

# DENSE MINORS OF GRAPHS WITH INDEPENDENCE NUMBER TWO

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ABSTRACT. Motivated by Hadwiger’s conjecture, we prove that every  $n$ -vertex graph  $G$  with no independent set of size three contains an  $\lceil n/2 \rceil$ -vertex simple minor  $H$  with

$$0.98688 \cdot \binom{|V(H)|}{2} - o(n^2)$$

edges.

## 1. INTRODUCTION

All the graphs in this paper are simple. In 1943 Hadwiger conjectured [Had43] that every graph with chromatic number at least  $t$  contains the complete graph  $K_t$  on  $t$  vertices as a minor. Hadwiger’s conjecture remains wide open for every  $t \geq 7$  and many of its relaxations have been investigated. (See [Sey16] for a survey.) In particular, over the years increasingly strong lower bounds have been obtained on the function  $f(t)$  such that every graph with chromatic number at least  $t$  is guaranteed to contain  $K_{f(t)}$  as a minor. The current record  $f(t) = \Omega(\frac{t}{\log \log t})$  is due to Delcourt and Postle [DP21].

Let  $\alpha(G)$  denote the maximum size of an independent set in  $G$ . Hadwiger’s conjecture implies that every graph  $G$  with  $\alpha(G) \leq 2$  contains  $K_{\lceil |V(G)|/2 \rceil}$  as a minor. It remains open and still very challenging in this case. A classical result of Duchet and Meyniel [DM82] implies that every graph  $G$  with  $\alpha(G) = 2$  contains  $K_{\lceil |V(G)|/3 \rceil}$  as a minor. A well-known open problem (see e.g. [Sey16, Question 4.8]) asks for a constant factor improvement of their bound.

We consider relaxing Hadwiger’s conjecture in a different direction, investigating maximum  $g(t)$  such that every graph with chromatic number at least  $t$  contains a minor with  $t$  vertices and  $g(t)$  edges. (Hadwiger’s conjecture asserts that  $g(t) = \binom{t}{2}$ .)

While it is not known whether  $f(t)$  is linear, the function  $g(t)$  is quadratic. Indeed, every graph with chromatic number at least  $t$  contains a subgraph with minimum degree at least  $t - 1$ . Mader [Mad68] showed that every graph with average degree at least  $t - 1$  contains a minor with at most  $t$  vertices and minimum degree at least  $t/2$ , and hence with at least  $t(t + 2)/8$  edges; and consequently there is a minor with exactly  $t$  vertices and at least  $t(t + 2)/8$  edges. It follows that  $g(t) \geq \frac{t(t+2)}{8} = \frac{1}{4} \binom{t}{2} + o(t^2)$ . The factor  $1/4$  can likely be significantly improved. However it appears to be quite challenging, for example, to show that every graph with chromatic number at least  $t$  contains a minor on  $t$  vertices with at most one tenth of all possible edges missing.

We concentrate on graphs  $G$  with  $\alpha(G) \leq 2$ . Our main result, stated below, guarantees in such graphs the existence of a minor with  $\lceil |V(G)|/2 \rceil$  vertices and with fewer than  $1/76$  of all possible edges missing.

**Theorem 1.1.** *Let  $G$  be a graph with  $\alpha(G) \leq 2$ . Then there exists a minor  $H$  of  $G$  with  $|V(H)| = \lceil |V(G)|/2 \rceil$  such that*

$$(1) \quad |E(H)| \geq (\gamma - o(1)) \binom{|V(H)|}{2},$$

where

$$\gamma = 1 - \max_{z \in [0, 1/4]} \frac{z^3(5 - 38z + 92z^2 - 80z^3)}{(1 - 3z + 3z^2)^2} = 0.986882\dots$$

## 2. PROOF OF THEOREM 1.1

We start with a brief outline of the proof. Let  $G$  be a graph with  $\alpha(G) \leq 2$ . We may assume that  $|V(G)|$  is even. Let  $\omega(G)$  denote the maximum size of cliques in  $G$ . If  $\omega(G) \geq |V(G)|/4$  then  $G$  contains a  $K_{\lceil |V(G)|/2 \rceil}$  minor by a result of Chudnovsky and Seymour [CS12], which we state below. Thus we assume  $\omega(G) < |V(G)|/4$ .

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A *seagull* in  $G$  is an induced three-vertex path. Note that every vertex of  $G$  has a neighbour in every seagull, and so by contracting the edges of a seagull, we obtain a vertex complete to the rest of the graph. Also note that the set of non-neighbours of every vertex forms a clique, and so every vertex of  $G$  has at most  $\omega(G)$  non-neighbours. We repeatedly use these observations.

We find a minor  $H$  satisfying the theorem by choosing an arbitrary maximum clique  $Z$ , partitioning the remaining vertices into a matching of size  $|V(G)|/2 - 2\omega(G)$  and a collection of seagulls, and contracting the edges of the matching and of the seagulls. We find the matching first, randomly, ensuring that there are not too many non-adjacencies between pairs of matching edges and between matching edges and  $Z$ . There are exactly  $3\omega(G)$  vertices of  $V(G) - Z$  that are left uncovered by the matching, and we partition them into seagulls.

We now move on to details. The last part of the above argument uses a characterization of graphs  $G$  with  $\alpha(G) \leq 2$  containing  $k$  pairwise disjoint seagulls due to Chudnovsky and Seymour [CS12]. To state this characterization we need a couple of definitions.

Let  $C$  be a clique in the graph  $G$ . Let  $X$  consist of all vertices in  $V(G) - C$  with a neighbour and a non-neighbour in  $C$ . The *capacity*  $\text{cap}(C)$  of  $C$  is defined to be equal to  $\frac{|V(G)-C|+|X|}{2}$ . A *five-wheel* is a six-vertex graph obtained from a cycle of length five by adding one new vertex adjacent to every vertex of the cycle. We denote by  $\bar{G}$  the complement of a graph  $G$ .

**Theorem 2.1** ([CS12, 1.6]). *Let  $G$  be a graph with  $\alpha(G) \leq 2$ , let  $k \geq 0$  be an integer. Then  $G$  contains  $k$  pairwise disjoint seagulls if and only if*

- (i):  $|V(G)| \geq 3k$ ,
- (ii):  $G$  is  $k$ -connected,
- (iii):  $\text{cap}(C) \geq k$  for every clique  $C$  in  $G$ ,
- (iv):  $\bar{G}$  contains a matching of size  $k$ ,
- (v): if  $k = 2$  then  $G$  is not a five-wheel.

We will use an easy-to-apply corollary of Theorem 2.1.

**Corollary 2.2.** *Let  $k \geq 0$  be an integer. Let  $G$  be a graph with  $\alpha(G) \leq 2$ ,  $|V(G)| = 3k$  and  $\omega(G) \leq k$ ; then  $G$  has  $k$  pairwise disjoint seagulls.*

*Proof.* We will verify that conditions (i)-(v) of Theorem 2.1 hold. Condition (i) is immediate.

Suppose for a contradiction that (ii) does not hold. Then there exists  $X \subseteq V(G)$  such that  $|X| < k$  and  $G \setminus X$  is not connected. It follows that  $G \setminus X$  is a union of two cliques. As  $|V(G \setminus X)| \geq 2k + 1$ , one of these cliques must have size at least  $k + 1$ . This gives the desired contradiction, as  $\omega(G) \leq k$ .

For every clique  $C$  in  $G$  we have  $|C| \leq k$ , and so  $\text{cap}(C) \geq \frac{|V(G)|-|C|}{2} \geq k$ . Thus (iii) holds.

Let  $M$  be a maximal matching in  $\bar{G}$ . Then the vertices of  $G$  incident with no edge of  $M$  form a clique, implying that  $|V(G)| - 2|M| \leq k$ . Thus  $|M| \geq k$  and (iv) holds.

Finally, (v) holds, as a five-wheel contains a clique of size three.  $\square$

Additionally, as mentioned above, we use another consequence of Theorem 2.1 established in [CS12].

**Theorem 2.3** ([CS12, 1.3]). *Let  $k \geq 0$  be an integer. Let  $G$  be a graph with  $\alpha(G) \leq 2$ , with an even number of vertices and with  $\omega(G) \geq |V(G)|/4$ . Then  $G$  contains a  $K_{|V(G)|/2}$  minor.*

Next we make a couple of observations about the procedure that we use to generate a random matching.

Let  $X$  be a finite set with  $|X|$  even. We denote by  $X^{(2)}$  the set of all two-element subsets of  $X$ . Let  $\mathcal{M}_X$ , or simply  $\mathcal{M}$  for brevity, denote a partition of  $X$  into pairs, chosen uniformly at random. Note that

$$(2) \quad \Pr[e \in \mathcal{M}_X] = \frac{1}{|X|-1} \quad \text{and} \quad \Pr[e, f \in \mathcal{M}_X] = \frac{1}{(|X|-1)(|X|-3)}$$

for every  $e \in X^{(2)}$  and every  $f \in X^{(2)}$  disjoint from  $e$ .

We need a bound on the concentration of  $|F \cap \mathcal{M}|$  for  $F \subseteq X^{(2)}$ .

**Lemma 2.4.** *Let  $X$  be a finite set with  $|X| \geq 4$  even. Then for every  $F \subseteq X^{(2)}$  and every  $\lambda > 0$  we have*

$$(3) \quad \Pr \left[ \left| |F \cap \mathcal{M}_X| - \frac{|F|}{|X|-1} \right| \geq \lambda \right] \leq \frac{|X|}{\lambda^2}.$$

*Proof.* We may assume that

$$(4) \quad |F| \leq \frac{|X|(|X| - 1)}{4}$$

by replacing  $F$  with  $X^{(2)} - F$  if needed. For  $e \in F$ , let  $Z_e$  be the indicator random variable with  $Z_e = 1$  if  $e \in \mathcal{M}_X$  and  $Z_e = 0$ , otherwise. Let  $Z = \sum_{e \in F} Z_e = |F \cap \mathcal{M}_X|$ . By (2) we have  $E[Z] = |F|/(|X| - 1)$  and

$$\begin{aligned} \text{Var}[Z] &= \sum_{(e,f) \in F \times F} \text{Cov}[Z_e, Z_f] \\ &\leq \sum_{e \in F} \text{Cov}[Z_e, Z_e] + \sum_{e \in F} \sum_{f \in F, f \cap e = \emptyset} \text{Cov}[Z_e, Z_f] \\ &\stackrel{(2)}{\leq} \frac{|F|}{|X| - 1} + |F|^2 \left( \frac{1}{(|X| - 1)(|X| - 3)} - \left( \frac{1}{|X| - 1} \right)^2 \right) \\ &\stackrel{(4)}{\leq} \frac{|X|}{4} + \frac{|X|^2}{8(|X| - 3)} \\ &\leq |X|. \end{aligned}$$

Thus by Chebyshev's inequality (see for instance [AS16, Theorem 4.1.1])

$$\Pr \left[ \left| |F \cap \mathcal{M}_X| - \frac{|F|}{|X| - 1} \right| \geq \lambda \right] \leq \Pr[|Z - E[Z]| \geq \lambda] \leq \frac{\text{Var}[Z]}{\lambda^2} \leq \frac{|X|}{\lambda^2},$$

as desired.  $\square$

We are now ready for the main technical lemma.

**Lemma 2.5.** *Let  $G$  be a graph with  $|V(G)| \geq 6$  and even, with  $\alpha(G) \leq 2$ , and with  $\omega(G) < |V(G)|/4$ ; and define  $n = |V(G)|/2$  and  $k = \omega(G)$ . Let  $Z$  be a clique of  $G$  with  $|Z| = k$ , and define  $a = \sum_{v \in Z} \deg_G(v)$  and  $b = |E(G \setminus Z)|$ . Let  $0 < \lambda \leq \frac{k-1}{2}$  with  $\lambda^2 > 2n$ , and let*

$$p = \frac{n - 2k}{n - \frac{k+1}{2} - \frac{b}{2n-k-2} - \lambda}.$$

Then there is a minor  $H$  of  $G$  with  $|V(H)| = n$  and

$$(5) \quad \binom{n}{2} - |E(H)| \leq \frac{1}{1 - \frac{2n}{\lambda^2}} \left( \frac{b(k-1)^2 p^2}{4(2n-k-2)(2n-k-4)} + \frac{a(k-1)p}{2(2n-k-2)} \right).$$

*Proof.* Let the graph  $G'$  be obtained from  $G$  by deleting  $Z$  and at most one other vertex, so that  $|V(G')| \geq 2n - k - 1$  is even. Let  $x = |V(G')|$ . Note that  $x \geq \lceil 3n/2 \rceil - 1 \geq 4$ .

Let  $\mathcal{A}$  be the set of all partitions  $M$  of  $V(G')$  into pairs that satisfy

$$(6) \quad |M \cap E(G')| \geq \frac{|E(G')|}{x-1} - \lambda.$$

Note that  $\deg_{G'}(v) \leq k$  for every  $v \in V(G)$ . Thus  $\deg_{G'}(v) \geq x - k - 1$  for every  $v \in V(G')$ , and so  $|E(G')| \geq \frac{x(x-k-1)}{2}$ . As  $\lambda \leq \frac{k-1}{2}$ , it follows that for every  $M \in \mathcal{A}$  we have

$$(7) \quad |M \cap E(G')| \geq \frac{x(x-k-1)}{2(x-1)} - \lambda \geq \frac{x}{2} - \frac{kx}{2(x-1)} - \frac{k-1}{2} \geq n - 2k$$

(since  $x \geq 2n - k - 1$  and  $x/(x-1) \leq 4/3$ ). Moreover, since  $|E(G')| + b \geq x(x-1)/2$ , it follows that for every  $M \in \mathcal{A}$

$$(8) \quad |M \cap E(G')| \geq \frac{x}{2} - \frac{b}{x-1} - \lambda \geq n - \frac{k+1}{2} - \frac{b}{2n-k-2} - \lambda.$$

Let  $\mathcal{M} = \mathcal{M}_{V(G')}$  be a partition of  $V(G')$  into pairs chosen uniformly at random. Let  $q = 1 - \frac{2n}{\lambda^2}$ . Thus  $q > 0$ . By Lemma 2.4, applied with  $X = V(G')$  and  $F = E(G')$ , we have

$$(9) \quad \Pr[\mathcal{M} \in \mathcal{A}] \geq 1 - \frac{x}{\lambda^2} \geq q.$$

Let  $\mathcal{M}^*$  be a random matching in  $G'$ , obtained by choosing  $M \in \mathcal{A}$  uniformly at random and then choosing a subset of  $n - 2k$  edges of  $M \cap E(G')$ , also uniformly at random. Such a selection

is possible by (7); and once  $M$  is chosen, every edge of  $M \cap E(G')$  is selected with probability at most  $\frac{n-2k}{|M \cap E(G')|} \leq p$  by (8). We have

$$\Pr[e \in \mathcal{M}] \geq \Pr[e \in \mathcal{M} \text{ and } \mathcal{M} \in \mathcal{A}] = \Pr[e \in M] \cdot \Pr[\mathcal{M} \in \mathcal{A}];$$

so using (2) and (9), we obtain

$$(10) \quad \Pr[e \in \mathcal{M}^*] \leq p \cdot \Pr[e \in M] \leq p \cdot \frac{\Pr[e \in \mathcal{M}]}{\Pr[\mathcal{M} \in \mathcal{A}]} \leq \frac{p}{q(x-1)}$$

for every  $e \in E(G')$ . Similarly, once  $M \in \mathcal{A}$  is chosen, a given pair of edges in  $M \cap E(G')$  is selected to be in  $\mathcal{M}^*$  with probability at most  $p^2$ , and thus

$$(11) \quad \Pr[e, f \in \mathcal{M}^*] \leq \frac{p^2}{q(x-1)(x-3)}$$

for every pair of distinct  $e, f \in E(G')$ .

Next we construct a minor  $H$  of  $G$  using  $\mathcal{M}^*$ . Let  $S$  be the set of all vertices in  $V(G) - Z$  incident with no edges of  $\mathcal{M}^*$ . Then  $|S| = 3k$ , and so there exists a collection  $\mathcal{S}$  of  $k$  pairwise disjoint seagulls in  $G$  with all vertices in  $S$ , by Corollary 2.2 applied to the subgraph of  $G$  induced by  $S$ . We now obtain  $H$  by contracting all the edges of  $\mathcal{M}^*$  and of the seagulls in  $\mathcal{S}$ . The rest of the proof consists of showing that the expectation of  $\binom{n}{2} - |E(H)| = |E(\bar{H})|$  satisfies (5).

We say that a subset  $Q \subseteq V(G')$  is a *bad quadruple* if  $|Q| = 4$  and the subgraph of  $G'$  induced on  $Q$  is a matching of size two. A subset  $T \subseteq V(G)$  is a *bad triple* if  $|T| = 3$ , there exists a unique vertex  $z \in T \cap Z$ , and  $z$  is not adjacent to the other vertices of  $T$ .

Every vertex of  $G$  has a neighbour in every seagull, and the vertices of  $Z$  are pairwise adjacent. Thus every pair of non-adjacent vertices of  $H$  corresponds to either a bad triple which includes an edge of  $\mathcal{M}^*$  or a bad quadruple which consists of the union of vertex sets of two edges of  $\mathcal{M}^*$ . We finish the proof by bounding above the expected number of bad triples and quadruples of this form.

First, we give an upper bound for the total number of bad quadruples by considering ordered sequences  $(u, v, w, z)$  of distinct vertices of  $G'$  such that  $\{u, v, w, z\}$  is a bad quadruple and  $uw, vz \in E(G')$ . Note that every bad quadruple corresponds to eight such ordered sequences. There are at most  $2|E(\bar{G}')| \leq 2b$  ways to choose the non-adjacent pair  $(u, v)$ , and for fixed  $(u, v)$ , the vertex  $w \in V(G') - \{u\}$  must be one of at most  $k-1$  remaining non-neighbours of  $v$ , and  $z$  must be chosen among at most  $k-1$  non-neighbours of  $v$ . Thus there are at most  $2b(k-1)^2$  ordered sequences  $(u, v, w, z)$  as above, and so at most  $b(k-1)^2/4$  bad quadruples. By (11) a given bad quadruple contains two edges of  $\mathcal{M}^*$  with probability at most  $\frac{p^2}{q(x-1)(x-3)}$ . It follows that the expected number of pairs of non-adjacent vertices of  $H$  coming from bad quadruples is at most

$$\frac{1}{q} \cdot \frac{b(k-1)^2 p^2}{4(x-1)(x-3)}.$$

Similarly, we give an upper bound for the total number of bad triples by considering ordered triples  $(z, v, w)$  such that  $z \in Z$  and  $v, w \in V(G')$  are distinct non-neighbours of  $z$ . There are  $a$  ways of choosing the pair  $(z, v)$ , and at most  $k-1$  ways of choosing the second non-neighbour  $w$  of  $z$ . As every bad triple corresponds to two such ordered triples, there are at most  $a(k-1)/2$  bad triples, and using (10) it follows that the expected number of pairs of non-adjacent vertices of  $H$  coming from bad triples is at most

$$\frac{1}{q} \cdot \frac{a(k-1)p}{2(x-1)}.$$

As  $x \geq 2n - k - 1$ , it follows that

$$\begin{aligned} \mathbb{E}[|E(\bar{H})|] &\leq \frac{1}{q} \left( \frac{b(k-1)^2 p^2}{4(x-1)(x-3)} + \frac{a(k-1)p}{2(x-1)} \right) \\ &\leq \frac{1}{1 - \frac{2n}{\lambda^2}} \left( \frac{b(k-1)^2 p^2}{4(2n-k-2)(2n-k-4)} + \frac{a(k-1)p}{2(2n-k-2)} \right), \end{aligned}$$

as desired.  $\square$

It remains to optimize the bound in Lemma 2.5 for large  $n$ .

*Proof of Theorem 1.1.* We may assume that the number of vertices of  $G$  is even, without loss of generality. To see this, suppose that the theorem holds for graphs with an even number of vertices, while the number of vertices of  $G$  is odd. Choose  $v \in V(G)$ ; there is a minor  $H$  of  $G \setminus v$  on

$\lceil |V(G)|/2 \rceil - 1$  vertices  $|E(H')| \geq (\gamma - o(1)) \binom{|V(H')|}{2}$ . By adding the vertex  $v$  to  $H'$  we obtain a minor  $H$  of  $G$  satisfying the theorem.

Let  $n = |V(G)|/2$  and let  $k = \omega(G)$ . If  $k \geq n/2$  then  $G$  contains a  $K_n$  minor by Theorem 2.3. Thus we may assume that  $k < n/2$  and apply Lemma 2.5 with  $\lambda = n^{2/3}$ . Let  $Z, a, b, p$  and  $H$  be as in Lemma 2.5. We will show that  $H$  satisfies the theorem.

Certainly  $\frac{2n}{\lambda^2} = o(1)$ . We have  $a + 2b = \sum_{v \in V(G)-Z} \deg_{\bar{G}}(v) \leq k(2n - k)$ , and so  $b \leq \frac{k(2n-k)-a}{2}$ , and

$$p \leq \frac{n - 2k}{n - k + \frac{a}{2(2n-k)}} + o(1).$$

Let  $z = \frac{k}{2n}$  and let  $\zeta = \frac{a}{4n^2}$ . It follows that  $b \leq 2n^2((1-z)z - \zeta)$ , and

$$p \leq \frac{1 - 4z}{1 - 2z + \frac{\zeta}{1-z}} + o(1).$$

Plugging the bounds above into (5) yields

$$\begin{aligned} \binom{n}{2} - |E(H)| &\leq \left( \frac{((1-z)z - \zeta)z^2}{2(1-z)^2} \left( \frac{1 - 4z}{1 - 2z + \frac{\zeta}{1-z}} \right)^2 + \frac{2\zeta z}{(1-z)} \left( \frac{1 - 4z}{1 - 2z + \frac{\zeta}{1-z}} \right) \right) n^2 + o(n^2) \\ (12) \quad &= \frac{z(1-4z)(z^2(1-5z+4z^2) + \zeta(4-13z+12z^2) + 4\zeta^2)}{(1+\zeta-3z+2z^2)^2} \binom{n}{2} + o(n^2) \end{aligned}$$

We have  $z \leq 1/4$ , as  $k \leq n/2$ . Moreover,  $a = \sum_{v \in Z} \deg_{\bar{G}}(v) \leq k^2$ , and so  $\zeta \leq z^2$ . It is straightforward to verify that the last expression in (12) increases with  $\zeta$  for  $0 \leq \zeta \leq z^2 \leq 1/16$ . Substituting  $\zeta = z^2$  yields

$$|E(H)| \geq \left( 1 - \frac{z^3(5 - 38z + 92z^2 - 80z^3)}{(1 - 3z + 3z^2)^2} \right) \binom{n}{2} - o(n^2).$$

The coefficient of  $\binom{n}{2}$  on the right side is minimized (for  $0 \leq z \leq 1/4$ ) when  $z \sim .193984$ , and its value there is approximately 0.986882, which implies (1).  $\square$

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