

Induced subgraphs of graphs with large chromatic number.
VII. Gyárfás' complementation conjecture

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Abstract

A class of graphs is χ -*bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every induced subgraph G of every graph in the class, where χ, ω denote the chromatic number and clique number of G respectively. In 1987, Gyárfás conjectured that for every c , if \mathcal{C} is a class of graphs such that $\chi(G) \leq \omega(G) + c$ for every induced subgraph G of every graph in the class, then the class of complements of members of \mathcal{C} is χ -bounded. We prove this conjecture. Indeed, more generally, a class of graphs is χ -bounded if it has the property that no graph in the class has $c + 1$ odd holes, pairwise disjoint and with no edges between them. The main tool is a lemma that if C is a shortest odd hole in a graph, and X is the set of vertices with at least five neighbours in $V(C)$, then there is a three-vertex set that dominates X .

1 Introduction

All graphs considered are finite and simple. The chromatic number of G is denoted by $\chi(G)$ and the cardinality of the largest clique by $\omega(G)$. A class \mathcal{C} of graphs is χ -bounded if there is a function f such that $\chi(H) \leq f(\omega(H))$ for all $G \in \mathcal{C}$ and every induced subgraph H of G . A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G ; and the class of perfect graphs is the primordial χ -bounded class. Lovász [5] proved that the complement of every perfect graph is perfect, and it is natural to ask whether an analogue of Lovász's theorem holds for other χ -bounded classes. Not always; for instance the class of all triangle-free graphs is not χ -bounded, yet the class of their complements is.

Let f be a function from the set of nonnegative integers into itself. We say that f is a *binding function* for a graph G if $\chi(H) \leq f(\omega(H))$ for every induced subgraph H of G . Let \mathcal{C}_f be the class of graphs G such that f is a binding function for G . We say f has a *complementary binding function* if the class of complements of members of \mathcal{C}_f is χ -bounded. Thus, Lovász's theorem implies that the function $f(x) = x$ has a complementary binding function, and we might ask which other functions do.

A. Gyárfás [4] proved that if f has a complementary binding function then $\inf f(x)/x = 1$, and proposed as a conjecture (conjecture 6.3 of [4]) that for all c the function $f(x) = x + c$ has a complementary binding function. That is our main result:

1.1 *Let $c \geq 0$, and let \mathcal{C} be a class of graphs such that $\chi(H) \leq \omega(H) + c$ for all $G \in \mathcal{C}$ and every induced subgraph H of G . Let \mathcal{C}' be the class of complements of members of \mathcal{C} . Then \mathcal{C}' is χ -bounded.*

Incidentally, one might wonder whether any classes with binding function $f(x) = x + c$ exist; but there are several of interest. For instance:

- perfect graphs have binding function $f(x) = x$;
- line graphs have a binding function $f(x) = x + 1$, by Vizing's theorem;
- the intersection graph of a set of Jordan curves in the plane, pairwise intersecting in at most one point, has a binding function $f(x) = x + 327$, by a theorem of Cames van Batenburg, Esperet and Müller [2]; and
- Esperet, Gonçalves and Labourel [3] conjecture that for some c , $f(x) = x + c$ is a binding function for intersection graphs of sets of non-crossing curves in the plane, pairwise intersecting in at most one point.

A *hole* in a graph G is an induced cycle of length at least four, and an *antihole* in G is a hole in the complement graph \overline{G} . Two induced subgraphs of G are *complete* if they are vertex-disjoint and every vertex of the first is adjacent in G to every vertex of the second; and they are *anticomplete* if they are disjoint and there are no edges of G joining them. If G is a graph such that $\chi(H) \leq \omega(H) + c$ for every induced subgraph H of G , then there do not exist $c + 1$ odd antiholes in G , pairwise complete. Thus the following (applied to \mathcal{C}') implies 1.1.

1.2 *Let $c \geq 0$, and let \mathcal{C} be a class of graphs G such that there do not exist $c + 1$ odd holes in G , pairwise anticomplete. Then \mathcal{C} is χ -bounded.*

Vaidy Sivaraman points out (private communication) that 1.2 also implies another conjecture of Gyárfás (conjecture 6.8 of [4]), the following, where $\alpha(H)$ denotes the cardinality of the largest stable subset of H :

1.3 *Let \mathcal{C} be the class of graphs G such that $\alpha(H)\omega(H) \geq |V(H)| - 1$ for every induced subgraph H of G . Then \mathcal{C} is χ -bounded.*

This is immediate from 1.2 since if H is the disjoint union of two odd cycles of length at least four, then $\alpha(H)\omega(H) = |V(H)| - 2$.

Let C be an odd hole in G . If C has minimum length among all odd holes in G we say C is a *shortest* odd hole. We denote the set of vertices in $V(G) \setminus V(C)$ with at least one neighbour in $V(C)$ by $N(C)$. If $X \subseteq V(G)$, we sometimes speak of $\chi(X)$ meaning $\chi(G[X])$. As we shall see, the next result implies 1.2:

1.4 *Let G be a graph such that for every vertex, its set of neighbours has chromatic number at most τ ; and let C be a shortest odd hole in G . Then $N(C)$ has chromatic number at most 21τ .*

We will also need the main result of [6], that:

1.5 *Let G be a graph with no odd hole. Then $\chi(G) \leq 2^{2^{\omega(G)+2}}$.*

Proof of 1.2, assuming 1.4. Define $\tau(\kappa, c) = \kappa$ for $\kappa = 0, 1$. For $\kappa \geq 2$ let $\tau(\kappa, 0) = 2^{2^{\kappa+2}}$, and let $\tau(\kappa, c) = \tau(\kappa, c-1) + 21\tau(\kappa-1, c)$ for $c > 0$. We prove by induction on $\kappa + c$ that $\chi(G) \leq \tau(\kappa, c)$ for every graph G with $\omega(G) \leq \kappa$ and with no $c+1$ odd holes, pairwise anticomplete. This is true if $\kappa \leq 1$, so we may assume that $\kappa \geq 2$. The claim is also true if $c = 0$, by 1.5, so we may assume that $c > 0$. Let G be a graph with $\omega(G) \leq \kappa$ and with no $c+1$ odd holes, pairwise anticomplete, and suppose that $\chi(G) > \tau(\kappa, c)$. For every vertex v of G , the set of neighbours of v in G has clique number at most $\kappa - 1$, and hence has chromatic number at most $\tau(\kappa - 1, c)$. Since $\tau(\kappa, c) \geq \tau(\kappa, 0) = 2^{2^{\kappa+2}}$, there is an odd hole in G by 1.5; choose a shortest odd hole C . Let G' be the graph obtained from G by deleting $N(C)$. By 1.4, $N(C)$ has chromatic number at most $21\tau(\kappa - 1, c)$; and so

$$\chi(G') \geq \chi(G) - 21\tau(\kappa - 1, c) > \tau(\kappa, c) - 21\tau(\kappa - 1, c) = \tau(\kappa, c - 1).$$

Since $\chi(G') > 3 = \chi(C)$ and C is a component of G' , it follows that $G' \setminus V(C)$ also has chromatic number more than $\tau(\kappa, c - 1)$. But $G' \setminus V(C)$ does not contain c pairwise anticomplete odd holes, a contradiction. This proves 1.1. ■

Thus, it remains to prove 1.4, and we will tackle this in the next two sections. Another way prove 1.2 is to prove directly the following strengthening of 1.5 (which also follows from 1.4):

1.6 *For all $\kappa, \tau \geq 0$ there exists $\tau' \geq 0$ such that if G is a graph with $\omega(G) \leq \kappa$ and $\chi(G) > \tau'$, then there is an odd hole C in G such that if X denotes the set of vertices of G that are not in C and have no neighbour in C , then $\chi(X) > \tau$.*

This can be proved by modifying slightly the proof of 1.5; we check that throughout the proof of 1.5, whenever we find an odd hole, it can be chosen such that the set of vertices with no neighbours in it still has large chromatic number. It follows that in every graph G with $\omega(G) \leq \kappa$ and $\chi(G)$ sufficiently large, there exist many odd holes pairwise anticomplete. To see this, choose an odd hole by 1.6, such that when we delete it and its neighbours the chromatic number stays large, and repeat many times. And so 1.2 would follow.

However, the proof via 1.4 is numerically better, and 1.4 may be of some independent interest. For instance, the difficult part of the proof of correctness of the algorithm to test whether a graph is Berge, given in [1], involved a closely related question which we discuss later.

2 C -minor vertices

Let C be a shortest odd hole in G . Let us say a vertex $v \in N(C)$ is C -minor if its neighbours in C all belong to some path of C of length two, and C -major otherwise. In this section we handle the C -minor vertices.

Let v be a C -minor vertex, and choose a minimal path of C of length at most two containing all neighbours of v in C . This path might have length zero, one or two, and we call this length the *type* of v . We need the following lemma (its proof is trivial since all vertices of K have degree at most $2|I|$):

2.1 *Let $I \subseteq \{1, \dots, n\}$, and let K be a graph with vertex set $\{1, \dots, n\}$, such that for all $i, j \in \{1, \dots, n\}$ with $i < j$, if i, j are adjacent in K then either $j - i \in I$ or $i + n - j \in I$. Then $\chi(K) \leq 2|I| + 1$.*

We deduce

2.2 *Let G be a graph such that for every vertex, its set of neighbours has chromatic number at most τ , and let C be a shortest odd hole in G . The set of C -minor vertices of type zero induces a subgraph with chromatic number at most 5τ .*

Proof. Let C have vertices c_1, \dots, c_n in order; and let X_i be the set of all C -minor vertices of type zero that are adjacent to c_i , for $1 \leq i \leq n$. It follows that $\chi(X_i) \leq \tau$ for $1 \leq i \leq n$. Let $1 \leq i < j \leq n$, and suppose that $u \in X_i$ is adjacent in G to $v \in X_j$, and c_i, c_j are not adjacent. One of the paths of C between c_i, c_j has even length, and its union with the path $c_i-u-v-c_j$ is an induced odd hole C' ; and since C is an odd hole of minimum length, C' has length at least that of C . Consequently the odd path of C between c_i, c_j has length three (if c_i, c_j are not adjacent) or one; and so either $j - i = 1$, or $j - i = 3$, or $i + n - j = 1$, or $i + n - j = 3$. Since this holds for all i, j , 2.1 (with $I = \{1, 3\}$) implies that the union of the sets X_i has chromatic number at most 5τ . This proves 2.2. ■

2.3 *Let G be a graph such that for every vertex, its set of neighbours has chromatic number at most τ , and let C be a shortest odd hole in G . The set of C -minor vertices of type one induces a subgraph with chromatic number at most 5τ .*

Proof. Let C have vertices c_1, \dots, c_n in order; and let X_i be the set of all C -minor vertices of type one that are adjacent to c_i, c_{i+1} , for $1 \leq i \leq n$, where c_{n+1} means c_1 . It follows that $\chi(X_i) \leq \tau$ for

$1 \leq i \leq n$. Let $1 \leq i < j \leq n$, and suppose that $u \in X_i$ is adjacent in G to $v \in X_j$. Let P be the path with vertices $c_{i+1} \cdots c_j$, and let Q be the path with vertices $c_{j+1} \cdots c_n c_1 \cdots c_i$. One of P, Q has odd length and one is even. If Q is odd then the union of P with the path $c_{i+1} u v c_j$ is an induced odd cycle C' . Thus either C' has length three or it has length at least that of C . In the first case $j - i = 1$, and in the second Q has length one, that is, $i + n - j = 2$. Similarly if P is odd then either $j - i = 2$ or $i + n - j = 1$. By 2.1 (with $I = \{1, 2\}$) the result follows. \blacksquare

2.4 *Let G be a graph such that for every vertex, its set of neighbours has chromatic number at most τ , and let C be a shortest odd hole in G . The set of C -minor vertices of type two induces a subgraph with chromatic number at most 5τ .*

Proof. Let C have vertices c_1, \dots, c_n in order; and for $1 \leq i \leq n$ let X_i be the set of all C -minor vertices of type two that are adjacent to c_{i-1} and to c_{i+1} (and possibly to c_i), where c_{n+1} means c_1 and c_{-1} means c_{n-1} . It follows that $\chi(X_i) \leq \tau$ for $1 \leq i \leq n$. Let $1 \leq i < j \leq n$, and suppose that $u \in X_i$ is adjacent in G to $v \in X_j$. It might be that $j = i + 1$ or $i + n = j + 1$; suppose not. Let P be the path with vertices $c_{i+1} \cdots c_{j-1}$, and let Q be the path with vertices $c_{j+1} \cdots c_n c_1 \cdots c_{i-1}$. One of P, Q has odd length and one is even. If Q is odd then the union of P with the path $c_{i+1} u v c_{j-1}$ is an induced odd cycle C' , and C' has length less than C . Thus C' has length three, that is, $j - i = 2$. Similarly if P is odd then $i + n - j = 2$. Thus we have shown that either $j - i = 1$, or $i + n - j = 1$, or $j - i = 2$, or $i + n - j = 2$. By 2.1 (with $I = \{1, 2\}$), the result follows. \blacksquare

Adding, we deduce from 2.2, 2.3 and 2.4 that:

2.5 *Let G be a graph such that for every vertex, its set of neighbours has chromatic number at most τ , and let C be a shortest odd hole in G . The set of C -minor vertices induces a subgraph with chromatic number at most 15τ .*

3 C -major vertices

If u, v are C -major vertices, we say they form a *skew pair* if they are nonadjacent, each has exactly three neighbours in C , and there is an odd path of C with vertices c_1, \dots, c_n in order, where $n \geq 6$, such that u is adjacent to c_1, c_2, c_{n-2} , and v to c_3, c_{n-1}, c_n (and hence u, v have no common neighbour in C). If $X \subseteq V(G)$, we say that Y *dominates* X if every vertex in X either belongs to Y or has a neighbour in Y . We will prove:

3.1 *Let C be a shortest odd hole in a graph G , and let X be the set of C -major vertices.*

- *If some pair of vertices in X is a skew pair, then there is a set of six vertices that dominates X .*
- *If there is no skew pair in X , but some vertex in X has at most four neighbours in $V(C)$, then there is a set of five vertices that dominates X .*
- *If every vertex in X has at least five neighbours in $V(C)$, then there is a set of three vertices that dominates X .*

Proof of 1.4, assuming 3.1. Let G, τ be as in 1.4, and let X be the set of C -major vertices. By 2.5, the set of C -minor vertices has chromatic number at most 15τ ; and by 3.1, the set of C -major vertices has chromatic number at most 6τ . Consequently, $N(C)$ has chromatic number at most 21τ , as required. \blacksquare

Some of the results of this section are related to results of [1]. In particular, one of the goals of [1] was to find a small set of vertices which together dominated all C -major vertices, where C is a shortest odd hole. There are some differences: for instance, in [1], it was sometimes assumed that the graph contained none of a list of “easily detectable subgraphs”, which we cannot, and in [1] the goal was to give (in polynomial time) a list of subsets of $V(G)$ guaranteed to include the set of C -major vertices (without knowledge of C), which we do not need. Nevertheless, some of the proofs here are derived from proofs in [1].

Let C be a cycle, and let $A \subseteq V(C)$. An A -gap is a subgraph of C composed of a component X of $C \setminus A$, the vertices of A with neighbours in X , and the edges between A and X . (So if some two vertices in A are nonadjacent, the A -gaps are the paths of C of length ≥ 2 , with both ends in A and no internal vertex in A .) If $A, B \subseteq V(C)$, an (A, B) -gap is a path P of C between a, b say, such that a is the unique vertex of P in A and b is the unique vertex of P in B . (Possibly $a = b$ and P has length 0.) The *length* of an A -gap or (A, B) -gap is the number of edges in it (so if A consists just of two adjacent vertices, there is a single A -gap, of length $|E(C)| - 1$). Every A -gap has length at least two, but an (A, B) -gap can have length 0 or 1 as well. We speak of cycles, A -gaps, and (A, B) -gaps being *odd* or *even* meaning that they have an odd (or even, respectively) number of edges. We say that A is *normal* in C if every A -gap is even. An A -edge is an edge of C with both ends in A . Thus every edge of C is either an A -edge or belongs to a unique A -gap (and not both).

To prove 3.1 we first need the following.

3.2 *Let C be an odd cycle, and let $A, B \subseteq V(C)$ be normal. Then every odd (A, B) -gap is disjoint from every even one. Moreover, if $A \cap B$ is not normal, then there is an odd (A, B) -gap and an even one.*

Proof. Let P, Q be odd and even (A, B) -gaps respectively, and suppose that there exists $v \in V(P \cap Q)$. Then v is an end of both P, Q , and belongs to one of A, B , say A ; but then $P \cup Q$ is an odd B -gap, a contradiction. This proves the first assertion of the theorem.

Now we assume that $A \cap B$ is not normal, and so there is an odd $A \cap B$ -gap R say. Every edge of R is either an A -edge or belongs to an A -gap, and since R is odd and all A -gaps are even it follows that an odd number of edges of R are A -edges. Since every A -edge of R belongs to a B -gap, some B -gap P included in R contains an odd number of A -edges. Since P is even, there is an odd number of edges of P that are not A -edges. Each such edge belongs either to a A -gap included in R , or to an (A, B) -gap included in R . The number of edges of R that belong to A -gaps included in R is even, since all A -gaps are even; so an odd number of edges of R belong to (A, B) -gaps included in R . Since there are exactly two (A, B) -gaps included in R , one of them is odd and one is even. This proves 3.2. \blacksquare

3.3 *Let C be a shortest odd hole in G , and let u, v be nonadjacent C -major vertices. Let A, B be the sets of neighbours in $V(C)$ of u, v respectively. Then either:*

- $A \cap B$ is normal; or
- u, v form a skew pair; or
- $A \cap B \neq \emptyset$ and $\min(|A|, |B|) = 3$; or
- there is a path $c_1 \cdots c_5$ of C such that $\{c_1, \dots, c_4\}$ is the set of neighbours of u in $V(C)$, and $\{c_2, \dots, c_5\}$ is the set of neighbours of v in $V(C)$.

Proof. Since u, v are C -major and C is a shortest odd hole it follows that A, B are normal. We will show that we may assume that:

(1) If P is an odd (A, B) -gap and Q is an even (A, B) -gap then $V(P \cap Q) = \emptyset$, and there is an edge e of C joining an end of P with an end of Q , and e is either an A -edge or a B -edge.

By 3.2, $V(P \cap Q) = \emptyset$. Suppose first that P, Q are anticomplete. Then the subgraph induced on $V(P \cup Q) \cup \{u, v\}$ is an odd hole, and so its length is at least that of C ; and hence at most two vertices of C do not belong to $P \cup Q$. Thus we can number the vertices of C in order as c_1, \dots, c_n where P is the subpath $c_1 c_2 \cdots c_r$, and Q is the subpath $c_{r+2} \cdots c_{n-1}$ (where $2 \leq r \leq n-3$, and possibly $r = n-3$). Since u, v both have only two neighbours in $P \cup Q$ it follows that $|A|, |B| \leq 4$. First suppose that not both c_{r+1}, c_n belong to $A \cap B$. Hence $\min(|A|, |B|) = 3$, and so we may assume that $A \cap B = \emptyset$, and therefore u, v are each adjacent to exactly one of c_{r+1}, c_n . From the symmetry we may assume that $c_{n-1} \in A$, and hence $c_{r+2} \in B$. Consequently $c_{r+1} \notin A$ since $c_{r+1} \cdots c_{n-1}$ is not an odd A -gap. Thus $c_{r+1} \in B$, and $c_n \in A$. Since u is C -major, $A \not\subseteq \{c_{n-1}, c_n, c_1\}$; and so $c_r \in A$ and $c_1 \in B$, and so u, v form a skew pair (the path $c_{n-1} c_n c_1 \cdots c_r c_{r+1} c_{r+2}$ corresponds to the path numbered $c_1 \cdots c_n$ in the definition of a skew pair).

We may therefore assume that $c_{r+1}, c_n \in A \cap B$. By exchanging A, B if necessary, we may assume that $c_{r+2} \in B$, and since $c_{r+2} \cdots c_n$ is not an odd B -gap, it follows that $r+2 = n-1$, and so $c_{n-1} \in A \cap B$. But then the last outcome of the theorem holds.

Thus we may assume that P, Q are not anticomplete. Hence some end p of P is adjacent to some end q of Q . If one of p, q is in $A \setminus B$ and the other in $B \setminus A$, then Q has positive length and the subgraph induced on $V(Q) \cup \{p\}$ is either an odd A -gap or an odd B -gap, in either case a contradiction. So one of A, B contains both of p, q . This proves (1).

We may assume that $A \cap B$ is not normal. By 3.2, there is an odd (A, B) -gap P and an even one Q . From (1), we may number the vertices of C as c_1, \dots, c_n in order such that P is $c_1 \cdots c_i$, and Q is $c_{i+1} \cdots c_j$, and one of A, B contains both of c_i, c_{i+1} . We may assume, exchanging A, B if necessary, that $c_1, c_j \in A$ and $c_i, c_{i+1} \in B$. (Possibly $i+1 = j$.) Now one of $c_{j+2}, \dots, c_n \in B$; for otherwise $c_{j+1} \in B$ (since $|B| \geq 3$), and $i+2 \neq j$ (since v is C -major), and $c_{i+1} \cdots c_{j+1}$ is an odd B -gap, which is impossible. Since $c_1 \in A \setminus B$, there is an (A, B) -gap S (possibly of length zero) that is a subpath of the path $c_{j+2} \cdots c_n c_1$. Now S is anticomplete to Q , since $c_i, c_{j+1} \notin V(S)$; so by (1) applied to S, Q , it follows that S is even. By (1) applied to P, S , $c_1 \notin V(S)$, and c_n is an end of S , and $c_n \in A$.

Suppose that both A, B have nonempty intersection with $\{c_{i+2}, \dots, c_{n-1}\}$. Then there is an (A, B) -gap R say with vertex set a subset of this set. Since R is anticomplete to P , (1) implies that R is odd. By (1), R is disjoint from both Q, S , and the union of the vertex sets of P, Q, R, S equals

$V(C)$, which is impossible since C is odd. Thus not both A, B have nonempty intersection with $\{c_{i+2}, \dots, c_{n-1}\}$. This implies that $\min(|A|, |B|) = 3$, and also that one of Q, S has length zero, so $A \cap B \neq \emptyset$ and the theorem holds. This proves 3.3. \blacksquare

We deduce:

3.4 *Let C be a shortest odd hole in a graph G , and let u, v be C -major vertices. Then either u, v are adjacent, or they form a skew pair, or some vertex in C is adjacent to both u, v .*

Proof. Let A, B be the sets of neighbours in $V(C)$ of u, v respectively. Thus, A, B are both normal, since C is a shortest odd hole and u, v are C -major. Suppose that u, v are nonadjacent and $A \cap B = \emptyset$ and u, v do not form a skew pair. In particular, $A \cap B$ is not normal, contrary to 3.3. This proves 3.4. \blacksquare

Next we need the following (compare theorem 7.6 of [1]):

3.5 *Let C be a shortest odd hole in G , and let u, v be nonadjacent C -major vertices, both with at least five neighbours in $V(C)$. Let A, B be the sets of neighbours of u, v in $V(C)$ respectively. Then every (A, B) -gap is even.*

Proof. Suppose that P is an odd (A, B) -gap. Let C have vertices c_1, \dots, c_n in order, where P is $c_1 \cdots c_r$, and r is even, with $2 \leq r < n$.

(1) *If Q is an even (A, B) -gap, then $V(P \cap Q) = \emptyset$ and there is an edge between $V(P), V(Q)$.*

For $P \cap Q$ is empty, by 3.2, so we assume there are no edges between P and Q . Then their union, together with u, v , forms an odd hole C' . Since $|A| \geq 5$, there are at least three vertices of C that do not belong to $P \cup Q$, and so C' is shorter than C , a contradiction. This proves (1).

(2) *There is an even (A, B) -gap.*

For if $A \cap B \neq \emptyset$ then there is an (A, B) -gap of length 0, so we assume that $A \cap B = \emptyset$. In particular $A \cap B$ is not normal, and the claim follows from 3.2. This proves (2).

By (1) and (2) we may assume that $c_{r+1} \cdots c_s$ is an even (A, B) -gap, Q say, for some odd s with $r + 1 \leq s \leq n$. Since $|A|, |B| \geq 5$, and only two vertices of $P \cup Q$ belong to A , and the same for B , it follows that $s \leq n - 3$, and both A, B meet the path $c_{s+2} \cdots c_{n-1}$, and so there is an (A, B) -gap contained in this path. It is not even, by (1), since there are no edges between it and P , and it is not odd, by (1), since there are no edges between it and Q , a contradiction. This proves 3.5. \blacksquare

We need theorem 7.4 of [1], the following:

3.6 *Let C be an odd cycle. Let $A_1, \dots, A_k \subseteq V(G)$ be normal, such that for $1 \leq i < j \leq k$, every (A_i, A_j) -gap is even. Then $A_1 \cap \dots \cap A_k$ is normal.*

By combining 3.5 and 3.6 we deduce:

3.7 Let C be a shortest odd hole in a graph G , and let X be a stable set of C -major vertices, each with at least five neighbours in $V(C)$. Then the set of vertices in $V(C)$ that are complete to X is normal.

3.8 Let C be a shortest odd hole in G , let z be a C -major vertex, and let X be a set of C -major vertices all different from and nonadjacent to z , and such that each vertex in $X \cup \{z\}$ has at least five neighbours in $V(C)$. Suppose that the set of common neighbours of $X \cup \{z\}$ in $V(C)$ is not normal. Then there exist adjacent $u, v \in X$, and a path c_1 - c_2 - c_3 - c_4 of C , such that every two vertices in $\{c_1, c_2, c_3, c_4, u, v, z\}$ are adjacent except for the pairs $c_3c_1, c_1c_4, c_4c_2, c_2u, uz, zv, vc_3$, which are nonadjacent.

Proof. Since the set of common neighbours of $X \cup \{z\}$ in $V(C)$ is not normal, at least one vertex in $X \cup \{z\}$ has a nonneighbour in $V(C)$, and since they each have at least five neighbours in C it follows that C has length at least six. We proceed by induction on $|X|$. Since the set of common neighbours of $X \cup \{z\}$ in $V(C)$ is not normal, 3.7 implies that some two vertices in X are adjacent, say u, v . Let A, B be the sets of common neighbours in $V(C)$ of $(X \setminus \{u\}) \cup \{z\}, (X \setminus \{v\}) \cup \{z\}$ respectively. From the inductive hypothesis we may assume that A, B are normal; and $A \cap B$ is not normal. Consequently $A \setminus B$ and $B \setminus A$ are both nonempty. If $a \in A \setminus B$ and $b \in B \setminus A$ then a is adjacent to v and not to u , and b is adjacent to u and not to v ; and since z - a - v - u - b - z is not a 5-hole (because C is a shortest odd hole and has length at least six), it follows that a, b are adjacent. Hence every vertex in $A \setminus B$ is adjacent to every vertex in $B \setminus A$; and since $A \setminus B$ and $B \setminus A$ are both nonempty, one of $A \setminus B, B \setminus A$ has cardinality one, say $A \setminus B$, and so we may number the vertices of C in order as c_1, \dots, c_n , such that $A \setminus B = \{c_2\}$, and $c_3 \in B \setminus A \subseteq \{c_1, c_3\}$. Since A is normal and hence $|A| \geq 2$, it follows that $A \cap B \neq \emptyset$. Since $A \cap B$ is not normal, there is an odd $A \cap B$ -gap R say. Since R is not an A -gap because A is normal, its interior contains a member of $A \setminus B$, and since $A \setminus B = \{c_2\}$, we deduce that c_2 belongs to the interior of R . One of the paths of R between c_2 and its ends is odd, and is therefore not an A -gap, and so has length one. Consequently one of c_1, c_3 is an end of R and therefore belongs to $A \cap B$. Since $c_3 \in B \setminus A$, it follows that $c_1 \in A \cap B$, and so $B \setminus A = \{c_3\}$. By the same argument with A, B exchanged, it follows that $c_4 \in A \cap B$, and then the theorem is satisfied. This proves 3.8. ■

3.9 Let C be a shortest odd hole in G , let z be a C -major vertex, and let X be a set of C -major vertices all different from and nonadjacent to z , and such that each vertex in $X \cup \{z\}$ has at least five neighbours in $V(C)$. Then there is an edge ab of C , such that a, b are both adjacent to z , and every vertex in X is adjacent to at least one of a, b .

Proof. We proceed by induction on $|X|$. If the set of common neighbours of $X \cup \{z\}$ is normal then there is an edge of C with both ends complete to $X \cup \{z\}$, which therefore satisfies the theorem. Consequently we may assume that the set of common neighbours of $X \cup \{z\}$ is not normal. By 3.8 there exist adjacent $u, v \in X$, and a path c_1 - c_2 - c_3 - c_4 of C , such that every two vertices in $\{c_1, c_2, c_3, c_4, u, v, z\}$ are adjacent except for the pairs $c_3c_1, c_1c_4, c_4c_2, c_2u, uz, zv, vc_3$, which are nonadjacent. From the inductive hypothesis there is an edge ab of C , such that a, b are both adjacent to z , and every vertex in $X \setminus \{u\}$ is adjacent to at least one of a, b . If u is also adjacent to one of a, b then the theorem is satisfied, so we assume not. Let v be adjacent to a say. Since u is not adjacent to a, b it follows that $a, b \neq c_1, c_3, c_4$; and therefore $a, b \neq c_2$ since a, b are consecutive in C . But then z - c_3 - u - v - a - z is a 5-hole, a contradiction. This proves 3.9. ■

We deduce 3.1, which we restate.

3.10 *Let C be a shortest odd hole in a graph G , and let X be the set of C -major vertices.*

- *If some pair of vertices in X is a skew pair, then there is a set of six vertices that dominates X .*
- *If there is no skew pair in X , but some vertex in X has at most four neighbours in $V(C)$, then there is a set of five vertices that dominates X .*
- *If every vertex in X has at least five neighbours in $V(C)$, then there is a set of three vertices that dominates X .*

Proof. Suppose first that some pair u, v of vertices in X is a skew pair. Consequently each has exactly three neighbours in C , and there is an odd path of C with vertices c_1, \dots, c_n in order, where $n \geq 6$, such that u is adjacent to c_1, c_2, c_{n-2} , and v to c_3, c_{n-1}, c_n . We claim that $\{c_0, c_1, c_2, c_3, c_n, u\}$ dominates X (where c_0 is the second vertex adjacent to c_1). For let $w \in X \setminus \{u\}$. If w is adjacent to u or has a common neighbour with u then one of c_1, c_2, c_{n-2}, u is adjacent to w , and otherwise by 3.4 u, w form a skew pair, and so w is adjacent to one of c_3, c_n . Thus the result holds if there is a skew pair.

Now we assume that there is no skew pair, but some vertex v in X has at most four neighbours in $V(C)$. By 3.4 the set consisting of v and its neighbours in $V(C)$ dominates X , and again the theorem holds. Thus we may assume that every vertex in X has at least five neighbours in $V(C)$.

Choose $z \in X$, and let X' be the set of vertices in $X \setminus \{z\}$ that are nonadjacent to z . By 3.9, there is an edge ab of C such that every vertex in X' is adjacent to one of a, b . But then $\{a, b, z\}$ dominates X . This proves 3.10. ■

It is an entertaining problem to attempt to replace the 21 in 1.4 by as small a number as possible. There are a number of tricks that give savings, and here we sketch some of them, without full details.

- In 3.10, the six of the first bullet can be reduced to five; the set $\{c_1, c_2, c_3, c_{n-2}, u\}$ dominates X (because u cannot be part of a skew pair on both sides, unless the shortest odd hole has length nine, and in this case we can replace 21 by 9).
- 3.9 can be extended to include the case when some C -major vertices have only four neighbours in C , provided they are not all four consecutive.
- In all cases of 3.10, there is a set of five vertices that dominates X including three consecutive vertices of C . To see this claim, we need to improve the second bullet of 3.10. When there is no skew pair, and some C -major vertex has three neighbours in C , the claim is true since two of its neighbours are consecutive; otherwise, if some v has only four neighbours and they are consecutive in C , then v plus the middle two of the four dominate X ; and if there is no such v , the conclusion of the third bullet applies (because of the strengthening of 3.9 mentioned above). This will help with colouring the C -minor vertices.
- Thus there are three consecutive vertices c_{n-2}, c_{n-1}, c_n of C as above, and we are going to pay 3τ to colour the C -major vertices adjacent to one of the three. We might as well colour the C -minor vertices adjacent to one of c_{n-2}, c_{n-1}, c_n at the same time; and so we need not bother to colour those C -minor vertices in the course of 2.2, 2.3, 2.4.

- For 2.2, we only need colour the union of the sets X_1, \dots, X_{n-4} . For $1 \leq i < j \leq n-4$, if there is an edge between X_i, X_j then $j-i \in \{1, 3\}$ and in particular, $j-i$ is odd; so the union of X_1, \dots, X_{n-4} has chromatic number at most 2τ . For 2.3, we only need colour the union of the sets X_1, \dots, X_{n-5} . Each X_i has neighbours in only two of the sets X_{i+1}, \dots, X_{n-5} , and so this union has chromatic number at most 3τ . Thus the 5τ of 2.3 can be reduced to 3τ ; and similarly the 5τ of 2.4 can be reduced to 3τ .
- There is another improvement possible; it is more efficient to handle the C -minor vertices of type zero and type two simultaneously. Let Y be the set of C -minor vertices of type zero or two that are nonadjacent to c_{n-2}, c_{n-1}, c_n . For $1 \leq i \leq n-3$ let X_i be the type zero vertices in Y adjacent to c_i , together with the type two vertices in Y adjacent to c_i, c_{i+2} . For $i < j$, if there is an edge between X_i, X_j then $j \leq i+3$; so the union of X_1, \dots, X_{n-3} has chromatic number at most 4τ . This shows we can colour all C -minor vertices of types zero or two nonadjacent to c_{n-2}, c_{n-1}, c_n with 4τ colours.

Thus we can colour all the C -minor vertices nonadjacent to c_{n-2}, c_{n-1}, c_n with 7τ colours. Handling the C -major vertices and the vertices adjacent to c_{n-2}, c_{n-1} or c_n takes 5τ more; so altogether we use 12τ colours. Thus the 21 of 1.4 can be reduced to 12.

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