Edge-colouring seven-regular planar graphs

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Abstract

A conjecture due to the fourth author states that every d-regular planar multigraph can be d-edge-coloured, provided that for every odd set X of vertices, there are at least d edges between X and its complement. For $d = 3$ this is the four-colour theorem, and the conjecture has been proved for all $d \leq 8$, by various authors. In particular, two of us proved it when $d = 7$; and then three of us proved it when $d = 8$. The methods used for the latter give a proof in the $d = 7$ case that is simpler than the original, and we present it here.

1 Introduction

Let G be a graph. (Graphs in this paper are finite, and may have loops or parallel edges.) If $X \subseteq V(G)$, $\delta_G(X) = \delta(X)$ denotes the set of all edges of G with an end in X and an end in $V(G) \setminus X$. We say that G is *oddly d-edge-connected* if $|\delta(X)| \geq d$ for all odd subsets X of $V(G)$. The following conjecture [8] was proposed by the fourth author in about 1973.

1.1. Conjecture. *If G is a d-regular planar graph, then G is d-edge-colourable if and only if G is oddly d-edge-connected.*

The “only if” part is true, and some special cases of the “if” part of this conjecture have been proved.

- For $d = 3$ it is the four-colour theorem, and was proved by Appel and Haken [1, 2, 7];
- for $d = 4, 5$ it was proved by Guenin [6];
- for $d = 6$ it was proved by Dvorak, Kawarabayashi and Kral [4];
- for $d = 7$ it was proved by the second and third authors and appears in the Master’s thesis [5] of the former;
- for $d = 8$ it was proved by three of us [3].

The methods of [3] can be adapted to yield a proof of the result for $d = 7$, that is shorter and simpler than that of [5]. Since in any case the original proof appears only in a thesis, we give the new one here. Thus, we show

1.2. *Every 7-regular oddly 7-edge-connected planar graph is 7-edge-colourable.*

All these proofs (for $d > 3$), including ours, assume the truth of the result for $d - 1$. Thus we need to assume the truth of the result for $d = 6$. Some things that are proved in [3] are true for all d, and we sometimes cite results from that paper.

2 An unavoidable list of reducible configurations.

Any 7-regular planar graph has parallel edges, and it is helpful to reformulate the problem in terms of the underlying simple graph; then we have have a number for each edge, recording the number of parallel edges it represents. Let us say a *d-target* is a pair (G, m) with the following properties (where for $F \subseteq E(G)$, $m(F)$ denotes $\sum_{e \in F} m(e)$):

- G is a simple graph drawn in the plane;
- $m(e) \geq 0$ is an integer for each edge e;
- $m(\delta(v)) = d$ for every vertex v; and
- $m(\delta(X)) \geq d$ for every odd subset $X \subseteq V(G)$.

In this language, 1.1 says that for every d-target (G,m), there is a list of d perfect matchings of G such that every edge e of G is in exactly $m(e)$ of them. (The elements of a list need not be distinct.) If there is such a list we call it a *d-edge-colouring*, and say that (G,m) is *d-edge-colourable.* For an edge $e \in E(G)$, we call $m(e)$ the *multiplicity* of e. If $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced on X. We need the following theorem from [3]:

2.1. *Let (G,m) be a d-target, that is not d-edge-colourable, but such that every d-target with fewer vertices is d-edge-colourable. Then*

- $|V(G)| \geq 6$*;*
- *for every $X \subseteq V(G)$ with $|X|$ odd, if $|X|, |V(G) \setminus X| \neq 1$ then $m(\delta(X)) \geq d+2$; and*
- *G is three-connected, and $m(e) \leq d-2$ for every edge e.*

A *triangle* is a region of G incident with exactly three edges. If a triangle is incident with vertices u, v, w, for convenience we refer to it as uvw, and in the same way an edge with ends u, v is called uv. Two edges are *disjoint* if they are distinct and no vertex is an end of both of them, and otherwise they *meet.* Let r be a region of G, and let $e \in E(G)$ be incident with r; let r' be the other region incident with e. We say that e is *i-heavy* (for r), where $i \geq 2$, if either $m(e) \geq i$ or r' is a triangle uvw where $e = uv$ and

$$m(uv) + \min(m(uw), m(vw)) \geq i.$$

We say e is a *door* for r if $m(e) = 1$ and there is an edge f incident with r' and disjoint from e with $m(f) = 1$. We say that r is *big* if there are at least four doors for r, and *small* otherwise. A *square* is a region with length four.

Since G is drawn in the plane and is two-connected, every region r has boundary some cycle which we denote by C_r. In what follows we will be studying cases in which certain configurations of regions are present in G. We will give a list of regions the closure of the union of which is a disc. For convenience, for an edge e in the boundary of this disc, we call the region outside the disc incident with e the "second region" for e; and we write $m^+(e) = m(e)$ if the second region is big, and $m^+(e) = m(e) + 1$ if the second region is small. This notation thus depends not just on (G,m) but on what regions we have specified, so it is imprecise, and when there is a danger of ambiguity we will specify it more clearly. If r is a triangle, incident with edges e, f, g, we define its *multiplicity* $m(r) = m(e) + m(f) + m(g)$. We also write $m^+(r) = m^+(e) + m^+(f) + m^+(g)$. A region r is *tough* if r is a triangle and $m^+(r) \geq 7$.

We will show that every 7-target (G,m) satisfying the conclusion of 2.1 (with $d = 7$) and such that $m(e) > 0$ for every edge e must contain one of a list of 16 reducible configurations. Let us say a 7-target is "prime" if it fails to satisfy this claim; that is, a 7-target (G,m) is *prime* if

- $m(e) > 0$ for every edge e;
- $|V(G)| \geq 6$;
- $m(\delta(X)) \geq 9$ for every $X \subseteq V(G)$ with $|X|$ odd and $|X|, |V(G) \setminus X| \neq 1$;
- G is three-connected, and $m(e) \leq 5$ for every edge e;

and in addition (G, m) contains none of of the following:

Conf(1): A triangle uvw, where u has degree three and its third neighbour x satisfies

$$m(ux) < m(uw) + m(vw).$$

Conf(2): Two triangles uvw, uwx with $m(uv) + m(uw) + m(vw) + m(ux) \geq 7$.

Conf(3): A square $uvwx$ where $m(uv) + m(vw) + m(ux) \geq 7$.

Conf(4): Two triangles uvw, uwx where $m^+(uv) + m(uw) + m^+(wx) \geq 6$.

Conf(5): A square $uvwx$ where $m^+(uv) + m^+(wx) \geq 6$.

Conf(6): A triangle uvw with $m^+(uv) + m^+(uw) = 6$ and either $m(uv) \geq 3$ or $m(uv) = m(vw) = m(uw) = 2$ or u has degree at least four.

Conf(7): A region r of length at least four, an edge e of C_r with $m^+(e) = 4$ where every edge of C_r disjoint from e is 2-heavy and not incident with a triangle with multiplicity three, and such that at most three edges disjoint from e are not 3-heavy.

Conf(8): A region r with an edge e of C_r with $m^+(e) = m(e) + 1 = 4$ and an edge f disjoint from e with $m^+(f) = m(f) + 1 = 2$, where every edge of $C_r \setminus \{f\}$ disjoint from e is 3-heavy with multiplicity at least two.

Conf(9): A region r of length at least four and an edge e of C_r such that $m(e) = 4$ and there is no door disjoint from e. Further for every edge f of C_r consecutive with e with multiplicity at least two, there is no door disjoint from f.

Conf(10): A region r of length four, five or six and an edge e of C_r such that $m(e) = 4$ and such that $m^+(f) \geq 2$ for every edge f of C_r disjoint from e.

Conf(11): A region r and an edge e of C_r, such that $m(e) = 5$ and at most five edges of C_r disjoint from e are doors for r, or $m^+(e) = m(e) + 1 = 5$ and at most four edges of C_r disjoint from e are doors for r.

Conf(12): A region r, an edge uv of C_r, and a triangle uvw such that $m(uv) + m(vw) = 5$ and at most five edges of C_r disjoint from v are doors for r.

Conf(13): A square $xuvy$ and a tough triangle uvz, where $m(uv) + m^+(xy) \geq 4$ and $m(xy) \geq 2$.

Conf(14): A region r of length five, an edge $f_0 \in E(C_r)$ with $m^+(e_0) \geq 2$ and $m^+(e) \geq 4$ for each edge $e \in E(C_r)$ disjoint from f_0.

Conf(15): A region r of length five, a 3-heavy edge $f_0 \in E(C_r)$ with $m(e_0) \geq 2$ and $m^+(e) \geq 3$ for each edge $e \in E(C_r)$ disjoint from f_0.

Conf(16): A region r of length six where five edges of C_r are 3-heavy with multiplicity at least two.

We will prove that no 7-target is prime (Theorem 3.1). To deduce 1.2, we will show that if there is a counterexample, then some counterexample is prime; but for this purpose, just choosing a counterexample with the minimum number of vertices is not enough, and we need a more delicate minimization. If (G, m) is a d-target, its *score sequence* is the $(d+1)$-tuple $(n_0, n_1, \dots, n_d)$ where n_i is the number of edges e of G with $m(e) = i$. If (G, m) and (G', m') are d-targets, with score sequences $(n_0, \dots, n_d)$ and $(n'_0, \dots, n'_d)$ respectively, we say that (G', m') is *smaller* than (G, m) if either

- $|V(G')| < |V(G)|$, or
- $|V(G')| = |V(G)|$ and there exists i with $1 \leq i \leq d$ such that $n'_i > n_i$, and $n'_j = n_j$ for all j with $i < j \leq d$, or
- $|V(G')| = |V(G)|$, and $n'_j = n_j$ for all j with $0 < j \leq d$, and $n'_0 < n_0$.

If some d-target is not d-edge-colourable, then we can choose a d-target (G, m) with the following properties:

- (G, m) is not d-edge-colourable
- every smaller d-target is d-edge-colourable.

Let us call such a pair (G, m) a *minimum d-counterexample.* To prove 1.2, we prove two things:

- No 7-target is prime (theorem 3.1), and
- Every minimum 7-counterexample is prime (theorem 4.1).

It will follow that there is no minimum 7-counterexample, and so the theorem is true.

3 Discharging and unavoidability

In this section we prove the following, with a discharging argument.

3.1. *No 7-target is prime.*

The proof is broken into several steps, through this section. Let (G, m) be a 7-target, where G is three-connected. For every region r, we define

$$\alpha(r) = 14 - 7|E(C_r)| + 2 \sum_{e \in E(C_r)} m(e).$$

To aid the reader's intuition, let us explain where this comes from. We start with a 7-regular three-connected planar graph H, and so some of its regions have boundary of length two; let us call them "digon regions". Let us give each region r of H a weight of $14 - 5|E(C_r)|$; then it is easy to check using Euler's formula that the sum of all region-weights is 28. If now we suppress the parallel edges, obtaining (G, m) say, and we want to preserve the property that the sum of all region-weights is 28, then we need to distribute the weights of all the lost digon-regions among the regions that remain, and we have done it in the simplest way. For each edge e of G, corresponding to $m(e)$ parallel edges of H, there are $m(e)-1$ digon-regions to account for, formed by pairs of these edges. Each digon-region has a region-weight of four, and so we have a total weight of $4(m(e)-1)$ to redistribute; and we share this equally between the two regions of G incident with e. (Thus the weight of each region r of G is increased by $2(m(e)-1)$ for every edge e of G incident with r.) The α function defined above is the region-weighting that results from this process.

We observe first:

3.2. *The sum of $\alpha(r)$ over all regions r is positive.*

Proof. Since (G, m) is a 7-target, $m(\delta(v)) = 7$ for each vertex v, and, summing over all v, we deduce that $2m(E(G)) = 7|V(G)|$. By Euler's formula, the number of regions R of G satisfies $|V(G)| - |E(G)| + R = 2$, and so $4m(E(G)) - 14|E(G)| + 14R = 28$. But $2m(E(G))$ is the sum over all regions r, of $\sum_{e \in E(C_r)} m(e)$, and $14R - 14|E(G)|$ is the sum over all regions r of $14 - 7|E(C_r)|$. It follows that the sum of $\alpha(r)$ over all regions r equals 28. This proves 3.2. ∎

Our goal is to define a region-weighting, with total sum positive, so that for any region of positive weight, there must be one of the reducible configurations close to it. The α function just defined does not work yet. We obtained it by splitting equally the weights on digon-regions, but it is better to share out these digon-region weights a little less equally; we should give more weight to big regions and less to small, when we have the chance. More exactly, when some edge e is incident with a big region and a small region, we should distribute the weight from the digon-regions represented by e unevenly, sending one more to the big region and one less to the small one (and sometimes not quite this either). We are about to define a β function that makes this adjustment.

For every edge e of G, define $\beta_e(s)$ for each region s as follows. Let r, r' be the two regions incident with e.

- If $s \neq r, r'$ then $\beta_e(s) = 0$.
- If r, r' are both big, or both tough, or both small and not tough, then $\beta_e(r), \beta_e(r') = 0$.

[β**0**]**:** If r' is tough, and r is small and not tough, then $\beta_e(r) = -\beta_e(r') = 1$.

Henceforth we assume that r is big and r' is small; let f, g be the edges of $C_{r'} \setminus e$ that share an end with e.

[β**1**]**:** If e is a door for r (and hence $m(e) = 1$) then $\beta_e(r) = \beta_e(r') = 0$.

[β**2**]**:** If r' is a triangle with $m(r') \geq 5$ then $\beta_e(r) = -\beta_e(r') = 2$.

[β**3**]**:** Otherwise $\beta_e(r) = -\beta_e(r') = 1$.

For each region r, define $\beta(r)$ to be the sum of $\beta_e(r)$ over all edges e. We see that the sum of $\beta(r)$ over all regions r is zero.

Let α, β be as above. Then the sum over all regions r of $\alpha(r) + \beta(r)$ is positive, and so there is a region r with $\alpha(r) + \beta(r) > 0$. Let us examine the possibilities for such a region. There now begins a long case analysis, and to save writing we just say "by Conf(7)" instead of "since (G, m) does not contain Conf(7)", and so on.

3.3. *If r is a big region and $\alpha(r)+\beta(r)>0$, then (G,m) is not prime.*

Proof. Suppose that (G,m) is prime. Let $C=C_r$. Suppose that $\alpha(r)+\beta(r)>0$; that is,

$$\sum_{e\in E(C)}(7-2m(e)-\beta_e(r))<14.$$

For $e\in E(C)$, define $\phi(e)=2m(e)+\beta_e(r)$, and let us say e is *major* if $\phi(e)>7$. If e is major, then since $\beta_e(r)\le 3$, it follows that $m(e)\ge 3$ and that e is 4-heavy. If $m(e)=3$ and e is major, then by Conf(1) the edges consecutive with e on C have multiplicity at most two. It follows that no two major edges are consecutive, since G has minimum degree at least three. Further when e is major, $\beta_e(r)$ is an integer from the β-rules, and therefore $\phi(e)\ge 8$.

Let D be the set of doors for C. Let

- $\xi=2$ if there are consecutive edges e,f in C such that $\phi(e)>9$ and f is a door for r,
- $\xi=3$ if not, but there are consecutive edges e,f in C such that $\phi(e)=9$ and f is a door for r,
- $\xi=4$ otherwise.

(1) *Let e,f,g be the edges of a path of C, in order, where e,g are major. Then*

$$(7-\phi(e))+2(7-\phi(f))+(7-\phi(g))\ge 2\xi|\{f\}\cap D|.$$

Let r_1,r_2,r_3 be the regions different from r incident with e,f,g respectively. Now $m(e)\le 5$ since G has minimum degree three, and if $m(e)=5$ then r_1 is big, by Conf(11), and so $\beta_e(r)=0$. If $m(e)=4$ then $\beta_e(r)\le 2$; and so in any case, $\phi(e)\le 10$. Similarly $\phi(g)\le 10$. Also, $\phi(e),\phi(g)\ge 8$ since e,g are major. Thus $\phi(e)+\phi(g)\in\{16,17,18,19,20\}$.

Since f is consecutive with a major edge, $m(f)\le 2$. Further if $m(f)=2$ then r_2 is not a triangle with multiplicity at least 5 by Conf(1) so rule $\beta 2$ does not apply. Therefore it follows from the β-rules that $\phi(f)\le 5$ and if $m(f)=1$ then $\phi(f)\le 4$.

First, suppose that one of $\phi(e),\phi(g)\ge 10$, say $\phi(e)=10$. In this case we must show that $2\phi(f)\le 18-\phi(g)-2\xi|\{f\}\cap D|$. It is enough to show that $2\phi(f)\le 8-2\xi|\{f\}\cap D|$. Now $m(e)\ge 4$ and e is 5-heavy by the β-rules, and so $m(f)=1$, since G is three-connected and by Conf(1). If f is a door then $\phi(f)=2$ by rule $\beta 1$ and $\xi=2$ so $2\phi(f)\le 8-2\xi|\{f\}\cap D|$. If f is not a door then since $\phi(f)\le 4$, it follows that $2\phi(f)\le 8-2\xi|\{f\}\cap D|$. So we may assume that $\phi(e),\phi(g)\le 9$.

Next, suppose that one of $\phi(e),\phi(g)=9$, say $\phi(e)=9$. By the β-rules, we have $m^+(e)=m(e)+1=5$. We must show that $2\phi(f)\le 19-\phi(g)-2\xi|\{f\}\cap D|$; it is enough to show $2\phi(f)\le 10-2\xi|\{f\}\cap D|$. Since $\phi(f)\le 5$ we may assume that f is a door. Thus $\phi(f)=2$ and $\xi\le 3$, so $4=2\phi(f)\le 19-\phi(g)-2\xi|\{f\}\cap D|$. We may therefore assume that $\phi(e)+\phi(g)=16$.

So, suppose that $\phi(e)+\phi(g)=16$ and so $\phi(e)=\phi(g)=8$. Now $\xi\le 4$ and we must show that $2\phi(f)\le 12-2\xi|\{f\}\cap D|$. Again, if f is not a door then $2\phi(f)\le 12$ as required. If f is a door then $2\phi(f)=4\le 12-2\xi|\{f\}\cap D|$. This proves (1).

(2) *Let e,f be consecutive edges of C, where e is major. Then*

$$(7-\phi(e))+2(7-\phi(f))\ge 2\xi|\{f\}\cap D|.$$

We have $\phi(e) \in \{8, 9, 10\}$. Suppose first that $\phi(e) = 10$. We must show that $2\phi(f) \leq 11 - 2\xi|\{f\} \cap D|$; but $m(f) = 1$ by Conf(1) since e is 5-heavy. Since $\phi(f) \leq 4$ we may assume that f is a door. Thus $\phi(f) = 2$ and $\xi \leq 2$, as needed.

Next, suppose that $\phi(e) \leq 9$; it is enough to show that $2\phi(f) \leq 12 - 2\xi|\{f\} \cap D|$. Now e is 4-heavy and $m(f) \leq 2$ so $\phi(f) \leq 6$ by the β-rules. We have $\xi \leq 4$. Since $\phi(f) \leq 6$, we may assume that f is a door. If f is a door, then $2\phi(f) = 4 \leq 12 - 2\xi|\{f\} \cap D|$. This proves (2).

For $i = 0, 1, 2$, let E_i be the set of edges $f \in E(C)$ such that f is not major, and f meets exactly i major edges in C. By (1), for each $f \in E_2$ we have

$$\frac{1}{2}(7 - \phi(e)) + (7 - \phi(f)) + \frac{1}{2}(7 - \phi(g)) \geq \xi|\{f\} \cap D|$$

where e, g are the major edges meeting f. By (2), for each $f \in E_1$ we have

$$\frac{1}{2}(7 - \phi(e)) + (7 - \phi(f)) \geq \xi|\{f\} \cap D|$$

where e is the major edge consecutive with f. Finally, for each $f \in E_0$ we have

$$7 - \phi(f) \geq \xi|\{f\} \cap D|$$

since $\phi(f) \leq 7$, and $\phi(f) = 2$ if $f \in D$. Summing these inequalities over all $f \in E_0 \cup E_1 \cup E_2$, we deduce that $\sum_{e \in E(C)}(7 - \phi(e)) \geq \xi|D|$. Consequently

$$14 > \sum_{e \in E(C)} (7 - 2m(e) - \beta_e(r)) \geq \xi|D|.$$

But $|D| \geq 4$ since r is big, and so $\xi \leq 3$. If $\xi = 3$, then $|D| = 4$, contrary to Conf(11). So $\xi = 2$ and $|D| \leq 6$. But then C_r has a 5-heavy edge with multiplicity at least four that is consecutive with a door and has at most five doors disjoint from it, contrary to Conf(11) and Conf(12). This proves 3.3. ∎

3.4. *If r is a triangle that is not tough, and $\alpha(r) + \beta(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime, and let $r = uvw$. Now $\alpha(r) = 2(m(uv) + m(vw) + m(uw)) - 7$, so

$$2(m(uv) + m(vw) + m(uw)) + \beta(r) > 7.$$

Let r_1, r_2, r_3 be the regions different from r incident with uv, vw, uw respectively. Since r is not tough, $m^+(r) \leq 6$, and so $m(r) \leq 6$ as well.

Suppose first that r has multiplicity six and hence $\beta(r) > -5$. Then r_1, r_2, r_3 are all big. Suppose that $m(uv) = 4$. Then rule $\beta 2$ applies to give $\beta(r) = -6$, a contradiction. Thus r has at least two edges with multiplicity at least two. Rules $\beta 2$ and $\beta 3$ apply giving $\beta(r) \leq -5$, a contradiction.

Suppose that r has multiplicity five and so $\beta(r) > -3$. Then at least two of r_1, r_2, r_3 are big, say r_2 and r_3, and so $\beta_{vw}(r) + \beta_{uw}(r) \leq -2$. Consequently $\beta_{uv}(r) > -1$ so we may assume that r_1 is a tough triangle uvx. By Conf(2), $m(ux) = m(vx) = 1$. Since uvx is tough, $m(uv) \geq 2$. Suppose that $m(uv) = 3$. Then by Conf(4), $m^+(ux) = m^+(vx) = 1$, contradicting the fact that uvx is tough.

So $m(uv) = 2$, $m(uvx) = 4$ and we may assume that $m(vw) = 2$. But by Conf(4), $m^+(ux) = 1$, contradicting the fact that uvx is tough.

Suppose that r has multiplicity four. Then $\beta(r) > -1$. Since $m^+(r) \leq 6$ we may assume that r_1 is big, so $\beta_{uv}(r) = -1$. Now if r_2 is tough then $\beta_{vw}(r) = 1$, and otherwise $\beta_{vw}(r) \leq 0$. Thus by symmetry we may assume that r_2 is a tough triangle vwx and r_3 is small. Suppose that $m(uv) = 2$. By Conf(4), $m^+(vx) + m(vw) + m(uw) + 1 \leq 5$. Also by Conf(4), $m(uv) + m(vw) + m^+(wx) \leq 5$. Since $m(uv) + m(vw) + m(uw) = 4$ it follows that $m^+(vx) + m(vw) + m^+(wx) \leq 5$, contradicting the fact that vwx is tough.

Therefore we may assume that r has multiplicity three. Now $\beta(r) > 1$. By the rules, if r_1 is tough then $\beta_{uv}(r) = 1$. If r_1 is big then $\beta_{uv}(r) = -1$. Otherwise $\beta_{uv}(r) = 0$. By symmetry, it follows that r_1, r_2, r_3 are all small and we may assume that r_1, r_2 are tough triangles uvx and vwy. It follows from Conf(4) that $m^+(vx), m^+(ux) \leq 2$. This contradicts the fact that uvx is tough. This proves 3.4. ∎

3.5. *If r is a tough triangle with $\alpha(r) + \beta(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime, and let $r = uvw$. Now $\alpha(r) = 2(m(uv)+m(vw)+m(uw))-7$, so

$$2(m(uv) + m(vw) + m(uw)) + \beta(r) > 7.$$

Let r_1, r_2, r_3 be the regions different from r incident with uv, vw, uw respectively. Since r is small and tough, observe from the rules that $\beta_e(r) \leq 0$ for $e = uv, vw, uw$.

Let $X = \{u, v, w\}$. Since (G, m) is prime, it follows that $|V(G) \setminus X| \geq 3$, and so $m(\delta(X)) \geq 9$. But

$$m(\delta(X)) = m(\delta(u)) + m(\delta(v)) + m(\delta(w)) - 2m(uv) - 2m(uw) - 2m(vw),$$

and so $9 \leq 7 + 7 + 7 - 2m(uv) - 2m(uw) - 2m(vw)$, that is, r has multiplicity at most six. Since $m^+(r) \geq 7$, r has multiplicity at least four.

We claim that no two tough triangles share an edge. For suppose that uvw and uvx are tough triangles. By Conf(4), $m^+(vx)+m(uv)+m^+(uw) \leq 5$. Also by Conf(4) $m^+(vw)+m(uv)+m^+(ux) \leq 5$. Since $m^+(vw) + m^+(uw) + m(uv) \geq 6$, $m^+(vx) + m^+(ux) + m(uv) \leq 4$, contradicting the fact that r_1 is tough.

Suppose first that r has multiplicity six and so $\beta(r) > -5$. By Conf(2), none of r_1, r_2, r_3 is a triangle. If $m(uv) = 4$ then by Conf(6), r_1, r_2, r_3 are all big, contradicting the fact that r is tough. If $m(uv) = 3$, we may assume without loss of generality that $m(vw) = 2$. Then by Conf(6), r_1 and r_2 are big, and rule $\beta 2$ applies, contradicting that $\beta(r) > -5$. By symmetry we may therefore assume that $m(uv) = m(vw) = m(uw) = 2$. By Conf(6) we can assume that r_1, r_2 are big and rule $\beta 2$ applies again. This contradicts that $\beta(r) > -5$.

Consequently r has multiplicity at most five. Then none of r_1, r_2, r_3 is tough and so $\beta(r) \leq -3$, contradicting that $2(m(uv) + m(vw) + m(uw)) + \beta(r) > 7$. This proves 3.5. ∎

3.6. *If r is a small region with length at least four and with $\alpha(r) + \beta(r) > 0$, then (G, m) is not prime.*

Proof. Suppose that (G, m) is prime. Let $C = C_r$. Since $\alpha(r) = 14 - 7|E(C)| + 2\sum_{e \in E(C)} m(e)$, it follows that

$$14 - 7|E(C)| + 2 \sum_{e \in E(C)} m(e) + \sum_{e \in E(C)} \beta_e(r) > 0,$$

that is,

$$\sum_{e \in E(C)} (2m(e) + \beta_e(r) - 7) > -14.$$

For each $e \in E(C)$, let

$$\phi(e) = 2m(e) + \beta_e(r),$$

(1) *For every $e \in E(C)$, $\phi(e) \in \{1, 2, 3, 4, 5, 6, 7\}$.*

Since r is not a triangle, $\beta_e(r) \in \{-1, 0, 1\}$. It follows from Conf(11) that $m(e) \leq 4$. Further, if $m(e) = 4$ then $m^+(e) = 4$ and $\beta_e(r) = -1$. This proves (1).

For each integer i, let E_i be the set of edges of C such that $\phi(e) = i$. From (1) $E(C)$ is the union of $E_1, E_2, E_3, E_4, E_5, E_6, E_7$.

Let e be an edge of C and denote by r' its second region. We now make a series of observations that are easily checked from the β-rules and the fact that $2m(e) - 1 \leq \phi(e) \leq 2m(e) + 1$, as well as Conf(6) which implies that if $m(e) = 3$ then r' is not tough.

(2) *$e \in E_1$ if and only if $m(e) = m^+(e) = 1$ and e is not a door for r'.*

(3) *$e \in E_2$ if and only if $m(e) = 1$ and either*

- *$m^+(e) = 1$ and e is a door for r', or*
- *$m^+(e) = 2$ and r' is not a tough triangle.*

(4) *$e \in E_3$ if and only if either*

- *$m(e) = 1$ and r' is a tough triangle, or*
- *$m(e) = m^+(e) = 2$.*

(5) *$e \in E_4$ if and only if $m(e) = 2$, $m^+(e) = 3$ and r' is not a tough triangle.*

(6) *$e \in E_5$ if and only if either*

- *$m(e) = 2$ and r' is a tough triangle, or*
- *$m(e) = m^+(e) = 3$.*

(7) $e \in E_6$ *if and only if* $m(e) = 3$ *and* $m^+(e) = 4$.

(8) $e \in E_7$ *if and only if* $m(e) = 4$ *and* $m^+(e) = 4$.

(9) *No edge in* E_7 *is consecutive with an edge in* $E_6 \cup E_7$.

Suppose that edges $e, f \in E(C)$ share an end v, and $e \in E_7$. Since v has degree at least three it follows that $m(e) + m(f) \leq 6$ so $f \notin E_6 \cup E_7$. This proves (9).

(10) *Let* e, f, g *be consecutive edges of* C. *If* $e, g \in E_7$ *then* $f \in E_1 \cup E_2 \cup E_3 \cup E_4$.

For by (9), $f \notin E_6$. Suppose that $f \in E_5$. Since $m(e) = m(g) = 4$ and G has minimum degree three, by (6) $m(f) = 2$ and the second region for f is a tough triangle r' with $m(r') = 4$. But $m^+(e) = m^+(g) = 4$, so r' is incident with two big regions; thus $m^+(r') = 5$, contradicting the fact that r' is tough. This proves (10).

For $1 \leq i \leq 7$, let $n_i = |E_i|$. Let $k = |E(C)|$.

(11) $5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 + k - n_7 \leq 13$.

Since

$$\sum_{e \in E(C)} (\phi(e) - 7) > -14,$$

we have $6n_1 + 5n_2 + 4n_3 + 3n_4 + 2n_5 + n_6 \leq 13$, that is,

$$5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 + k - n_7 \leq 13,$$

since $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = k$, proving (11).

(12) $4n_1 + 3n_2 + 2n_3 + n_4 + k \leq 12$ *and* $n_1 + n_2 \leq 2$.

By (9) we have $n_1 + n_2 + n_3 + n_4 + n_5 \geq n_7$. Suppose that $n_1 + n_2 + n_3 + n_4 + n_5 = n_7$. By Conf(7), the edges of C cannot all be in E_6, so $n_7 > 0$. Then k is even and every second edge of C is in E_7, so by (10), $n_5 = n_6 = 0$; and hence $n_1 + n_2 + n_3 + n_4 = \frac{k}{2}$ and $n_7 = \frac{k}{2}$. By (11) $3n_1 + 2n_2 + n_3 + \frac{3}{2}k \leq 13$. Therefore, either $n_1 + n_2 \leq 1$, or $k = 4$, or $n_1 + n_2 = 2$ and $k = 6$. But by Conf(9), every edge in E_7 is disjoint from an edge in $E_1 \cup E_2$, a contradiction. This proves that $n_1 + n_2 + n_3 + n_4 + n_5 \geq n_7 + 1$. The first inequality follows from (11) and the second from the fact that $k \geq 4$. This proves (12).

Case 1: $n_1 + n_2 = 2$.

Suppose that $k + n_1 \geq 6$. By (12), $n_3 = n_4 = 0$. By Conf(9), every edge in E_7 is disjoint from an edge in $E_1 \cup E_2$, and therefore, by (9), is consecutive with an edge in E_5. Further, by (10) no edge in E_5 meets two edges in E_7, and so $n_5 \geq n_7$, contradicting (11). This proves that $k + n_1 \leq 5$.

Suppose that $k = 5$. Then $n_2 = 2$, and so by (12), $n_3 = 0$ and $n_4 \leq 1$. Also $n_4 + n_5 + n_6 + n_7 = 3$. By (11), $n_7 \geq 2n_4 + n_5$. Suppose that $n_6 = 3$; then by (7), C has three edges of multiplicity three, each of whose second region is small. Further, by (3), if the edges in E_2 are consecutive, they are both incident with small regions. This contradicts Conf(14). Therefore $n_6 \leq 2$, and so $n_7 \geq 1$. By Conf(10), one of the edges in E_2 must be incident with a big region, and by (3), it must be a door for that region. Since $n_3 = 0$, it follows that the two edges in E_2 are disjoint. It follows that $n_7 = 1$. By (11), $n_4 = 0$ and $n_6 \geq 1$. Let $e \in E_6$. Then e must be consecutive with both edges in E_2, for it is not consecutive with the edge in E_7. But then e is disjoint only from edges in $E_7 \cup E_5$, contrary to Conf(7).

Suppose that $k = 4$. Then $n_1 \leq 1$. By Conf(10) and (3), $n_1 \geq n_7$. Therefore by (11), $3n_3 + 2n_4 + n_5 \leq 1$, and so $n_3 = n_4 = 0$ and $n_5 \leq 1$. Since $n_5 + n_6 + n_7 = 2$, and edges in E_5, E_6, E_7 have multiplicity at least two, three, four, respectively, Conf(3) implies $n_7 = 0$ and $n_6 \leq 1$. Hence $n_5 = n_6 = 1$. From (11) it follows that $n_1 = 0$. By Conf(5) the edge disjoint from the edge in E_6 must have multiplicity one and be incident with a big region. By (3) this edge must be in E_1, a contradiction. This proves that Case 1 does not apply.

Case 2: $n_1 + n_2 = 1$.

Let $e_0 \in E_1 \cup E_2$. We claim that neither edge consecutive with e_0 is in $E_6 \cup E_7$. For let e_1 be an edge consecutive with e_0 on C and suppose that $e_1 \in E_6 \cup E_7$; then by (7), $m^+(e_1) = 4$. But all edges disjoint from e_1 on C are not in $E_1 \cup E_2$ and therefore are 2-heavy and their second regions are not triangles with multiplicity three. Therefore Conf(7) implies that at least four edges disjoint from e_0 are not 3-heavy and hence $n_3 + n_4 \geq 4$ and that $k \geq 7$, contradicting (11). This proves that all edges in $E_6 \cup E_7$ are disjoint from e_0, and so $n_3 + n_4 + n_5 \geq 2$. We consider two cases:

Subcase 2.1: $n_7 \geq 1$.
Let $f \in E_7$. By Conf(9), if an edge e_1 meets both e_0 and f then $m(e_1) = 1$ and so $e_1 \in E_3$. By (10) an edge meeting two edges in E_7 is in $E_3 \cup E_4$. Summing over the edges meeting $E_7 \cup \{e_0\}$ it follows that $2n_3 + 2n_4 + n_5 \geq 2(n_7 + 1)$. From (11) we deduce $5n_1 + 4n_2 + n_3 + n_7 + k \leq 11$; thus $k + n_1 + n_3 + n_7 \leq 7$. By Conf(10), $m^+(e_0) = 1$, so by (3), either $e_0 \in E_1$ or there is an edge of multiplicity one disjoint from e_0. Since $n_1 + n_2 = 1$, such an edge would be in E_3; it follows that $n_1 + n_3 \geq 1$. We deduce that $k \leq 5$. If $k = 5$ then by Conf(9) the edge meeting e_0 and f is in E_3, and so $n_1 + n_3 \geq 2$, a contradiction.

Thus $k = 4$. Then by Conf(10) and (3), $e_0 \in E_1$. By Conf(3) the two edges consecutive with e_0 are in E_3. But then $k + n_1 + n_3 + n_7 = 8$, a contradiction.

Subcase 2.2: $n_7 = 0$.
Let $e_0, \ldots, e_{k-1}$ denote the edges of C listed in consecutive order. Since $n_3 + n_4 + n_5 \geq 2$, (11) implies $k \leq 7$.

Suppose that $k = 7$. Then the inequality in (11) is tight, and we have $n_2 = 1$, $n_5 = 2$ and $n_6 = 4$. Consequently $n_1 = n_3 = n_4 = 0$. Then $e_1, e_6 \in E_5$, and so by (6) and (7) are 3-heavy with multiplicity at least two, and $e_2, e_3, e_4, e_5 \in E_6$. This is a contradiction by Conf(8).

Suppose that $k = 6$. We know $e_1, e_5 \notin E_6$. By (11), $n_1 + 3n_3 + 2n_4 + n_5 \leq 3$, but $n_3 + n_4 + n_5 \geq 2$ so $n_3 = 0$ and consequently $n_4 + n_5 + n_6 = 5$. Also $n_1 + 2n_4 + n_5 \leq 3$. In particular $n_4 \leq 1$. Suppose that $n_4 = 1$, then $n_6 = 3$ and $n_5 = 1$ and $e_2, e_3, e_4 \in E_6$. It follows from Conf(8) that $m^+(e_0) = 1$, and so $n_1 = 1$, contradicting that $n_1 + 2n_4 + n_5 \leq 3$. Thus $n_4 = 0$. It follows that $n_5 + n_6 = 5$. This contradicts Conf(16).

Next suppose that $k = 5$. We know $e_1, e_4 \notin E_6$. By (11), $n_1 + 3n_3 + 2n_4 + n_5 \leq 4$. Suppose that $2n_3 + n_4 \geq 2$. Then $n_1 + n_3 + n_4 + n_5 \leq 2$, and so $n_2 + n_6 \geq 3$. Since $n_6 \leq 2$ we may assume that $e_2, e_3 \in E_6$ and $e_0 \in E_2$, contrary to Conf(14). It follows that $2n_3 + n_4 \leq 1$. Consequently $n_3 = 0$ and $n_5 + n_6 \geq 3$. Thus we may assume that $m^+(e_1), m^+(e_2), m^+(e_3), m^+(e_4) \geq 3$, and e_1 is 3-heavy. This contradicts Conf(15).

Finally, suppose that $k = 4$. By (11), $n_1 + 3n_3 + 2n_4 + n_5 \leq 5$. By Conf(5), at least one of $m^+(e_1), m^+(e_3) \leq 2$, so we may assume that $e_1 \in E_3$ and so $n_3 = 1$. Since $m^+(e_1) = 2$, Conf(8) implies $e_3 \notin E_6$, and so $e_3 \in E_5$. Suppose that $e_0 \in E_1$. Then $2n_4 + n_5 \leq 1$, and so $n_4 = 0$ and $n_5 \leq 1$. Since $e_2 \notin E_5$, $e_2 \in E_6$. Since $m(e_2) = 3$ by (7), it follows from Conf(3) that $m(e_1) = 1$, $m(e_3) = 2$ and from (4) and (6) that e_1, e_3 are incident with tough triangles v_1v_2x and v_3v_0y. This contradicts Conf(13).

Thus $e_0 \in E_2$ and so $m^+(e_0) = 2$. By Conf(8), $e_2 \notin E_6$. Hence $e_2 \in E_4 \cup E_5$. Since $2n_4 + n_5 \leq 2$ and $e_3 \in E_5$, it follows that $e_2 \in E_5$. By Conf(13), the second region for e_1 is not a tough triangle, and so $m(e_1) = 2$. Since $m(e_2), m(e_3) \geq 2$, Conf(3) tells us $m(e_3) = 2$ and the second region for e_3 is a tough triangle v_0v_3x. But this contradicts Conf(13). We conclude that Case 2 does not apply.

Case 3: $n_1 + n_2 = 0$.

In this case, C has no doors, so by Conf(9) $n_7 = 0$. Suppose that $n_6 \geq 1$ and let $e \in E_6$. Then by Conf(7), there are at least four edges disjoint from e that are not 3-heavy. Therefore $n_3 + n_4 \geq 4$ and $k \geq 7$, contradicting (11). It follows that $n_1 = n_2 = n_6 = n_7 = 0$, and so $n_3 + n_4 + n_5 = k$. By (11), $3n_3 + 2n_4 + n_5 + k \leq 13$, and $k \leq 6$. Further, $3n_3 + 2n_4 + 2n_5 + k \leq 13 + n_5$, and so $n_5 - n_3 \geq 3k - 13$.

Suppose first that $k = 6$; then $n_5 \geq 5$, so by (6) C has five 3-heavy edges, each with multiplicity two or three, contrary to Conf(16). Suppose that $k = 5$; then $3n_3 + 2n_4 + n_5 \leq 8$, and so, since $n_3 + n_4 + n_5 = 5$, $n_3 \leq 1$. Also $n_5 \geq 1$, and if $n_3 = 1$ then $n_4 \leq 1$. Consequently we may assume that there is an ordering $e_0, \ldots, e_4$ of $E(C)$, where $e_0 \in E_5$ and $e_2, e_3 \in E_4 \cup E_5$, contrary to Conf(15).

Finally, suppose that $k = 4$; then $3n_3 + 2n_4 + n_5 \leq 9$. Since, by (5) and (6), every edge $f \in E_4 \cup E_5$ has $m^+(f) \geq 3$, Conf(5) tells us there are two consecutive edges in E_3, say e_0 and e_1. Hence $n_5 \geq 1$ and $n_4 + n_5 = 2$. We may assume that $e_2 \in E_4 \cup E_5$ and $e_3 \in E_5$. Since $m(e_2) \geq 2$, Conf(3) implies that $m(e_1) + m(e_3) \leq 4$. Thus by (4) and (6), either the second region for e_1 is a tough triangle, or the second region for e_3 is a tough triangle and $m(e_1) = 2$. Further, $m^+(e_1) + m^(e_3) = 5$. This contradicts Conf(13). We conclude that Case 3 does not apply.

This completes the proof of 3.6. ∎

Proof of 3.1. Suppose that (G, m) is a prime 7-target, and let α, β be as before. Since the sum over all regions r of $\alpha(r) + \beta(r)$ is positive, there is a region r with $\alpha(r) + \beta(r) > 0$. But this is contrary to one of 3.3, 3.4, 3.5, 3.6. This proves 3.1. ∎

4 Reducibility

Now we begin the second half of the paper, devoted to proving the following.

4.1. *Every minimum 7-counterexample is prime.*

Again, the proof is broken into several steps. Clearly no minimum 7-counterexample (G, m) has an edge e with $m(e) = 0$, because deleting e would give a smaller 7-counterexample; and by 2.1, every minimum 7-counterexample satisfies the conclusions of 2.1. Thus, it remains to check that (G, m) contains none of Conf(1)–Conf(16). In [3] we found it was sometimes just as easy to prove a result for general d instead of $d = 8$, and so the following theorem is proved there.

4.2. *If (G, m) is a minimum d-counterexample, then every triangle has multiplicity less than d.*

It turns out that Conf(1) is a reducible configuration for every d as well; this follows easily from 2.1 and is proved in [3].

4.3. *No minimum d-counterexample contains Conf(1).*

If (G, m) is a d-target, and x, y are distinct vertices both incident with some common region r, we define $(G, m) + xy$ to be the d-target (G', m') obtained as follows:

- If x, y are adjacent in G, let $(G', m') = (G, m)$.
- If x, y are non-adjacent in G, let G' be obtained from G by adding a new edge xy, extending the drawing of G to one of G' and setting $m'(e) = m(e)$ for every $e \in E(G)$ and $m'(xy) = 0$.

Let (G, m) be a d-target, and let x-u-v-y be a three-edge path of G, where x, y are incident with a common region. Let (G', m') be obtained as follows:

- If x, y are adjacent in G, let $G' = G$, and otherwise let G' be obtained from G by adding the edge xy and extending the drawing of G to one of G'.
- Let $m'(xu) = m(xu) - 1$, $m'(uv) = m(uv) + 1$, $m'(vy) = m(vy) - 1$, $m'(xy) = m(xy) + 1$ if $xy \in E(G)$ and $m'(xy) = 1$ otherwise, and $m'(e) = m(e)$ for all other edges e.

If (G, m) is a minimum d-counterexample, then because of the second statement of 2.1, it follows that (G', m') is a d-target. We say that (G', m') is obtained from (G, m) by *switching on the sequence x-u-v-y*. If (G', m') admits a d-edge-colouring, we say that the path x-u-v-y is *switchable.*

4.4. *No minimum 7-counterexample contains Conf(2) or Conf(3).*

Proof. To handle both cases at once, let us assume that (G, m) is a 7-target, and uvw, uwx are triangles with $m(uv) + m(uw) + m(vw) + m(ux) \geq 7$, (where possibly $m(uw) = 0$); and either (G, m) is a minimum 7-counterexample, or $m(uw) = 0$ and deleting uw gives a minimum 7-counterexample (G_0, m_0) say. Let (G, m') be obtained by switching (G, m) on u-v-w-x.

(1) *(G, m') is not smaller than (G, m).*

Because suppose it is. Then it admits a 7-edge-colouring; because if (G, m) is a minimum 7-counterexample this is clear, and otherwise $m(uw) = 0$, and (G', m') is smaller than (G_0, m_0). Let $F'_1, \ldots, F'_7$ be a 7-edge-colouring of (G', m'). Since

$$m'(uv) + m'(uw) + m'(vw) + m'(ux) \geq 8,$$

one of $F'_1, \ldots, F'_7$, say F'_1, contains two of uv, uw, vw, ux and hence contains vw, ux. Then

$$(F'_1 \setminus \{vw, ux\}) \cup \{uv, wx\}$$

is a perfect matching, and it together with $F'_2, \ldots, F'_7$ provide a 7-edge-colouring of (G, m), a contradiction. This proves (1).

From (1) we deduce that $\max(m(ux), m(vw)) < \max(m(uv), m(wx))$. Consequently,

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 6,$$

by (1) applied with u, w exchanged; and

$$m(uv) + m(ux) + m(wx) + m(uw) \leq 6,$$

by (1) applied with v, x exchanged. Consequently $m(ux) > m(wx)$, and hence $m(ux) \geq 2$; and $m(vw) > m(wx)$, and so $m(vw) \geq 2$. Since $m(uv) + m(uw) + m(vw) + m(wx) \leq 6$ and $m(vw) \geq 2$, it follows that $m(uv) \leq 3$; and since $\max(m(ux), m(vw)) < \max(m(uv), m(wx))$, it follows that $m(uv) = 3$, $m(vw) = m(ux) = 2$ and $m(wx) = 1$. But this is contrary to (1), and so proves 4.4. ▮

5 Guenin's cuts

Next we introduce a method of Guenin [6]. Let G be a three-connected graph drawn in the plane, and let G^* be its dual graph; let us identify $E(G^*)$ with $E(G)$ in the natural way. A *cocycle* means the edge-set of a cycle of the dual graph; thus, $Q \subseteq E(G)$ is a cocycle of G if and only if Q can be numbered $\{e_1, \ldots, e_k\}$ for some $k \geq 3$ and there are distinct regions $r_1, \ldots, r_k$ of G such that $1 \leq i \leq k$, e_i is incident with r_i and with r_{i+1} (where r_{k+1} means r_1). Guenin's method is the use of the following theorem, a proof of which is given in [3].

5.1. *Suppose that $d \geq 1$ is an integer such that every $(d-1)$-regular oddly $(d-1)$-edge-connected planar graph is $(d-1)$-edge-colourable. Let (G, m) be a minimum d-counterexample, and let x-u-v-y be a path of G with x, y on a common region. Let (G', m') be obtained by switching on x-u-v-y, and let $F_1, \ldots, F_d$ be a d-edge-colouring of (G', m'), where $xy \in F_k$. Then none of $F_1, \ldots, F_d$ contain both uv and xy. Moreover, let $I = \{1, \ldots, d\} \setminus \{k\}$ if $xy \notin E(G)$, and $I = \{1, \ldots, d\}$ if $xy \in E(G)$. Then for each $i \in I$, there is a cocycle Q_i of G' with the following properties:*

- *for $1 \leq j \leq d$ with $j \neq i$, $|F_j \cap Q_i| = 1$;*
- *$|F_i \cap Q_i| \geq 5$;*
- *there is a set $X \subseteq V(G)$ with $|X|$ odd such that $\delta_{G'}(X) = Q_i$; and*
- *$uv, xy \in Q_i$ and $ux, vy \notin Q_i$.*

By the result of [4], every 6-regular oddly 6-edge-connected planar graph is 6-edge-colourable, so we can apply 5.1 when $d = 7$.

5.2. *No minimum 7-counterexample contains Conf(4) or Conf(5).*

Proof. To handle both at once, let us assume that (G, m) is a 7-target, and uvw, uwx are two triangles with $m^+(uv) + m(uw) + m^+(wx) \geq 6$; and either (G, m) is a minimum 7-counterexample, or $m(uw) = 0$ and deleting uw gives a minimum 7-counterexample. We claim that u-x-w-v-u is switchable. For suppose not; then we may assume that $m(vw) > max(m(uv), m(wx))$ and $m(vw) \geq m(ux)$. Now we do not have Conf(2) or Conf(3) by 4.4 so

$$m(uv) + m(uw) + m(vw) + m(wx) \leq 6,$$

and yet $m(uv) + m(uw) + m(wx) \geq 4$ since $m^+(uv) + m(uw) + m^+(wx) \geq 6$; and so $m(vw) \leq 2$. Consequently $m(uv), m(wx) = 1$, and $m(ux) \leq 2$. Since u-x-w-v-u is not switchable, it follows that $m(ux) = m(vw) = 2$; and since $m^+(uv) + m(uw) + m^+(wx) \geq 6$, it follows that $m(uw) \geq 2$ giving Conf(2), contrary to 4.4. This proves that u-x-w-v-u is switchable.

Let r_1, r_2 be the second regions incident with uv, wx respectively, and for $i = 1, 2$ let D_i be the set of doors for r_i. Let $k = m(uv) + m(uw) + m(wx) + 2$. Let (G, m') be obtained by switching on u-x-w-v-u, and let $F_1, \ldots, F_7$ be a 7-edge-colouring of (G, m'), where F_i contains one of uv, uw, wx for $1 \leq i \leq k$. For $1 \leq i \leq 7$, let Q_i be as in 5.1.

(1) *For $1 \leq i \leq 7$, either $F_i \cap Q_i \cap D_1 \neq \emptyset$, or $F_i \cap Q_i \cap D_2 \neq \emptyset$; and both are nonempty if either $k = 7$ or $i = 7$.*

For let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = wx$, $e_2 = uw$, and $e_3 = uv$. Since F_j contains one of e_1, e_2, e_3 for $1 \leq j \leq k$, it follows that none of $e_4, \ldots, e_n$ belongs to any F_j with $j \leq k$ and $j \neq i$, and, if $k = 6$ and $i \neq 7$, that only one of them is in F_7. But since at most one of e_1, e_2, e_3 is in F_i and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 7$; so either e_4, e_5 belong only to F_i, or e_n, e_{n-1} belong only to F_i, and both if $k = 7$ or $i = 7$. But if e_4, e_5 are only contained in F_i, then they both have multiplicity one, and are disjoint, so e_4 is a door for r_1 and hence $e_4 \in F_i \cap Q_i \cap D_1$. Similarly if e_n, e_{n-1} are only contained in F_i then $e_n \in F_i \cap Q_i \cap D_2$. This proves (1).

Now $k \leq 7$, so one of r_1, r_2 is small since $m^+(uv) + m(uw) + m^+(wx) \geq 6$; and if $k = 7$ then by (1) $|D_1|, |D_2| \geq 7$, a contradiction. Thus $k = 6$, so both r_1, r_2 are small, but from (1) $|D_1| + |D_2| \geq 8$, again a contradiction. This proves 5.2. ∎

5.3. *No minimum 7-counterexample contains Conf(6).*

Proof. Let (G, m) be a minimum 7-counterexample, and suppose that uvw is a triangle with $m^+(uv) + m^+(uw) = 6$ and either $m(uv) \geq 3$ or $m(uv) = m(vw) = m(uw) = 2$ or u has degree at least four. Let r_1, r_2 be the second regions for uv, uw respectively, and for $i = 1, 2$ let D_i be the set of doors for r_i. Since we do not have Conf(4) by 5.2, neither of r_1, r_2 is a triangle. Let tu be the edge incident with r_2 and u different from uw. It follows from 4.3 that we do not have Conf(1) so $m(tu) \leq 2$, since $m(uv) + m(uw) \geq 4$ and $m(vw) + max(m(uv), m(uw)) \geq 4$. By 4.2, $m(vw) \leq m(uv)$. Thus the path t-u-v-w is switchable. Note that t, w are non-adjacent in G, since r_2 is not a triangle.

Let (G', m') be obtained by switching on this path, and let $F_1, \ldots, F_7$ be a 7-edge-colouring of it. Let $k = m(uv) + m(uw) + 2$; thus $k \geq 6$, since $m(uv) + m(uw) \geq 4$. By 5.1 we may assume that for $1 \leq j < k$, F_j contains one of uv, uw, and $tw \in F_k$.

Let $I = \{1, \ldots, 7\} \setminus \{k\}$, and for each $i \in I$, let Q_i be as in 5.1. Now let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$, $e_2 = uw$, and $e_3 = tw$. Since F_j contains one of e_1, e_2, e_3 for $1 \le j \le k$ it follows that none of $e_4, \ldots, e_n$ belong to any F_j with $j \le k$; and if $k = 6$ and $i \ne 7$, only one of them belongs to F_7. Since F_i contains at most one of e_1, e_2, e_3 and $|F_i \cap Q_i| \ge 5$, it follows that $n \ge 7$, and so either e_4, e_5 belong only to F_i, or e_n, e_{n-1} belong only to F_i; and both if either $k = 7$ or $i = 7$. Thus either $e_4 \in F_i \cap Q_i \cap D_2$ or $e_n \in F_i \cap Q_i \cap D_1$, and both if $k = 7$ or $i = 7$. Since $k \le 7$, one of r_1, r_2 is small since $m^+(uv) + m^+(uw) = 6$; and yet if $k = 7$ then $|D_1|, |D_2| \ge |I| = 6$, a contradiction. Thus $k = 6$, so r_1, r_2 are both small, and yet $|D_1| + |D_2| \ge 7$, a contradiction. This proves 5.3. ∎

5.4. *No minimum 7-counterexample contains Conf(7).*

Proof. Let (G, m) be a minimum 7-counterexample, with an edge uv with $m^+(uv) \ge 4$ incident with regions r_1 and r_2 and r_1 has length at least four. Suppose further that every edge e of C_{r_1} disjoint from uv is 2-heavy and not incident with a triangle with multiplicity three. It is enough to show that there are at least four edges on C_{r_1} disjoint from uv that are not 3-heavy. By 5.8 and 5.6 we do not have Conf(11) or Conf(9). Hence $m(uv) = 3$ and r_2 is small.

Let x-u-v-y be a path of C_r. By 5.2 we do not have Conf(5), so x and y are not adjacent in G. Since G has minimum degree three, $m(uv) \ge m(ux), m(vy)$ so x-u-v-y is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \ldots, F_7$ be a 7-edge-colouring of (G', m').

Since $m'(uv) + m'(xy) = 5$ we may assume by 5.1 that $uv \in F_i$ for $1 \le i \le 4$ and $xy \in F_5$. Let $I = \{1, \ldots, 7\} \setminus \{5\}$ and for $i \in I$, let the edges of Q_i in order be $e^i_1, \ldots, e^i_n, e^i_1$, where $e^i_1 = uv$ and $e^i_2 = xy$.

Since $|F_i \cap Q_i| \ge 5$ and F_i contains at most one of e^i_1, e^i_2, it follows that $n \ge 6$. Let D_2 denote the set of doors for r_2.

(1) *Let $i \in I$. If $i > k$ then $F_i \cap D_2$ is nonempty. Further, if $F_i \cap D_2$ is empty, or $i > k$ then e^i_3 is not 3-heavy, and either*

- *e^i_3 belongs to F_i, or*
- *e^i_4 belongs to F_i and $m(e^i_3) = m(e^i_4) = 1$ and e^i_3, e^i_4 belong to a triangle.*

For $1 \le j \le 5$, F_j contains one of e^i_1, e^i_2; and hence $e^i_3, \ldots, e^i_n \notin F_j$ for all $j \in \{1, \ldots, 5\}$ with $j \ne i$. Therefore $e^i_3, \ldots, e^i_n$ belong only to F_i, F_6, F_7. Since e^6_3 is 2-heavy, one of e^6_3, e^6_4 does not belong to F_6 and therefore belongs to F_7. It follows that $e^6_n, e^6_{n-1} \notin F_7$ so $F_6 \cap D_2$ is nonempty. Similarly, $F_7 \cap D_2$ is nonempty. This proves the first assertion.

Suppose that $F_i \cap D_2$ is empty, or $i > 5$; we have $|\{e^i_n, e^i_{n-1}\} \cap (F_6 \cup F_7)| \ge 1$. Without loss of generality say $|\{e^i_n, e^i_{n-1}\} \cap F_6| \ge 1$. It follows that e^i_3, e^i_4 belong only to F_i, F_7, so e^i_3 is not 3-heavy. On the other hand, e^i_3 is 2-heavy by hypothesis, so if $e^i_3 \notin F_i$, then e^i_3 has multiplicity one, $e^i_3 \in F_7$, e^i_4 belongs to F_i, has multiplicity one. Since e^i_3 is 2-heavy, e^i_3 and e^i_4 belong to a triangle. This proves (1).

Let I_1 denote the indices $i \le 6, i \ne 5$ such that e^i_3 is not 3-heavy and either $e^i_3 \in F_i$, or $e^i_4 \in F_i$ and e^i_3, e^i_4 have multiplicity one and belong to a triangle incident with r_1. From (1) and because r_2 is small, $|I_1| \ge 4$. Suppose that for $i \ne i' \in I_1$, the corresponding edges e^i_3 and $e^{i'}_3$ are the same. We may assume that $i' \le 4$. If $e^i_3 \in F_{i'}$, this is a contradiction. Otherwise $m(e^i_3) = m(e^i_4) = 1$ and e^i_3, e^i_4

belong to a triangle incident with r_1. It follows that $e_4^i = e_4^{i'}$ since e_3^i is not incident with a triangle of multiplicity three, and so $e_4^i \in F_{i'}$, a contradiction.

It follows that there are at least four edges of C_r disjoint from uv that are not 3-heavy. This proves 5.4. ∎

5.5. *No minimum 7-counterexample contains Conf(8).*

Proof. Let (G, m) be a minimum 7-counterexample, with an edge uv with multiplicity three, incident with regions r and r_1 where r_1 is small. Suppose that there is an edge f disjoint from e with $m^+(f) = m(f) + 1 = 2$, where every edge of $C_r \setminus \{f\}$ disjoint from e is 3-heavy with multiplicity at least two. Since e and f are disjoint r has length at least four. Let x-u-v-y be a path of C_r. By 5.2 we do not have Conf(5), so x and y are not adjacent in G. Since G has minimum degree at least three, it follows that $m(uv) \geq m(ux), m(vy)$ so x-u-v-y is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \dots, F_7$ be a 7-edge-colouring of (G', m'). Since $m'(uv) + m'(xy) = 5$ we may assume by 5.1 that $uv \in F_i$ for $1 \leq i \leq 4$ and $xy \in F_5$. Let $I = \{1, \dots, 7\} \setminus \{5\}$ and for $i \in I$, let Q_i be as in 5.1.

For $i \in I$, let the edges of Q_i in order be $e_1, \dots, e_n, e_1$, where $e_1 = uv$ and $e_2 = xy$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 6$. For $1 \leq j \leq 5$, F_j contains one of e_1, e_2; and hence for all $j \in \{1, \dots, 5\}$, $e_3, \dots, e_n \notin F_i$, and so $e_3, \dots, e_n$ belong only to F_i, F_6 or F_7. In particular when $i \in \{6, 7\}$, e_3 is not 3-heavy and so $e_3 = f$. It follows f belongs only to F_6, F_7; we assume without loss of generality $f \in F_6$. Let D_1 denote the set of doors for r_1. Denote by r_2 the second region for f and by D_2 its set of doors.

(1) *Let $i \in I$. At least one of $F_i \cap Q_i \cap D_1$, $F_i \cap Q_i \cap D_2$ is nonempty, and if $i = 7$ then both are nonempty.*

Suppose that $i = 7$. Then $e_3 = f \in F_6$ and $e_4, \dots, e_n$ belong only to F_7, and so e_4 is a door for r_2 and e_n is a door for r_1. Now suppose that $i < 7$. If $e_3 = f$, then since F_i contains at most one of e_1, e_2, e_3 and $|F_i \cap Q_i| \geq 5$, it follows that $n \geq 7$. It follows that $e_4, \dots, e_n$ belong only to F_7 or F_i, and so either e_4 is a door for r_2 or e_n is a door for r_1 as required. If $e_3 \neq f$ then e_3 is 3-heavy, and so F_i, F_6, F_7 each contain one of e_3, e_4. Therefore e_{n-1}, e_n belong only to F_i, and so e_n is a door for r_1. This proves (1).

By (1), $|D_1| + |D_2| \geq 7$, but r_1 and r_2 are both small, a contradiction. This proves 5.5. ∎

5.6. *No minimum 7-counterexample contains Conf(9).*

Proof. Let (G, m) be a minimum 7-counterexample, and suppose that some edge uv with $m(uv) = 4$ is incident with a region r of length at least four. Let x-u-v-y be a path of C_{r_1}. If x and y are adjacent, then since we do not have Conf(5) by 5.2, xy is incident with a big region. Therefore we may assume that x and y are nonadjacent.

We will show r has a door f disjoint from uv, and that if $m(xu) \geq 2$ then f is also disjoint from xu (and similarly for vy.)

Since $m(e) \geq 4$, this path is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \dots, F_7$ be a 7-edge-colouring of (G', m').

Thus we may assume that $uv \in F_i$ for $1 \le i \le 5$, and $xy \in F_6$. Further, if $m(xu) \ge 2$ then $xu \in F_7$ and simlarly for vy. Let $I = \{1, \dots, 7\} \setminus \{6\}$. For $i \in I$, let Q_i be as in 5.1. Since Q_i contains both uv, xy for each $i \in I$, it follows that for $1 \le j \le 7$, F_j contains at most one of uv, xy.

Consider now Q_7, and let the edges of Q_7 in order be $e_1, \dots, e_n, e_1$ where $e_1 = uv$ and $e_2 = xy$. For $1 \le j \le 6$, F_j contains one of e_1, e_2, and hence $e_3, \dots, e_n$ belong only to F_7. Since $e_3 \in C_r \setminus \{xu, uv, vy\}$ by the choice of the switchable path, e_3 is a door for r disjoint from uv. Further if $m(xu) \ge 2$ then e_3 is disjoint from xu, and similarly for vy.

This proves 5.6. ∎

5.7. *No minimum 7-counterexample contains Conf(10).*

Proof. Let (G, m) be a minimum 7-counterexample, and suppose that there is a region r of length between four and six incident with an edge uv with multiplicity four, and suppose that $m^+(e) \ge 2$ for every edge e of C_r disjoint from uv. Let x-u-v-y be a path of C_r. By 5.2, we do not have Conf(5) so x and y are not adjacent in G (and r has length five or six). Since $m(uv) = 4$, the path x-u-v-y is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \dots, F_7$ be a 7-edge-colouring of (G', m'). By 5.1 we may assume that $uv \in F_i$ for $1 \le i \le 5$, and $xy \in F_6$. Let $I = \{1, \dots, 7\} \setminus \{6\}$ and for $i \in I$, let Q_i be as in 5.1.

Define $\ell = |F_7 \cap E(C_r) \setminus \{xu, uv, vy\}|$. Suppose that $\ell = 0$; then let the edges of Q_7 in order be $e_1, \dots, e_n, e_1$, where $e_1 = uv$ and $e_2 = xy$. Since $|F_i \cap Q_i| \ge 5$ and F_i contains at most one of e_1, e_2, it follows that $n \ge 6$. For $1 \le j \le 6$, F_j contains one of e_1, e_2; and hence $e_3, \dots, e_n$ belong only to F_7. But e_3 is an edge of $E(C_r) \setminus \{xu, uv, vy\}$ by the choice of the switchable path, a contradiction. Thus $\ell \ge 1$. Fix an edge $f \in F_7 \cap E(C_r) \setminus \{xu, uv, vy\}$ and let I_1 denote the indices $i \in I$ for which $f \in Q_i$.

(1) $|I_1| \le 3$.

Denote by r_2 the second region for f and denote by D_2 the set of doors for r_2. Suppose that $|I_1| \ge 4$. For $i \in I_1$, let the edges of Q_i in order be $e_1, \dots, e_n, e_1$, where $e_1 = uv$, $e_2 = xy$ and $e_3 = f$. Since $|F_i \cap Q_i| \ge 5$ and F_i contains at most one of e_1, e_2,e_3, it follows that $n \ge 7$. For $1 \le j \le 7$, F_j contains one of e_1, e_2, e_3; and hence $e_4, \dots, e_n$ belong only to F_i. Further, e_4 is incident with r_2 and therefore is a door for r_2. But then $|D_2| \ge 4$, so $m^+(f) = 1$, a contradiction. This proves (1).

Since r has length at most six, there are two cases:

Case 1: $\ell = 1$. Let $f \in F_7 \cap E(C_r) \setminus \{xu, uv, vy\}$, denote by r_2 the second region for f and denote by D_2 the set of doors for r_2. Since the edges of $C_r \setminus \{xu, uv, vy, f\}$ each belong to F_j for some $j \ne 7$, there are at most two indices $i \in I$ for which $f \notin Q_i$. But then we have $|I_1| \ge 4$, contradicting (1).

Case 2: $\ell = 2$. Let $f, f' \in F_7 \cap E(C_r) \setminus \{xu, uv, vy\}$. If $m(f') \ge 2$, then $f' \in F_j$ for some $j \ne 7$, and so there are at most two values of $i \in I$ for which $f \notin Q_i$. Then $|I_1| \ge 4$, contradicting (1). So $m(f') = 1$ and by symmetry, $m(f) = 1$. There is at most one value of $i \in I$ for which $f, f' \notin Q_i$. Therefore, without loss of generality we may assume that there at least three indices $i \in I$, $f \in Q_i$, and so $|I_1| = 3$. Denote by r_2 the second region for f and D_2 the set of doors for r_2. For each $i \in I_1$, it follows that $e_4, \dots, e_n$ belong only to F_i, and e_4 is incident with r_2 and therefore is a door for r_2. Further, since f and f' are disjoint and have multiplicity one, f is a door for r_2. If follows that $|D_2| \ge 4$, so $m^+(f) = 1$, a contradiction.

This completes the proof of 5.7. ∎

5.8. *No minimum 7-counterexample contains Conf(11).*

Proof. Let (G, m) be a minimum 7-counterexample, and suppose that some edge uv is incident with regions r_1, r_2 where either $m(uv) = 4$ and r_2 is small, or $m(uv) \geq 5$. By exchanging r_1, r_2 if necessary, we may assume that if r_1, r_2 are both small, then the length of r_1 is at least the length of r_2. Suppose that r_1 is a triangle. By 5.2 we do not have Conf(4), and so r_2 is not a triangle and therefore r_2 is big. Then by hypothesis, $m(uv) \geq 5$, contradicting 4.2. Thus r_1 is not a triangle.

Let x-u-v-y be a path of C_{r_1}. By 5.2 we do not have Conf(5) so x, y are non-adjacent in G. Since $m(e) \geq 4$, this path is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \ldots, F_7$ be a 7-edge-colouring of (G', m'). Let $k = m(uv) + 2 \geq 6$. By 5.1 we may assume that $uv \in F_i$ for $1 \leq i \leq k-1$, and $xy \in F_k$, and so $k \leq 7$. Let $I = \{1, \ldots, 7\} \setminus \{k\}$ and for $i \in I$, let Q_i be as in 5.1.

Let D_1 be the set of doors for r_1 that are disjoint from e, and let D_2 be the set of doors for r_2.

(1) *For each $i \in I$, one of $F_i \cap Q_i \cap D_1, F_i \cap Q_i \cap D_2$ is nonempty, and if $k = 7$ or $i > k$ then both are nonempty.*

Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = uv$ and $e_2 = xy$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 6$. Suppose that $k = 7$. Then for $1 \leq j \leq 7$, F_j contains one of e_1, e_2; and hence $e_3, \ldots, e_n \notin F_j$ for all $j \in \{1, \ldots, 7\}$ with $j \neq i$. It follows that e_n, e_{n-1} belong only to F_i and hence $e_n \in F_i \cap Q_i \cap D_2$. Since this holds for all $i \in I$, it follows that $|D_2| \geq |I| \geq 6$. Hence r_2 is big, and so by hypothesis, $m(uv) \geq 5$. Since $xy \notin E(G)$, e_3 is an edge of C_{r_1}, and since e_3, e_4 belong only to F_i, it follows that e_3 is a door for r_1. But $e_3 \neq ux, vy$ from the choice of the switchable path, and so $e_3 \in F_i \cap Q_i \cap D_1$. Hence in this case (1) holds.

Thus we may assume that $k = 6$ and so $I = \{1, \ldots, 5, 7\}$; we have $m(e) = 4$, and r_2 is small, and $uv \in F_1, \ldots, F_5$, and $xy \in F_6$. If $i = 7$, then since $uv, xy \in Q_i$ and F_j contains one of e_1, e_2 for all $j \in \{1, \ldots, 6\}$, it follows as before that $e_3 \in F_i \cap Q_i \cap D_1$ and $e_n \in F_i \cap Q_i \cap D_2$. We may therefore assume that $i \leq 6$. For $1 \leq j \leq 7$ with $j \neq i$, $|F_j \cap Q_i| = 1$, and for $1 \leq j \leq 6$, F_j contains one of e_1, e_2. Hence $e_3, \ldots, e_n$ belong only to F_i and to F_7, and only one of them belongs to F_7. If neither of e_n, e_{n-1} belong to F_7 then $e_n \in F_i \cap Q_i \cap D_2$ as required; so we assume that F_7 contains one of e_n, e_{n-1}; and so $e_3, \ldots, e_{n-2}$ belong only to F_i. Since $n \geq 6$, it follows that $e_3 \in F_i \cap Q_i \cap D_1$ as required. This proves (1).

If $k = 7$, then (1) implies that $|D_1|, |D_2| \geq 6$ as required. So we may assume that $k = 6$ and hence $m(e) = 4$ and $xy \notin E(G)$; and r_2 is small. Suppose that there are three values of $i \in \{1, \ldots, 5\}$ such that $|F_i \cap D_1| = 1$ and $F_i \cap D_2 = \emptyset$, say $i = 1, 2, 3$. Let $f_i \in F_i \cap D_1$ for $i = 1, 2, 3$, and we may assume that f_3 is between f_1 and f_2 in the path $C_{r_1} \setminus \{uv\}$. Choose $X \subseteq V(G')$ such that $\delta_{G'}(X) = Q_3$. Since only one edge of $C_{r_1} \setminus \{e\}$ belongs to Q_3, one of f_1, f_2 has both ends in X and the other has both ends in $V(G') \setminus X$; say f_1 has both ends in X. Let Z be the set of edges with both ends in X. Thus $(F_1 \cap Z) \cup (F_2 \setminus Z)$ is a perfect matching, since $e \in F_1 \cap F_2$, and no other edge of $\delta_{G'}(X)$ belongs to $F_1 \cup F_2$; and similarly $(F_2 \cap Z) \cup (F_1 \setminus Z)$ is a perfect matching. Call them F_1', F_2' respectively. Then $F_1', F_2', F_3, F_4, \ldots, F_7$ form a 7-edge-colouring of (G', m'), yet the only edges of $D_1 \cup D_2$ included in $F_1' \cup F_2'$ are f_1, f_2, and neither of them is in F_2', contrary to (1). Thus there are no three such values of i; and similarly there are at most two such that $|F_i \cap D_2| = 1$ and $F_i \cap D_1 = \emptyset$. Thus there are at least two values of $i \in I$ such that $|F_i \cap D_1| + |F_i \cap D_2| \geq 2$ (counting

$i = 7$), and so $|D_1| + |D_2| \geq 8$. But $|D_2| \leq 3$ since r_2 is small, so $|D_1| \geq 5$. This proves 5.8. ∎

5.9. *No minimum 7-counterexample contains Conf(12).*

Proof. Let (G, m) be a minimum 7-counterexample, and suppose that some edge uv is incident with a triangle uvw with $m(uv) + m(vw) = 5$, and suppose that uv is also incident with a region r_1 that has at most five doors disjoint from v. Let tv be the edge incident with r_1 and v different from uv. By 4.3, we do not have Conf(1) so $m(tv) = 1$, and by 4.2, $m(uw) = 1$. By 4.4 we do not have Conf(3), u and t are nonadjacent in G. It follows that the path u-w-v-t is switchable; let (G', m') be obtained from (G, m) by switching on it, and let $F_1, \ldots, F_7$ be a 7-edge-colouring of (G', m'). Since $m'(uv) + m'(uw) + m'(ut) = 7$, we may assume by 5.1 that $ut \in F_7$, and F_j contains one of uv, vw for $1 \leq j \leq 6$ Let $I = \{1, \ldots, 6\}$ and for $i \in I$, let Q_i be as in 5.1.

Let D_1 be the set of doors for r_1 that are disjoint from v. Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_n, e_1$, where $e_1 = vw, e_2 = uv$ and $e_3 = ut$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, e_3, it follows that $n \geq 7$. For $1 \leq j \leq 7$, F_j contains one of e_1, e_2, e_3; and hence $e_3, \ldots, e_n \notin F_j$ for all $j \in \{1, \ldots, 7\}$ with $j \neq i$. It follows that e_4, e_5 belong only to F_i. By the choice of the switchable path $e_4 \neq tv$ and hence $e_4 \in F_i \cap Q_i \cap D_1$. Since this holds for all $i \in I$, it follows that $|D_1| \geq |I| \geq 6$, a contradiction. This proves 5.9. ∎

5.10. *Let (G, m) be a minimum 7-counterexample, let x-u-v-y be a three-edge path of G, and let (G, m') obtained by switching on x-u-v-y. If (G, m) is not smaller than (G, m'), and (G, m') contains one of Conf(1)–Conf(12) then x-u-v-y is switchable.*

Proof. Suppose that x-u-v-y is not switchable. Then, since (G, m') is a 7-counterexample and (G, m) is not smaller than (G, m'), the latter is a minimum counterexample. But by 4.3–5.9, no minimum 7-counterexample contains any of Conf(1)–Conf(12), a contradiction. This proves 5.10. ∎

5.11. *No minimum 7-counterexample contains Conf(13).*

Proof. Let (G, m) be a minimum 7-counterexample, with a square $xuvy$ and a tough triangle uvz, where $m(uv) + m^+(xy) \geq 4$ and $m(xy) \geq 2$. Since (G, m) does not contain Conf(5) by 5.2, we have $m(uv) + m^+(xy) = 4$. Suppose that $m(uv) \geq 3$; then since $xuvy$ is small and (G, m) does not contain Conf(6) by 5.3, we have $m(uv) = 3$ and $m^+(uz) = m^+(vz) = 1$, contradicting the fact that uvz is tough. Thus $m(uv) \leq 2$.

Since (G, m) does not contain Conf(3) by 4.4, it follows that $m(ux) + m(vy) \leq 4$. Thus the cycle x-u-v-y-x is switchable; let (G, m') be obtained from (G, m) by switching on it, and let $F_1, \ldots, F_7$ be a 7-edge-colouring of (G', m'). Let $k = m'(uv) + m'(xy) \in \{5, 6\}$. By 5.1 we may assume that $uv \in F_i$ for $1 \leq i \leq m'(uv)$, and $xy \in F_i$ for $m'(uv) < i \leq k$. Let $I = \{1, \ldots, 7\}$ and for $i \in I$, let Q_i be as in 5.1. Denote by r_1, r_2, the second regions for vz, xy, respectively, and by D_1, D_2 their respective sets of doors.

(1) *One of $m^+(uz), m^+(vz) = 1$.*

Let $i \in I$, and let the edges of Q_i in order be $e^i_1, \ldots, e^i_{n_i}, e^i_1$, where $e^i_1 = uv$, $e^i_2 = xy$ and $e^i_{n_i} \in \{uz, vz\}$.

Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e^i_1, e^i_2, it follows that $n_i \geq 6$. For $1 \leq j \leq k$, F_j contains one of e^i_1, e^i_2; and hence $e^i_3, \ldots, e^i_{n_i} \notin F_j$ for all $j \in \{1, \ldots, k\}$ with $j \neq i$.

Suppose that $k = 6$. We may assume by symmetry that $vz \in Q_7$, and so $m(vz) = 1$ and $vz \in F_7$. Also, $uz \in F_i$ for some $m'(uv) < i \leq k$, say $uz \in F_6$. Let $i \in I \setminus \{6, 7\}$. Then since uz and xy both belong to F_6, $vz \in Q_i$. Then since $e^i_{n_i} = vz$ and $vz \notin F_i$, we have $n_i \geq 7$ and $e^i_3, \ldots, e^i_{n_i-1}$ belong only to F_i. It follows that $F_i \cap Q_i \cap D_1$ is nonempty, and so r_1 is big. Hence $m^+(vz) = 1$ as required.

Suppose that $k = 5$. Then by hypothesis, $m(uv) = 1$, $m(xy) = 2$, and r_2 is small. We have $uv \in F_1, F_2$ and $xy \in F_3, F_4, F_5$. Suppose that $uz \in Q_7$ and $m(uz) \geq 2$. Then uz belongs to both F_7 and F_6. Further $vz \notin F_1, F_2, F_6, F_7$ and so by symmetry we can assume that $vz \in F_5$. Consequently when $i \in I \setminus \{5\}$, we have $uz \in Q_i$, $n_i \geq 7$ and $e^i_3, \ldots, e^i_{n-1}$ belong only to F_i. Further, $m(uz) = 2$. But then $F_i \cap Q_i \cap D_3$ is nonempty, contradicting the fact that r_3 is small. By the same argument if $m(vz) \geq 2$ then $vz \notin Q_7$.

Since uvz is tough, by symmetry we may assume that $m^+(uz) \geq 3$. Thus $uz \notin Q_7$, and so $vz \in Q_7$ and $m(vz) = 1$. Since $m(uz) \geq 2$, uz belongs to two of F_3, F_4, F_5, F_6; by symmetry say $uz \in F_5$. Thus for $i \in I \setminus \{5\}$, $vz \in Q_i$, $e^i_3, \ldots, e^i_{n_i-1}$ belong only to F_i, F_6. It follows that at least one of $F_i \cap Q_i \cap D_1$, $F_i \cap Q_i \cap D_2$ is nonempty, and if $i = 6$ then both are nonempty. Thus $|D_1| + |D_2| \geq 7$, and since r_2 is small $|D_1| \geq 4$. It follows that $m^+(vz) = 1$, as required. This proves (1).

By (1) we may assume that $m^+(vz) = 1$. Since uvz is tough, (1) implies $m^+(uz) + m^+(uv) \geq 6$. Since (G, m) does not contain Conf(6) by 5.3, it follows that $m(uv) = 2$, $m(uz) = 2$ and $m(ux) \geq 3$. But (G, m) does not contain Conf(3) by 4.4, a contradiction. This proves 5.11. ∎

5.12. *No minimum 7-counterexample contains Conf(14).*

Proof. Let (G, m) be a minimum 7-counterexample, with a region r bounded by a cycle $C_r = v_0, \ldots, v_4$. Denote the edge $v_i v_{i+1}$ by f_i for $0 \leq i \leq 4$ (taking indices modulo 5) and suppose that $m^+(e_0) \geq 2$, and that $m^+(f_2), m^+(f_3) \geq 4$. Since G has minimum degree at least three, $m(f_2) = m(f_3) = 3$.

Let (G', m') be obtained by switching on the path v_4-v_0-v_1-v_2; since $m(f_2), m(f_3) \geq 3$, (G', m') contains a triangle $v_2v_3v_4$ with $m'(v_2v_3v_4) \geq 7$. Since (G, m) is a 7-target, $m(\delta_G(\{u, v, x\})) \geq 9$ and it follows that $m'(\delta_{G'}(\{u, v, x\})) \geq 7$. Since $m'(uv) + m'(ux) + m'(vx) \geq 7$, it follows that $m'(\delta(\{u, v, x\})) = 7$. Hence by 2.1, (G', m') is 7-edge colourable. Let $F_1, \ldots, F_7$ be a 7-edge colouring of (G', m'). Let $k = m'(v_0v_1) + m'(v_2v_4) \geq 3$. By 5.1 we may assume that $v_0v_1 \in F_i$ for $1 \leq i \leq m'(v_0v_1)$, and $v_2v_4 \in F_k$. Let $I = \{1, \ldots, 7\} \setminus \{k\}$ and for $i \in I$, let Q_i be as in 5.1. Let $i \in I$, and let the edges of Q_i in order be $e_1, \ldots, e_{n_i}, e_1$, where $e_1 = v_0v_1$ and $e_2 = v_2v_4$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n_i \geq 6$. For $1 \leq j \leq 6$, F_j contains one of e_1, e_2; and hence $e_3, \ldots, e_n \notin F_j$ for all $j \in \{1, \ldots, k\}$ with $j \neq i$. By the choice of the switchable path, $e_3 \in \{f_2, f_3\}$. By setting $i = 7$, without loss of generality we may say $f_2 \in Q_7$; it follows that f_2 does not belong to $F_1, \ldots, F_k$ and $k \leq 4$. Thus f_2 belongs to three of $F_{k+1}, \ldots, F_7$, say f_2 belongs to F_5, F_6, F_7. Further f_3 belongs to three of $F_1, \ldots, F_4$. Let r_2 denote the second region for f_2 and let D_2 denote its set of doors.

It follows that $f_2 \in Q_i$ for each $i \in I$. Suppose that $k = 4$. Then for each $i \in I$, the edges of $Q_i \setminus \{f_0, f_2\}$ belong only to F_i. Thus $F_i \cap Q_i \cap D_2$ is nonempty, contradicting the fact that r_2 is small. Thus $k = 3$, and so $m(f_1) = 1$. Denote by r_1 the second region for f_0 and D_1 its set of doors. For each $i \in I$, $n_i \geq 7$ and the edges of $Q_i \setminus \{f_0, f_2\}$ belong only to F_i, F_4. Consequently at least one of $F_i \cap Q_i \cap D_1$, $F_i \cap Q_i \cap D_2$ is nonempty, and both are nonempty if $i = 4$. Thus $|D_1| + |D_2| \geq 7$, but since r_1 is small, $|D_2| \geq 4$, a contradiction. This proves 5.12. ∎

5.13. *No minimum 7-counterexample contains Conf(15).*

Proof.

Let (G, m) be a minimum 7-counterexample, with a region r bounded by a cycle $C_r = v_0, \dots, v_4$. Denote the edge v_iv_{i+1} by f_i for $0 \leq i \leq 4$ (taking indices modulo 5) and suppose that $m^+(f_0) \geq 3$, and that $m^+(f_2), m^+(f_3) \geq 3$.

(1) *Suppose that either f_0 is 3-heavy, or both f_2, f_3 are 3-heavy. Then the path v_4-v_0-v_1-v_2 is not switchable.*

Suppose that the path v_4-v_0-v_1-v_2 is switchable; let (G', m') be obtained by switching on it and let $F_1, \dots, F_7$ be a 7-edge colouring. Let $k = m'(v_0v_1) + m'(v_2v_4) \geq 4$. By 5.1 we may assume that $v_0v_1 \in F_i$ for $1 \leq i \leq m'(v_0v_1)$, and $v_2v_4 \in F_k$. Let $I = \{1, \dots, 7\} \setminus \{k\}$ and for $i \in I$, let Q_i be as in 5.1.

Since $k \geq 4$ and $m(f_2), m(f_3) \geq 2$, we may assume without loss of generality that both f_0, f_3 belong to F_1. Consequently, $f_2 \in Q_i$ for each $i \in I \backslash \{1\}$ and f_2 belongs to at least two of $F_{k+1}, \dots, F_7$, say f_2 belongs to F_6, F_7, and so $k \leq 5$. Let $i \in I \backslash \{1\}$, and let the edges of Q_i in order be $e_1, \dots, e_n, e_1$, where $e_1 = v_0v_1, e_2 = v_2v_4$ and $e_3 = f_2$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 7$. For $1 \leq j \leq 6$, F_j contains one of e_1, e_2; and hence $e_4, \dots, e_n \notin F_j$ belong only to F_i, and possibly F_7.

Denote by r_1, r_2 the second regions for f_0, f_2, respectively and denote by D_1, D_2 their respective sets of doors. Suppose that $k + m(f_2) = 7$, and so $m(f_0) + m(f_2) \leq 5$. Then for each $i \in I \setminus \{1\}$, both $F_i \cap Q_i \cap D_1$, $F_i \cap Q_i \cap D_2$ are nonempty. It follows that both r_1 and r_2 are big, a contradiction.

Thus $k + m(f_2) \leq 6$, and so $k \leq 4$. For each $i \in I \setminus \{1\}$, at least one of $F_i \cap Q_i \cap D_1$, $F_i \cap Q_i \cap D_2$ is nonempty, and both are nonempty if $i = 5$. Since at least one of r_1, r_2 is a triangle, one of $|D_1|, |D_2| \leq 2$, and so $k + m(f_2) \leq 6$. $|D_1| + |D_2| \geq |I| = 6$. But $k \geq 4$ and $m^+(f_2) \geq 3$ and so r_1, r_2 are both small, a contradiction. This proves (1).

Now, suppose that (G, m) contains Conf(15), and so f_0 is 3-heavy. By (1), the path v_4-v_0-v_1-v_2 is not switchable, and $m(f_0) = 2$, and by symmetry we may assume that $m(f_4) \geq 3$. It follows that $m(f_2) \leq 2$, for otherwise we could relabel the vertices of C_r to contradict (1). Further by (1) the path v_1-v_2-v_3-v_4 is not switchable. Similarly f_1 is not 3-heavy. Since v_1-v_2-v_3-v_4 is not switchable, and $m(f_1), m(f_2) \leq 2$, it follows that $m(f_3) \geq 3$. Further the 7-target obtained by switching on v_1-v_2-v_3-v_4 contains Conf(2), and so by 5.10 it follows that $m(f_1) \geq 2$. Now, the path v_2-v_3-v_4-v_0 is switchable; let (G', m') be obtained by switching on it and let $F_1, \dots, F_7$ be a 7-edge-colouring. Since $m'(v_3v_4) + m'(v_0v_2) = 5$, we may assume by 5.1 that v_3v_4 belongs to F_i for $1 \leq i \leq 4$ and $v_0v_2 \in F_5$. Also by symmetry v_2v_3 and v_4v_0 both belong to F_6, and so f_0, f_1 do not belong to F_6. Let $I = \{1, \dots, 7\} \setminus \{5\}$ and for $i \in I$ let Q_i be as in 5.1. Let the edges of Q_6 in order be $e_1, \dots, e_n, e_1$, where $e_1 = v_3v_4$ and $e_2 = v_4v_0$. Since $|F_i \cap Q_6| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 6$. For $1 \leq j \leq 6$, F_j contains one of e_1, e_2; and hence $e_3, \dots, e_n \notin F_j$ for all $j \in \{1, \dots, k\}$ with $j \neq 6$. It follows that $e_3, \dots, , e_n$ belong only to F_6, F_7. By the choice of the switchable path,

$e_3 \in \{f_0, f_1\}$, and so $m(e_3) \geq 2$. Hence e_3 belongs to both F_6, F_7, a contradiction. This proves 5.13. ∎

5.14. *No minimum 7-counterexample contains Conf(16).*

Proof. Let (G, m) be a minimum 7-counterexample, with a region r bounded by a cycle $C_r = v_0, \ldots, v_5$. Denote the edge $v_i v_{i+1}$ by f_i for $0 \leq i \leq 5$ (taking indices modulo 6) and suppose that f_1, f_2, f_3, f_4, f_5 are 3-heavy with multiplicity at least two.

(1) *The path v_0-v_1-v_2-v_3 is not switchable.*

Suppose that v_0-v_1-v_2-v_3 is switchable. Let (G', m') be obtained by switching on it and let $F_1, \ldots, F_7$ be a 7-edge-colouring of (G', m'). Let $k = m'(v_1v_2) + m'(v_0v_3) \geq 4$. We may assume by 5.1 that $v_1v_2 \in F_i$ for $1 \leq i < k$ and $v_0v_3 \in F_k$. Let $I = \{1, \ldots, 7\} \setminus \{k\}$ and for $i \in I$, let Q_i be as in 5.1.

For $i \in I$, let the edges of Q_i in order be $e_1^i, \ldots, e_{n_i}^i, e_1^i$, where $e_1^i = v_1v_2$ and $e_2^i = v_0v_3$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1^i, e_2^i, it follows that $n \geq 6$. Let $i \in I$. For $1 \leq j \leq k$, F_j contains one of e_1^i, e_2^i; and hence $e_3^i, \ldots, e_{n_i}^i \notin F_j$ for all $j \in \{1, \ldots, k\}$ with $j \neq i$. By the choice of the switchable path $e_3^7 \in \{f_3, f_4, f_5\}$, and so e_3^7 is 3-heavy; thus one of $e_3^7 e_4^7$ must belong to one of $F_1, \ldots, F_5$.

Thus $k = 4$ and the second region for v_1v_2 is a triangle v_1v_2x. Choose $i \in \{5, 6, 7\}$ such that neither of $\{v_1x, v_2x\}$ is an edge of multiplicity one belonging to F_i. Now, $e_3^i, \ldots, e_{n_i}^i$ do not belong to $F_1, \ldots, F_4$. By the choice of the switchable path, e_3^i is 3-heavy, and so $e_{n_i}^i$ has multiplicity one and belongs only to F_i, a contradiction. This proves (1).

Now $m(v_0v_1) \leq 2$, for otherwise the vertices of C_r could be relabeled to contradict (1). By (1), v_0-v_1-v_2-v_3 is not switchable. It follows that $m(v_1v_2) = 2$ and the second region for v_1v_2 is a triangle and $m(v_2v_3) \geq 3$. By symmetry, $m(v_5v_0) = 2$, the second region for v_5v_0 is a triangle, and $m(v_4v_5) \geq 3$. The 7-target (G, m) obtained by switching on v_0-v_1-v_2-v_3 contains Conf(3), so by 5.10 (G, m) is smaller than (G', m'). It follows that $m(v_0v_1) + m(v_2v_3) \geq 5$. Similarly $m(v_0v_1) + m(v_4v_5) \geq 5$.

Since $m(v_2v_3) \geq 3$, the path v_1-v_2-v_3-v_4 is switchable. Let (G', m') be obtained by switching on it and let $F_1, \ldots, F_7$ be a 7-edge-colouring. Let $k = m'(v_2v_3) + m'(v_1v_4) \in \{5, 6\}$. We may assume by 5.1 that $v_2v_3 \in F_i$ for $1 \leq i < k$ and $v_1v_4 \in F_k$. By symmetry we may assume that $v_1v_2 \in F_{k+1}$. Let $I = \{1, \ldots, 7\} \setminus \{k\}$ and for $i \in I$, let Q_i be as in 5.1. Let the edges of Q_7 in order be $e_1, \ldots, e_n, e_1$, where $e_1 = v_2v_3$ and $e_2 = v_1v_4$. Since $|F_i \cap Q_i| \geq 5$ and F_i contains at most one of e_1, e_2, it follows that $n \geq 6$. For $1 \leq j \leq k$, F_j contains one of e_1, e_2; and hence $e_3, \ldots, e_n \notin F_j$ for all $j \in \{1, \ldots, k\}$ with $j \neq i$.

Suppose that $k = 6$. Then $e_3, \ldots, e_n$ belong only to F_7, and so e_3 has multiplicity one. By the choice of the switchable path, $e_3 = f_0$. But $f_0 \notin F_7$ since $f_1 \in F_7$, a contradiction. Thus $k = 5$, and so $m(f_2) = 3$ and $m(f_0) \geq 2$. Now $e_3, \ldots, e_n$ belong only to F_6, F_7, and so e_3 is not 3-heavy. It follows from the choice of the switchable path that $e_3 = f_0$. But $m(f_0) \geq 2$ and $f_0 \notin F_6$ since $f_1 \in F_6$, a contradiction. This proves 5.14. ∎

This completes the proof of 4.1 and hence of 1.2.

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