p$ for $1 \leq i \leq k$ and for each edge $e \in E(G)$

we say the problem is *$\mathbb{Z}/p$–feasible* (or $\mathbb{Z}$–feasible if $p = 1$) and we call deciding $\mathbb{Z}/p$–feasibility the *(k,p)-commodity flow problem.* To include the original $\mathbb{R}$–feasibility problem in this language, let us set $p = \infty$; thus, $\mathbb{Z}/\infty$–feasibility means $\mathbb{R}$–feasibility.

From the algorithmic point of view, checking $\mathbb{Z}$–feasibility and checking $\mathbb{Z}/2$–feasibility behave similarly, in that both problems are solvable in polynomial time when k is fixed [17], and both are NP-complete if k is not fixed [3, 12]. Nevertheless, there is a significant difference between the two problems, as we shall see.

If a k-commodity flow problem is feasible, and we contract some edge, this results in a new k-commodity flow problem that is also feasible. (Contraction may make loops or parallel edges.) Thus, given G and $s_1, t_1, \ldots, s_k, t_k$ as before, let us say $(G, s_1, t_1, \ldots, s_k, t_k)$ is *$\mathbb{Z}/p$–critical* if

- the corresponding k-commodity flow problem is not $\mathbb{Z}/p$–feasible;
- for every edge e, if we contract e then the k-commodity flow problem becomes $\mathbb{Z}/p$–feasible; and
- no vertex different from $s_1, t_1, \ldots, s_k, t_k$ has degree zero.

(Again, $\mathbb{Z}/\infty$–criticality and $\mathbb{R}$–criticality mean the same thing.)

Consequently, a k-commodity flow problem is not $\mathbb{Z}/p$–feasible if and only if it can be reduced to a $\mathbb{Z}/p$–critical instance by contracting edges and removing isolated vertices. The following is a version of the main result of this paper:

1.1 *For all integers $k \geq 0$ there exists $n(k)$ such that for all $p > 1$ (including $p = \infty$), if $(G, s_1, t_1, \ldots, s_k, t_k)$ is a $\mathbb{Z}/p$–critical instance, then $|V(G)| + |E(G)| \leq n(k)$.*

This just says that $n(k)$ exists, but the same methods could be used construct such a number $n(k)$ if desired. It would be large, of the order of

$$\exp(\exp(\exp(\exp(\exp(k^c)))))$$

for a constant c.

We cannot extend this to include $p = 1$, even when $k = 2$. For instance, let $n \geq 2$, and let G have vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, and edges $a_i a_{i+1}$ and $b_i b_{i+1}$ for $1 \leq i < n$, and $a_i b_i$ for $1 \leq i \leq n$. (We call this graph the n-ladder.) Then (G, a_1, b_n, a_n, b_1) is $\mathbb{Z}$–critical; and yet $|V(G)|$ can be arbitrarily large.

Our result implies the following, which was known for $k = 2$ (when taking $p = 2$ works), but seems to be new even for $k = 3$:

1.2 *For all integers $k \geq 0$ there exists $p > 0$ such that every k-commodity flow problem that is $\mathbb{R}$–feasible is also $\mathbb{Z}/p$–feasible.*

Proof (assuming 1.1.) Choose $n(k)$ as in 1.1. Choose an integer p so large that p is a multiple of the determinant of every nonsingular $d \times d$ matrix with all entries in $\{-1, 0, 1\}$, for all d with $1 \leq d \leq kn(k)$. We claim that p satisfies the theorem; so we need to show that if $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{R}$–feasible then it is $\mathbb{Z}/p$–feasible. We prove this by induction on $|V(G)| + |E(G)|$, and therefore may assume that no vertex v different from $s_1, t_1, \ldots, s_k, t_k$ has degree zero.

Suppose first that $|V(G)| + |E(G)| \leq n(k)$. We can write this $\mathbb{R}$-feasibility problem as a linear programme, where the constraint matrix has only $k|E(G)| \leq kn(k)$ columns; and since the entries in the constraint matrix are all from $\{-1, 0, 1\}$, it follows that every nonsingular submatrix of the contraint matrix has determinant that divides p. Since the instance is $\mathbb{R}$–feasible, it follows from Cramer's rule that there is a solution in which all values are multiples of $1/p$, that is, the instance is $\mathbb{Z}/p$–feasible, as required.

Thus we may assume that $|V(G)| + |E(G)| > n(k)$. For every edge e, the instance obtained from $(G, s_1, t_1, \ldots, s_k, t_k)$ by contracting e is $\mathbb{R}$–feasible, and therefore $\mathbb{Z}/p$–feasible from the inductive hypothesis. Since $(G, s_1, t_1, \ldots, s_k, t_k)$ is not $\mathbb{Z}/p$–critical (because $|V(G)| + |E(G)| > n(k)$), we deduce that it is $\mathbb{Z}/p$–feasible. This proves 1.2. ∎

1.1 implies that:

1.3 *For all integers $k \geq 0$ there exists $n(k)$ such that for all $p > 1$, a k-commodity flow problem $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}/p$–feasible if and only if for every partition $(X_1, \ldots, X_m)$ of $V(G)$ into non-empty sets with $m \leq n(k)$, the k-commodity flow problem obtained by identifying the members of each X_i to a single vertex is $\mathbb{Z}/p$–feasible.*

Taking $m = 2$ in the above is exactly the so-called "cut condition", that

- for every $X \subseteq V(G)$, the number of edges of G between X and $V(G) \setminus X$ is at least the number of values of $i \in \{1, \ldots, k\}$ such that X contains exactly one of s_i, t_i.

This is always necessary for $\mathbb{Z}/p$–feasibility (and indeed for $\mathbb{R}$–feasibility); and it is sufficient when all the pairs (s_i, t_i) are the same, and it was heavily investigated in early work on the multicommodity flow problem. It turns out that the cut condition is not sufficient for $\mathbb{Z}$–feasibility for any $k > 1$, and not sufficient for $\mathbb{R}$–feasibility for any $k \geq 3$, and yet in some interesting circumstances the cut condition is necessary and sufficient for $\mathbb{Z}/2$–feasibility. For instance, this is the case when

- there are only two distinct pairs (s_i, t_i) (this is Hu's two-commodity flow theorem [6]), or more generally, the set $\{s_1, t_1, \ldots, s_k, t_k\}$ has at most four elements [21]
- the graph obtained from G by adding edges to make all the pairs s_i, t_i adjacent is planar [22], or
- G is planar, and can be drawn so that $s_1, t_1, \ldots, s_k, t_k$ all belong to the infinite face [14],

and more (see [9]).

Our result 1.3 is a sort of generalization of these, and has the same complexity consequences. For instance, to check if $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}/p$–feasible (where $p > 1$), it suffices to check if there is some partition of $V(G)$ into a bounded number of sets with the property that after identifying the vertices in each set, the problem is not $\mathbb{Z}/p$–feasible. We can do this in polynomial time; for let P be the set of edges with ends in different sets of this partition. We may assume that $|P|$ is bounded, because if there are more than k edges between any two of the sets we could replace the two sets by their union. So we can try all possibilities for P in polynomial time; and having selected P, we contract all edges not in P, and then read off whether the desired partition exists. This is of no real interest, because as we said before, for fixed k all these problems are polynomial-time solvable; but this shows that there is a simple algorithm, not using the methods of Graph Minors.

Actually we can do much better than this. Our methods yield a simple and fast algorithm to test $\mathbb{Z}/p$–feasibility, for any fixed $p > 1$. We explain this in section 4, and use some of the same lemmas to prove our main result in the remainder of the paper.

In particular, for fixed k and p there are only finitely many $\mathbb{Z}/p$–critical k-commodity instances, and for very small k it might be of interest to figure out what they are. For example, it is easy to check that

1.4 *For all $p > 1$, the only $\mathbb{Z}/p$–critical 2-commodity flow problem with more than two vertices is (G, s_1, t_1, s_2, t_2), where G is the four-vertex cycle, and $\{s_1, t_1\}$ and $\{s_2, t_2\}$ are its pairs of non-adjacent vertices (and this is $\mathbb{Z}/p$–critical only when p is finite and odd).*

It is more difficult to prove the following (unpublished joint work with Katherine Edwards):

1.5 *If $(G, s_1, t_2, s_2, t_2, s_3, t_3)$ is $\mathbb{Z}/3$–critical and $(s_2, t_2) = (s_3, t_3)$ then $|V(G)| \le 6$.*

It turns out that there is one such instance with six vertices, two with four vertices, and one with two.

Let us return to 1.1. This asserts the boundedness of the size of all $\mathbb{Z}/p$–critical instances $(G, s_1, t_1, \ldots, s_k, t_k)$, but only with k fixed. How much can we relax the hypotheses before the result becomes false? Suppose then that p is fixed, and we have some class of (k, p)-commodity flow problems defined by some restriction on permitted sequences $(s_1, t_1, \ldots, s_k, t_k)$, but with k unbounded. (One could put restrictions on the graphs G instead, but that is not considered here.) For instance, we might restrict the size of the set $W = \{s_1, t_1, \ldots, s_k, t_k\}$, or the number k_0 of *distinct* pairs (s_i, t_i). If $|W| \le 2$ this is just the Menger problem, and all critical instances have two vertices; but even with $|W| \le 3$ we are in trouble; if p is odd there are arbitrarily large $\mathbb{Z}/3$–critical instances with $|W| = 3$. For instance, let $n \ge 0$ be an integer, let G be the complete bipartite graph $K_{3,9n+2}$, and let u, v, w be its three vertices of degree $9n + 2$. Let $k = 12n + 3$. For $1 \le i \le k$, let

$$(s_i, t_i) = \begin{cases} (u, v) \text{ if } 1 \le i \le k/3 \\ (v, w) \text{ if } k/3 < i \le 2k/3 \\ (w, u) \text{ if } 2k/3 < i \le k; \end{cases}$$

then $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}/3$–critical. These can be converted to unbounded $\mathbb{Z}/3$–critical instances with $k_0 = 2$, by adding a new vertex adjacent to u, v. There are similar examples for all odd $p > 1$. (Incidentally, when $|W| = 3$, checking $\mathbb{Z}/p$-feasibility is polynomial-time solvable [20] with no bound on k.)

Let us try $p = 2$ instead; and in that case setting $|W| \le 4$ gives us no problem, as all critical instances have two vertices [21]. But we cannot go much further. If we hope for boundedness of the critical instances in the class, it is necessary that deciding $\mathbb{Z}/2$-feasibility in the class is polynomially-solvable; and Middendorf and Pfeiffer (published in [15]) showed that checking $\mathbb{Z}/2$-feasibility is NP-complete when $k_0 = 3$.

Incidentally, a recent result of Moitra [13] implies that there are $\mathbb{R}$–critical instances with an unbounded number of vertices when $k_0 = 3$, although $\mathbb{R}$-feasibility is checkable in polynomial time (even with k_0 unbounded).

In summary:

	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{R}$
k fixed	P, unbounded	P, bounded	P, bounded	P, bounded
$\lvert W\rvert = 3$	P, unbounded	P, 2	P, unbounded	P, 2
$k_0 = 2$	NPC, unbounded	P, 2	open, unbounded	P, 2
$k_0 \ge 3$, fixed	NPC, unbounded	NPC, unbounded	open, unbounded	P, unbounded
k_0 unrestricted	NPC, unbounded	NPC, unbounded	open, unbounded	P, unbounded

Table 1: Complexity, and largest critical instances

Let us say that $(G, s_1, t_1, \ldots, s_k, t_k)$ is *eulerian* if for each $v \in V(G)$, the degree of v in G has the same parity as the number of pairs (s_i, t_i) where one of s_i, t_i equals v. (In other words, if we add k new "demand" edges joining the pairs s_i, t_i, then all vertices have even degree in the new graph.) In all the theorems listed earlier where the cut condition is necessary and sufficient for $\mathbb{Z}/2$–feasibility, it is also true that when the input is eulerian, the cut condition is necessary and sufficient for $\mathbb{Z}$–feasibility. This phenomenon persists with criticality. Let the *oddness* of G be the number of vertices with odd degree, and let the *skewness* of G be the number of non-loop edges not parallel to other edges. We will show:

1.6 *For all integers $k, s \geq 0$ there exists n such that if $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}$–critical and G has oddness or skewness at most s, then $|V(G)| + |E(G)| \leq n$.*

Thus, although for fixed k there are arbitrarily large $\mathbb{Z}$–critical instances, these must have arbitrarily many vertices of odd degree and arbitrarily many edges that are not parallel to other edges.

Let $(G, s_1, t_1, \ldots, s_k, t_k)$ be $\mathbb{Z}$–critical. Since oddness zero and skewness zero both imply that G has no edge-cut of cardinality three (assuming that $|E(G)| > 3$), and much of the proof of 1.6 uses only that there are no edge-cuts of cardinality three, it is tempting to conjecture that if $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}$–critical and G has no edge-cut of cardinality three then $|V(G)|$ has size bounded by a function of k; but that is false. Here is a counterexample: let $n \geq 2$ be an integer, and let G have $2n$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$, and edges as follows:

- an edge between x_i, y_i for $1 \leq i \leq n$;
- two edges between x_i, x_{i+1} for $1 \leq i < n$;
- two edges between y_i, y_{i+1} for $1 \leq i < n$.

Let $s_1 = s_2 = x_1$, $s_3 = s_4 = y_1$, $t_1 = t_3 = x_n$ and $t_2 = t_4 = y_n$. Then G has no edge-cut of cardinality three, and it is easy to check that $(G, s_1, t_1, \ldots, s_4, t_4)$ is $\mathbb{Z}$–critical.

One could formulate more complicated (and esoteric) versions of the multicommodity flow problem, and again ask whether there is a bound on the size of critical instances. For instance, currently all edges have "capacity" one; what if we permit arbitrary capacities, all at least one? Or what if for different $s_i - t_i$ pairs we ask for flows with values in $\mathbb{Z}/p$ for different integers p_i? What if for each pair $s_i - t_i$ we insist that all the flow between them travels on one path, but the demand between the pair is at most $1/2$? What if, instead of specifying *pairs* of vertices between which we require flow, we specify *sets* of vertices, and try to pack edge-disjoint connected subgraphs each including one of the sets? Our methods give a general approach to all of these; we prove that in every sufficiently large graph, there is one of two kinds of subgraph, and, for the (k, p)-commodity flow problem examined in this paper, there are edges of these subgraphs that can be contracted without changing feasibility, and therefore the instance is not critical if it is sufficiently large. I expect that the same two kinds of subgraph would serve to handle the other versions of multicommodity flow problem just mentioned (or at least the ones that are true), but have not worked out the details.

Also, here is a counterexample that handles some of these extensions: let G be the n-ladder, with notation as before, and let $p, q \geq 1$ be integers, relatively prime. In G we cannot choose p paths $P_1, \ldots, P_p$ between a_1, b_n and q paths between a_n, b_1, such that for every edge e, $\frac{x(e)}{p} + \frac{y(e)}{q} \leq 1$, where $x(e)$ denotes the number of $P_1, \ldots, P_p$ that contain e, and $y(e)$ is defined similarly; yet if we contract any edge then such a choice of paths exists.

Incidentally, let P be the set of integers p such that $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}/p$–feasible. What can we say about P? Clearly if $1 \in P$ then P contains all positive integers, but there are other dependencies that are not so obvious; for instance, if $2, 3 \in P$, then P contains all integers greater than one. More generally, if $p_1, p_2 \in P$ then $p_1+p_2 \in P$. (This can be shown be taking an appropriate linear combination of the solutions for p_1 and p_2.) It is not easy to find instances with $P \neq \emptyset$ and $2 \notin P$. But for all $p \geq 1$, there exist $(G, s_1, t_1, \ldots, s_k, t_k)$ where P consists of all positive multiples of p (and indeed the solution is unique), as follows:

- take $2p + 1$ disjoint sets of vertices $A_1, \ldots, A_{2p+1}$, each of cardinality $p + 2$, and let $A_i = \{a_i^1, \ldots, a_i^p, u_i, v_i\}$ for $1 \leq i \leq 2p + 1$;
- let a_i^j be adjacent to a_{i+1}^j for all i, j (where a_{2p+2}^j means a_1^j), and let u_i, v_i be adjacent to $a_i^1, \ldots, a_i^p$ for all i;
- let $k = p(2p + 1)$, and let the list $(s_1, t_1), \ldots, (s_k, t_k)$ consist of (u_i, v_{i+p}) for $1 \leq i \leq 2p + 1$ (reading subscripts modulo $2p + 1$), together with $p - 1$ copies of (u_i, v_i) for $1 \leq i \leq 2p + 1$.

A more complicated construction due to Lomonosov [10] (see [19] for a simplified version) shows more-or-less the same thing, with $k_0 = 3$ in addition.

Let us return to 1.1, and dispose of the case $p = \infty$ before we start on the serious proofs.

1.7 *If $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{R}$–critical, then there exists an integer $p > 1$ (depending on G) such that $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}/p$–critical. Consequently, to prove 1.1 when $p = \infty$ it suffices to prove 1.1 for all finite $p > 1$.*

Proof. Since $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{R}$–critical, it follows that for every edge e of G, there are k flows in G/e satisfying the requirements of the multicommodity flow problem. Since these flows are the solution of a linear program with rational constraints, it follows that they can be chosen such that the value of each flow on each edge is rational. Hence there is an integer $p(e) \geq 2$ such that the value of each flow on each edge belongs to $\mathbb{Z}/p(e)$. Let p be the least common multiple of all the $p(e)$'s; then $(G, s_1, t_1, \ldots, s_k, t_k)$ is $\mathbb{Z}/p$–critical. This proves the first assertion. For the second, note that the value of $n(k)$ in 1.1 does not depend on p, and the second assertion follows. This proves 1.7. ∎

Our main result 1.1 has some resemblance to theorems about "mimicking networks". Let $v_1, \ldots, v_k$ be distinct vertices of a graph G, and let $c : E(G) \to \mathbb{R}_+$ be a capacity function. A *mimicking network* (for G, $v_1, \ldots, v_k$ and c) is a graph H, distinct vertices $w_1, \ldots, w_k$ of H, and a map $d : E(H) \to \mathbb{R}_+$, satisfying the following. For every $I \subseteq \{1, \ldots, k\}$ and every $r \in \mathbb{R}_+$, there exists $A \subseteq V(G)$ satisfying

- for $1 \leq i \leq k$, $v_i \in A$ if and only if $i \in I$
- $\sum_{e \in \delta_G(A)} c(e) \leq r$

if and only if there exists $B \subseteq V(H)$ satisfying

- for $1 \leq i \leq k$, $w_i \in B$ if and only if $i \in I$
- $\sum_{e \in \delta_H(B)} d(e) \leq r$.

($\delta_G(A)$ or $\delta(A)$ is the set of non-loop edges of G with exactly one end in A.) The objective is to find mimicking networks with as few vertices as possible; and Hagerup, Katajainen, Nishimura, and Ragde [5] showed that for every choice of $G, v_1, \ldots, v_k, c$ there is a mimicking network with at most 2^{2^k} vertices. But the flow problem that mimicking networks handle is the one-commodity multiterminal flow problem, not the multicommodity flow problem, so our result is different, and more general (except that we do not permit capacities).

2 Demand systems

It is helpful to reformulate the various problems a little. First, let us add a new vertex v_0 and $2k$ new edges, each incident with v_0, and respectively incident with $s_1, t_1, \ldots, s_k, t_k$. (Thus if two of $s_1, t_1, \ldots, s_k, t_k$ are equal we add two parallel edges.) These new edges come naturally in pairs (one incident with s_i is paired with one incident with t_i), and the multicommodity flow problem asks for a flow of total value one between s_i and t_i, which we can express in terms of this pair of edges. But there is now no advantage in this pairing; we might as well be requesting flows between *every* pair of vertices both adjacent to v_0. And that being so, there is no reason to insist that v_0 has even degree. Thus, from now on v_0 will have degree k instead of $2k$. Furthermore, since from 1.7 we no longer have to concern ourselves with the $p = \infty$ case, we might as well scale all the demands by p to make them integers. Thus, now we are looking for integer-valued flows, summing to at most p on each edge. And since every such flow dominates a sum of unit flows along paths, we can recast our problem now as a search for paths of $G \setminus v_0$ (or cycles through v_0) with constraints on how many of them can use any edge.

The following then becomes the situation. (We denote by $\delta(v)$ the set of edges incident with a vertex v.)

- $p \geq 1$ is a positive integer.
- v_0 is a vertex of degree k of a graph G, and no loop is incident with v_0.
- D is a symmetric matrix of non-negative integers, with rows and columns indexed by $\delta(v_0)$, with zero diagonal, and with all row and columns sums at most p.

We call D a *demand matrix*. Let us call (G, v_0, D, p) a *demand system* of *degree k*. Let $\mathcal{C}$ denote the set of cycles of G that contain v_0. A demand system (with this notation) is *feasible* if there is a map $\phi : \mathcal{C} \to \mathbb{Z}_+$ (the set of non-negative integers) such that

- for all $e, f \in \delta(v_0)$, $D_{e,f}$ equals the sum of $\phi(C)$ over all cycles C containing e, f
- for every edge e of G, the sum of $\phi(C)$ over all $C \in \mathcal{C}$ containing e is at most p

and we call such a map ϕ a *solution*.

If e is an edge of G, G/e denotes the graph obtained by contracting e. A demand system (G, v_0, D, p) is *critical* if it is not feasible, but for every edge e of G not incident with v_0, the demand system $(G/e, v_0, D, p)$ is feasible, and no vertex of G different from v_0 has degree zero. We will prove the following, which implies 1.1 and 1.6.

2.1 *For all integers $k, s \geq 0$ there exists $n \geq 0$ with the following property. Let (G, v_0, D, p) be a critical demand system of degree at most k, such that either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and G has oddness at most s, or*
- *$p = 1$ and G has skewness at most s.*

Then $|V(G)| + |E(G)| \leq n$.

It is tempting to try to prove the $p > 1$ case of 2.1 by replacing each edge by p parallel edges and applying the $p = 1$ case when every edge is parallel to another. This does not work, because then the number n provided by 2.1 would depend on p (because its input k would be scaled by p), and it is important that the number n of 2.1 is independent of p. We have not been able to unify these three alternate hypotheses, and we have to give them separate proofs. Nevertheless, the three proofs have a great deal in common, and we will describe them simultaneously as far as we can.

We have stressed that the n provided by 2.1 is independent of p. Thus in any demand system (G, v_0, D, p) of degree at most k, if $|V(G)|+|E(G)| > n$ then there is an edge e that can be contracted without changing $\mathbb{Z}/p$-feasiblity (or an isolated vertex). But more than that is true; the choice of e also does not depend on p. The following is a strengthening of 2.1, and is the main result of the paper:

2.2 *For all integers $k, s \geq 0$ there exists $n \geq 0$ with the following property. Let G be a graph with $|V(G)| + |E(G)| > n$, and let v_0 be a vertex of degree at most k, such that no vertex different from v_0 has degree zero. Then there is an edge e of G, such that for every demand system (G, v_0, D, p), if either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and G has oddness at most s, or*
- *$p = 1$ and G has skewness at most s,*

and $(G/e, v_0, D, p)$ is feasible, then (G, v_0, D, p) is feasible.

3 Porosity and contractibility

Let G be a graph and $X \subseteq V(G)$, with $X \neq V(G)$. Let H be the graph obtained from G by identifying all vertices in $V(G) \setminus X$ into one vertex v (and deleting any loops that result). Let $p \geq 1$ be an integer. We say that X is *p-porous* (in G) if every demand system (H, v, D, p) is feasible.

If (G, v_0, D, p) is a demand system, we say a set F of edges of G is *contractible* if none of them is incident with v_0, and $(G/F, v_0, D, p)$ is feasible if and only if (G, v_0, D, p) is feasible. Hence every subset of a contractible set is also contractible.

We denote by $G|Y$ the subgraph of G induced on Y, and remind the reader that $\delta(Y)$ denotes the set of edges of G with one end in Y and the other in $V(G) \setminus Y$.

3.1 *Let (G, v_0, D, p) be a demand system, and let $Y \subseteq V(G) \setminus \{v_0\}$, such that Y is p-porous. Then $E(G|Y)$ is contractible.*

Proof. Let $F = E(G|Y)$. Since contracting edges preserves feasibility, it suffices to show that if $(G/F, v_0, D, p)$ is feasible then so is (G, v_0, D, p). Thus, let ϕ' be a solution for $(G/F, v_0, D, p)$. For each pair of edges e, f of $\delta(Y)$, let $D'(e, f)$ be the sum of $\phi'(C)$ over all cycles C of G/F that contain v_0, e and f. Let H be obtained from G by identifying all vertices of $V(G) \setminus Y$ into one vertex v. Then (H, v, D', p) is a demand system, and is therefore feasible since Y is p-porous. But the solution for (H, v, D', p) can be combined with ϕ' in the natural way to yield a solution for (G, v_0, D, p). This proves 3.1. ▮

We deduce:

3.2 *Let (G, v_0, D, p) be a demand system, and let $v \in V(G) \setminus \{v_0\}$. Choose $Y \subseteq V(G)$ containing v and not v_0, with $|\delta(Y)|$ minimum. Then Y is p-porous in G, and hence $E(G|Y)$ is contractible. Moreover if $|\delta(Y)| = |\delta(v_0)|$, then (G, v_0, D, p) is feasible.*

Proof. Let $|\delta(Y)| = t$ say. Note that the minimality of $|\delta(Y)|$ implies that there are t edge-disjoint paths in G between v_0 and v, each with exactly one edge in $\delta(Y)$; and in particular, $t \leq |\delta(v_0)|, |\delta(v)|$. It follows that there are edge-disjoint paths $P_1, \ldots, P_t$, each with first vertex v, and each with last vertex in $V(G) \setminus Y$, and every interior vertex in Y. Hence Y is p-porous, and the result follows from 3.1. This proves 3.2. ▮

4 An algorithm for $\mathbb{Z}/p$–feasibility with $p > 1$

The idea of the proof of 2.2 is as follows. It is easy, using 3.2, to find the required contractible edge if some vertex has degree larger than k, so we may assume that all vertices have degree at most k. We prove that in *every* graph G with sufficiently many vertices, with maximum degree k, and for every vertex v_0 of G with degree k, there is a subset $X \subseteq V(G)$ not containing v_0 of one of two special types (let us call them types 1 and 2 for the moment). Then we observe that if (G, v_0, D, p) is a demand system and $X \subseteq V(G) \setminus \{v_0\}$ is of one of these two types, then X is p-porous, and consequently the set of edges with both ends in X is contractible (and there are such edges), as required.

There is a useful byproduct of this, giving a simple algorithm to check $\mathbb{Z}/p$–feasibility with $p > 1$. There is a third type of subset (let us call them type 3). Since the set of edges within a type 1 subset is contractible, we deduce that the set within a type 3 subset is also contractible (since every type 3 subset is in fact a subset of a type 1 subset). And there is a simple algorithm that, given (G, v_0, D, p), finds either a type 3 subset, or find a kind of tree-decomposition of bounded width (but using edge-cutsets instead of the usual vertex-cutsets), and this leads to a simple algorithm to check $\mathbb{Z}/p$–feasibility with $p > 1$. We present the algorithm first, before the more complicated material about subsets of type 1 and 2; and so we will assume for the moment that type 3 subsets have the property that we just said.

The existence of a polynomial-time algorithm for this is not new; what is new is the simplicity of the algorithm. The result of [17] gives a polynomial-time algorithm to solve the k edge-disjoint paths problem for fixed k, but it is very complicated. It breaks into three parts (actually the algorithm is for *vertex*-disjoint paths, but we apply it in the line graph):

- what to do if the tree-width (of the line graph) is small

- what to do if we have found a large clique minor (of the line graph), and
- what to do if, in the line graph, the tree-width is big but we have not found a large clique minor.

These are in increasing order of complication; the third part in particular is very tricky. But for our problem we only need the first part, as we shall see. Independently, Kawarabayashi and Kobayashi [7] also have an algorithm to check $\mathbb{Z}/p$–feasibility for $p = 2$, certainly much simpler than the algorithm from [17], eliminating the third part above but using the first and second.

If $X, Y \subseteq V(G)$ are disjoint, $\delta(X, Y)$ denotes the set of edges with an end in X and an end in Y. We say $Z \subseteq V(G)$ is *robust* if

$$|\delta(X, Y)| \geq \min(|\delta(X, V(G) \setminus Z)|, |\delta(Y, V(G) \setminus Z)|)$$

for every partition (X, Y) of Z. (Robust sets Z with $|\delta(Z)|$ large are what we were calling type 3 at the start of this section.) Both the algorithm of this section, and our main result 2.2, rely on the following lemma.

4.1 *For all $k, s \geq 0$ there exists K with the following property. Let (G, v_0, D, p) be a demand system of degree at most k, such that either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and G has oddness at most s, or*
- *$p = 1$ and G has skewness at most s.*

Let $Y \subseteq V(G) \setminus \{v_0\}$, where Y is robust and $|\delta(Y)| \geq K$. Then $E(G|Y)$ is contractible.

We prove 4.1 later in the paper, but for the moment, we assume its correctness, and deduce the correctness of the algorithm. Let T be a tree, in which every vertex has degree one or three, and let $L(T)$ denote the set of leaves of T. (A *leaf* is a vertex of degree one.) Let G be a graph, let $v_0 \in V(G)$, and let ϕ be a surjective map from $V(G)$ to $L(T)$, such that for some $t_0 \in L(T)$ and for all $v \in V(G)$, $\phi(v) = t_0$ if and only if $v = v_0$. We call t_0 the *root* (it is unique). We call (T, ϕ) a *partial carving* of (G, v_0). For $t \in L(T)$, $\phi^{-1}(t)$ denotes the set of $v \in V(G)$ with $\phi(v) = t$. If $|\phi^{-1}(t)| = 1$ for every $t \in L(T) \setminus \{t_0\}$, we call (T, ϕ) an *carving*. For each edge $f \in E(T)$, let $n(f)$ be the number of edges $uv \in E(G)$ such that $\phi(u), \phi(v)$ belong to different components of $T \setminus f$; the *width* of a partial carving (T, ϕ) is the maximum of $n(f)$ over all edges $f \in E(T)$.

We denote by $\Delta(G)$ the maximum degree of the vertices of G. Now let $0 \leq k \leq K$, and let G be a graph with $\Delta(G) \leq k$, and let $v_0 \in V(G)$. A partial carving (T, ϕ) with root t_0 of (G, v_0) is *(k, K)-optimal* if

- (T, ϕ) has width at most K,
- for every $t \in L(T) \setminus \{t_0\}$, $\phi^{-1}(t)$ is robust, and
- for every $t \in L(T) \setminus \{t_0\}$, if $|\phi^{-1}(t)| \geq 2$ then $|\delta(\phi^{-1}(t))| > K - k$.

We need to show that any fixed sufficiently large K, a (k, K)-optimal partial carving always exists, and that we can find one efficiently. Both are implied by the following.

4.2 *For all integers $0 \le k \le K$, there is an algorithm as follows:*

- **Input:** *A graph G with $|V(G)| \ge 2$ and with $\Delta(G) \le k$, and a vertex $v_0 \subseteq V(G)$.*
- **Output:** *A (k,K)-optimal partial carving of (G, v_0).*
- **Running time:** $O(|V(G)|^2)$.

Proof. Let (T_1, ϕ_1) be the partial carving where T_1 has two vertices t_0, t_1 and $\phi^{-1}(t_0) = \{v_0\}$. This has width at most $k \le K$, since $|\delta(v_0)| \le k$; and $|L(T_1)| = 2$. Now, for $i \ge 1$, suppose we have some partial carving (T_i, ϕ_i) of (G, v_0), of width at most K, with $|L(T_i)| = i+1$. Let t_0 be the root.

For each $t \in L(T_i) \setminus \{t_0\}$, let $Z_t = \phi^{-1}(t)$; we check whether Z_t is robust, and if not, find some partition (X,Y) of Z_t not satisfying the corresponding inequality. For each t, this takes time $O(|Z_t|)$. To show this, we proceed as follows. Since $|\delta(Z_t)| \le K$, and K is a constant, we can enumerate all partitions (P,Q) of $\delta(Z_t)$ in constant time. For each such (P,Q), let X_0, Y_0 be the sets of ends in Z_t of the edges in P, Q respectively; then we test whether there is a partition (X,Y) of $V(G)$ such that $X_0 \subseteq X$ and $Y_0 \subseteq Y$, with $|\delta(X,Y)| \le \min(|P|, |Q|)$. (This is a max-flow problem solvable in time $O(|Z_t|)$.) If there is a such a partition, we find one, and otherwise Z_t is robust.

The total running time for this step is the sum, over all t, of the running time for the algorithm applied to Z_t, and since the sets Z_t are pairwise disjoint and non-empty, it follows that the total running time for this step is $O(|V(G)|)$.

Suppose that for some t we find a partition (X,Y) of Z_t with

$$|\delta(X,Y)| \le \min(|\delta(X, V(G) \setminus Z_t)|, |\delta(Y, V(G) \setminus Z_t)|).$$

It follows that $X, Y \neq \emptyset$. Let T_{i+1} be obtained from T_i by adding two new vertices t_1, t_2 to T_i, both adjacent to t. For $v \in V(G)$, define

$$\phi_{i+1}(v) = \begin{cases} t_1 & \text{if } v \in X \\ t_2 & \text{if } v \in Y \\ \phi_i(v) & \text{if } v \notin Z_t. \end{cases}$$

Then (T_{i+1}, ϕ_{i+1}) is a partial carving. Moreover, its width is at most K; because

$$|\delta(X)| = |\delta(X,Y)| + |\delta(X, V(G) \setminus Z_i| \le |\delta(Y, V(G) \setminus Z_i)| + |\delta(X, V(G) \setminus Z_i)| = |\delta(Z_t)| \le K,$$

and similarly $|\delta(Y)| \le K$. Moreover, $|L(T_{i+1})| = i+2$, and the iteration is complete.

Thus we may assume that each Z_t is robust. Next we check whether for every $t \in L(T) \setminus \{t_0\}$, if $|Z_t| \ge 2$ then $|\delta(Z_t)| > K - k$. (This takes linear time.) Suppose that this is false for some t; thus $t \in L(T) \setminus \{t_0\}$, $|Z_t| \ge 2$ and $|\delta(Z_t)| \le K - k$. Choose $z \in Z_t$, arbitrarily. Let T_{i+1} be obtained from T_i by adding two new vertices t_1, t_2 to T_i, both adjacent to t. For $v \in V(G)$, define

$$\phi_{i+1}(v) = \begin{cases} t_1 & \text{if } v = z \\ t_2 & \text{if } v \in Z_t \setminus \{z\} \\ \phi_i(v) & \text{if } v \notin Z_t. \end{cases}$$

Then (T_{i+1}, ϕ_{i+1}) is a partial carving. Moreover, its width is at most K; for $|\delta(z)| \le \Delta(G) \le k \le K$, and

$$|\delta(Z_t \setminus \{z\})| = |\delta(Z_t \setminus \{z\}, V(G) \setminus Z_t)| + |\delta(Z_t \setminus \{z\}, \{z\})| \le |\delta(Z_t)| + |\delta(z)| \le |\delta(Z_t)| + k \le K.$$

Thus again the iteration is complete.

Finally, if there is no such t then (T_i, ϕ_i) is (k, K)-optimal and we output it. Note that this must happen within at most $|V(G)|$ iterations, since every partial carving (T, ϕ) satisfies $|L(T)| \le |V(G)|$ (because ϕ is a surjection). Consequently the total running time is $O(|V(G)|^2)$. This proves 4.2. ∎

The main result of this section is the following.

4.3 *For all integers $k \ge 0$, there is an algorithm as follows:*

- **Input:** *A demand system (G, v_0, D, p) of degree at most k, such that either*
  - *$p \ge 2$ and $s = 0$, or*
  - *$p = 1$ and G has oddness at most s, or*
  - *$p = 1$ and G has skewness at most s.*
- **Output:** *Decides whether the demand system is feasible.*
- **Running time:** $O(|V(G)|^2)$.

Proof. Here is the algorithm. If there is a vertex $v \ne v_0$ with degree larger than k, we choose $X \subseteq V(G)$ containing v and not v_0, with $|\delta(X)|$ minimum. (This is a max-flow problem, solvable in linear time.) We contract all edges with both ends in X. (By 3.2 this does not change feasibility.) We repeat until every vertex in Y has degree at most k. (This takes total time $O(|V(G)|^2)$.)

Let K be as in 4.1, and let $K' = K + k$. Now we find a (k, K')-optimal partial carving (T, ϕ) with respect to v_0, using 4.2; this takes time $O(|V(G)|^2)$. For each $t \in L(T) \setminus \{t_0\}$ (where t_0 is the root) we contract all edges with both ends in $\phi^{-1}(t)$. (We show below that this does not change feasibility.) This results in a carving (not partial any more) of width at most K', and so the problem can now be solved by standard dynamic programming methods.

We must show that contracting edges with both ends in $\phi^{-1}(t)$ does not change feasibility. Let $Z_t = \phi^{-1}(t)$. If $|Z_t| = 1$ then this contraction has no effect, and if $|Z_t| > 1$ then by the definition of (k, K')-optimal, $|\delta(Z_t)| \ge K' - k = K$, and 4.1 implies that $E(G|Z_t)$ is contractible; and the claim follows. (Note that the robustness of Z_t is not changed by contracting the edges with both ends in other Z_j's.) This proves 4.3. ∎

5 Porous subsets from robustness

Now we return to the proof of our main result 2.2. We need to do three things:

- explain what we mean by subsets of types 1 and 2 (used at the start of the previous section)
- prove that subsets of type 1 and 2 are p-porous

- prove that there is a subset of type 1 or 2 in every sufficiently large graph.

In this section, we prove that subsets of type 1 are p-porous (for all $p > 1$). Let $K \geq 0$ be an integer. We say a subset $X \subseteq V(G)$ is *K-supported* if there exists $Y \subseteq X$ such that Y is robust, $|\delta(Y)| \geq K$, and $|\delta(Z)| \geq |\delta(X)|$ for every Z with $Y \subseteq Z \subseteq X$. For K sufficiently large, the K-supported sets X with $|\delta(X)| \leq k$ are what we were calling subsets of type 1. We will prove the following.

5.1 *For all $k, s \geq 0$ there exists K such that, if either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and G has oddness at most s, or*
- *$p = 1$ and G has skewness at most s*

and $X \subseteq V(G)$ is K-supported with $|\delta(X)| \leq k$, then X is p-porous.

First, we observe that this will imply 4.1.

Proof of 4.1, assuming 5.1. Let $k \geq 0$, and let K be as in the first statement of 5.1. We claim that K satisfies 4.1. For let (G, v_0, D, p) be a demand system of degree at most k, with $p > 1$, and let $Y \subseteq V(G) \setminus \{v_0\}$, where Y is robust and $|\delta(Y)| \geq K$. Choose $X \subseteq V(G)$ with $Y \subseteq X \subseteq V(G) \setminus \{v_0\}$, with $\delta(X)$ minimum. Then X is K-supported, and hence p-porous by 5.1. But then $E(G|X)$ is contractible by 3.1, and hence so is $E(G|Y)$. This proves 4.1. ∎

There are three alternative hypotheses in 5.1, and the first is handled by applying a lemma proved in [11]. The second is handled by another method, using a lemma from [1] (and the third could be done either way, and we choose to use the second method for it). We begin with the first method. A *separation* of *order* k in a graph G is a pair (A, B) of subgraphs of G such that $A \cup B = G$, $E(A \cap B) = \emptyset$, and $|V(A \cap B)| = k$.

If $\theta \geq 1$ is an integer, a *tangle* of *order* θ in a graph G is a set $\mathcal{T}$ of separations of G, each of order less than θ, such that

- for every separation (A, B) of order less than θ, $\mathcal{T}$ contains at least one of $(A, B), (B, A)$
- if $(A_i, B_i) \in \mathcal{T}$ for $i = 1, 2, 3$, then $A_1 \cup A_2 \cup A_3 \neq G$
- if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Let G, H be graphs, where H is simple. A *model* of H in G is a map η with domain $V(H) \cup E(H)$, where

- for every $v \in V(H)$, $\eta(v)$ is a non-null connected subgraph of G, all pairwise vertex-disjoint
- for every edge $e = uv$ of H, $\eta(e)$ is an edge of G with one end in $V(\eta(u))$ and the other in $V(\eta(v))$.

The following is the main result of [11]:

5.2 *Let H be a planar graph, drawn in the plane, and let $u_1, \dots, u_m$ be distinct vertices of H, each incident with the infinite region. Then there exists K with the following property. Let $\mathcal{T}$ be a tangle of order at least $K/3$ in a graph G, and let $F \subseteq V(G)$ with $|F| = m$ such that there is no separation $(A, B) \in \mathcal{T}$ of order less than m with $F \subseteq V(A)$. Then there is a model η of H in G such that for $1 \le i \le m$, $\eta(u_i)$ contains a vertex of F.*

We deduce one third of 5.1, the following:

5.3 *For all $k \ge 0$ there exists K such that, if G is a graph and $p \ge 2$, and $X \subseteq V(G)$ is K-supported with $|\delta(X)| \le k$, then X is p-porous.*

Proof. For $0 \le m \le k$, let H_m be the graph with vertex set

$$\{u_1, \dots, u_m\} \cup \{v_{ij} \ : 1 \le i \le 6m, 1 \le j \le 3m^2\}$$

and edge set as follows:

- for $1 \le i \le m$, u_i is adjacent to $v_{6i-5,1}, v_{6i-4,1}, \dots, v_{6i,1}$
- for $1 \le i, i' \le 6m$ and $1 \le j, j' \le 3m^2$, v_{ij} is adjacent to $v_{i'j'}$ if $|i' - i| + |j' - j| = 1$.

This graph H_m is planar, and can be drawn in the plane such that $u_1, \dots, u_m$ are incident with the infinite region. Choose K to satisfy 5.2 (with H_m for H) for all choices of m with $0 \le m \le k$. We claim this satisfies 5.3.

For let G be a graph, let $p \ge 2$, and let $X \subseteq V(G)$ be K-supported, with $|\delta(X)| \le k$. Thus, there exists $Y \subseteq X$ such that

- Y is robust,
- $|\delta(Y)| \ge K$, and
- there is no subset Z with $Y \subseteq Z \subseteq X$ with $|\delta(Z)| < |\delta(X)|$.

We may assume that $X \ne V(G)$, for otherwise the result is trivial; and we may assume that there is a unique vertex v_0 in $V(G) \setminus X$ and $\delta_G(X) = \delta_G(v_0)$, by identifying all vertices in this set and deleting any loops we create. Let (G, v_0, D, p) be a demand system; we must show that it is feasible.

Let L be the graph with vertex set $E(G)$, setting e, f adjacent in L if some vertex $v \in X$ is incident with both e, f. Let $F = \delta_G(X)$, $m = |F|$ and $W = \delta_G(Y)$. Then

- there is no separation (A, B) of L of order less than $\min(|V(A) \cap W|, |V(B) \cap W|)$,
- the set of all separations (A, B) of L of order less than $K/3$ such that $|V(A) \cap W| \le |V(A \cap B)|$ is a tangle of order $\lceil K/3 \rceil$, and
- there is no separation (A, B) of L of order less than $|F|$ with $F \subseteq V(A)$ and $|V(B) \cap W| \ge |F|$.

(The first assertion follows since Y is robust in G; the second is an easy exercise that we leave to the reader, and the third holds because there is no $Z \subseteq X$ with $Y \subseteq Z$ and with $|\delta_G(Z)| < |\delta_G(X)| = |F|$.) From 5.2, there is a model η of H_m in L such that for $1 \le i \le m$, $\eta(u_i)$ contains a vertex of F. We deduce that there is a map ζ with domain $V(H_m)$, such that

- for each $v \in V(H_m)$, $\zeta(v)$ is a nonempty set of edges of G, with $\zeta(v) \subseteq V(L)$ and inducing a connected subgraph of L,
- for all distinct $u, v \in V(H_m)$, $\zeta(u) \cap \zeta(v) = \emptyset$
- for all $u, v \in V(H_m)$ adjacent in H_m, some vertex in X is incident with an edge in $\zeta(u)$ and with an edge in $\zeta(v)$
- F can be numbered $\{f_1, \ldots, f_m\}$ such that for $1 \leq i \leq m$, $f_i \in \zeta(u_i)$.

Now D is a demand matrix (d_{ij} $1 \leq i, j \leq m$), say, and each d_{ij} is a non-negative integer. Moreover, each row sum $\sum_{1 \leq j \leq m} d_{ij}$ is at most p.

(1) *We can write $D = D^1 + D^2 + D^3$ where D^1, D^2, D^3 are all $m \times m$ matrices of non-negative integers, and for $n = 1, 2, 3$ the row sums and the column sums of D^n are at most $p/2$. (We do not require that D^1, D^2, D^3 are symmetric.)*

Let J be the bipartite graph with vertex set $\{(i, r) : 1 \leq i \leq m, 1 \leq r \leq 2\}$, in which there are d_{ij} edges between $(i, 1)$ and $(j, 2)$ (and therefore also between $(j, 1)$ and $(i, 2)$) for $1 \leq i, j \leq m$. This graph has maximum degree at most p, and so can be p-edge-coloured; let $M_1, \ldots, M_p$ be matchings of J, pairwise disjoint and with union $E(J)$. For $1 \leq n \leq 3$ and $1 \leq i, j \leq m$, we define $D^n = (d^n_{ij})$ as follows. If p is even, for $1 \leq i, j \leq m$, let d^1_{ij} be the number of edges between $(i, 1)$ and $(j, 2)$ in $M_1 \cup \cdots \cup M_{p/2}$, let d^2_{ij} be the number of edges between $(i, 1)$ and $(j, 2)$ in $M_{p/2+1} \cup \cdots \cup M_p$, and let $d^3_{ij} = 0$. If p is odd, say $p = 2q + 1$, let d^1_{ij} be the number of edges between $(i, 1)$ and $(j, 2)$ in $M_1 \cup \cdots \cup M_q$, let d^2_{ij} be the number of edges between $(i, 1)$ and $(j, 2)$ in $M_{q+2} \cup \cdots \cup M_{2q+1}$, and let d^3_{ij} be the number of edges between $(i, 1)$ and $(j, 2)$ in M_{q+1}. Evidently $\sum_j d^1_{ij}, \sum_j d^2_{ij} \leq p/2$; and $\sum_j d^3_{ij} \leq p/2$ since $\sum_j d^3_{ij} \leq 1$. Thus all row sums of D^1, D^2, D^3 are at most $p/2$, and similarly so are the column sums. (This is the only place in the proof we use that $p > 1$.) This proves (1).

For $1 \leq i \leq m$ and $1 \leq n \leq 6$ let R^n_i be the path of H_m with vertices

$$u_i, v_{6i-6+n,1}, v_{6i-6+n,2}, \ldots, v_{6i-6+n,3m^2}$$

in order. For $1 \leq r \leq m^2$ and $1 \leq n \leq 3$, let C^n_r be the path of H_m with vertices

$$v_{1,3r-3+n}, v_{2,3r-3+n}, v_{3m,3r-3+n}$$

in order. For $1 \leq n \leq 3$ and $1 \leq i < j \leq m$, let P^n_{ij} be the path of H_m between u_i and u_j included in $R^{n+3}_i \cup R^n_j \cup C^n_{m(i-1)+j}$.

Let $1 \leq n \leq 3$ and $1 \leq i < j \leq m$, and let P^n_{ij} have vertices $a_1, \ldots, a_t$ in order. Each of the sets $\zeta(a_s)$ (for $1 \leq s \leq t$) is a non-empty subset of $E(G)$, inducing a connected subgraph of L; and since for $1 \leq s < t$ some vertex of X is incident in G with an edge in $\zeta(a_s)$ and with an edge in $\zeta(a_{s+1})$, it follows that the union $\zeta(P^n_{ij})$ say of the sets $\zeta(a_s)$ ($1 \leq s \leq t$) also induces a connected subgraph of L. Since this set includes the edges f_i, f_j of G, it follows that there is a path of $G|X$ between the end of f_i in X and the end of F_j in X, and all edges of this path belong to $\zeta(P^n_{ij})$. Together with the edges f_i, f_j, this forms a cycle of G, Q^n_{ij} say, containing v_0.

For every cycle C of G containing v_0, let f_i, f_j be the edges of C incident with v_0 where $i < j$, and define $\phi^n(C) = d^n_{ij}$ if $C = Q^n_{ij}$, and otherwise $\phi^n(C) = 0$. Let $\phi(C) = \phi^1(C) + \phi^2(C) + \phi^3(C)$; we claim that ϕ is a solution for the demand system (G, v_0, D, p).

We must check that

- for all $e, f \in \delta(v_0)$, D_{ef} equals the sum of $\phi(C)$ over all cycles C containing e, f
- for every edge e of G, the sum of $\phi(C)$ over all $C \in \mathcal{C}$ containing e is at most p.

For the first assertion, let $1 \leq i < j \leq m$. We must show that d_{ij} equals the sum of $\phi(C)$ over all cycles C containing e, f. But the latter is $d^1_{ij} + d^2_{ij} + d^3_{ij} = d_{ij}$ as required.

For the second assertion, let $e \in E(G)$. If e belongs to none of the sets $\zeta(v)$ $(v \in V(H_m))$ then e belongs to none of the cycles Q^n_{ij} and the claim holds. Thus we may assume that $e \in \zeta(v)$ for some (necessarily unique) $v \in V(H_m)$. First suppose that $v = u_h$ for some h with $1 \leq h \leq m$. For $1 \leq i < j \leq m$ and $1 \leq n \leq 3$, if e belongs to Q^n_{ij} then u_h belongs to P^n_{ij}, and so one of i, j equals h; and so either $h = i < j$, or $i < j = h$. We must check then that

$$\sum_{1 \leq n \leq 3} \sum_{h < j \leq m} d^n_{hj} + \sum_{1 \leq n \leq 3} \sum_{1 \leq i < h} d^n_{ih} \leq p.$$

But the first sum equals

$$\sum_{h < j \leq m} d_{hj} = \sum_{h < j \leq m} d_{jh}$$

since D is symmetric, and the second sum is $\sum_{1 \leq i < h} d_{ih}$, so together they add to $\sum_{1 \leq i < m} d_{ih}$ and hence to at most p, as required.

We may therefore assume that $v = v_{6r-6+a, 3s-3+b}$ for some r, s, a, b with $1 \leq r \leq m$ and $1 \leq s \leq m^2$ and $1 \leq a \leq 6$ and $1 \leq b \leq 3$. Now for $1 \leq i < j \leq m$ and $1 \leq n \leq 3$, e belongs to Q^n_{ij} for some n, i, j only if v belongs to P^n_{ij}, and so $v \in V(R^{n+3}_i \cup R^n_j \cup C^n_{m(i-1)+j})$. Thus, either

- $i = r$, $n = a - 3$ and $r < j \leq m$, or
- $j = r$, $n = a$ and $1 \leq i < r$, or
- $n = b$, and (i, j) satisfy $m(i-1) + j = s$ and $1 \leq i < j \leq m$ (and hence i, j are unique, say $(i, j) = (i', j')$).

The first can only occur if $a > 3$, and the second only if $a \leq 3$, and so it is enough to check that if $a \leq 3$ then

$$\sum_{1 \leq i < r} d^a_{ir} + d^b_{i'j'} \leq p$$

and if $a > 3$ then

$$\sum_{r < j \leq m} d^{a-3}_{rj} + d^b_{i'j'} \leq p.$$

But $\sum_{1 \leq i < r} d^a_{ir}$ is at most the rth column sum of D^a and hence at most $p/2$; and $d^b_{i'j'}$ is at most the i'th row sum of D^b and hence at most $p/2$, so the first inequality holds. The second follows similarly. This proves that ϕ is indeed a solution, and so X is p-porous. This proves 5.3. ∎

Now we turn to the proof of the remaining two-thirds of 5.1. First, we observe the following.

5.4 *Let $Y \subseteq V(G)$ be robust, and let $W \subseteq V(G)$. Then one of $|\delta(W \cup Y)|, |\delta(W \setminus Y)| \leq |\delta(W)|$.*

Proof. Let $U = V(G) \setminus W$. We must show that $\min(|\delta(U \cap Y)|, |\delta(W \cap Y)|) \leq |\delta(W)|$, so there is symmetry between U, W. Since Y is robust, it follows that

$$|\delta(W \cap Y, U \cap Y)| \geq \min(|\delta(W \cap Y, V(G) \setminus Y)|, |\delta(U \cap Y, V(G) \setminus Y)|),$$

and by exchanging U, W if necessary we may assume that $|\delta(W \cap Y, U \cap Y)| \geq |\delta(W \cap Y, V(G) \setminus Y)|$. Since every edge in $\delta(W \setminus Y) \setminus \delta(W)$ belongs to $\delta(W \cap Y, V(G) \setminus Y)$, and every edge in $\delta(W \cap Y, U \cap Y)$ belongs to $\delta(W) \setminus \delta(W \setminus Y)$, it follows that $|\delta(W \setminus Y)| \leq |\delta(W)|$. This proves 5.4. ∎

We need the following, a special case of theorem 5.3 of [17]:

5.5 *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = 2p$. Let $t \geq 8p$, and let $G_1, \ldots, G_t$ be subgraphs of G, mutually vertex-disjoint, such that*

- *for $1 \leq i \leq t$, G_i is connected;*
- *for $1 \leq i < j \leq t$, there is an edge of G between G_i and G_j; and*
- *for $1 \leq i \leq t$, there is no separation (A, B) of G of order less than $2p$, such that $Z \subseteq V(A)$ and $A \cap G_i$ is null.*

Let $Z = \{a_1, b_1, \ldots, a_p, b_p\}$. Then there are p paths $P_1, \ldots, P_p$ of G, pairwise vertex-disjoint, such that P_i has ends a_i, b_i for $1 \leq i \leq p$.

Let $h \geq 2$ be even. An *elementary wall of height h* is a graph whose vertex set can be numbered

$$\{v_{ij}\ ; 1 \leq i \leq h+1, 1 \leq j \leq 2h+2, (i,j) \neq (1, 2h+2), (h+1, 1)\}$$

where distinct vertices $v_{ij}, v_{i'j'}$ are adjacent if either

- $i = i'$ and $|j' - j| = 1$, or
- $j = j'$ and $|i' - i| = 1$ and $\min(i, i') + j$ is even.

For $1 \leq i \leq h+1$, we call the path with vertex set

$$\{v_{ij}\ ; 1 \leq j \leq 2h+2, (i,j) \neq (1, 2h+2), (h+1, 1)\}$$

a *row* of the elementary wall, and the vertices $v_{i,2i}$ $(2 \leq i \leq h)$ are its *diagonal vertices.* A *wall of height h* is a subdivision of an elementary wall of height h. We define its rows and diagonal vertices analogously. For $g \geq 2$, the $g \times g$ grid $\mathcal{G}_g$ is a graph with vertex set $\{v_{ij} : 1 \leq i, j \leq g\}$, where v_{ij} is adjacent to $v_{i'j'}$ if $|i - i'| + |j - j'| = 1$.

An *immersion* of a loopless graph H in G is a map η, with domain $V(H) \cup E(H)$, mapping each vertex of H to a vertex of G, and each edge of H to a path of G, satisfying the following:

- $\eta(u) \neq \eta(v)$ for all distinct $u, v \in V(H)$

- for each $e \in E(H)$ with distinct ends u, v, $\eta(e)$ is a path of G with ends $\eta(u), \eta(v)$
- for $v \in V(H)$ and $e \in E(H)$, if e is not incident with v in H then $\eta(v) \notin V(\eta(e))$
- for all distinct $e, f \in E(H)$, $E(\eta(e) \cap \eta(f)) = \emptyset$.

We also need the main theorem of [1], the following:

5.6 *For all $g > 1$ there exists $b \geq 0$, with the following property. Let W be a wall in a graph G, and let S be a set of diagonal vertices of W, pairwise 4-edge-connected in G, and with $|S| \geq b$. Then there is an immersion η of $\mathcal{G}_g$ in G such that $\eta(v) \in S$ for each $v \in V(\mathcal{G}_g)$.*

The next result will complete the proof of 5.1.

5.7 *For all $k, s \geq 0$ there exists K such that, if either*

- *G has oddness at most s, or*
- *G has skewness at most s,*

and $X \subseteq V(G)$ is K-supported with $|\delta(X)| \leq k$, then X is 1-porous.

Proof. We may assume that $k \geq 1$. Let $g = 5k$, and let b be as in 5.6. Let $K = 6k \cdot 20^{64(b+s+1)^5}$. Now let G, X be as in the theorem. We proceed by induction on $|V(G)| + |E(G)|$. Consequently we may assume that there is a unique vertex v_0 say not in X (by identifying all vertices of G not in X), and no loop is incident with v_0. Since G is K-supported, there exists $Y \subseteq X$ such that

- Y is robust,
- $|\delta(Y)| \geq K$, and
- there is no subset Z with $Y \subseteq Z \subseteq X$ such that $|\delta(Z)| < |\delta(X)|$.

Let $(G, v_0, D, 1)$ be a demand system; we must show it is feasible. Suppose then that it is not.

(1) *For every edge e of G not incident with v_0, either $e \in \delta(Y)$, or $(G/e, v_0, D, 1)$ is feasible.*

For suppose that $e \notin \delta(Y)$. The oddness of G/e is at most that of G, and the same holds for skewness. Moreover, Y (or Y/e, if e has both ends in Y) is robust in G/e; so by the inductive hypothesis, $(G/e, v_0, D, 1)$ is feasible. This proves (1).

(2) *Every vertex has degree at most $2k$.*

For let $v \in V(G)$, and suppose that v has degree larger than $2k$. Certainly v_0 has degree at most k, so $v \neq v_0$. Choose $W \subseteq V(G)$ with $v \in W$ and $v_0 \notin W$, with $|\delta(W)|$ minimum. Let $|\delta(W)| = k'$ say. Thus $k' \leq |\delta(X)| \leq k$. Since $(G, v_0, D, 1)$ is not feasible, 3.2 implies that $k' < |\delta(X)|$. By 3.2, $E(G|W)$ is contractible. By (1), every edge of $G|W$ belongs to $\delta(Y)$, for every such choice of W. Since $Y \subseteq Y \cup W \cup X$, and X is K-supported, there is no Z with $Y \subseteq Z \subseteq X$ such that $|\delta(Z)| < |\delta(X)|$, it follows that $|\delta(Y \cup W)| > k'$. Since Y is robust, 5.4 implies that $|\delta(W \setminus Y)| \leq k'$. Suppose that $v \in W \setminus Y$; then $W \setminus Y$ is an alternative choice of W, and so no edge has both ends in $W \setminus Y$. In particular, every edge incident with v belongs to $\delta(W \setminus Y)$, and so there are at most $k' \leq k$ such edges, a contradiction. Thus $v \in Y$. Now every edge e of G incident with v has its second end not in $Y \cap W$, since $\delta(Y)$ contains every edge of G with both ends in W. But at most k' such edges have second end not in W, since $|\delta(W)| = k'$; and at most k' have second end in $W \setminus Y$, since $|\delta(W \setminus Y)| \leq k'$ as we already saw. Consequently there are at most $2k'$ edges incident with v, a contradiction. This proves (2).

Let $W = \delta_G(Y)$. Thus $|W| \geq K$. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order less than $K/(6k)$ such that $|E(B) \cap W|) \geq |W|/2$.

(3) *$\mathcal{T}$ is a tangle of order $K/(6k)$.*

To check the first of the three tangle axioms, we observe that if (A, B) is a separation of G of order less than $K/(6k)$ then since every member of W belongs to one of $E(A), E(B)$, it follows that one of $(A, B), (B, A) \in \mathcal{T}$.

For the second, we claim that if $(A, B) \in \mathcal{T}$ then $|E(A) \cap W| < |W|/3$. For suppose not. Let $P = V(A) \cap Y$ and $Q = Y \setminus V(A)$. Let there be w_1 edges in $E(B) \cap W$ incident with a vertex in $Y \cap V(A \cap B)$, and w_2 edges in $E(B) \cap W$ incident with a vertex in Q. Thus $w_1 + w_2 \geq |W|/2$. Since Y is robust, it follows that

$$|\delta(P, Q)| \geq \min(|\delta(P, V(G) \setminus Y)|, |\delta(Q, V(G) \setminus Y)|) \geq \min(|W|/3, w_2).$$

But since every vertex in $A \cap B$ has degree at most $2k$, and every edge in $\delta(P, Q)$ is incident with such a vertex, and so is every edge in $E(B) \cap W$ incident with a vertex in $Y \cap V(A \cap B)$, it follows that

$$|\delta(P, Q)| + w_1 \leq 2k|V(A \cap B)|.$$

Consequently

$$2k|V(A \cap B)| \geq w_1 + \min(|W|/3, w_2) \geq |W|/3$$

since $w_1 + w_2 \geq |W|/2$. It follows that $|V(A \cap B)| \geq |W|/(6k) \geq K/(6k)$, a contradiction. This proves our claim that if $(A, B) \in \mathcal{T}$ then $|E(A) \cap W| < |W|/3$, and the second tangle axiom follows.

For the third, let $(A, B) \in \mathcal{T}$, and suppose that $V(A) = V(G)$. Since $|E(B) \cap W| \geq |W|/2$, it follows that at least $|W|/2$ edges have both ends in $V(A \cap B)$; but there are at most $k|V(A \cap B)|$ such edges since every vertex has degree at most $2k$, and so $k|V(A \cap B)| \geq |W|/2 \geq K/2$, a contradiction. This proves (3).

Since $K/(6k) = 20^{64(b+s+1)^5}$, (3) and the main theorem of [18] imply that there is a wall M of height $b + s + 1$ in G, such that for every $(A, B) \in \mathcal{T}$ of order at most $b + s + 1$, B includes a row of the wall. (The theorem of [23] gives a grid minor rather than a wall subgraph, so we adjusted the numbers to get a $2(b + s + 3) \times 2(b + s + 3)$ grid minor, which gives a wall of height $b + s + 1$.) This wall has $b + s$ diagonal vertices.

(4) *There are at least b diagonal vertices of M that are pairwise four-edge-connected.*

We recall that by hypothesis, either there are at most s edges of G that are not parallel to other edges, or there are at most s vertices in G that have odd degree. Let us say a *branch* of M is a path P of M with distinct ends, both of degree three in M, and such that every internal vertex of P has degree two in M. If v is a diagonal vertex of M, let $N_M(v)$ denote the set of vertices u of M such that either $u = v$ or u is an internal vertex of a branch of M incident with v. A vertex w of G is *local to* a diagonal vertex v if either

- $w \in N_M(v)$ or
- $w \notin V(M)$, and if C denotes the component of $G \setminus V(M)$ containing w, then some vertex in $N_M(v)$ has a neighbour in $V(C)$, but no vertex in $V(M) \setminus N_M(v)$ has a neighbour in $V(C)$.

Let us say a diagonal vertex v is *bad* if there are edges e_1, e_2, e_3 of M, one from each of the three branches of M incident with v, such that

- for each branch of M incident with v, one of its edges is not parallel to any other edge in G, and
- there is a vertex w with odd degree in G that is local to v.

It follows that there are at most s bad diagonal vertices, and so at least b that are not bad.

Next, we claim that if u, v are diagonal vertices that are not bad, then u, v are 4-edge-connected to one another in G. For suppose not; then there exists $L \subseteq V(G)$ with $v \in L$ and $u \notin L$ such that $\delta_G(L) \leq 3$, and with $G|L$ connected. Since u, v are three-edge-connected in M, it follows that $\delta_G(L) \subseteq E(M)$. Since the only three-edge cuts of M separating u and v consist of either one edge from each of the three branches incident with v, or the same for u, we may assume the former from the symmetry. In particular, each of the three edges of $\delta_G(L)$ is not parallel in G to another edge, and so each branch of M incident with v has an edge not parallel to another edge in G. But also, since $|\delta_G(L)|$ is odd, it follows that some vertex $w \in L$ has odd degree in G, and w is local to v since $\delta_G(L) \subseteq E(M)$. But then v is bad, a contradiction. This proves that u, v are four-edge-connected, and hence completes the proof of (4).

By (4) and 5.6, there is an immersion η of the $g \times g$ grid $\mathcal{G}_g$ in G, such that each vertex $\eta(v)$ ($v \in V(\mathcal{G}_g)$) belongs to a distinct row of M. Consequently there are g connected subgraphs $C_1, \ldots, C_g$ of G, pairwise edge-disjoint and each with at least one edge, such that every pair of them have a vertex in common, and such that each C_i has non-empty intersection with at least g rows of M. At most k of them contain v_0, since v_0 has degree at most k and $C_1, \ldots, C_g$ are pairwise edge-disjoint; so we may assume that $C_1, \ldots, C_{4k}$ do not contain v_0.

Let L be as in the proof of 5.3; that is, its vertex set is $E(G)$, and two edges are adjacent in L if some vertex in X is incident with them both. We see that the edge sets of $C_1, \ldots, C_g$ form the vertex sets of g pairwise disjoint connected subgraphs of L, and for every two of them, some edge of L has ends in both.

(5) *There is no separation (A, B) of L of order less than $|\delta(X)|$, such that $\delta(X) \subseteq V(A)$, and $E(C_i) \subseteq V(B) \setminus V(A)$ for some $i \in \{1, \ldots, 4k\}$.*

For suppose that there is such a separation (A, B), and $E(C_1) \subseteq V(B) \setminus V(A)$, say. Let Q be the set of vertices of G incident with an edge of G in $V(B) \setminus V(A)$, and $P = V(G) \setminus Q$. Thus $v_0 \in P$, and C_1 is a subgraph of $G|Q$, and $\delta(P) \subseteq V(A \cap B)$. In particular

$$|\delta(P)| \leq |V(A \cap B)| < |\delta(X)| \leq k.$$

Since C_1 has non-empty intersection with at least g rows of M, and $g > k > |\delta(P)|$, it follows that some row of M is a subgraph of $G|Q$. Hence, since there are $b+3$ vertex-disjoint paths between every two rows of M, no row of M is a subgraph of $G|P$. Let D be the set of vertices in P incident with an edge in $\delta_G(P)$; thus $|D| \leq |\delta(Q)|$. Let (A', B') be the separation of G with $V(A' \cap B') = D$ and $A' = G|P$ and $V(B') = Q \cup D$. Thus (A', B') has order $|D| \leq |\delta(Q)|$, and so $(B', A') \notin \mathcal{T}$ since A' includes no row of M. Consequently $(A', B') \in \mathcal{T}$, and so there are at least $|W|/2$ edges of $\delta(Y)$ in $E(B')$, and hence with at least one end in Q. Now $|\delta(Y \cup Q)| \geq |\delta(X)| > |\delta(Q)|$ by hypothesis, and so 5.4 implies that $|\delta(Q \setminus Y)| \leq |\delta(Q)|$. But every edge in $\delta(Y)$ with at least one end in Q belongs to one of $\delta(Q)$, $\delta(Q \setminus Y)$, and so $|W|/2 \leq |\delta(Q)| + |\delta(Q \setminus Y)| \leq 2k$, a contradiction. This proves (5).

From (5) and 5.5 applied to L, it follows that $(G, v_0, D, 1)$ is feasible. This proves 5.7. ∎

6 Routing in a tree

Next we move on to subsets of type 2, and for that we first need a lemma that we prove in this section. Let T be a tree with vertex set $\{1, \dots, t\}$ say, fixed throughout this section. Let d be a $2t \times 2t$ symmetric matrix of non-negative integers, with $d_{ii} = 0$ for $1 \leq i \leq 2t$ and $d_{ij} = 0$ if $i, j > t$. We call such a matrix d a *stage*.

To explain what is going on, let G be a graph which is the cartesian product of T with an n-vertex path P. Thus H consists of n copies of T, with a matching between each copy and the next making copies of the same vertex adjacent. Let us add a new vertex v_0 adjacent to every vertex of the first copy T_1 and last copy T_n of T, forming G. We need to prove that if n is large enough then $V(G) \setminus \{v_0\}$ is p-porous. Thus, suppose that (G, v_0, D, p) is a demand system. The pairs of edges in $\delta(v_0)$ are of three kinds; both with ends in T_1, both with ends in T_n, and the pairs of edges one with an end in T_1 and the other with an end in T_n. It turns out to suffice to show feasibility if there are no pairs of the second type; so every pair has at least one edge with an end in T_1. To show this is feasible, we use the many copies of T to do a little bit of the routing at a time; from each copy of T we will only use one edge. Thus, we can achieve some small step towards the desired demand matrix using T_1, and then we have a slightly simpler demand matrix that is passed on to the second copy, and so on until it becomes trivial. These successive demand matrices are what we call the sequence of stages below, and the simplications we make using one copy of T are the "moves" described below.

Now let $p \geq 0$ be an integer. A stage d is *p-bounded* if $\sum_{1 \leq j \leq 2t} d_{ij} \leq p$ for $1 \leq i \leq 2t$. A *final stage* is a stage d with $d_{ij} = 0$ for all $i, j \in \{1, \dots, 2t\}$ with $|j - i| \neq t$.

Given an initial stage, we wish to transform it to a final stage by means of a sequence of intermediate stages, each obtained from the previous by a "move", and all of them p-bounded. Let d be a p-bounded stage. There are three *moves*, leading from d to a stage d', described next. In each case we choose $h, i \in \{1, \dots, t\}$, adjacent in T.

- **Exchange:** Choose distinct $j, k \in \{1, \ldots, 2t\} \setminus \{h, i\}$, and choose an integer $r > 0$ with $d_{hj} \geq r$ and $d_{ik} \geq r$. Let $d' = d$ except that

$$\begin{aligned} d'_{hj} = \ & d'_{jh} = & d_{hj} - r \\ d'_{ij} = \ & d'_{ji} = & d_{ij} + r \\ d'_{ik} = \ & d'_{ki} = & d_{ik} - r \\ d'_{hk} = \ & d'_{kh} = & d_{hk} + r. \end{aligned}$$

  We call this an (h, i, j, k)-*exchange move with value* r.

- **Extension:** Choose $j \in \{1, \ldots, 2t\}$, such that $j \neq h, i$. Choose an integer $r > 0$, with $d_{hj} \geq r$. Let $d' = d$ except that $d'_{hj} = d'_{jh} = d_{hj} - r$ and $d'_{ij} = d'_{ji} = d_{ij} + r$. Choose r such that d' is p-bounded (equivalently, such that $\sum_{1 \leq k \leq 2t} d'_{ik} \leq p$). We call this an (h, i, j)-*extension move with value* r.

- **Delivery:** Define $d' = d$ except that $d'_{hi} = d'_{ih} = 0$.

Under the first and third moves, the outcome d' is automatically p-bounded. It is easy to see (and not needed) that some sequence of moves will transform any initial stage to a final stage; but we care that the total number of moves is at most some function of $t = |V(T)|$, independent of p. (We will prove that $4t^3$ moves is enough.) Our method is, first we eliminate all the demands d_{ij} with $i, j < t$, paying no regard to the demands d_{ij} with $j > t$; and then we eliminate the latter demands, being careful not to reintroduce the former.

Let d be a p-bounded stage. We denote by B_d the set of $i \in \{1, \ldots, t\}$ such that $d_{ij} > 0$ for some $j \in \{1, \ldots, t\}$. We denote by S_d the minimal subtree of T including every vertex of B_d (or the null graph if $B_d = \emptyset$.) If $i \in B_d$, we denote by $S_d(i)$ the subtree of S_d formed by the union of the paths between v_i, v_j, for all $j \in \{1, \ldots, t\}$ with $d_{ij} > 0$. We begin with:

6.1 *Let d be a p-bounded stage, and let $i \in B_d$. Then there is a sequence of at most $2t$ moves that transforms d to a stage d' with $S_{d'} \subseteq S_d$, and such that either $i \notin B_{d'}$ or $S_{d'}(i)$ is a proper subtree of $S_d(i)$.*

Proof. Let us assume $i = 1$ for definiteness. Since $1 \in B_d$, there exists $j \in \{2, \ldots, t\}$ such that $d_{ij} > 0$; choose j such that the path of T between 1 and j is maximal, and let $j = 2$ say. If $1, 2$ are adjacent in T, then one delivery move achieves the desired outcome. Thus we assume that $1, 2$ are not adjacent; let 3 (say) be the vertex of T on the path between $1, 2$ adjacent to 2. Let there be $x(d)$ values of $j \in \{2, \ldots, 2t\}$ such that $d_{3,j} > 0$. We prove by induction on $x(d)$ that the desired stage d' can be achieved in at most $1 + x(d)$ moves.

Suppose that there exists $j \in \{2, \ldots, 2t\}$ with $d_{3,j} > 0$. Thus $j \neq 3$, and if $j = 2$ then a delivery move reduces $x(d)$ by one and the result follows. Thus we may assume that $j \neq 1, 2, 3$. Let $r = \min(d_{1,2}, d_{3,j})$. Let d' be obtained by applying a $(2, 3, 1, j)$-exchange move with value r. Now if $j \leq t$ then $j \in B_d$, and if $j > t$ then $j \notin B_{d'}$, so $B_{d'} \subseteq B_d \cup \{3\}$; and since 3 lies on the path between $1, 2$ and hence is a vertex of S_d, it follows that $S_{d'}$ is a subtree of S_d. Moreover, $S_{d'}(1)$ is a subtree of $S_d(1)$, for the same reason. If $r = d_{1,2}$, then 2 belongs to $S_d(1)$ and not to $S_{d'}(1)$ (because of the maximality of the path between $1, 2$), and so $S_{d'}(1)$ is a proper subtree of $S_d(1)$ as required.

If $r = d_{3,j} < d_{1,2}$, then $x(d)$ is reduced by one and the result follows from the inductive hypothesis on $x(d)$.

Thus we may assume that $x(d) = 0$. Let d' be obtained by applying a $(2,3,1)$-extension move with value $d_{1,2}$. We must check that d' is p-bounded, and it suffices to show that $\sum_{1 \le j \le 2t} d'_{3,j} \le p$. But $x(d) = 0$, so

$$\sum_{1 \le j \le 2t} d'_{3,j} = d'_{3,1} = d_{3,1} + d_{2,1} \le \sum_{1 \le j \le 2t} d_{j,1} \le p.$$

Thus d' is p-bounded. But $S_{d'}$ is a subtree of S_d, and $S_{d'}(1)$ is a subtree of $S_d(1)$, for the same reasons as before. Moreover 2 is not a vertex of $S_{d'}(1)$, so $S_{d'}(1)$ is a proper subtree of $S_d(1)$ as required.

This completes the inductive proof that $x(d) + 1$ moves suffice. Since $x(d) \le 2t - 1$, it follows that $2t$ moves suffice. This proves 6.1. ∎

We deduce:

6.2 *Let d be a p-bounded stage. If $B_d \neq \emptyset$, there is a sequence of at most $2t^2$ moves that transforms d to a stage d' with $|V(S_{d'})| < |V(S_d)|$.*

Proof. Since $B_d \neq \emptyset$, it follows that $|B_d| > 1$ and hence there is a vertex (i say) in B_d with a unique neighbour in S_d. Since $|V(S_d)| \le t$, it follows by at most t applications of 6.1 that there is a sequence of at most $2t^2$ moves that transforms d to a stage d' with $S_{d'} \subseteq S_d$ and $i \notin B_{d'}$. From the choice of i it follows that $i \notin V(S_{d'})$, and so $|V(S_{d'})| < |V(S_d)|$. This proves 6.2. ∎

Let us say that a stage d is *semifinal* if $d_{ij} = 0$ for all i, j with $1 \le i, j \le t$. By at most t applications of 6.2, we deduce:

6.3 *Let d be a p-bounded stage. There is a sequence of at most $2t^3$ moves that transforms d to a semifinal stage.*

A semifinal stage is *p-packed* if $\sum_{1 \le j \le 2t} d_{ij} = p$ for $1 \le i \le 2t$. Starting from a semifinal stage and finding a sequence of moves to a final stage is technically easier if we start from a p-packed stage, though in principle it contains the whole problem, since every p-bounded semifinal stage is "dominated" (defined below) by a p-packed stage. So we do p-packed stages first, and then reduce the general question to the p-packed case.

Given a p-packed semifinal stage d, let A_d be the set of all $i \in \{1, \dots, t\}$ such that $d_{i,i+t} < p$. Thus, $i \in A_d$ if and only if $d_{i,j+t} > 0$ for some $j \in \{1, \dots, t\}$, and also if and only if $d_{j,i+t} > 0$ for some $j \in \{1, \dots, t\}$. We denote by R_d the minimal subtree of T with vertex set including A_d (or the null graph if $A_d = \emptyset$); and for $i \in A_d$ we denote by $R_d(i)$ the minimal subtree of T that contains i and all $j \in \{1, \dots, t\} \setminus \{i\}$ such that $d_{j,i+t} > 0$.

6.4 *Let d be a p-packed semifinal stage, and let $i \in A_d$. Then there is a sequence of at most t exchange moves that transforms d to a p-packed semifinal stage d' with $R_{d'}$ a subtree of R_d, and such that either $i \notin A_{d'}$ or $R_{d'}(i)$ is a proper subtree of $R_d(i)$.*

Let us assume $i = 1$ for definiteness. Since $1 \in A_d$, there exists $j \in \{2, \dots, t\}$ such that $d_{j,1+t} > 0$; choose j such that the path in T between 1 and j is maximal, and let $j = 2$ say. Since $d_{2,1+t} > 0$, it follows that $d_{2,2+t} < p$ and so $2 \in A_d$. Let h be the vertex of T on the path between $1, 2$ adjacent to 2 (possibly $h = 1$). It follows that h is a vertex of R_d. Let there be $x(d)$ values of $k \in \{2, \ldots, t\}$ such that $d_{h,k+t} > 0$. We prove by induction on $x(d)$ that the desired stage d' can be achieved in at most $x(d) + 1$ moves.

Suppose first that $x(d) = 0$. Since d is p-packed, it follows that

$$p = \sum_{1 \le j \le t} d_{h,j+t} = d_{h,1+t} < d_{h,1+t} + d_{2,1+t} \le \sum_{1 \le j \le t} d_{j,1+t} = p,$$

a contradiction. Thus $x(d) > 0$, and so there exists $k \in \{2, \ldots, t\}$ such that $d_{h,k+t} > 0$. Thus $2, h, 1+t, k+t$ are all different; let d' be obtained from d by applying a $(2, h, 1+t, k+t)$-exchange move with value $r = \min(d_{h,k+t}, d_{2,1+t})$. Then d' is p-packed semifinal, and $A_{d'} \subseteq A_d \cup \{h\}$, and so $R_{d'}$ is a subtree of R_d. Moreover, $R_{d'}(1)$ is a subtree of $R_d(1)$, since h lies on the path of T between 1 and 2. If $r = d_{2,1+t}$, then 2 belongs to $R_d(1)$ and not to $R_{d'}(1)$, because of the maximality of the path between 1 and 2, so d' is the desired stage; while if $r = d_{h,k+t} < d_{2,1+t}$, then $x(d') = x(d) - 1$ and the result follows by induction.

This proves our claim that $x(d)$ moves suffice. Since $x(d) \le t - 1$, this proves 6.4. ∎

From 6.4 and the analogues of 6.2 and 6.3 we deduce:

6.5 *Let d be a p-packed semifinal stage. There is a sequence of at most t^3 moves that transforms d to a p-packed final stage.*

Now we use 6.5 to deduce a similar statement for p-bounded semifinal stages. If d, d' are stages, we say d *dominates* d' if $d_{ij} \ge d'_{ij}$ for all $i, j \in \{1, \ldots, 2t\}$. We need the following lemma.

6.6 *Let a, a' be p-packed semifinal stages, such that a' is obtained from a by an exchange move. For every semifinal stage d dominated by a, there is a semifinal stage d' dominated by a', such that d' can be obtained from d by a sequence of at most two moves.*

Proof. From the definition of exchange move, and since a is semifinal, there exist adjacent $h, i \in \{1, \ldots, t\}$ and distinct $j, k \in \{t+1, \ldots, 2t\}$, and an integer $r > 0$ with $a_{hj} \ge r$ and $a_{ik} \ge r$, such that

$$\begin{aligned} a'_{hj} = \quad a'_{jh} = &\quad a_{hj} - r \\ a'_{ij} = \quad a'_{ji} = &\quad a_{ij} + r \\ a'_{ik} = \quad a'_{ki} = &\quad a_{ik} - r \\ a'_{hk} = \quad a'_{kh} = &\quad a_{hk} + r. \end{aligned}$$

and otherwise $a' = a$. By exchanging h with i and j with k if necessary, we may assume that $d_{ik} \le d_{hj}$. Let $r' = \min(r, d_{ik})$; if $r' > 0$ let c be obtained from d by an (h, i, j, k)-exchange move of value r', and if $r' = 0$ let $c = d$.

We may assume that a' does not dominate c, for otherwise we may take $d' = c$; and consequently $a'_{uv} < c_{uv}$, where uv is one of the pairs hj, ij, ik, hk. Now $a'_{ij} = a_{ij} + r \ge d_{ij} + r' = c_{ij}$, so uv is not ij and similarly not hk. Consequently $a'_{uv} = a_{uv} - r$, and $c_{uv} = d_{uv} - r'$, and so $a_{uv} - r < d_{uv} - r'$. Since $a_{uv} \ge d_{uv}$, it follows that $r' < r$, and so

$$r' = d_{ik}.$$

We deduce that $c_{ik} = 0$, and therefore uv is not ik; and so uv is hj, and

$$a_{hj} - r = a'_{hj} < c_{hj} = d_{hj} - r'.$$

Consequently $r'' > 0$, where

$$r'' = d_{hj} - r' - a_{hj} + r.$$

Now $c_{hj} = d_{hj} - r' \geq r''$; let d' be obtained from c by an (h,i,j)-extension move of value r''. We claim that a' dominates d'. (We also have to show that d' is p-bounded, to check that it was constructed by a valid extension move; but this will follow if we show it is dominated by a', since a' is p-packed.) To show this it suffices to check that $d'_{xy} \leq a'_{xy}$ for all pairs $xy \in \{hj, ij, ik, hk\}$. The arguments are:

- $d'_{hj} = c_{hj} - r'' = d_{hj} - r' - r'' = a_{hj} - r = a'_{hj}$.
- $d'_{hk} = c_{hk} = d_{hk} + r' \leq a_{hk} + r = a'_{hk}$.
- $d'_{ij} = c_{ij} + r'' = d_{ij} + r' + r'' = d_{ij} + d_{hj} - a_{hj} + r \leq a_{ij} + r = a'_{ij}$.
- $d'_{ik} = c_{ik} = d_{ik} - r' = 0 \leq a'_{ik}$.

Thus a' dominates d', and this proves 6.6. ∎

We deduce:

6.7 *Let d be a p-bounded semifinal stage. There is a sequence of at most $2t^3$ moves that transforms d to a final stage.*

Proof. Choose a p-packed semifinal stage that dominates d, say $a(1)$. (It is easy to see that this exists.) By 6.5 there is a sequence $a(1), a(2), \ldots, a(n)$ for some $n \leq t^3 + 1$ of p-packed semifinal stages, each obtained from its predecessor by an exchange move, where $a(n)$ is final. Let $d(1) = d$. By 6.6, we may inductively define $d(2), \ldots, d(n)$, such that for each i, $d(i)$ is a p-bounded semifinal stage, $d(i)$ is dominated by $a(i)$, and $d(i)$ can be obtained from $d(i-1)$ by a sequence of at most two moves. In particular, $d(n)$ is final, since it is dominated by the final stage $a(n)$. This proves 6.7. ∎

Combining 6.3 and 6.7 yields the main result of this section:

6.8 *Let d be a p-bounded stage. Then there is a sequence of at most $4t^3$ moves that transforms d to a final stage.*

In the definition of an exchange move, we choose an integer r with certain properties. Let us call such a move a *small exchange move* if either $p = 1$ or $r \leq p/2$. Any exchange move can be replaced by a sequence of at most three small exchange moves; because if $p > 1$ then any number r with $1 \leq r \leq p$ is the sum of at most three numbers each between 1 and $p/2$. Let us say a *small move* is either an extension or delivery move, or a small exchange move. From 6.8 we deduce

6.9 *Let d be a p-bounded stage. Then there is a sequence of at most $12t^3$ small moves that transforms d to a final stage.*

Let the *(T,p)-height* of a p-bounded stage d be the minimum number of small moves that transform d to a final stage.

7 Bangles and porosity

So far, our demand systems have been based on one vertex v_0, and sometimes we obtain this vertex by identifying a set of vertices. In this section it becomes more convenient not to make the identification. Thus we would like to be able to speak of the subgraphs that would become cycles through v_0 if we identified into v_0 all the vertices in some subset, and that motivates the following definition.

Let G be a graph and $X \subseteq V(G)$. An *X-route* is a subgraph P of G such that either

- P is a path of G with at least two edges, its ends are in $V(G) \setminus X$ and every internal vertex is in X; or
- P is a cycle of G with at least two edges, and with exactly one vertex not in X.

A *pseudo-X-route* is a subgraph P of G such that $P|X$ is connected and non-null, and P is connected, and there are exactly two edges of P in $\delta_G(X)$. Thus every pseudo-X-route includes an X-route. The vertices of P not in X are called its *end(s)*, and the two edges of P in $\delta_G(X)$ are its *end-edges*.

If $X \subseteq V(G)$, an *X-demand system* is a pair (D, p), where D is a symmetric matrix of non-negative integers, with rows and columns indexed by $\delta(X)$, with zero diagonal, and with all row and columns sums at most p. Let $\mathcal{C}$ be the set of all X-routes. We say the X-demand system is *feasible* if there is a map $\phi : \mathcal{C} \to \mathbb{Z}_+$ such that

- for all $e, f \in \delta(X)$, $D_{e,f}$ equals the sum of $\phi(C)$ over all $C \in \mathcal{C}$ containing e, f
- for every edge e of G, the sum of $\phi(C)$ over all $C \in \mathcal{C}$ containing e is at most p

and we call such a map ϕ a *solution*. If instead we set $\mathcal{C}$ be to the set of all pseudo-X-routes, and ϕ satisfies the same conditions, we call ϕ a *pseudo-solution*. It is easy to see that there is a solution if and only if there is a pseudo-solution, since every pseudo-X-route includes an X-route, but sometimes it is easier to work with a pseudo-solution.

Let $t \geq 0$ and $n \geq 1$ be integers. A *t-bangle* in a graph G of *length* n is a sequence

$$(X_0, X_1, \ldots, X_n, X_{n+1})$$

of subsets of $V(G)$, together with a set $\{P_1, \ldots, P_t\}$ of paths of G, with the following properties:

- $X_0, X_1, \ldots, X_n, X_{n+1}$ are pairwise disjoint nonempty subsets of $V(G)$ with union $V(G)$
- for $1 \leq m \leq n$, $G|X_m$ is connected
- for $0 \leq m \leq n$, $|\delta(X_m, X_{m+1})| = t$
- for $0 \leq m < m' \leq n+1$, if $\delta(X_m, X_{m'}) \neq \emptyset$ then either $m' = m+1$ or $(m, m') = (0, n+1)$
- $P_1, \ldots, P_t$ are pairwise edge-disjoint
- for $1 \leq i \leq t$, P_i has one end in X_0 and the other in X_{n+1}, and has at least two edges, and all its internal vertices are in $X_1 \cup \cdots \cup X_n$.

We call $X_1 \cup \cdots \cup X_n$ the *interior* of the t-bangle. It follows that each P_i uses exactly one edge in $\delta(X_m, X_{m+1})$ for $0 \leq m \leq n$. In this section we prove the following.

7.1 *For all $s, t \geq 0$ and $p \geq 1$, if $\mathcal{B}$ is a t-bangle of length at least $24(s+1)t^{t+1}$ in a graph G, with interior W, and either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and at most s vertices in W have odd degree, or*
- *$p = 1$ and at most s edges of $G|W$ are not parallel to other edges,*

then W is p-porous.

Before we prove this we prove a weaker statement, and before that we need a few definitions. Let $(X_0, X_1, \ldots, X_n, X_{n+1})$ and $\{P_1, \ldots, P_t\}$ form a t-bangle in a graph G. For $1 \leq m \leq n$, let T_m be the graph with vertex set $\{1, \ldots, t\}$, in which distinct i, j are adjacent if there is a path P of $G|X_m$, with first vertex in $V(P_i)$ and last vertex in $V(P_j)$, such that for $1 \leq h \leq t$, at most one vertex of P is in P_h and no internal vertex of P is in P_h. (We accept the one-vertex path P, so i, j are adjacent if some vertex of X_m belongs to both P_i and P_j.) We call T_m the *graph of jumps at X_m*. Since $G|X_m$ is connected it follows easily that T_m is connected.

Up to this point in the paper, we have followed the convention that the entries in our matrix D are called d_{ij}. This not going to be true in what follows.

Again, let $(X_0, X_1, \ldots, X_n, X_{n+1})$ and $\{P_1, \ldots, P_t\}$ form a t-bangle in a graph G, with interior W. Let $\delta(W, X_0 \cup X_{n+1}) = \{e_1, \ldots, e_{2t}\}$, where $\delta(X_0, X_1) = \{e_1, \ldots, e_t\}$ and $\delta(X_n, X_{n+1}) = \{e_{t+1}, \ldots, e_{2t}\}$, numbered such that for $1 \leq i \leq t$, e_i and e_{t+i} are the first and last edges of P_i. Let $d_{ij} = D_{e_i, e_j}$ for $1 \leq i, j \leq 2t$. (Thus the matrix D is a function with domain $\delta(W) \times \delta(W)$, and the matrix d with entries d_{ij} for $i, j \in \{1, \ldots, 2t\}$ has domain $\{1, \ldots, 2t\} \times \{1, \ldots, 2t\}$.) If in addition $d_{ij} = 0$ for all $i, j > t$, then d is a p-bounded stage, and hence has a (T, p)-height, for any choice of tree T with $V(T) = \{1, \ldots, t\}$ and any integer $p \geq 1$.

7.2 *Let $k \geq 0$, and let $(X_0, X_1, \ldots, X_n, X_{n+1})$ and $\{P_1, \ldots, P_t\}$ form a t-bangle in a graph G. Let W be its interior, and let (D, p) be a W-demand system. Define the matrix d as above. Suppose that*

- *there is a tree T with vertex set $\{1, \ldots, t\}$ such that T is a subgraph of the graph of jumps at X_m for all $m \in \{1, \ldots, n\}$;*
- *if $p = 1$ then either every vertex in W has even degree, or every edge of $G|W$ is parallel to another edge;*
- *$D_{e,f} = 0$ for all $e, f \in \delta(X_n, X_{n+1})$*
- *n is at least the (T, p)-height of the p-bounded stage d.*

Then (D, p) is feasible.

Proof. We prove the result by induction on the (T, p)-height of d. Let $e_1, \ldots, e_{2t}$ be defined as above. If the height is zero, and d is therefore a final stage, then for $1 \leq i, j \leq 2t$, d_{ij} is nonzero only if $|j - i| = t$ and so a solution ϕ is given by defining $\phi(P_i) = d_{i,i+t}$ for $1 \leq i \leq t$, and $\phi(P) = 0$ for all other W-routes P. Thus we may assume that d has positive (T, p)-height. Hence there is a p-bounded stage d', obtained from d by one small move, with (T, p)-height smaller than that of d.

For $1 \le i \le t$, let f_i be the edge of P_i in $\delta(X_1, X_2)$, and let P'_i be the minimal subpath of P_i between X_1 and X_{n+1}. Thus P'_i has first edge f_i and last edge e_{t+i}. Let $W' = X_2 \cup \cdots \cup X_n$. For $1 \le i, j \le 2t$, let $a = f_i$ if $i \le t$, and e_i otherwise, and $b = f_j$ if $j \le t$, and e_j otherwise, and let $D'_{ab} = d'_{ij}$. Thus (D', p) is a W'-demand system. Now

$$(X_0 \cup X_1, X_2, \ldots, X_n, X_{n+1})$$

and the paths $\{P'_1, \ldots, P'_t\}$ form a k-bangle of length $n-1$, unless $n = 1$. From the inductive hypothesis if $n > 1$, and trivially if $n = 1$, it follows that the p-bounded W'-demand system (D', p) is feasible. Let ϕ' be a solution. Extending this solution to a solution for (D, p) depends on the small move that takes d to d'. We do the three possible small moves in reverse order, which is the order of increasing difficulty. For $1 \le i \le t$, let P''_i be the minimal subpath of P_i between X_0 and X_2.

For each small move, there are $h, i \in \{1, \ldots, t\}$, adjacent in T, and to save on variables let us assume that $h = 1$ and $i = 2$. So there is a path Q of $G|X_1$, with first vertex a in $V(P_1)$ and last vertex b in $V(P_2)$, with no other vertices in $P_1 \cup P_2$, and with no internal vertex and at most one end-vertex in P_i for $1 \le i \le t$. Let K_1, L_1 denote the minimal subpaths of P_1 between X_0 and a, and between a and X_2, respectively; and define K_2, L_2 similarly. Thus $P''_i = K_i \cup L_i$, for $i = 1, 2$.

If P' is a W'-route, its *natural extension* is the pseudo-W-route P obtained as follows: P is the union of P' and P''_i for each $i \in \{1, \ldots, t\}$ such that f_i is an end-edge of P'. (Note that this is only a pseudo-W-route since possibly both end-edges of P' lie in $\{f_1, \ldots, f_t\}$, say f_i, f_j, and P''_i and P''_j may not be vertex-disjoint.)

Case 1: delivery move. Thus $d'_{1,2} = d'_{2,1} = 0$, and otherwise $d' = d$. For each W'-route P' let P be its natural extension, and define $\phi(P) = \phi'(P')$. For the W-route $P = K_1 \cup Q \cup K_2$ define $\phi(P) = d_{1,2}$, and for all other pseudo-W-routes P let $\phi(P) = 0$. Then ϕ is a pseudo-solution.

Case 2: extension move. Thus there exist $j \in \{3, \ldots, 2t\}$ and an integer $r > 0$, with $d_{1,j} \ge r$. We have $d'_{1,j} = d'_{j,1} = d_{1,j} - r$ and $d'_{2,j} = d'_{j,2} = d_{2,j} + r$, and otherwise $d' = d$.

Let $g = f_j$ if $j \le t$, and $g = e_j$ if $j > t$. Thus, $g \ne f_1, f_2$. For each W'-route P' with an end-edge different from f_2, g, let P be its natural extension and define $\phi(P) = \phi'(P')$. Let $\mathcal{C}$ be the set of W'-routes containing f_2 and g. Since $d'_{2,j} \ge r$ and therefore $\sum_{P' \in \mathcal{C}} \phi'(P') \ge r$, it follows that we may choose non-negative maps $\phi_1, \rho : \mathcal{C} \to \mathbb{Z}_+$ such that $\phi_1(P') + \rho(P') = \phi'(P')$ for each $P' \in \mathcal{C}$, and $\sum_{P' \in \mathcal{C}} \rho(P') = r$. For each $P' \in \mathcal{C}$, let $\phi(P) = \rho(P')$, where $P = K_1 \cup Q \cup L_2 \cup P'$, and let $\phi(P) = \phi_1(P')$ where P is the natural extension of P'. Let $\phi(P) = 0$ for all other pseudo-W-routes P. Then this is a pseudo-solution.

Case 3: small exchange move. Thus there exist distinct $j, k \in \{3, \ldots, 2t\}$, and an integer $r > 0$, with $r \le p/2$ if $p > 1$, such that

$$\begin{aligned} d'_{1,j} = \; & d'_{j,1} = & d_{1,j} - r \\ d'_{2,j} = \; & d'_{j,2} = & d_{2,j} + r \\ d'_{2,k} = \; & d'_{k,2} = & d_{2,k} - r \\ d'_{1,k} = \; & d'_{k,1} = & d_{1,k} + r, \end{aligned}$$

and otherwise $d' = d$. Let $g_1 = f_k$ if $k \le t$, and $g_1 = e_k$ otherwise; and $g_2 = f_j$ if $j \le t$, and $g_2 = e_j$ otherwise. For $i = 1, 2$, let $\mathcal{C}_i$ be the set of all W'-routes with end-edges f_i, g_i. As in case two, for $i = 1, 2$ there exist ϕ_i, ρ_i such that $\phi_i(P') + \rho_i(P') = \phi'(P')$ for each $P' \in \mathcal{C}_i$, and $\sum_{P' \in \mathcal{C}_i} \rho_i(P') = r$. For $i = 1, 2$ and each $P' \in \mathcal{C}_i$, let $\phi(P) = \phi_i(P')$ where P is the natural extension of P'.

We claim that there is a path Q' of $G|X_1$ between the ends a, b of Q, edge-disjoint from $P_1, \ldots, P_t$, and edge-disjoint from Q if $p = 1$. If $p > 1$ or $a = b$ we may take $Q' = Q$, so we assume that $p = 1$ and $a \neq b$. Consequently either every vertex in W has even degree, or every edge of $G|W$ is parallel to another edge. If every edge of $G|X_1$ is parallel to another edge the claim is clear (because none of $P_1, \ldots, P_t$ has two vertices in Q), so we assume that every vertex in X_1 has even degree. Let H be the subgraph of G with vertex set X_1 and edge set all edges of $G|X_1$ that are not in any of $P_1, \ldots, P_k, Q$. Then every vertex of H has even degree except a, b, and so there is a path Q' of H between a, b. This proves our claim that Q' exists.

For each $P' \in \mathcal{C}_1$ let $\phi(P) = \rho_1(P')$ where $P = K_1 \cup Q \cup L_2 \cup P'$, and $\phi(P) = \phi_1(P')$ where P is the natural extension of P'; and for each $P' \in \mathcal{C}_2$ let $\phi(P) = \rho_1(P')$ where $P = K_2 \cup Q' \cup L_1 \cup P'$, and $\phi(P) = \phi_2(P')$ where P is the natural extension of P'. Let $\phi(P) = 0$ for all other pseudo-W-routes P. Then this is a pseudo-solution.

We have omitted the proofs that these maps ϕ are indeed pseudo-solutions, and perhaps it would help if we give some hints. We need to check that

- for all $e, f \in \delta(W)$, $D_{e,f}$ equals the sum of $\phi(C)$ over all $C \in \mathcal{C}$ containing e, f
- for every edge e of G, the sum of $\phi(C)$ over all $C \in \mathcal{C}$ containing e is at most p

where $\mathcal{C}$ is the set of all pseudo-W-routes. The first is easy, but the second is a little less obvious. (Let us call it the "capacity condition".) Nothing is changing for the edges in $G|W'$ and for the edges in $\delta(X_1, X_2)$; and for the edges in $\delta(X_0, X_1)$, they obey the capacity condition since (D, p) is a W-demand system. The edges in P''_i are, in all cases, used at most as much as the edges e_i, f_i, and so obey the capacity condition. (Note that in case 3, we were careful that all the edges in P''_i are used exactly the same number of times for $i = 1, 2$.) Finally, we have to check the edges of Q, and of the path Q' in case 3; but these obey the capacity condition because $r \leq p/2$ when Q, Q' are not edge-disjoint. This proves 7.2. ∎

Now we strengthen this to:

7.3 *Let $k \geq 0$, and let $(X_0, X_1, \ldots, X_n, X_{n+1})$ and $\{P_1, \ldots, P_t\}$ form a t-bangle in a graph G, with interior W. Suppose that*

- *there is a tree T with vertex set $\{1, \ldots, t\}$ such that T is a subgraph of the graph of jumps at X_m for all $m \in \{1, \ldots, n\}$;*
- *if $p = 1$ then either every vertex in W has even degree, or every edge of $G|W$ is parallel to another edge;*
- *$n \geq 24t^3$.*

Then W is p-porous.

Proof. Let (D, p) be a W-demand system; we must show that it is feasible. Let $m = 12t^3$, let $\delta(X_0, X_1) = \{e_1, \ldots, e_t\}$, let $\delta(X_m, X_{m+1}) = \{f_1, \ldots, f_t\}$, and let $\delta(X_n, X_{n+1}) = \{e_{t+1}, \ldots, e_{2t}\}$, where e_i, f_i, and e_{t+i} belong to P_i for $1 \leq i \leq t$. Let $W_1 = X_1 \cup \cdots \cup X_m$, and $W_2 = X_{m+1} \cup \cdots \cup X_n$. For $1 \leq i \leq t$, let $r_i = \sum_{t+1 \leq j \leq 2t} D_{e_i, e_j}$. Let A be the symmetric $\delta(W_1) \times \delta(W_1)$ matrix defined by:

- for $e, f \in \delta(X_0, X_1)$, $A_{ef} = D_{ef}$
- for $e, f \in \delta(X_m, X_{m+1})$, $A_{ef} = 0$
- for $1 \leq i, j \leq t$, $A_{e_i, f_j} = r_i$ if $i = j$, and otherwise $A_{e_i, f_j} = 0$

and let B be the symmetric $\delta(W_2) \times \delta(W_2)$ matrix defined by:

- for $e, f \in \delta(X_n, X_{n+1})$, $B_{ef} = D_{ef}$
- for $e, f \in \delta(X_m, X_{m+1})$, $B_{ef} = 0$
- for $1 \leq i \leq t$ and $t + 1 \leq j \leq 2t$, $B_{f_i, e_j} = D_{e_i, e_j}$.

Then (A, p) is a W_1-demand system, and (B, p) is a W_2-demand system. Since

$$(X_0, X_1, \dots, X_m, W_2 \cup X_{n+1})$$

and the appropriate subpaths of $P_1, \dots, P_t$ form a t-bangle of length $12t^3$, 6.9 and 7.2 imply that it is feasible. Similarly so is the W_2-demand system (B, p), using

$$(X_{n+1}, X_n, \dots, X_{m+1}, W_1 \cup X_0).$$

Combining the solutions in the natural way yields a solution for (D, p). This proves 7.3. ∎

Let $(X_0, X_1, \dots, X_n, X_{n+1})$ and $\{P_1, \dots, P_t\}$ form a t-bangle $\mathcal{B}$ in G, with $n \geq 2$. Let $0 \leq k \leq n$, and for $0 \leq m \leq n$ let

$$Y_m = \begin{cases} X_m \text{ if } m < k \\ X_m \cup X_{m+1} \text{ if } m = k \\ X_{m+1} \text{ if } m > k. \end{cases}$$

For $1 \leq i \leq t$, let P'_i be the minimal subpath of P_i between Y_0 and Y_n. (Thus $P'_i = P_i$ unless $k = 0$ or $k = n$.) Then $(Y_0, \dots, Y_n)$ and $\{P'_1, \dots, P'_k\}$ form a t-bangle. Any t-bangle obtained by repeating this process is called a *compression* of $\mathcal{B}$.

Now we can prove the main result of this section, 7.1, which we restate:

7.4 *For all $t, s \geq 0$ and $p \geq 1$, if $\mathcal{B}$ is a t-bangle of length at least $24(s+1)t^{t+1}$ in a graph G, with interior W, and either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and at most s vertices in W have odd degree, or*
- *$p = 1$ and at most s edges of $G|W$ are not parallel to other edges,*

then W is p-porous.

Proof. We observe, first, that

(1) *There is a compression $\mathcal{B}_1$ of $\mathcal{B}$ of length at least $24(s+1)t^3$, and a tree T with vertex set $\{1,\dots,t\}$, such that T is a subgraph of each graph of jumps of $\mathcal{B}_1$.*

Since each $G|X_m$ is connected, its graph of jumps has a spanning tree, with vertex set $\{1,\dots,t\}$; and since there are only t^{t-2} such trees, there is a tree T such that it is a spanning tree of the graph of jumps at X_m, for at least $24(s+1)t^3$ values of m with $1 \le m \le n$. But then (1) follows.

(2) *There is a compression $\mathcal{B}_2$ of $\mathcal{B}$ of length at least $24t^3$, with interior W_2 say, such that if $p=1$ then either every vertex in W_2 has even degree, or every edge in $G|W_2$ is parallel to another edge.*

If $p \neq 1$ the result is trivial, so we assume that $p=1$. Let $\mathcal{B}_1$ be $(X_0, X_1, \dots, X_n, X_{n+1})$ and $\{P_1,\dots,P_k\}$ say. For $1 \le m \le n$, let us say m is *exceptional* if some vertex in X_m has odd degree and some edge e is not parallel to another edge, where either e has both ends in X_m or e belongs to $\delta(X_m, X_{m+1})$. By hypothesis since $p=1$, there are at most s exceptional values of m; and so there are at least $(n-s)/(s+1) \ge 24t^3$ consecutive values of m that are not exceptional. But then the claim follows.

From 7.3, it follows that W_2 is p-porous. But then W is also p-porous (because there are $2t$ edge-disjoint paths between $\delta(W)$ and $\delta(W_2)$). This proves 7.4. ∎

8 Unavoidability

The t-bangles of the previous section will provide the promised subsets of type 2. It remains to prove that in every sufficiently large graph there is a subset of type 1 or 2, and thereby finish the proof of 2.2. The idea is, if the graph does not admit a carving of bounded width, we can easily obtain the appropriate robust subset, and if it does admit such a carving, and the graph is large enough, then the tree of the carving has a long path, which corresponds to a nested sequence of edge-cutsets of the graph, all of bounded cardinality and all different. We need to show that such a nested sequence of edge-cutsets always gives us a (sufficiently long) t-bangle with some bounded t.

Thus, if $n \ge 1$, a *chain of length* n in a graph G is a sequence $(X_1,\dots,X_n)$ of subsets of $V(G)$, such that:

- $X_1,\dots,X_n$ are pairwise disjoint and nonempty; and
- for $2 \le m \le n-1$, $\delta(X_0, X_m) = \emptyset$, where X_0 denotes $V(G) \setminus (X_1 \cup \dots \cup X_n)$.

We call $X_1 \cup \dots \cup X_n$ the *support* of the chain, and $X_2 \cup \dots \cup X_{n-1}$ is its *interior*. If in addition $t \ge 0$ is an integer and

- for $1 \le m < n$, $|\delta(X_1 \cup \dots \cup X_m, X_{m+1} \cup \dots \cup X_n)| \le t$

we call the chain a *t-chain*.

This differs significantly from a t-bangle. Not all the edge-cutsets are the same size, but more importantly, there may be edges between any pairs of terms X_i, X_j. We need to clean it up. Let us say a chain *encloses* another if the support of the second is a subset of the support of the first, and the interior of the second is a subset of the interior of the first. For $s \geq 2$, a t-chain is *s-strong* if it does not enclose any $(t-1)$-chain of length at least s. (Thus every 0-chain is s-strong, for all $s \geq 2$.)

First we observe that if we have a long enough t-chain, we can get a long s-strong t'-chain for some $t' \leq t$, with s as large as we like.

8.1 *Let $t \geq 0$, and for $0 \leq i \leq t$ let $n_i \geq 2$. Let $n_{-1} = 2$. Let $(X_1, \dots, X_n)$ be a t-chain of length at least n_t in a graph G. Then there exists i with $0 \leq i \leq t$, such that $(X_1, \dots, X_n)$ encloses an n_{i-1}-strong i-chain of length at least n_i.*

Proof. We proceed by induction on t. Let $(X_1, \dots, X_n)$ be a t-chain of length at least n_t. If it is n_{t-1}-strong then the result holds. If not, then $t > 0$, and it encloses a $(t-1)$-chain of length at least n_{t-1}, and the result follows by induction. This proves 8.1. ∎

We need several lemmas about s-strong t-chains. If $(X_1, \dots, X_n)$ is a t-chain, we say that $i \in \{1, \dots, n-1\}$ is *t-thin* if

$$|\delta(X_1 \cup \cdots \cup X_i, X_{i+1} \cup \cdots \cup X_n)| < t$$

and the number of such values of i is the *t-thinness* of the chain. If $(X_1, \dots, X_n)$ is a chain and $1 \leq i < n$, for $1 \leq m \leq n-1$, let

$$Y_m = \begin{cases} X_m \text{ if } m < i \\ X_i \cup X_{i+1} \text{ if } m = i \\ X_{m+1} \text{ if } m > i. \end{cases}$$

Then $(Y_1, \dots, Y_{n-1})$ is a t-chain enclosed by $(X_1, \dots, X_n)$, and we say it is obtained by *merging* X_i and X_{i+1}.

8.2 *Every s-strong t-chain has t-thinness at most $s-2$.*

Proof. Suppose not, and choose an s-strong t-chain with t-thinness at least $s-1$, with minimum length, say $(X_1, \dots, X_n)$. Thus $n \geq s \geq 2$. If some $i \in \{1, \dots, n-1\}$ is not t-thin, then merging X_i and X_{i+1} yields an s-strong t-chain with length $n-1$, still with t-thinness at least $s-1$, a contradiction. Thus every $i \in \{1, \dots, n-1\}$ is t-thin, and so $(X_1, \dots, X_n)$ is a $(t-1)$-chain of length $n \geq s$, contradicting that it is an s-strong t-chain. This proves 8.2. ∎

A chain $(X_1, \dots, X_n)$ is *t-linked* if

- it is a t-chain, and has t-thinness zero
- for $2 \leq i \leq n-1$ there are t edge-disjoint paths of G from $X_1 \cup \cdots \cup X_{i-1}$ to $X_{i+1} \cup \cdots \cup X_n$, each with all internal vertices in X_i.

8.3 *Every s-strong t-chain of length n encloses a t-linked chain of length at least $n/(s-1)$.*

Proof. Let us say a t-chain $(X_1, \ldots, X_n)$ is *refined* if for $2 \le i \le n-1$, there is no partition of X_i into two nonempty sets Y, Y' such that

$$(X_1, X_2, \ldots, X_{i-1}, Y, Y', X_{i+1}, \ldots, X_n)$$

is a t-chain. Evidently every t-chain encloses a refined t-chain with length at least as great. Thus it suffices to show that if $(X_1, \ldots, X_n)$ is a refined t-chain then it encloses a t-linked chain of length at least $n/(s-1)$. Let W be the support of this chain.

(1) *For $2 \le i \le n-1$, if $i-1$ and i are not t-thin, then there are t edge-disjoint paths of G from $X_1 \cup \cdots \cup X_{i-1}$ to $X_{i+1} \cup \cdots \cup X_n$, each with all internal vertices in X_i.*

For if not, then by Menger's theorem there is a partition (A, B) of W with $X_1, \ldots, X_{i-1} \subseteq A$ and $X_{i+1}, \ldots, X_n \subseteq B$, such that $|\delta(A, B)| < t$. Since $i-1$ is not t-thin it follows that $A \cap X_i \neq \emptyset$, and similarly $B \cap X_i \neq \emptyset$, contradicting that $(X_1, \ldots, X_n)$ is refined. This proves (1).

By 8.2, there are at most $s-2$ t-thin values of $i \in \{1, \ldots, n-1\}$; and so there are at least $(n-s+1)/(s-1)$ consecutive values of $i \in \{1, \ldots, n-1\}$ that are not t-thin, say $i, i+1, \ldots, i+r$ where $r = \lceil (n-s+1)/(s-1) \rceil$ - 1. But then

$$(X_1 \cup \cdots \cup X_i, X_{i+1}, \ldots, X_{i+r}, X_{i+r+1} \cup \cdots \cup X_n)$$

is a t-chain enclosed by $(X_1, \ldots, X_n)$, and we claim it is t-linked. It has t-thinness zero, from the choice of $i, \ldots, i+r$. Moreover if $j \in \{i+1, \ldots, i+r\}$, (1) implies that there are t edge-disjoint paths of G from $X_1 \cup \cdots \cup X_{j-1}$ to $X_{j+1} \cup \cdots \cup X_n$, each with all internal vertices in X_j. Consequently this t-chain is t-linked. But it has length $r + 2 \ge n/(s-1)$. This proves 8.3. ▮

8.4 *Let $(X_1, \ldots, X_n)$ be an s-strong t-chain. If $1 \le h < j \le n$, and $\delta(X_h, X_j) \neq \emptyset$, then $j - h \le s$.*

Proof. Choose $e \in \delta(X_h, X_j)$. The sequence $(X_h, X_{h+1}, \ldots, X_{j-1})$ is a chain of length $j-h$ enclosed by $(X_1, \ldots, X_n)$. Moreover, for $h \le m \le j-2$,

$$\delta(X_h \cup \cdots \cup X_m, X_{m+1} \cup \cdots \cup X_{j-1}) \subseteq \delta(X_1 \cup \cdots \cup X_m, X_{m+1} \cup \cdots \cup X_n);$$

and the inclusion is proper since e belongs to the set on the right of the inclusion and not to the one on the left. We deduce that the set on the left side has cardinality at most $t-1$, for all such m, and so $(X_h, X_{h+1}, \ldots, X_{j-1})$ is a $(t-1)$-chain. Since $(X_1, \ldots, X_n)$ is s-strong, we deduce that $j - h < s$. This proves 8.4. ▮

8.5 *Let $(X_1, \ldots, X_n)$ be an s-strong t-linked chain. Let $2 \le h < j \le n-1$ with $j - h \ge 2s - 2$; then there is a component D of $G|(X_h \cup \cdots \cup X_j)$ such that every edge in $\delta(X_1 \cup \cdots \cup X_{h-1}, X_h \cup \cdots \cup X_n)$ has an end in $V(D)$, and so does every edge in $\delta(X_1 \cup \cdots \cup X_j, X_{j+1} \cup \cdots \cup X_n)$.*

Proof. Let $A = X_1 \cup \cdots \cup X_{h-1}$, $B = X_h \cup \cdots \cup X_j$, and $C = X_{j+1} \cup \cdots \cup X_n$. By 8.4, $\delta(A, C) = \emptyset$, since $j-h > s-2$. Thus we must show that there is a component D of $G|B$ such that $\delta(V(D)) = \delta(B)$. Since $(X_1, \ldots, X_n)$ is t-linked, there are t edge-disjoint paths $P_1, \ldots, P_t$ of G from A to C, such that each has all internal vertices in B. Since $\delta(A, C) = \emptyset$, $P_1, \ldots, P_t$ each have an internal vertex in B. Since every edge of $\delta(A, B)$ and $\delta(B, C)$ belongs to one of $P_1, \ldots, P_t$, it suffices to show that the internal vertices of $P_1, \ldots, P_t$ all belong to the same component of $G|B$. Suppose not, and take a partition of B into two nonempty sets Y, Z, such that $\delta(Y, Z) = \emptyset$, and one of $P_1, \ldots, P_t$ has internal vertices in Y, and one of them has internal vertices in Z. For $h \leq i \leq j$, let $Y_i = Y \cap X_i$ and $Z_i = Z \cap X_i$. Now the sequence $(Y_h, Y_{h+1}, \ldots, Y_j)$ has $j - h + 1$ terms but it may not be a chain, since some of its terms may be empty. Let $\mathcal{Y}$ be the sequence obtained from $(Y_h, Y_{h+1}, \ldots, Y_j)$ by removing the empty terms, and define $\mathcal{Z}$ similarly from $(Z_h, \ldots, Z_j)$. Since for $h \leq i \leq j$, one of Y_i, Z_i is nonempty, it follows that the sum of the lengths of $\mathcal{Y}$, $\mathcal{Z}$ is at least $j - h + 1$, and so one of them has length at least $\lceil (j - h + 1)/2 \rceil \geq s$, and from the symmetry we may assume that $\mathcal{Y}$ has length at least s. Choose $k \in \{1, \ldots, t\}$ such that some internal vertex of P_k is in Z. It follows that no internal vertex of P_k belongs to Y. Now let $h \leq i < j$, and let e be the edge of P_k in $\delta(X_1 \cup \cdots \cup X_i, X_{i+1} \cup \cdots \cup X_n)$. Since neither end of e is in Y (because any end of e in B is an internal vertex of P_k and hence belongs to Z), it follows that $e \notin \delta(Y_h \cup \cdots \cup Y_i, Y_{i+1} \cup \cdots \cup Y_j)$. Thus $\delta(Y_h \cup \cdots \cup Y_i, Y_{i+1} \cup Y_j)$ is a proper subset of $\delta(X_1 \cup \cdots \cup X_i, X_{i+1} \cup \cdots \cup X_n)$; and hence has cardinality at most $t - 1$. This proves that $\mathcal{Y}$ is a $(t-1)$-chain of length at least s, a contradiction since $(X_1, \ldots, X_n)$ is s-strong. This proves 8.5. ∎

A chain $(X_1, \ldots, X_n)$ is *taut* if $\delta(X_i, X_j) = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ with $j \geq i + 2$, and $G|X_i$ is connected for $2 \leq i \leq n - 1$.

8.6 *Let $(X_1, \ldots, X_n)$ be an s-strong t-linked chain. Then it encloses a taut t-linked chain of length at least $\lfloor (n-2)/(2s) \rfloor$.*

Proof. Let $n' = \lfloor (n-2)/(2s) \rfloor$. For $1 \leq i \leq n'$, let

$$Y_i = \cup (X_m : 1 + 2s(i-1) < m \leq 1 + 2si).$$

Then $(Y_1, \ldots, Y_{n'})$ is a t-chain enclosed by $(X_1, \ldots, X_n)$. Moreover, 8.5 implies that for $1 \leq i \leq n'$, there is a component D_i of $G|Y_i$ such that $\delta(V(D_i)) = \delta(Y_i)$. But then $(V(D_1), \ldots, V(D_{n'}))$ is a taut t-linked chain enclosed by $(X_1, \ldots, X_n)$. This proves 8.6. ∎

But a taut t-linked chain is almost the same as a t-bangle. Putting these results together, we deduce:

8.7 *For all $s \geq 2$ and $t \geq 0$ and $m \geq 1$, if $(X_1, \ldots, X_n)$ is an s-strong t-chain in G, with $n \geq 2(s-1)(s(m+2)+1)$, then there is a t-bangle $(Y_0, Y_1, \ldots, Y_m, Y_{m+1})$ in G with interior a subset of the support of $(X_1, \ldots, X_n)$.*

Proof. By 8.3, $(X_1, \ldots, X_n)$ encloses a t-linked chain of length at least $n/(s-1) \geq 2s(m+2)+2$, which is therefore also s-strong. By 8.6, the latter chain encloses a taut t-linked chain of length $m+2$, say $(Y_1, \ldots, Y_{m+2})$. Let this have support W. Let $P_1, \ldots, P_t$ be edge-disjoint paths of G from Y_1 to Y_{m+2}, each with every internal vertex in $Y_2 \cup \cdots \cup Y_{m+1}$; then $(Y_1 \cup (V(G) \setminus W), Y_2, \ldots, Y_{m+2})$ together with $\{P_1, \ldots, P_t\}$ form a t-bangle of length m. This proves 8.7. ∎

Combining this with 8.1, we have:

8.8 *Let $t \geq 0$, and for $0 \leq i \leq t$ let $m_i \geq 1$. Then there exists n_t with the following property. Let $(X_1, \ldots, X_n)$ be a t-chain of length at least n_t in a graph G. Then there exists i with $0 \leq i \leq t$, such that there is an i-bangle of length m_i with interior a subset of the support of $(X_1, \ldots, X_n)$.*

Proof. For $0 \leq i \leq t$, let $n_i = 2(n_{i-1} - 1)(n_{i-1}(m_i + 2) + 1)$, where $n_{-1} = 2$. Now let $(X_1, \ldots, X_n)$ be a t-chain of length at least n_t in a graph G. By 8.1, there exists i with $0 \leq i \leq t$ such that $(X_1, \ldots, X_n)$ encloses an n_{i-1}-strong i-chain $(Y_1, \ldots, Y_{n_i})$ say, of length n_i. By 8.7, there is an i-bangle with length m_i in G with interior a subset of the support of $(Y_1, \ldots, Y_{n_i})$, and hence of the support of $(X_1, \ldots, X_n)$. This proves 8.8. ∎

Now back to critical demand systems. We have two kinds of contractible subsets: the edges within a robust set Y with $|\delta(Y)|$ sufficiently large (by 4.1), and the edges within the support of a t-bangle of sufficient length (by 7.1). We need to prove that one of them must be present in every sufficiently large graph. More precisely, we show the following:

8.9 *Let $0 \leq k \leq K$, and for $0 \leq i \leq K + k$ let $m_i \geq 1$. Then there exists N such that if G is a graph with at least N vertices, with $\Delta(G) \leq k$, and $v_0 \in V(G)$, then either*

- *there is a robust set $Y \subseteq V(G) \setminus \{v_0\}$ with $|\delta(Y)| \geq K$, or*
- *for some i with $0 \leq i \leq K + k$ there is an i-bangle in G with length m_i, such that v_0 is not in its interior.*

Proof. Let $t = K + k$. By 8.8, there exists M such that for every t-chain $(X_1, \ldots, X_n)$ of length at least M, there exists i with $0 \leq i \leq t$, such that there is an i-bangle of length m_i with interior a subset of the support of $(X_1, \ldots, X_n)$. Let $N = 2^{M-1} + 1$, and we claim that N satisfies the theorem.

For let G be a graph with at least N vertices and $\Delta(G) \leq k$. By 4.2, there is a (k, t)-optimal partial carving (T, ϕ) of (G, v_0), with root t_0 say. If there exists $v \in L(T) \setminus \{t_0\}$ with $|\phi^{-1}(v)| \geq 2$, then $|\delta(\phi^{-1}(v))| > t - k = K$ and setting $Y = \delta(\phi^{-1}(v))$ satisfies the first outcome of the theorem. Thus we may assume that (T, ϕ) is a carving of width at most t.

Since ϕ is a bijection, it follows that T has $|V(G)|$ leaves. Since every internal vertex of T has degree three, there is a path P of T with one end t_0 and with n edges, where $2^{n-1} \geq |V(G)| - 1 \geq N - 1 = 2^{M-1}$, and so $n \geq M$. Let $t_0, t_1, \ldots, t_n$ be the vertices of P in order. For $0 \leq i \leq n$, let T_i be the component containing t_i of the forest obtained from T by deleting all the edges of P; and let $X_i = \phi^{-1}(V(T_i))$.

Then each of the sets $X_1, \ldots, X_n$ is nonempty, since $T_1, \ldots, T_n$ each contain a leaf of T (because $t_1, \ldots, t_{n-1}$ all have degree three in T, and so does t_n unless it is a leaf). Consequently $(X_1, \ldots, X_n)$ is a chain of length n; it is a t-chain since (T, ϕ) has width at most t; and v_0 is not in its support since $v_0 \in X_0$. Since $n \geq M$, the choice of M implies that there exists i with $0 \leq i \leq t$, such that there is an i-bangle of length m_i with interior a subset of the support of $(X_1, \ldots, X_n)$. But then the second outcome of the theorem holds. This proves 8.9. ∎

Now at last we can prove our main result 2.2, which we restate, slightly modified:

8.10 *For all integers $k, s \geq 0$ there exists $N \geq 0$ with the following property. Let G be a graph with either $|V(G)| > N$ and $|E(G)| > k|V(G)|/2$, and let v_0 be a vertex of degree at most k, such that no vertex different from v_0 has degree zero. Then there is an edge e of G, such that for every demand system (G, v_0, D, p), if either*

- *$p \geq 2$ and $s = 0$, or*
- *$p = 1$ and G has oddness at most s, or*
- *$p = 1$ and G has skewness at most s,*

and $(G/e, v_0, D, p)$ is feasible, then (G, v_0, D, p) is feasible.

Proof. Choose $K > k$ to satisfy 4.1. Let $t = K + k$, let $m_0 = 1$, and for $1 \leq i \leq t$ let $m_i = 24(s+1)i^{i+2}$. Now let N satisfy 8.9; and we claim that it satisfies the theorem. For let G, v_0 be as in the theorem. If some vertex v of G has degree more than k, choose $Y \subseteq V(G)$ containing v and not v_0, with $|\delta(Y)|$ minimum. Since v_0 has degree at most k, it follows that $|\delta(Y)| \leq k$. By 3.2 $E(G|Y)$ is contractible (for all choices of D, p to make a demand system (G, v_0, D, p)), and it is non-empty since there are more than k edges incident with v, and not all of them belong to $\delta(Y)$. Any edge in $E(G|Y)$ satisfies the theorem.

We may therefore assume that $\Delta(G) \leq k$, and so $|E(G)| \leq k|V(G)|/2$, and hence $|V(G)| > N$. By 8.9, one of the two outcomes of 8.9 holds.

If there is a robust set $Y \subseteq V(G) \setminus \{v_0\}$ with $|\delta(Y)| \geq K$, then $E(G|Y)$ is nonempty (since $K > k$ and $\Delta(G) \leq k$), and contractible by 4.1 (for all choices of D, p), and again the theorem holds.

Thus we may assume that for some i with $0 \leq i \leq K + k$ there is a i-bangle in G with length m_i, such that v_0 is not in its support. Suppose first that $i > 0$. Then for all $p \geq 2$ its support is p-porous by 7.1, and hence is contractible (for all choices of D, p); and since some edge has both ends in this support (since $i > 0$) the theorem holds. Thus we may assume that $i = 0$, and so there is a 0-bangle, say $(X_0, \ldots, X_{r+1})$, with v_0 not in its interior. Now $r \geq 1$ from the definition of a t-bangle, and $\delta(X_1) = 0$, and so X_1 is contractible (for all choices of D, p). But G has no vertex of degree zero different from v_0, and therefore $G|X_1$ has an edge, and the theorem holds. This proves 8.10. ∎

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