Even Pairs In Berge Graphs

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Abstract

Our proof (with Robertson and Thomas) of the strong perfect graph conjecture ran to 179 pages of dense matter; and the most impenetrable part was the final 55 pages, on what we called "wheel systems". In this paper we give a replacement for those 55 pages, much easier and shorter, using "even pairs". This is based on an approach of Maffray and Trotignon.

1 Introduction

All graphs in this paper are finite and simple. The *complement* $\overline{G}$ of a graph G has the same vertex set as G, and distinct vertices u, v are adjacent in $\overline{G}$ just when they are not adjacent in G. A *hole* of G is an induced subgraph of G which is a cycle of length at least 4. An *antihole* of G is an induced subgraph of G whose complement is a hole in $\overline{G}$. A graph G is *Berge* if every hole and antihole of G has even length.

A *clique* in G is a subset X of $V(G)$ such that every two members of X are adjacent. A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H equals the size of the largest clique of H. With Robertson and Thomas [3], we proved Berge's celebrated "strong perfect graph conjecture" [1], the following:

1.1 *A graph is perfect if and only if it is Berge.*

The proof was very long (179 pages), and it would be good to find a shorter one. Our goal in this paper is to replace the final 55 pages with a new easy proof.

In [3] we proved more than just 1.1; we proved:

1.2 *Every Berge graph either belongs to one of five "basic" classes, or admits one of four kinds of decomposition.*

(We have omitted the definitions needed to make this precise, because we shall not need them any more in this paper.) This result 1.2 implies 1.1 because the smallest counterexample to 1.1 cannot belong to any of the basic classes and cannot admit any of the decompositions, and therefore cannot satisfy 1.2. *We do not have a shortened proof of 1.2;* only of its corollary 1.1.

We need some more definitions before we can state more precisely what we do in this paper. If $X \subseteq V(G)$ then $G|X$ denotes the subgraph of G induced on X, and $G \setminus X$ denotes $G|(V(G) \setminus X)$. A *path in G* is an induced subgraph of G that is a path, and an *antipath in G* is the complement of a path in $\overline{G}$. The *interior* of a path or antipath is the set of all its vertices that are not its ends, and the members of the interior are called *internal vertices*. We denote the interior of a path or antipath P by P^*. The *length* of a path or hole is the number of edges in it, and the *length* of an antipath or antihole is the length of its complement path or hole. If G is a graph, a subset $T \subseteq V(G)$ is *connected* if $G|T$ is connected, and T is *anticonnected* if $\overline{G}|T$ is connected. For $T \subseteq V(G)$, we define $N(T)$ to be the set of all vertices in $V(G) \setminus T$ that are adjacent to every member of T. An *odd wheel* (C, T) in G consists of a hole C of length at least six, and a nonempty anticonnected subset $T \subseteq V(G) \setminus V(C)$, such that at least three vertices of C belong to $N(T)$, and there is a path P of C with odd length at least three, such that its ends are not in $N(T)$ and all its internal vertices belong to $N(T)$. A *long prism* in G is an induced subgraph H with three paths P_1, P_2, P_3, satisfying:

- P_1, P_2, P_3 are pairwise disjoint and every vertex of H belongs to one of them
- P_1, P_2, P_3 all have length at least one, and at least one of them has length greater than one
- the ends of P_i are a_i, b_i (for $1 \leq i \leq 3$), and $a_i a_j$ and $b_i b_j$ are edges and there are no other edges between $V(P_i)$ and $V(P_j)$ (for $1 \leq i < j \leq 3$).

A *double diamond* in G is an induced subgraph with eight vertices $a_1, \dots, a_4, b_1, \dots, b_4$ and with the following adjacencies: every two a_i's are adjacent except a_3a_4, every two b_i's are adjacent except b_3b_4, and a_ib_i is an edge for $1 \le i \le 4$. Let us say that G is *impoverished* if G is Berge, and G and $\overline{G}$ both contain no odd wheel, long prism or double diamond.

A *star cutset* in G is a subset $X \subseteq V(G)$ such that $V(G) \setminus X$ is not connected and some vertex in X is adjacent to every other vertex in X. An *even pair* in G is a pair (x, y) of nonadjacent vertices x, y such that every path in G between x, y has even length (we recall that, by definition, a path in G means an induced subgraph of G that is a path). A *dominant pair* in G is a pair (x, y) of nonadjacent vertices such that every other vertex of G is adjacent to at least one of x, y. Our main result is the following:

1.3 *If G is impoverished, then either G admits a star cutset or an even pair or a dominant pair, or G is a complete graph.*

Part of the proof of 1.3 is a variant of a proof of Maffray and Trotignon [6], which was the basis for all this research. Let us see that 1.3 can be used to replace the end of [3].

Proof of 1.1. Suppose that 1.1 is false, and let G be a counterexample with $|V(G)|$ minimum, and subject to that with $|E(G)|$ maximum. Thus G is Berge and not perfect. The first 124 pages of [3] show that G is impoverished. By [2, 4, 7], G does not contain a star cutset or an even pair. Thus by 1.3, G contains a dominant pair (x, y). Let G' be obtained from G by making x, y adjacent.

(1) *G' is Berge.*

For suppose first that G' has an odd hole C. Then $x, y \in V(C)$ since G is Berge; and so there is a path in G between x, y of length at least four. But this contradicts that (x, y) is a dominant pair. Next suppose that C is an odd antihole of G'. Then again $x, y \in V(C)$, and so there is an induced subgraph H of $\overline{G}$ that is obtained from an odd cycle of length at least five by adding one edge xy. Consequently the edge xy belongs to an induced cycle C' of H with odd length. Since G and hence H is Berge, it follows that C' has length three; and so some vertex of C is nonadjacent in G to both x, y. But this contradicts that (x, y) is a dominant pair. This proves (1).

(2) *G' is not perfect.*

Let k be the maximum cardinality of stable sets of G'; then G has a stable set of cardinality k, and so $V(G)$ cannot be partitioned into k cliques, since G and hence $\overline{G}$ are both minimal imperfect graphs by the weak perfect graph theorem [5]. Suppose that G' is perfect; then there is a partition $(A_1, \dots, A_k)$ of $V(G')$ into cliques $A_1, \dots, A_k$ of G', where k is the maximum cardinality of stable sets of G'. We may assume that $\{x, y\} \subseteq A_1 \cup A_2$. Now the complement H of $G|(A_1 \cup A_2)$ is obtained from a bipartite graph by adding one edge not in a triangle; and consequently H is also bipartite (since it is Berge). Let A'_1, A'_2 be disjoint cliques of G with $A'_1 \cup A'_2 = A_1 \cup A_2$; then $V(G)$ is the union of k cliques of G, namely $A'_1, A'_2, A_3, \dots, A_k$, a contradiction. This proves (2).

But then G' is Berge and not perfect, contrary to the choice of G. This proves 1.1. ∎

2 Lemmas from [3]

We need a few results from [3], that we restate here. If $T \subseteq V(G)$ and $v \in V(G) \setminus T$ is adjacent to every vertex in T, we say that v is *T-complete*; and if v is nonadjacent to every member of T we say it is *T-anticomplete*. First, a fundamental lemma of Roussel and Rubio [8], in the special case when G contains no long prism, and theorem 2.2 of [3]:

2.1 *Let G be Berge containing no long prism, let P be a path in G of odd length at least three, and let $T \subseteq V(G) \setminus V(P)$ be anticonnected, and such that the ends of P are T-complete and the internal vertices are not. Then P has length three and every T-complete vertex in G has a neighbour in P^*.*

Second, we need a special case of theorem 15.7 of [3], the following.

2.2 *Let G be impoverished, and let C be a hole and D an antihole, both of length at least six. Then $|V(C) \cap V(D)| \leq 2$.*

THird, we need the following. A *triangle* in G is a set of three pairwise adjacent vertices of G, and a set $F \subseteq V(G)$ is said to *catch* a triangle $\{a_1, a_2, a_3\}$ if F is connected and $a_1, a_2, a_3 \notin F$ and each of a_1, a_2, a_3 has a neighbour in F. Theorem 17.1 of [3] implies:

2.3 *Let A be a triangle in an an impoverished graph G, and let $F \subseteq V(G)$ catch A, such that every vertex of F has at most one neighbour in A. Then F contains a triangle B such that $G|(A \cup B)$ is an antihole of length six.*

Finally we need a slight weakening of theorem 17.2 of [3], the following.

2.4 *Let G be impoverished, and let v_1-v_2-v_3-v_4 be a 3-edge path in G. Let $X, Y \subseteq V(G) \setminus \{v_1, v_2, v_3, v_4\}$ be disjoint, such that Y is anticonnected and X is connected. Suppose that v_2, v_3 are Y-complete and X-anticomplete, and v_1, v_4 are not Y-complete and not X-anticomplete. Then some vertex of Y is X-anticomplete.*

3 The main proof

Now we prove 1.3. Let G be impoverished. We prove by induction on $|V(G)|$ that either G is complete, or has an even pair, a star cutset or a dominant pair. If G is disconnected then it has an even pair (choose x, y in different components), so we may assume that G is connected. We may also assume that G is not a complete graph, and consequently some vertex of G has two nonadjacent neighbours. Hence there is a nonempty anticonnected set $T \subseteq V(G)$ such that $N(T)$ is not a clique. Choose T maximal with this property. Throughout this section G, T are as just described. Let us say an *outer path* is a path of length at least two, with both ends in $N(T)$ and every internal vertex in $V(G) \setminus (T \cup N(T))$. We shall prove the following three statements:

- If there is no outer path then G contains an even pair, a star cutset or a dominant pair.
- If some outer path has odd length then G contains a star cutset or a dominant pair.
- If there is an outer path and every outer path has even length then G contains an even pair.

We observe first:

3.1 *If there is no outer path then G contains an even pair, a star cutset or a dominant pair.*

Proof. First suppose that $T \cup N(T) = V(G)$. Now $G|N(T)$ is not complete, and T is nonempty, so from the inductive hypothesis $G|N(T)$ contains an even pair, a star cutset or a dominant pair. If (x, y) is an even pair or dominant pair in $G|N(T)$ then it is also an even pair or dominant pair in G, since every vertex in T is adjacent to both x, y. If $X \subseteq N(T)$ is a star cutset of $G|N(T)$, and say $x \in X$ is adjacent to every other member of X, then $X \cup T$ is a star cutset of G, since x is adjacent to every member of $X \cup T$ and

$$(G|N(T)) \setminus X = G \setminus (X \cup T).$$

Thus we may assume that $T \cup N(T) \neq V(G)$. Let F be a component of $G \setminus (T \cup N(T))$, and let Y be the set of all vertices in $T \cup N(T)$ with a neighbour in F. Since there is no outer path, it follows that $Y \cap N(T)$ is a clique. Choose $y \in N(T)$, with $y \in Y$ if $Y \cap N(T) \neq \emptyset$. Since $N(T)$ is not a clique, there exists $z \in N(T) \setminus Y$ with $z \neq y$. Then y is adjacent to every member of $Y \setminus \{y\}$. Hence Y is a star cutset (separating F from z). The result follows. ∎

Second, we observe:

3.2 *For every vertex $v \in V(G) \setminus (T \cup N(T))$, the set of neighbours of v in $N(T)$ is a clique; and so every outer path has length at least three.*

Proof. If the set of neighbours of v in $N(T)$ is not a clique, we can add v to T contradicting the maximality of T. This proves 3.2. ∎

We need an easy lemma:

3.3 *Let P be a path with vertices p_0-$\cdots$-p_n in order, where $n \geq 4$, $p_0, p_n \in N(T)$ and $p_1, p_2 \notin N(T)$. Then some vertex in T is nonadjacent to p_1, p_2, and indeed every antipath between p_1, p_2 with interior in T has length two.*

Proof. There is an antipath S joining p_1, p_2 with interior in T, and it can be completed to an antihole via p_2-p_n-p_1 since $n \geq 4$. Thus S has even length, and so the antipath p_1-S-p_2-p_0 has odd length. All the internal vertices of the latter antipath have neighbours in the connected set $\{p_3, \ldots, p_n\}$, and its ends p_0, p_1 do not; so by 2.1 applied in the complement, this antipath has length three, that is, S has length two. The result follows. ∎

3.4 *If some outer path has odd length then G admits a star cutset or a dominant pair.*

Proof. Let Z be the set of all vertices in $N(T)$ that have a nonneighbour in $N(T)$. We claim first that

(1) *If Q is an outer path of odd length, then Q has length three, and every vertex in Z is adjacent to exactly one of the two internal vertices of Q.*

For the ends of Q are T-complete and its internal vertices are not, and so 2.1 implies that Q has length three, and therefore the first assertion holds. Let q_1, q_2 be the two internal vertices of Q. Let $v \in Z$; by 2.1, v is adjacent to at least one of q_1, q_2. Since $v \in Z$, it is nonadjacent to some $u \in Z$, and by the same argument u is adjacent to at least one of q_1, q_2. Hence there is an outer path joining u, v with interior in $\{q_1, q_2\}$, which therefore has length three by 3.2; and so u, v are both adjacent to exactly one of q_1, q_2. This proves (1).

(2) *Every outer path has length three.*

Suppose not, and let P be an outer path of even length, and Q one of odd length, chosen with $V(P) \cup V(Q)$ minimal. By 3.2, P has length at least four; let its vertices be $p_0\text{-}\cdots\text{-}p_{k+1}$ say in order, where $k \geq 3$. By (1), Q has length three; let its internal vertices be q_1, q_2. By (1), p_0, p_{k+1} are each adjacent to one of q_1, q_2, and since there is no outer path of length two, we may assume that $p_0\text{-}q_1\text{-}q_2\text{-}p_{k+1}$ is a path, and indeed this is the path Q. Suppose first that $q_1, q_2 \notin V(P)$. Since $P \cup Q$ is not an odd hole, at least one of q_1, q_2 is adjacent to one of $p_1, \ldots, p_k$, say q_1. Choose $i \in \{1, \ldots, k+1\}$ maximum such that q_1, p_i are adjacent. Suppose that $i > 1$. Then $p_0\text{-}q_1\text{-}p_i\text{-}p_{i+1}\text{-}\cdots\text{-}p_{k+1}$ is an outer path R say. Since $V(Q) \cup V(R)$ is a proper subset of $V(P) \cup V(Q)$ it follows from the minimality of $V(P) \cup V(Q)$ that the length of R is not even, and similarly it is not odd, a contradiction. This proves that $i = 1$. If also q_2 is adjacent to one of $p_1, \ldots, p_k$, then similarly its only neighbour is p_k, and then $q_1\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}q_2\text{-}q_1$ is an odd hole, a contradiction. Thus q_2 is nonadjacent to all of $p_1, \ldots, p_k$. Now there is an antipath S joining q_1, p_1 with interior in T. Since it can be completed to an antihole via $p_1\text{-}p_{k+1}\text{-}q_1$, and this antihole shares three vertices (namely p_1, q_1, p_{k+1}) with the hole $q_1\text{-}p_1\text{-}\cdots\text{-}p_{k+1}\text{-}q_2\text{-}q_1$, it follows from 2.2 that the antihole has length at most four; and so S has length two. Consequently there exists $t \in T$ nonadjacent to both q_1, p_1. Then $F = \{p_2, \ldots, p_{k+1}, q_2, t\}$ is connected and catches the triangle $\{p_0, p_1, q_1\}$; the only neighbour of p_1 in F is p_2, the only neighbour of p_0 is t, and the only neighbour of q_1 is q_2; and since p_2, q_2 are nonadjacent this contradicts 2.3.

This proves that one of $q_1, q_2 \in V(P)$; so we may assume that $q_1 = p_1$ say. Hence $q_2 \notin V(P)$. By 3.3 some vertex $t \in T$ is nonadjacent to both p_1, p_2, and indeed every antipath between p_1, p_2 with interior in T has length two. There is also an antipath S between p_1, q_2 with interior in T, and S can be completed to an antihole via $q_2\text{-}p_0\text{-}p_{k+1}\text{-}p_1$. Let this antihole be D; then D has length at least six. If p_2 has a nonneighbour in S^*, then there is an antipath between p_1, p_2 with interior in S^*, and so we can choose t to be the vertex of S nonadjacent to p_1; then if we choose $i \in \{1, \ldots, k+1\}$ minimum such that t, p_i are adjacent, the hole $t\text{-}p_0\text{-}p_1\text{-}\cdots\text{-}p_i\text{-}t$ has length at least six and shares three vertices with D (namely p_0, p_1, t), contrary to 2.2. This proves that p_2 is complete to S^*. Let i be as before, such that $t\text{-}p_0\text{-}p_1\text{-}\cdots\text{-}p_i\text{-}t$ is a hole C say.

We have already seen that p_2 is S^*-complete, and in particular $t \notin V(S)$. Let S have vertices $p_1\text{-}s_1\text{-}s_2\text{-}\cdots\text{-}s_n\text{-}s_{n+1} = q_2$ in order. If t is nonadjacent to any of $s_2, \ldots, s_{n+1}$, choose j maximum such that t is nonadjacent to s_j; then

$$p_1\text{-}t\text{-}s_j\text{-}s_{j+1}\text{-}\cdots\text{-}s_{n+1}$$

is an antipath between p_1, q_2 such that p_2 is nonadjacent to one of its internal vertices, a contradiction. Thus t is adjacent to $s_2, \ldots, s_n, q_2$.

Suppose that t is nonadjacent to s_1; so $p_2\text{-}t\text{-}s_1\text{-}\cdots\text{-}s_n$ is an antipath U say, of odd length. If also p_2 is nonadjacent to q_2, then U can be completed to an odd antihole via $s_n\text{-}q_2\text{-}p_2$, a contradiction;

while if p_2 is adjacent to q_2, then p_2-U-s_n-q_2-p_0-p_2 is an antihole of length at least six, sharing three vertices (namely p_0, p_2, t) with the hole C, contrary to 2.2.

This proves that t is adjacent to s_1, and so p_2-t-p_1-s_1-$\cdots$-s_n is an antipath V say, of even length. If also q_2, p_2 are adjacent, then p_2-V-s_n-q_2-p_0-p_2 is an odd antihole, while if q_2, p_2 are nonadjacent then p_2-V-s_n-q_2-p_2 is an antihole of length at least six sharing three vertices (namely p_1, p_2, t) with C, in either case a contradiction. Thus we cannot choose P, Q with these properties. This proves (2).

For each $v \in V(G) \setminus (T \cup N(T))$, let Z_v be the set of neighbours of v in Z. Let W be the set of all edges xy such that some outer path has interior $\{x, y\}$. By (1) it follows that if $xy \in W$ then Z_x, Z_y are disjoint and have union Z.

(3) *If $uv, xy \in W$ then $\{Z_u, Z_v\} = \{Z_x, Z_y\}$.*

For Z_u, Z_v are both cliques, by 3.2. Similarly Z_x, Z_y are cliques. From the symmetry we may assume that there exists $a \in Z_u \cap Z_x$. Since a has a nonneighbour b say in Z, and Z_u, Z_x are cliques, it follows that $b \in Z_v \cap Z_y$. If $Z_u \cap Z_y, Z_v \cap Z_x$ are both empty then the claim holds, so from the symmetry we may assume that there exists $c \in Z_u \cap Z_y$; and since c has a nonneighbour $d \in Z$, it follows similarly that $d \in Z_v \cap Z_x$. Since $Z_u \neq Z_x$ we deduce that $u \neq x$, and similarly u, v, x, y are all distinct. If u, x are adjacent then c-u-x-d is an outer path and yet Z_u, Z_x are not disjoint, a contradiction; so u, x are nonadjacent, and similarly u, v are both nonadjacent to both x, y. But then a-x-y-b-v-u-a is a hole C say, and $(C, \{c\})$ is an odd wheel, a contradiction. This proves (3).

Now by hypothesis, W is nonempty; so there are two disjoint nonempty cliques A, B with union Z such that for every $xy \in W$, one of Z_x, Z_y is A and the other is B. Let C be the set of all vertices in $V(G) \setminus (T \cup N(T))$ that are adjacent to every member of A and to no member of B, and define D similarly (with A, B exchanged). Thus by (2), every outer path has length three, and has vertex set a-c-d-b in order, for some $a \in A, b \in B, c \in C$ and $d \in D$.

(4) *If $T \cup N(T) \cup C \cup D = V(G)$ then G has a dominant pair.*

For choose $x, y \in N(T)$, nonadjacent. Then $x, y \in Z$, and so one is in A and the other in B, say $x \in A$ and $y \in B$. Every vertex in $T \cup (A \setminus \{x\}) \cup (N(T) \setminus Z) \cup C$ is adjacent to x, and every vertex in $(B \setminus \{y\}) \cup D$ is adjacent to y, and so (x, y) is a dominant pair. This proves (4).

From (4) we may assume the graph $G \setminus (T \cup N(T) \cup C \cup D)$ has at least one component, F say. Let Y be the set of all vertices in $T \cup N(T) \cup C \cup D$ that have a neighbour in F.

Suppose first that $Y \cap C, Y \cap D$ are both empty. Since every outer path contains a vertex of C, every vertex in $Y \cap A$ is adjacent to every vertex in $Y \cap B$, and so $Y \cap N(T)$ is a clique. Choose $y \in Y \cap N(T)$ if $Y \cap N(T) \neq \emptyset$, and otherwise let $y \in N(T)$ be arbitrary; then y is adjacent to every other vertex in Y, and so $Y \cup \{y\}$ is a star cutset (separating F from all members of $C \cup D$). From the symmetry we may therefore assume that $Y \cap C \neq \emptyset$. If there exists $b \in Y \cap B$, choose $a \in A$ nonadjacent to b; then there is a Y-path between a, b with interior in $F \cup C$, a contradiction. Thus $Y \cap B = \emptyset$. Next suppose that $Y \cap D$ is empty, and choose $a \in A$; then a is adjacent to every member of $Y \setminus \{a\}$, and so Y is a star cutset (separating F from every member of D). Hence we may assume that $Y \cap D \neq \emptyset$, and so $Y \cap A = \emptyset$ by a similar argument. Choose $c \in Y \cap C$ and $d \in Y \cap D$; and choose $a \in A$ and $b \in B$, nonadjacent. Then there is an outer path between a, b with interior in $F \cup \{c, d\}$, so by (2) it follows that c, d are adjacent. But then 2.4 (applied in the complement) implies that some vertex of F is T-complete, a contradiction. This proves 3.4. ∎

To complete the proof of 1.3, it remains to show the following, a variant of a result of Maffray and Trotignon [6].

3.5 *If there is an outer path and every outer path has even length then G contains an even pair.*

Proof. Choose an outer path R of minimum length, with vertices r_0-r_1-r_2-$\cdots$-r_k-r_{k+1} in order, say; thus $r_0, r_{k+1} \in N(T)$ and $r_1, \dots, r_k \notin T \cup N(T)$, and $k \geq 3$ is odd. Let A be the set of all vertices in $N(T)$ that are adjacent to r_1 and to none of $r_2, \dots, r_k$. (Thus $r_0 \in A$.) Similarly let B be the set of all vertices in $N(T)$ adjacent to r_k and to none of $r_1, \dots, r_{k-1}$. From 3.2, A, B are cliques; and if $a \in A$ and $b \in B$ then since a-r_1-$\cdots$-r_k-b-a is not an odd hole, it follows that a, b are nonadjacent. Thus A is anticomplete to B.

(1) *If $v \in N(T) \setminus (A \cup B)$ then either v has no neighbour in R^* or v is complete to $A \cup B$.*

Certainly $v \notin R^*$ since no vertex of R^* is in $N(T)$. Suppose that v has a neighbour in R^*, and has a nonneighbour $a \in A$ say. Choose $i \in \{1, \dots, k\}$ minimum such that v, r_i are adjacent. Then $i < k$ since $v \notin B$; and so a-r_1-$\cdots$-r_i-v is an outer path of length less than that of R, a contradiction. This proves (1).

(2) *Every path between A and B with no internal vertex in $A \cup B$ has even length.*

For suppose that P is a path between A and B of odd length, with no internal vertex in $A \cup B$. Then $V(P) \cap T = \emptyset$, since every vertex in T is adjacent to both a, b. Let P have vertices p_0-$\cdots$-p_{n+1} in order, where $p_0 \in A$ and $p_{n+1} \in B$. Let us say a *segment* is a maximal subpath of P such that all its vertices belong to $N(T)$. Thus an edge of P belongs to a segment if and only both its ends are in $N(T)$, and then it belongs to a unique segment; and otherwise it belongs to a unique outer path included in P. Since all outer paths have even length and P has odd length, it follows that some segment has odd length, say p_i-$\cdots$-p_j. By (1) $p_i, \dots, p_j$ are not in R^*, and $\{p_i, \dots, p_j\} \cap P^*$ is anticomplete to R^*. Choose $i' \leq i$ maximum such that p'_i has a neighbour in R^* (this is possible since p_0 has a neighbour in R^*); and similarly choose $j' \geq j$ minimum such that p'_j has a neighbour in R^*. Then $j' - i' \geq j - i + 1 \geq 2$. Choose a path Q between p'_i, p'_j with interior in R^*, and let P' be the subpath of P between p'_i, p'_j. Then $P' \cup Q$ is a hole C say, and p_i-$\cdots$-p_j is a maximal subpath of $C|N(T)$. If $j > i + 1$ then C has length at least six and (C, T) is an odd wheel, a contradiction, so $j = i + 1$. Suppose first that $i \geq 1$ and $j \leq n$. Then $i \geq 2$ and $j \leq n - 1$, and $i' < i$ and $j < j'$, and the path p_{i-1}-p_i-p_{i+1}-p_{i+2} ($= S$ say) is an induced subgraph of C. Both ends of S have neighbours in the connected set $V(R) \cup \{p_0, \dots, p_{i-2}, p_{i+3}, \dots, p_{n+1}\}$ ($= F$ say) and the internal vertices of S do not; and the ends of S have nonneighbours in T, and the internal vertices are T-complete. By 2.4, we deduce that some vertex of T is anticomplete to F, a contradiction. Thus we may assume that $i = 0$i and $j = 1$. Hence $p_1 \in N(T)$, and therefore p_1 is anticomplete to R^* by (1). Suppose that r_1 is nonadjacent to p_2. Then r_1-p_0-p_1-p_2 is a path S say; its ends have neighbours in the connected set

$\{r_2, \ldots, r_{k+1}\} \cup \{p_3, \ldots, p_{n+1}\}$, and its internal vertices do not; and its ends have nonneighbours in T, and its internal vertices are T-complete. Again this is contrary to 2.4. Thus it follows that r_1, p_2 are adjacent.

Choose i with $1 \leq i \leq k+1$ maximum such that p_2, r_i are adjacent. Then p_1-p_2-r_i-r_{i+1}-$\cdots$-p_{n+1} is a path; its ends are T-complete, its internal vertices are not, and r_0 is a T-complete vertex with no neighbour in the interior of this path. From 2.1, this path has even length, and since k is odd it follows that i is even. In particular $i \geq 2$. If $i \geq 3$ then p_0-r_1-p_2-r_i-r_{i+1}-$\cdots$-r_k-p_{n+1} is an outer path with odd length, a contradiction; so $i = 2$. By 3.3 there exists $t \in T$ nonadjacent to r_1, r_2. Then $\{p_0, t, p_1, r_3, \ldots, r_k\}$ ($= F$ say) catches the triangle $\{r_1, r_2, p_2\}$; the only neighbour of r_1 in F is p_0; the only neighbour of r_2 in F is r_3; and p_0, r_3 are nonadjacent to each other, and both nonadjacent to p_2, contrary to 2.3. This proves (2).

For $x, y \in A$, distinct, let us say an (x, y)*-path* is a path between y and some vertex in B, such that x is nonadjacent to every vertex of this path except y. If there is an (x, y)-path, we write $x \rightarrow y$. We need:

(3) *If $x, y \in A$ are distinct, then not both $x \rightarrow y$ and $y \rightarrow x$.*

For suppose that P is an (x, y)-path and Q is a (y, x)-path. We may assume that both P, Q have only their final vertex in B. Moreover, since x has no neighbours in P except y, and A is a clique, it follows that y is the only vertex of P in A, and similarly x is the only vertex of Q in A. By (2), both P, Q have even length. Let P have vertices y-p_1-$\cdots$-p_m in order, and let Q have vertices x-q_1-$\cdots$-q_n in order. Suppose first that p_1, q_1 are nonadjacent. Then p_1-y-x-q_1 is a three-edge path. Let F be the union of the vertex sets of the three paths $P \setminus \{y, p_1\}$, $Q \setminus \{x, q_1\}$ and $R \setminus \{r_0, r_1\}$. Then F is connected (since all three paths contain a vertex in B); and p_1, q_1 have neighbours in F, and x, y do not. Moreover, $T \cup \{r_1\}$ is anticonnected ($= T'$ say); and x, y are T'-complete, and p_1, q_1 are not. (This last holds since if $p_1 \in N(T)$ then p_1 is nonadjacent to r_1 by 3.2.) Consequently, some vertex of T' is anticomplete to F, by 2.4. But every vertex in T is adjacent to $r_{k+1} \in F$, and r_1 is adjacent to $r_0 \in F$, a contradiction.

This proves that p_1, q_1 are adjacent. Choose i with $1 \leq i \leq m$ maximum such that q_1, p_i are adjacent. Then x-q_1-p_i-p_{i+1}-$\cdots$-p_m is a path with one end in A and the other in B, and with no internal vertex in $A \cup B$; so by (2) this path has even length. Since m is even, it follows that i is even, and in particular $i \geq 2$. If $i > 2$ then y-p_1-q_1-p_i-p_{i+1}-$\cdots$-p_m is a path of odd length, contrary to (2). Thus $i = 2$.

Suppose that $p_1 \notin N(T)$. By 3.2, $q_1, p_2 \notin N(T)$. By 3.3 some vertex $t \in T$ is nonadjacent to p_1, p_2. But then $\{p_3, \ldots, p_m, t, x, y\}$ ($= F$ say) is connected and catches the triangle $\{p_1, p_2, q_1\}$; the only neighbour of p_2 in F is p_3; the only neighbour of p_1 in F is y; and p_3, y are nonadjacent to each other, and both nonadjacent to q_1, contrary to 2.3. This proves that $p_1 \in N(T)$, and from the symmetry between P, Q it follows that $q_1 \in N(T)$.

Since $p_1, q_1 \notin A \cup B$, (1) implies that p_1, q_1 have no neighbours in $V(R) \setminus \{r_0\}$. But then $(V(R) \setminus \{r_0, r_1\}) \cup (V(P) \setminus \{y\}) \cup \{q_1\}$ ($= F$ say) catches the triangle $\{r_1, x, y\}$; the only neighbour of y in F is p_1; the only neighbour of x in F is q_1; and both p_1, q_1 are nonadjacent to r_1. Hence by 2.3 it follows that some vertex in $F \setminus \{p_1, q_1\}$ is adjacent to p_1, q_1, r_1, and this must be p_2 since p_1 has no neighbour in $V(R) \setminus \{r_0\}$. By 3.2, $p_2 \notin N(T)$. Let S be an antipath between p_2, r_1 with interior in T. Then one of p_2-S-r_1-p_m-p_2, p_2-S-r_1-p_1-x-p_2 is an odd antihole, a contradiction. This proves (3).

(4) *If* $x, y, z \in A$ *and* $x \to y$ *and* $y \to z$ *then* $x \to z$.

For let P be a (y, z)-path, with vertices z-p_1-$\cdots$-p_n say. If x is adjacent to one of $p_1, \ldots, p_n$, choose $i \leq n$ maximum such that x, p_i are adjacent; then x-p_i-p_{i+1}-$\cdots$-p_n is a (y, x) path, contrary to (3). Thus x is nonadjacent to all of $p_1, \ldots, p_n$, and so P is an (x, z)-path, and therefore $x \to z$. This proves (4).

From (3) and (4) there exists $a \in A$ such that $a \not\to a'$ for all $a' \in A \setminus \{a\}$; that is, every path from a to B contains no vertex of A except a. Similarly choose $b \in B$ such that every path from A to b contains no vertex of B except b. Suppose that P is a path between a, b; then no internal vertex of P belongs to $A \cup B$ from the choice of a, b, and so P has even length by (2). Consequently (a, b) is an even pair. This completes the proof of 3.5 and hence of 1.3. ∎

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