Claw-free Graphs. II. Non-orientable prismatic graphs

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Abstract

A graph is *prismatic* if for every triangle T, every vertex not in T has exactly one neighbour in T. In a previous paper we gave a complete description of all 3-colourable prismatic graphs, and of a slightly more general class, the "orientable" prismatic graphs. In this paper we describe the non-orientable ones, thereby completing a description of all prismatic graphs.

Since complements of prismatic graphs are claw-free, this is a step towards the main goal of this series of papers, providing a structural description of all claw-free graphs (a graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours).

1 Introduction

Let G be a graph. (All graphs in this paper are simple, and finite unless we say otherwise.) A *clique* in G is a set of pairwise adjacent vertices, and a *triangle* is a clique with cardinality three. We say G is *prismatic* if for every triangle T, every vertex not in T has exactly one neighbour in T. Our objective in this paper is to describe all prismatic graphs.

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. The main goal of this series of papers is to give a structure theorem describing all claw-free graphs. Complements of prismatic graphs are claw-free, and we find it best to handle such graphs separately from the general case, since they seem to require completely different methods.

Let $T = \{a, b, c\}$ be a set with a, b, c distinct. There are two cyclic permutations of T, and we use the notation $a \to b \to c \to a$ to denote the cyclic permutation mapping a to b, b to c and c to a. (Thus $a \to b \to c \to a$ and $b \to c \to a \to b$ mean the same permutation.)

Let G be a prismatic graph. If S, T are triangles of G with $S \cap T = \emptyset$, then since every vertex of S has a unique neighbour in T and vice versa, it follows that there are precisely three edges of G between S and T, forming a 3-edge matching. An *orientation* $\mathcal{O}$ of G is a choice of a cyclic permutation $\mathcal{O}(T)$ for every triangle T of G, such that if $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are triangles with $S \cap T = \emptyset$, and s_it_i is an edge for $1 \leq i \leq 3$, then $\mathcal{O}(S)$ is $s_1 \to s_2 \to s_3 \to s_1$ if and only if $\mathcal{O}(T)$ is $t_1 \to t_2 \to t_3 \to t_1$. We say that G is *orientable* if it admits an orientation, and *non-orientable* otherwise.

A complete description of all orientable prismatic graphs was given in [2]. Our goal in this paper is to describe the non-orientable prismatic graphs.

2 Rigidity

We begin with some more definitions. If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G|X$, and $G \setminus X$ means $G|(V(G) \setminus X)$. If $Y \subseteq V(G)$ and $x \in V(G) \setminus Y$, we say that x is *complete to* Y if x is adjacent to every member of Y; and x is *anticomplete to* Y if x is adjacent to no member of Y. If $X, Y \subseteq V(G)$ are disjoint, we say that X is *complete* to Y if every vertex of X is adjacent to every vertex of Y, and X is *anticomplete* to Y if X is complete to Y in the complement graph of G. If G is prismatic, its *core* is the union of all triangles of G.

Vertices in the core are quite tightly structured, as we shall see, but vertices outside the core are less under control. This is for two reasons. First, if v is a vertex not in the core, in some prismatic graph G, then we can "replicate" v; replace v by several vertices, all with the same set of neighbours as v (and nonadjacent to each other), and the graph we construct will still be prismatic. Second, if G is prismatic and e is an edge joining two vertices not in the core, then deleting e yields another graph that is still prismatic.

Let us say a prismatic graph G with core W is *rigid* if

- there do not exist distinct $u, v \in V(G) \setminus W$ adjacent to precisely the same vertices in W, and
- every two nonadjacent vertices have a common neighbour in W.

We observe the following convenient lemma:

2.1 *Let G be a non-orientable prismatic graph. Then either*

- *there exist nonadjacent u, v, not in the core, and with no common neighbour, or*

- *G can be obtained from a rigid non-orientable prismatic graph by replicating vertices not in the core.*

Proof. Let W be the core. Suppose that there are two nonadjacent vertices u, v with no common neighbour in W. If $u \in W$, then since v has a neighbour in a triangle containing u, it follows that u, v have a common neighbour in W, a contradiction. Thus $u \notin W$ and similarly $v \notin W$. If they have no common neighbour at all then the claim holds, so we suppose that $w \in V(G) \setminus W$ is adjacent to both u, v. Let N_u, N_v, N_w be the sets of vertices in W adjacent to u, v, w respectively. Since $u, v, w \notin W$, it follows that N_u, N_v, N_w are stable. By hypothesis, $N_u \cap N_v = \emptyset$. Since $u, v, w \notin W$, it follows that $N_u \cap N_w = \emptyset$ and $N_v \cap N_w = \emptyset$, and so N_u, N_v, N_w are pairwise disjoint. For every triangle T of G, since $T \subseteq W$, it follows that N_u, N_v, N_w each intersect T, and so $T \subseteq N_u \cup N_v \cup N_w$. Hence $W = N_u \cup N_v \cup N_w$, and so $G|W$ is 3-colourable, contradicting that G is non-orientable (by theorem 4.1 of [2]).

Thus we may assume that every two nonadjacent vertices have a common neighbour in W. Hence the second condition in the definition of "rigid" is satisfied. But also, every two vertices in $V(G) \setminus W$ are adjacent if and only if they have no common neighbour in W (for the "only if" part is clear). It follows that any two vertices in $V(G) \setminus W$ with the same set of neighbours in W have the same set of neighbours in $V(G)$, and so G is obtained from a rigid prismatic graph by replicating vertices. This proves 2.1. ■

To understand the structure of all non-orientable prismatic graphs, it suffices to understand the rigid non-orientable prismatic graphs, because the previous result implies that

2.2 *Every non-orientable prismatic graph can be obtained from a rigid non-orientable prismatic graph by replicating vertices not in the core, and then deleting edges between vertices not in the core.*

Proof. Let G be a non-orientable prismatic graph with core W. We proceed by induction on the number of nonadjacent pairs of vertices of G. By 2.1, we may assume that there exist nonadjacent u, v, not in the core, and with no common neighbour; but then adding the edge uv yields a prismatic graph with the same core, and the result follows from the inductive hypothesis applied to this graph. This proves 2.2. ■

3 Operations on prismatic graphs

Thus, the goal of this paper is to describe explicitly all the rigid non-orientable prismatic graphs. We will prove that every such graph belongs to one of several possible families of graphs. Before we can give a precise statement of the main theorem, we need to list these families; and to simplify that, it is helpful first to introduce some operations on prismatic graphs.

First, let H be prismatic, and let $X \subseteq V(H)$. For each $x \in X$ let A_x be a finite set of new vertices, pairwise disjoint. Let $A = \bigcup_{x \in X} A_x$, and let ϕ be a map from A to the set of integers, injective on A_x for each $x \in X$. Let G be the graph with vertex set $(V(H) \setminus X) \cup A$ and with edges as follows. Let $v, v' \in V(G)$ be distinct.

- If $v, v' \in V(H) \setminus X$ then v, v' are adjacent in G if and only if they are adjacent in H.

- If $v \in V(H) \setminus X$ and $v' \in A_x$ where $x \in X$, then v, v' are adjacent in G if and only if v, x are adjacent in H.
- If $v, v' \in A_x$ where $x \in X$, then v, v' are nonadjacent in G.
- If $v \in A_x$ and $v' \in A_{x'}$ where $x, x' \in X$ are distinct, then v, v' are adjacent in G if and only if $\phi(v) \neq \phi(v')$.

We say that G is obtained from H by *multiplying* X. (This operation does not always produce a prismatic graph, and we only use it in special circumstances.) For $x \in X$, we call the set A_x the *set of new vertices corresponding to* x, and we call ϕ the *corresponding integer map*.

Here is a special case when the result of multiplication is always prismatic. Let T be a triangle of a prismatic graph H, say $T = \{a, b, c\}$. We say T is a *leaf triangle at* c if a, b both only belong to one triangle of H (namely, T). Suppose that T is a leaf triangle at c. Then any graph obtained from H by multiplying $\{a, b\}$ is prismatic (we leave checking this to the reader).

We need a variant of this. It can only be applied to leaf triangles that have a certain additional property. Thus, let H be prismatic with core W, and let $T = \{a, b, c\}$ be a leaf triangle at c in H. Define subsets D_1, D_2, D_3 of the set of neighbours of c as follows. If v is adjacent to c in H, let

- $v \in D_1$ if v belongs to a triangle that does not contain c;
- $v \in D_2$ if $v \in W \setminus T$, and every triangle containing v also contains c (and hence is unique);
- $v \in D_3$ if $v \notin W$.

Thus the four sets $D_1, D_2, D_3, \{a, b\}$ are pairwise disjoint and have union the set of neighbours of c in H. Suppose that D_1, D_2 are both stable. Let A, B, C be three pairwise disjoint sets of new vertices, and let G be obtained from H by deleting a, b and adding the new vertices $A \cup B \cup C$, with adjacency as follows:

- A, B and C are stable;
- every vertex in A has at most one neighbour in B, and vice versa;
- every vertex in $V(H) \setminus \{a, b\}$ adjacent to a in H is complete to A in G, and every vertex in $V(H) \setminus \{a, b\}$ nonadjacent to a in H is anticomplete to A in G;
- every vertex in $V(H) \setminus \{a, b\}$ adjacent to b in H is complete to B in G, and every vertex in $V(H) \setminus \{a, b\}$ nonadjacent to b in H is anticomplete to B in G;
- every vertex in C is complete to $D_1 \cup D_3$, and anticomplete to $V(H) \setminus (D_1 \cup D_3 \cup \{a, b\})$;
- every vertex in C is adjacent to exactly one end of every edge between A and B, and adjacent to every vertex in $A \cup B$ with no neighbour in $A \cup B$.

We say that G is obtained from H by *exponentiating* the leaf triangle $\{a, b, c\}$.

4 A menagerie of prismatic graphs

In this section we list the classes of prismatic graphs that we need to state the main result.

Schläfli-prismatic graphs

Let G have 27 vertices $\{r_j^i, s_j^i, t_j^i : 1 \leq i, j \leq 3\}$, with adjacency as follows. Let $1 \leq i, i', j, j' \leq 3$.

- If $i \neq i'$ and $j \neq j'$ then r_j^i is adjacent to $r_{j'}^{i'}$, and s_j^i is adjacent to $s_{j'}^{i'}$, and t_j^i is adjacent to $t_{j'}^{i'}$; while if either $i = i'$ or $j = j'$ (and not both) then the same three pairs are nonadjacent.
- If $j = i'$ then r_j^i is adjacent to $s_{j'}^{i'}$, and s_j^i is adjacent to $t_{j'}^{i'}$, and t_j^i is adjacent to $r_{j'}^{i'}$; while if $j \neq i'$ then the same three pairs are nonadjacent.

This graph is the complement of the Schläfli graph, and is much more symmetrical than is apparent from this description — see [1], for instance. (Our thanks to Adrian Bondy, who identified this mysterious graph for us.) All induced subgraphs of G are prismatic, and we call any such graph *Schläfli-prismatic*.

Fuzzily Schläfli-prismatic graphs

Let $\{a, b, c\}$ be a leaf triangle on c in a Schläfli-prismatic graph H. If G can be obtained from H by multiplying $\{a, b\}$, and A, B are the two sets of new vertices corresponding to a, b respectively, we say that G is *fuzzily Schläfli-prismatic (at (A, B, c))*. (Note that this operation is not iterated: we only permit one leaf triangle to be multiplied.)

Graphs of parallel-square type

Let X be the edge-set of some 4-cycle of the complete bipartite graph $K_{3,3}$, and let z be the edge of $K_{3,3}$ disjoint from all the edges in X. Thus X induces a cycle of the line graph H of $K_{3,3}$. Any graph obtained from H by multiplying X, and possibly deleting z, is prismatic, and is called a graph of *parallel-square type*.

Graphs of skew-square type

Let K be a graph with five vertices a, b, c, s, t, where $\{s, a, c\}$ and $\{t, b, c\}$ are triangles and there are no more edges. Let H be obtained from K by multiplying $\{a, b, c\}$, let A, B, C be the sets of new vertices corresponding to a, b, c respectively, and let ϕ be the corresponding integer map. Add three more vertices d_1, d_2, d_3 to H, with adjacency as follows:

- d_1, d_2, d_3, s, t are pairwise nonadjacent
- for $1 \leq i \leq 3$ and $v \in A \cup B$, d_i is adjacent to v if and only if $1 \leq \phi(v) \leq 3$ and $\phi(v) \neq i$
- for $1 \leq i \leq 3$ and $v \in C$, d_i is nonadjacent to v if and only if $1 \leq \phi(v) \leq 3$ and $\phi(v) \neq i$.

Any such graph is prismatic, and is called a graph of *skew-square type*. Note that these are closely related to graphs of parallel-square type. For instance, if we take a graph of skew-square type and delete one of the vertices d_1, d_2, d_3, we obtain a graph of parallel-square type.

The class $\mathcal{F}_1$.

Let G be a graph with vertex set the disjoint union of sets $\{s, t\}, R, A, B$, where $|R| \leq 1$, and with edges as follows:

- s, t are adjacent, and both are complete to R;
- s is complete to A; t is complete to B;
- every vertex in A has at most one neighbour in A, and every vertex in B has at most one neighbour in B;
- if $a, a' \in A$ are adjacent and $b, b' \in B$ are adjacent, then the subgraph induced on $\{a, a', b, b'\}$ is a cycle;
- if $a, a' \in A$ are adjacent, and $b \in B$ has no neighbour in B, then b is adjacent to exactly one of a, a';
- if $b, b' \in B$ are adjacent and $a \in A$ has no neighbour in A, then a is adjacent to exactly one of b, b';
- if $a \in A$ has no neighbour in A, and $b \in B$ has no neighbour in B, then a, b are adjacent.

We define $\mathcal{F}_1$ to be the class of all such graphs G.

The class $\mathcal{F}_2$.

Take the line graph of $K_{3,3}$, with vertices numbered s^i_j $(1 \leq i, j \leq 3)$, where s^i_j and $s^{i'}_{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. Let H be obtained from this by multiplying $\{s^1_2, s^1_3, s^2_1, s^3_1\}$; thus, H is of parallel-square type. Let $A^1_2, A^1_3, A^2_1, A^3_1$ be the sets of new vertices corresponding to $s^1_2, s^1_3, s^2_1, s^3_1$ respectively, and let ϕ be the corresponding integer map. Suppose that

- there do not exist $u \in A^3_1$ and $v \in A^1_3$ with $\phi(u) = \phi(v)$;
- there exist $a^1_2 \in A^1_2$ and $a^2_1 \in A^2_1$ such that $\phi(a^1_2) = \phi(a^2_1) = 1$;
- $\phi(v) \neq 1$ for all $v \in A^3_1 \cup A^1_3$.

Let G be obtained from H by exponentiating the leaf triangle $\{a^1_2, a^2_1, s^3_3\}$. We define $\mathcal{F}_2$ to be the class of all such graphs G (they are all prismatic).

The class $\mathcal{F}_3$.

Take the line graph of $K_{3,3}$, with vertices numbered s^i_j $(1 \leq i, j \leq 3)$, where s^i_j and $s^{i'}_{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. Let H be obtained from this by deleting the vertex s^2_2 and possibly s^1_1, and then multiplying $\{s^1_2, s^1_3, s^2_1, s^3_1\}$; thus, H is of parallel-square type. Let $A^1_2, A^1_3, A^2_1, A^3_1$ be the sets of new vertices corresponding to $s^1_2, s^1_3, s^2_1, s^3_1$ respectively, and let ϕ be the corresponding integer map. Suppose that

- there exist $a_2^1 \in A_2^1$ and $a_1^3 \in A_1^3$ such that $\phi(a_2^1) = \phi(a_1^3) = 1$;
- $\phi(v) \neq 1$ for all $v \in A_3^1 \cup A_1^2$;
- there exist $a_3^1 \in A_3^1$ and $a_1^2 \in A_1^2$ such that $\phi(a_3^1) = \phi(a_1^2) = 2$;
- $\phi(v) \neq 2$ for all $v \in A_2^1 \cup A_1^3$.

Let G be obtained from H by exponentiating the leaf triangles $\{a_2^1, a_1^3, s_3^2\}$ and $\{a_3^1, a_1^2, s_2^3\}$. We define $\mathcal{F}_3$ to be the class of all such graphs G (they are all prismatic).

The class $\mathcal{F}_4$.

Take the complement of the Schläfli graph, with vertices numbered r_j^i, s_j^i, t_j^i as usual. Let H be the subgraph induced on

$$Y \cup \{s_j^i : (i,j) \in I\} \cup \{t_1^1, t_2^2, t_3^3\},$$

where $\emptyset \neq Y \subseteq \{r_1^3, r_2^3, r_3^3\}$ and $I \subseteq \{(i,j) : 1 \leq i,j \leq 3\}$ with $|I| \geq 8$ and including

$$\{(i,j) : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 2\}.$$

Let G be obtained from H by exponentiating the leaf triangle $\{t_1^1, t_2^2, t_3^3\}$. We define $\mathcal{F}_4$ to be the class of all such graphs G (they are all prismatic).

The class $\mathcal{F}_5$.

Take the complement of the Schläfli graph, with vertices numbered r_j^i, s_j^i, t_j^i as usual. Let H be the subgraph induced on

$$\{r_j^i : (i,j) \in I_1\} \cup \{s_j^i : (i,j) \in I_2\} \cup \{t_j^i : (i,j) \in I_3\},$$

where $I_1, I_2, I_3 \subseteq \{(i,j) : 1 \leq i,j \leq 3\}$ are chosen such that

- $(1,1), (3,1), (3,2), (3,3) \in I_1$ and $(2,2), (2,3) \notin I_1$
- $(1,1) \notin I_2$
- $(1,2), (1,3), (2,3), (3,3) \in I_3$ and $(2,1), (3,1) \notin I_3$.

Let G be obtained from H by adding the edge $r_1^1 t_2^1$. We define $\mathcal{F}_5$ to be the class of all such graphs G (they are all prismatic).

The class $\mathcal{F}_6$.

Graphs of the previous class sometimes admit a leaf triangle, that we can multiply. More precisely, take the complement of the Schläfli graph, with vertices numbered r_j^i, s_j^i, t_j^i as usual. Let H be the subgraph induced on

$$\{r_j^i : (i,j) \in I_1\} \cup \{s_j^i : (i,j) \in I_2\} \cup \{t_j^i : (i,j) \in I_3\},$$

where

- $I_1 = \{(1,1),(1,2),(3,1),(3,2),(3,3)\}$
- $I_2 = \{(1,2),(2,1),(2,2),(3,3)\}$
- $I_3 = \{(1,2),(2,2),(1,3),(2,3),(3,3)\}$.

Let G be obtained from H by adding the edge $r_1^1 t_2^1$, and then multiplying $\{r_3^3, t_3^3\}$. We define $\mathcal{F}_6$ to be the class of all such graphs G (they are all prismatic).

The class $\mathcal{F}_7$.

The *six-vertex prism* is the graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and edges

$$a_1a_2, a_1a_3, a_2a_3, b_1b_2, b_1b_3, b_2b_3, a_1b_1, a_2b_2, a_3b_3.$$

Let K be a graph with six vertices, with the six-vertex prism as a subgraph. Construct a new graph G as follows. The vertex set of G consists of $E(K)$ and some of the vertices of K, so $E(K) \subseteq V(G) \subseteq E(K) \cup V(K)$; two edges of K are adjacent in G if they have no common end in K; an edge and a vertex of K are adjacent in G if they are incident in K; and two vertices of H are adjacent in G if they are nonadjacent in K. The class of all such graphs G is called $\mathcal{F}_7$ (they are all prismatic).

The class $\mathcal{F}_8$.

Let H be the graph with nine vertices $v_1, \dots, v_9$ and with edges as follows: $\{v_1, v_2, v_3\}$ is a triangle, $\{v_4, v_5, v_6\}$ is complete to $\{v_7, v_8, v_9\}$, and for $i = 1, 2, 3$, v_i is adjacent to v_{i+3}, v_{i+6}. Let G be obtained from H by multiplying $\{v_4, v_7\}$, $\{v_5, v_8\}$ and $\{v_6, v_9\}$. We define $\mathcal{F}_8$ to be the class of all such graphs G (they are all prismatic).

The class $\mathcal{F}_9$.

Take the complement of the Schläfli graph, with vertices numbered r_j^i, s_j^i, t_j^i as usual. Let H be the subgraph induced on

$$\{r_j^i : (i,j) \in I_1\} \cup \{s_j^i : (i,j) \in I_2\} \cup \{t_j^i : (i,j) \in I_3\},$$

where $I_1, I_2, I_3 \subseteq \{(i,j) : 1 \le i,j \le 3\}$ satisfy

- $(2,1),(3,1),(3,2),(3,3) \in I_1$, and I_1 contains at least one of $(1,2),(1,3)$, and $(1,1),(2,2),(2,3) \notin I_1$
- $(1,1),(2,2),(3,3) \in I_2$ and $(1,2),(1,3) \notin I_2$
- $(1,3),(2,3),(3,3) \in I_3$, and I_3 contains at least one of $(1,2),(2,2),(3,2)$, and $(1,1),(2,1),(3,1) \notin I_3$
- either $(1,2),(1,3) \in I_1$, or I_3 contains $(1,2)$ and at least one of $(2,2),(3,2)$.

Let G be obtained from H by adding a new vertex z adjacent to r_2^3, r_3^3, s_1^1, and to t_2^2 if $(2,2) \in I_3$, and to t_2^3 if $(3,2) \in I_3$. We define $\mathcal{F}_9$ to be the class of all such graphs G (they are all prismatic).

That concludes the list of "basic" prismatic graphs. Let us say that G is *in the menagerie* if either G is of parallel-square or skew-square type, or is Schläfli-prismatic or fuzzily Schläfli-prismatic, or $G \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_9$. Now we can state the main theorem properly.

4.1 *Every rigid non-orientable prismatic graph is in the menagerie.*

The proof will occupy all the remainder of this paper.

5 Wickets

A *wicket* in G is a triple (r, s, t) of distinct vertices such that

- s, t are adjacent, and r is nonadjacent to both s, t
- every triangle in G contains one of r, s, t
- the set of vertices nonadjacent to both r, s is stable, and
- the set of vertices nonadjacent to both r, t is stable.

It is helpful to study prismatic graphs with wickets separately from those without, because they lead to different structures.

5.1 *Let G be a rigid prismatic graph containing a wicket. Then $G \in \mathcal{F}_2 \cup \mathcal{F}_3$.*

Proof. Let (r, s, t) be a wicket. Define subsets A, B, C, R, S, T, U of $V(G) \setminus \{r, s, t\}$ as follows. For $v \in V(G) \setminus \{r, s, t\}$, let

- $v \in A$ if v is adjacent to r, s and not to t
- $v \in B$ if v is adjacent to r, t and not to s
- $v \in C$ if v is adjacent to both s, t
- $v \in R$ if v is adjacent to r and to neither of s, t
- $v \in S$ if v is adjacent to s and to neither of r, t
- $v \in T$ if v is adjacent to t and to neither of r, s
- $v \in U$ if v is adjacent to none of r, s, t.

Thus every vertex in $V(G) \setminus \{r, s, t\}$ belongs to exactly one of these seven sets.

(1) *$|C| \leq 1$, and C is complete to R, U, and anticomplete to A, B, S, T.*

For let $c \in C$. Any other member of C would have two neighbours in the triangle $\{c, s, t\}$, and so $C = \{c\}$. If $x \in R \cup U$, then since x has a neighbour in the triangle $\{c, s, t\}$ it follows that x, c are adjacent. Thus c is complete to $R \cup U$. If $y \in A \cup B \cup S \cup T$, then since y is adjacent to one of s, t and has only one neighbour in the triangle $\{c, s, t\}$, it follows that c, y are nonadjacent. This proves (1).

(2) *$A \cup R, B \cup R, S \cup U, T \cup U$ are stable.*

If $x, y \in A \cup R$ are adjacent, then t has no neighbour in the triangle $\{x, y, r\}$, which is impossible. Thus $A \cup R$ is stable, and similarly so is $B \cup R$. Since (r, s, t) is a wicket and therefore the set of vertices nonadjacent to both r, s is stable, it follows that $T \cup U$ is stable, and similarly $S \cup U$ is stable. This proves (2).

Because r is complete to $A \cup B$, it follows that every vertex in A has at most one neighbour in B and vice versa. Because s is complete to $A \cup S$, every vertex in A has at most one neighbour in S and vice versa; and similarly every vertex in B has at most one neighbour in T and vice versa. Let A_1 be the set of all members of A with a neighbour in S, and let A_2 be the set of vertices in A with a neighbour in B. Let $A_0 = A \setminus (A_1 \cup A_2)$, let $A_3 = A_1 \setminus A_2$, and let A_4 be the set of vertices in $A_2 \setminus A_1$ whose neighbour in B has no neighbour in T. Define $B_0, B_1, B_2, B_3, B_4 \subseteq B$ similarly. Let S_1 be the set of all members of S with a neighbour in A, and let S_3 be the set of vertices in S_1 with a neighbour in A_3. Let $S_0 = S \setminus S_1$. Define $T_0, T_1, T_3 \subseteq T$ similarly.

(3) *$A_0 \cup A_4$ is complete to T, and anticomplete to $(B \setminus B_4) \cup R \cup S$.*

For let $a \in A_0 \cup A_4$. If $a \in A_0$ and $v \in T$, then a, v have no common neighbour, and so rigidity implies that a, v are adjacent. Thus we may assume that $a \in A_4$. Let $b \in B$ be adjacent to a; then b has no neighbour in T, from the definition of A_4, and so b, v are nonadjacent. Since v has a neighbour in the triangle $\{r, a, b\}$, it follows that a, v are adjacent. Thus $A_0 \cup A_4$ is complete to T. The other part of the claim is immediate from the definition of A_4, B_4. This proves (3).

(4) *A_3 is complete to $T_1 \cup U$ and anticomplete to $B \cup R \cup (S \setminus S_3)$.*

For let $a \in A_3$, and let $v \in T_1 \cup U$. Let $x \in S$ be adjacent to a. Since v has a neighbour in the triangle $\{a, s, x\}$ and v, s are nonadjacent, it follows that v is adjacent to one of a, x. If $v \in U$, then (2) implies that v is adjacent to a as required, so we assume that $v \in T_1$. Let $b \in B$ be adjacent to v. Now a, b are nonadjacent since $a \in A_3$, and since a has a neighbour in the triangle $\{b, t, v\}$, it follows that a, v are adjacent. This proves that A_3 is complete to $T_1 \cup U$. The second part of (4) is immediate. This proves (4).

(5) *U is complete to $A_1 \cup A_0$, and to $B_1 \cup B_0$, and U is anticomplete to $A_2 \setminus (A_1 \cup A_4)$, and to $B_2 \setminus (B_1 \cup B_4)$.*

For let $u \in U$ and $a \in A_1 \cup A_0$. If $a \in A_1$, u has a neighbour in the triangle $\{a, s, x\}$, where $x \in S$ is adjacent to a, and so a, u are adjacent by (2). If $a \in A_0$, then a, u have no common neighbour, and therefore are adjacent by rigidity. This proves the first claim. For the second, let $u \in U$ and $a \in A_2 \setminus (A_1 \cup A_4)$. Let $b \in B$ be adjacent to a. Since $a \notin A_1$ and $a \notin A_4$, it follows that

$b \in B_1$, and so b, u are adjacent (by the first claim). This proves (5).

Now there are two cases, depending whether R and C are both nonempty or not.

(6) *If R and C are both nonempty then $|C| = |R| = 1$.*

This is immediate since G is prismatic.

(7) *If R, C are both nonempty then S is anticomplete to T, and A_1 is anticomplete to B_1.*

For let $c \in C$ and $r_1 \in R$. Since every vertex in $S \cup T$ has a neighbour in the triangle $\{c, r, r_1\}$, it follows that r_1 is complete to $S \cup T$, and since every triangle contains one of r, s, t, it follows that S is anticomplete to T. This proves the first claim. For the second, let $a \in A_1$ and $b \in B_1$. Let $x \in S$ be adjacent to a, and let $y \in T$ be adjacent to b. Then y has a neighbour in the triangle $\{a, s, x\}$, and since S is anticomplete to T, it follows that y is adjacent to a, and therefore a, b are nonadjacent. This proves (7).

(8) *If R, C are both nonempty, and $a \in A$ and $v \in T$, then a, v are adjacent unless they have a common neighbour in B.*

This is immediate from rigidity, because every common neighbour of v, a belongs to B.

(9) *If R, C are both nonempty, then $G \in \mathcal{F}_2$.*

To see this, let $C = \{s_1^1\}$ and $R = \{s_2^2\}$, and rename r, s, t by s_3^3, s_3^2, s_2^3 respectively. The sets called $A_2^1, A_3^1, A_1^2, A_1^3$ in the definition of $\mathcal{F}_2$ are $\{a_2^1\} \cup (A \setminus A_4), T, \{a_1^2\} \cup (B \setminus B_4), S$ respectively, where a_2^1, a_1^2 are new vertices; and (A_4, B_4, U) is the triple of new vertices introduced by exponentiating $\{a_2^1, a_1^2, s_3^3\}$. This proves (9).

In view of (9) we henceforth assume that one of R, C is empty.

(10) *There is a 4-cycle with vertices a-b-y-x-a in order, where $a \in A_1, b \in B_1, y \in T_1$ and $x \in S_1$. Consequently $R = U = \emptyset$.*

For suppose there is no such r-cycle. Since one of R, C is empty, every triangle of G containing r consists of r, a vertex of A and a vertex of B. Define $\mathcal{O}$ as follows:

- For every triangle D containing s and not t, let $D = \{a, s, x\}$ where $a \in A$ and $x \in S$, and define $\mathcal{O}(D)$ to be $s \to x \to a \to s$.
- For every triangle D containing t and not s, let $D = \{b, t, y\}$ where $b \in B$ and $y \in T$, and define $\mathcal{O}(D)$ to be $t \to b \to y \to t$.
- For every triangle D containing r, let $D = \{r, a, b\}$ where $a \in A$ and $b \in B$, and define $\mathcal{O}(D)$ to be $r \to a \to b \to r$.
- If there exists $c \in C$ and D is the triangle $\{c, s, t\}$, define $\mathcal{O}(D)$ to be $c \to s \to t \to c$.

It is easy to check (since there is no 4-cycle as in the statement of (10)) that $\mathcal{O}$ is an orientation, a contradiction. This proves the first claim. Let a-b-y-x-a be vertices in order of this 4-cycle, with $a \in A, b \in B$, $y \in T$ and $x \in S$. If $u \in U$, then by (5), $\{u, a, b\}$ is a triangle containing none of r, s, t, a contradiction. Thus $U = \emptyset$. If $v \in R$, then since v has a neighbour in the triangle $\{a, s, x\}$, (2) implies that v, x are adjacent, and similarly so are v, y; but then $\{v, x, y\}$ is a triangle containing none of r, s, t, a contradiction. Thus $R = \emptyset$. This proves (10).

But from (10) it follows that $G \in \mathcal{F}_3$. (To see this, map r, s, t and $c \in C$ (if it exists) to $s_3^3, s_3^2, s_2^3, s_1^1$ respectively. Take the sets $A_2^1, A_1^2, A_3^1, A_1^3$ to be

$$(A \setminus A_3) \cup \{a_2^1\}, (B \setminus B_3) \cup \{a_1^2\}, T^3 \cup \{a_3^1\}, S^3 \cup \{a_1^3\}$$

respectively, where $a_2^1, a_1^2, a_3^1, a_1^3$ are new vertices. Take $(A_3, S_3, T \setminus T_1)$ and $(T_3, B_3, S \setminus S_1)$ to be the triples of new vertices introduced by exponentiating the leaf triangles $\{a_2^1, a_1^3, s_3^2\}$ and $\{a_3^1, a_1^2, s_2^3\}$ respectively.) This proves 5.1. ■

6 Minimal non-orientable prismatic graphs

Here are two graphs that will be important in the remainder of the paper.

- **A rotator.** Let G have nine vertices $v_1, v_2, \ldots, v_9$, where $\{v_1, v_2, v_3\}$ is a triangle, $\{v_4, v_5, v_6\}$ is complete to $\{v_7, v_8, v_9\}$, and for $i = 1, 2, 3$, v_i is adjacent to v_{i+3}, v_{i+6}, and there are no other edges.
- **A twister.** Let G have ten vertices $u_1, u_2, v_1, \ldots, v_8$, where u_1, u_2 are adjacent, for $i = 1, \ldots, 8$ v_i is adjacent to $v_{i-1}, v_{i+1}, v_{i+4}$ (reading subscripts modulo 8), and for $i = 1, 2$, u_i is adjacent to $v_i, v_{i+2}, v_{i+4}, v_{i+6}$, and there are no other edges. We call u_1, u_2 the *axis* of the twister.

It is easy to see that both these graphs are prismatic and not orientable. But more important is the converse, the following:

6.1 *Let G be prismatic. Then G is orientable if and only if no induced subgraph of G is a twister or rotator.*

Proof. The "only if" part is clear, since these two graphs are not orientable. To prove the "if" part, suppose that G is prismatic and not orientable. Let H be the graph with vertex set the set of all triangles of T, in which two triangles are adjacent if they are disjoint in G. For each triangle T of G, choose an arbitrary cyclic permutation $\beta(T)$ of T. For each edge ST of H, we define its "sign" as follows. Let $\beta(S)$ be $s_1 \to s_2 \to s_3 \to s_1$ say, and for $i = 1, 2, 3$, let $t_i \in T$ be adjacent to s_i. Then the two cyclic permutations of T are $t_1 \to t_2 \to t_3 \to t_1$ and $t_1 \to t_3 \to t_2 \to t_1$, and so $\beta(T)$ is one of them. If $\beta(T)$ is the first of these permutations, we say the edge ST has positive sign, and otherwise it has negative sign.

Suppose that every cycle of H has an even number of edges with negative sign. It follows that if we contract all the edges of H with positive sign, we obtain a bipartite graph; and so there is a partition (X, Y) of $V(H)$ such that for every edge ST of H, this edge has negative sign if and only if exactly one of S, T belongs to X. Consequently, we may construct an orientation of G by defining

$\mathcal{O}(T) = \beta(T)$ if $T \in Y$, and $\mathcal{O}(T)$ to be the cyclic permutation of T different from $\beta(T)$ if $T \in X$, a contradiction since G is not orientable. Consequently there is a cycle of H so that an odd number of its edges have negative sign.

Let T_1-$\cdots$-T_n-T_1 be the shortest such cycle of H. It follows that it is an induced cycle of H; that is, for $1 \le i < j \le n$, $T_i \cap T_j = \emptyset$ if and only if $j = i+1$ or $(i,j) = (1,n)$. In particular, $T_4, \ldots, T_{n-1}$ all meet both of T_1, T_2. Now every triangle that meets both of T_1, T_2 contains both ends of one of the three edges between T_1 and T_2, and for each such edge there is at most one triangle that contains both its ends. It follows that each of the triangles $T_4, \ldots, T_{n-1}$ contains a unique vertex of T_1, and no two of these triangles contain the same vertex of T_1. Since no edge of G belongs to two triangles, it follows that $T_4, \ldots, T_{n-1}$ are pairwise disjoint. If $n \ge 7$ then T_4 meets T_6, a contradiction; and so $n \le 6$.

Let $W = T_1 \cup \cdots \cup T_n$. We claim that $G|W$ is not orientable. For suppose that $\mathcal{O}$ is an orientation of $G|W$. Let $X = \{T \in \{T_1, \ldots, T_n\} :\ \beta(T) \ne \mathcal{O}(T)\}$, and $Y = \{T_1, \ldots, T_n\} \setminus X$. Let ST be an edge of the cycle. Since the matching of G between S and T maps $\mathcal{O}(S)$ to $\mathcal{O}(T)$, this matching maps $\beta(S)$ to $\beta(T)$ if and only if X contains both or neither of S, T. Hence there are an even number of edges ST of the cycle such that the matching between S and T does not map $\beta(S)$ to $\beta(T)$; in other words, an even number of edges of the cycle have negative sign, a contradiction. This proves that $G|W$ is not orientable.

Suppose that $n = 6$. Since T_1, T_3, T_5 pairwise intersect, and no two vertices belong to two triangles, it follows that there is a vertex $v_1 \in T_1 \cap T_3 \cap T_5$. Similarly there exists $v_2 \in T_2 \cap T_4 \cap T_6$. Since T_1, T_3, T_5 are each disjoint from one of T_2, T_4, T_6, it follows that $v_2 \notin T_1 \cup T_3 \cup T_5$, and similarly $v_1 \notin T_2 \cup T_4 \cup T_6$. Let $T_i = \{v_1, a_i, b_i\}$ for $i = 1,3,5$, and $T_i = \{v_2, a_i, b_i\}$ for $i = 2,4,6$. Since $T_1 \cap T_4 \ne \emptyset$, we may assume that $b_1 = b_4$, and similarly $b_3 = b_6$ and $b_5 = b_2$. Now $v_1 b_3$ and $b_1 v_2$ are edges, and since a_6 has a neighbour in T_1, it follows that a_1, a_6 are adjacent. Similarly a_i, a_j are adjacent for all odd i and even j with $1 \le i, j \le 6$ and $|j - i| \ne 3$. Define $\mathcal{O}(T_i)$ to be $v_1 \to a_i \to b_i \to v_1$ for $i = 1,3,5$, and to be $v_2 \to b_i \to a_i \to v_2$ for $i = 2,4,6$; then it is easy to see that $\mathcal{O}$ is an orientation of $G|W$, a contradiction. Thus $n \le 5$.

Suppose that $n = 5$. Since no three of $T_1, \ldots, T_5$ pairwise intersect, and yet every nonconsecutive pair of triangles in the sequence $T_1, \ldots, T_5, T_1$ intersect, we may label W as $\{v_1, \ldots, v_5, u_1, \ldots, u_5\}$ where for $i = 1, \ldots, 5$, $T_i = \{u_i, v_{i+2}, v_{i+3}\}$ (reading subscripts modulo 5). Let $1 \le i \le 5$. Now v_i has a unique neighbour in T_{i+1}, and so v_i, v_{i+3} are not adjacent; and similarly v_i, v_{i+2} are not adjacent. Since v_i has a neighbour in T_i, it follows that u_i, v_i are adjacent. Define $\mathcal{O}(T_i)$ to be $u_i \to v_{i+2} \to v_{i+3} \to u_i$ for $1 \le i \le 5$; then $\mathcal{O}$ is an orientation of $G|W$, a contradiction. Thus $n \le 4$.

Suppose that $n = 4$. Choose $u_1 \in T_1 \cap T_3$, and $u_2 \in T_2 \cap T_4$. Now $T_1 \cup T_3$ is disjoint from $T_2 \cup T_4$, so $|W| = 10$ and we may label $W \setminus \{u_1, u_2\}$ as $\{v_1, \ldots, v_8\}$ where $T_i \setminus \{u_1, u_2\} = \{v_i, v_{i+4}\}$ for $i = 1,2,3,4$ (reading subscripts modulo 8). Suppose first that u_1, u_2 are nonadjacent. Then we may assume that u_1 is adjacent to v_2, v_4, and u_2 to v_1, v_3. Since v_5 has a neighbour in T_2, and each vertex of T_2 has a unique neighbour in T_1, it follows that v_5, v_6 are adjacent, and similarly $\{v_5, v_7\}$ is complete to $\{v_6, v_8\}$. Define $\mathcal{O}(T_i)$ to be $u_1 \to v_i \to v_{i+4} \to u_1$ if $i = 1,3$ and to be $u_2 \to v_{i+4} \to v_i \to u_2$ if $i = 2,4$; then $\mathcal{O}$ is an orientation of $G|W$, a contradiction. This proves that u_1, u_2 are adjacent. Now since every vertex of T_1 has a unique neighbour in T_2 and vice versa, we may assume that $v_1 v_2$ and $v_5 v_6$ are edges. Similarly (by exchanging v_3, v_7 if necessary) we may assume that $v_2 v_3$ and $v_6 v_7$ are edges; and (exchanging v_4, v_8 if necessary) that $v_3 v_4$ and $v_7 v_8$ are edges. It remains to determine the edges between T_1 and T_4. If $v_1 v_4$ and $v_5 v_8$ are edges, then $G|W$ is orientable (define $\mathcal{O}(T_i)$ to be $u_1 \to v_i \to v_{i+4} \to u_1$ for $i = 1, 3$, and to be $u_2 \to v_i \to v_{i+4} \to u_2$ for $i = 2, 4$), a contradiction. Thus v_1v_8 and v_4v_5 are edges; but then $G|W$ is a twister, and the theorem holds. Thus we may assume that $n = 3$.

Now T_1, T_2, T_3 are pairwise disjoint, and so $|W| = 9$, and we may label $T_j = \{t_j^1, t_j^2, t_j^3\}$ for $j = 1, 2, 3$. Since there is a 3-edge matching between T_1 and T_2, we may assume that $t_1^i t_2^i$ is an edge for $i = 1, 2, 3$, and similarly $t_2^i t_3^i$ is an edge for $i = 1, 2, 3$. It remains to determine the edges between T_1, T_3. Up to isomorphism, there are three distinct possibilities for these edges:

- $t_1^1t_3^1, t_1^2t_3^2, t_1^3t_3^3$
- $t_1^1t_3^2, t_1^2t_3^3, t_1^3t_3^1$
- $t_1^1t_3^1, t_1^2t_3^3, t_1^3t_3^2$

In the first two cases, $G|W$ admits an orientation (we leave this to the reader), and in the third case $G|W$ is a rotator. This completes the proof of 6.1. ■

7 Excluding a rotator

To understand all non-orientable prismatic graphs, it suffices therefore to understand those that contain a rotator, and those that contain a twister and no rotator. The second is the goal of this section. We begin with a lemma.

7.1 *Let G be prismatic, and suppose that $G|W$ is a twister. Let $W = \{u_1, u_2, v_1, \ldots, v_8\}$, as in the definition of a twister, and suppose that for $1 \leq i \leq 8$, there is no vertex adjacent to both v_i, v_{i+1} (reading subscripts modulo 8). Then $G \in \mathcal{F}_1$.*

Proof. We claim that every vertex is adjacent to at least one of u_1, u_2. For let $v \in V(G)$, and suppose it is nonadjacent to both u_1, u_2. Since it has a neighbour in the triangle $\{u_1, v_i, v_{i+4}\}$ for $i = 1, 3$, we may assume from the symmetry that v is adjacent to v_1, v_3. But it has a neighbour in $\{u_2, v_4, v_8\}$, and so is adjacent either to v_1 and v_8, or to v_3 and v_4, in either case a contradiction. This proves that every vertex is adjacent to one of u_1, u_2. The remainder of the proof is easy and we leave it to the reader. ■

The main result of this section is the following.

7.2 *Let G be a rigid non-orientable prismatic graph, such that no induced subgraph of G is a rotator. Then either G is Schäfli-prismatic, or G belongs to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$.*

Proof. Let G be as in the theorem. By 6.1, G contains a twister as an induced subgraph, and so there is a set $W = \{a_0, b_0, v_1, \ldots, v_8\}$, such that a_0, b_0 are adjacent, for $i = 1, \ldots, 8$ v_i is adjacent to $v_{i-1}, v_{i+1}, v_{i+4}$ (reading subscripts modulo 8), a_0 is adjacent to v_1, v_3, v_5, v_7, and b_0 is adjacent to v_2, v_4, v_6, v_8. Define A, B, C, D as follows: A is the set of all vertices adjacent to a_0 and not to b_0; B is the set adjacent to b_0 and not to a_0; C is the set of (at most one) vertex adjacent to both a_0, b_0; and D is the set adjacent to neither of a_0, b_0.

(1) *For each* $d \in D$*, there exists* $i \in \{1,\dots,8\}$ *such that the neighbours of* d *in* $\{v_1,\dots,v_8\}$ *are* $v_i, v_{i+1}, v_{i+3}, v_{i+6}$.

For d has a unique neighbour in each of the triangles $\{a_0, v_1, v_5\}, \{a_0, v_3, v_7\}$, and so from the symmetry we may assume that d is adjacent to v_5, v_7 and nonadjacent to v_1, v_3. Since $\{d, v_6\}$ is a subset of at most one triangle, d is not adjacent to v_6; and since it has a unique neighbour in each of $\{b_0, v_2, v_6\}, \{b_0, v_4, v_8\}$, it follows that d is adjacent to v_2, and to one of v_4, v_8. But then the result holds with $i = 4$ or $i = 7$. This proves (1).

For each $d \in D$, let $i(d)$ be the (unique) value of i that satisfies (1).

(2) *If* $d, d' \in D$ *are distinct, then* $i(d') = i(d) \pm 1$ *or* $i(d) \pm 2$*; and consequently* D *is stable.*

For let $d, d' \in D$ be distinct. We may assume that $i(d) = 1$, and let T be the triangle $\{d, v_1, v_2\}$. Since d' has a unique neighbour in T it follows that $i(d') \neq 1$. Suppose that $i(d') = 5$. Then since d' has a neighbour in T, it follows that d, d' are adjacent; but then the subgraph induced on

$$\{v_1, v_2, v_5, v_6, v_7, a_0, d, d'\}$$

is a rotator, a contradiction. Next suppose that $i(d') = 4$. Since d' is adjacent to v_2 and has a unique neighbour in T, it follows that d, d' are nonadjacent; but then the subgraph induced on

$$\{v_1, v_2, v_4, v_5, v_7, a_0, d, d'\}$$

is a rotator, a contradiction. Thus $i(d') \neq 4$ and similarly $i(d') \neq 6$. Consequently $i(d')$ is one of $2, 3, 7, 8$, and so d' is adjacent to one of v_1, v_2. Since d' has a unique neighbour in T, it follows that d, d' are nonadjacent. This proves (2).

For $1 \leq i \leq 8$, let X_i be the set of all vertices in $V(G) \setminus W$ that are adjacent to v_{i-1} and to v_{i+1}, and are nonadjacent to all other v_j for $j \in \{1,\dots,8\}$. It follows that $X_i \subseteq A$ if i is odd, and $X_i \subseteq B$ if i is even.

(3) $A = \{v_1, v_3, v_5, v_7\} \cup X_1 \cup X_3 \cup X_5 \cup X_7$*, and* $B = \{v_2, v_4, v_6, v_8\} \cup X_2 \cup X_4 \cup X_6 \cup X_8$.

For let $x \in X_1$ say. Since x has a neighbour in the triangle $\{a_0, v_1, v_5\}$, it follows that x is adjacent to a_0; and since it has only one neighbour in $\{b_0, v_2, v_6\}$, it follows that x, b_0 are nonadjacent. Thus $x \in A$, and similarly $X_i \subseteq A$ for i odd, and $X_i \subseteq B$ for i even. For the reverse inclusion, let $x \in A \setminus W$ say. Then x is nonadjacent to v_i for i odd (since it has only one neighbour in $\{a_0, v_i, v_{i+4}\}$). Since x has a unique neighbour in each of the triangles $\{b_0, v_2, v_6\}, \{b_0, v_4, v_8\}$, it follows that x belongs to one of X_1, X_3, X_5, X_7. This proves (3).

(4) *If* $d \in D$ *and* $x \in X_i$*, then* $i \neq i(d) + 4, i(d) + 5$*; and* x, d *are adjacent if and only if* $i = i(d) + 3$ *or* $i(d) + 6$.

For let $i(d) = 1$ say. Now x has a unique neighbour in the triangle $\{d, v_1, v_2\}$, and so x, d are nonadjacent if and only if x is nonadjacent to both v_1, v_2, that is, if and only if $i \in \{4, 5, 6, 7\}$. This proves the second claim. For the first, suppose that $i \in \{5,6\}$; then from the symmetry (exchanging a_0, b_0 if necessary) we may assume that $i = 5$. Therefore x, d are adjacent; but then the subgraph induced on

$$\{v_1, v_2, v_4, v_5, v_6, a_0, b_0, d, x\}$$

is a rotator, a contradiction. This proves the first claim, and so proves (4).

If C is nonempty, let c be its unique member, and otherwise c is undefined.

(5) *If T is a triangle of G, then either*

- *c exists and $T = \{a_0, b_0, c\}$, or*
- *$T = \{a_0, x, y\}$ for some $x \in X_i \cup \{v_i\}$ and $y \in X_{i+4} \cup \{v_{i+4}\}$, for some odd $i \in \{1, \ldots, 8\}$, or*
- *$T = \{b_0, x, y\}$ for some $x \in X_i \cup \{v_i\}$ and $y \in X_{i+4} \cup \{v_{i+4}\}$, for some even $i \in \{1, \ldots, 8\}$, or*
- *$T = \{d, v_{i(d)}, v_{i(d)+1}\}$ for some $d \in D$.*

For if both $a_0, b_0 \in T$, then its third member is c, so we may assume that $b_0 \notin T$. Suppose first that $a_0 \in T$, and let $T = \{a_0, t_1, t_2\}$ say. Then t_1, t_2 are adjacent to a_0 and not to b_0, and therefore belong to A; and if one of them is in W, then so is the other and the claim holds, by (3); and so we may assume that $t_1, t_2 \notin W$. By (3), $t_1, t_2 \in X_1 \cup X_3 \cup X_5 \cup X_7$. Now v_2, v_4, v_6, v_8 all have a neighbour in T, and they are nonadjacent to a_0, and so $t_1 \in X_i$ and $t_2 \in X_{i+4}$ for some i, and the claim holds. We may therefore assume that $a_0, b_0 \notin A$. Since D is stable by (1), at most one member of T is in D; and since a_0, b_0 have unique neighbours in T, it follows that $D = \{x, y, d\}$ where $x \in A, y \in B$ and $d \in D$. From the symmetry we may assume that $i(d) = 1$. Suppose that $\{x, y\} \neq \{v_1, v_2\}$; then $\{d, v_1, v_2\}$ and $\{d, x, y\}$ are distinct triangle, and so $\{x, y\} \cap \{v_1, v_2\} = \emptyset$ and $\{x, y\}$ is anticomplete to $\{v_1, v_2\}$. But then the subgraph induced on

$$\{v_1, v_2, v_5, v_6, a_0, b_0, d, x, y\}$$

is a rotator, a contradiction. Thus $\{x, y\} = \{v_1, v_2\}$ and the claim holds. This proves (5).

(6) *If there exist $d_1, d_3 \in D$ with $i(d_1) = 1$ and $i(d_3) = 3$, then G is Schläfli-prismatic.*

For then by (3), either $D = \{d_1, d_3\}$, or $D = \{d_1, d_2, d_3\}$ where $i(d_2) = 2$. Let d_2 be the third member of D if it exists, and otherwise d_2 is undefined. By (4), X_5, X_6, X_7, X_8 are all empty, and so from (5), the core of G is $\{a_0, b_0, v_1, \ldots, v_8\} \cup C \cup D$. By rigidity, it follows that $|X_i| \leq 1$ for $i = 1,2,3,4$, and X_i is complete to X_{i+1} for $i = 1,2,3$, and X_1 is complete to X_4. Moreover, X_1 is anticomplete to X_3 by (5) since X_1, X_3 are both complete to v_2, and X_2 is anticomplete to X_4 similarly. Let H be the complement of the Schläfli graph, with vertices labelled r^i_j, s^i_j, t^i_j $1 \leq i,j \leq 3$, as in the definition of the Schläfli graph. We define a map ϕ from $V(G)$ to $V(H)$ as follows. The images of $v_6, c, v_3, d_2, v_7, b_0, a_0, v_2$ are respectively $r^1_1, r^1_2, r^1_3, r^2_1, r^2_2, r^2_3, r^3_1, r^3_2$; the images of d_3, v_8, v_1 are s^2_1, s^2_2, s^1_3, and the images of v_5, d_1, v_4 are t^1_1, t^2_1, t^3_2. For $x \in X_1, X_2, X_3, X_4$ we define $\phi(x)$ to be $s^3_3, t^3_1, s^1_3, t^3_3$, respectively. We leave the reader to verify that ϕ is an isomorphism between G and an induced subgraph of H. This proves (5).

In view of (2) and (6) we may assume that $i(d') = i(d) \pm 1$ for all distinct $d, d' \in D$, and in particular $|D| \le 2$.

(7) *If* $|D| = 2$ *then* $G \in \mathcal{F}_4$.

For let $D = \{d_1, d_2\}$; we may assume that $i(d_1) = 1$ and $i(d_2) = 2$. By (4), X_5, X_6, X_7 are empty. Since no vertex in the core has a neighbour in $X_1 \cup X_3$ and a neighbour in X_2, rigidity implies that X_2 is complete to $X_1 \cup X_3$, and $|X_2| \le 1$. But then $G \in \mathcal{F}_4$. (To see this, map the vertices $a_0, b_0, d_1, d_2, v_1, \ldots, v_8$ and c (if it exists) to

$$s_3^1, t_3^3, s_2^1, s_1^1, s_1^3, s_3^2, s_2^3, t_2^2, s_2^2, r_2^3, s_1^2, t_1^1, r_1^3$$

respectively. The three sets of "new" vertices are $X_4 \cup \{v_4\}, X_8 \cup \{v_8\}$ and $X_1 \cup X_3$.)

(8) *If* $|D| = 1$ *then* $G \in \mathcal{F}_2 \cup \mathcal{F}_3$.

For let $D = \{d\}$ say; we may assume that $i(d) = 1$. Then X_5, X_6 are empty, by (4). The set of vertices nonadjacent to both d, a_0 is $\{v_6, v_8\} \cup X_2 \cup X_8$, and since X_2 is anticomplete to $X_8 \cup \{v_8\}$, it follows that this set is stable. Similarly the set of vertices nonadjacent to both d, b_0 is stable, and so (a_0, b_0, d) is a wicket and the result follows from 5.1. This proves (8).

If $D = \emptyset$, then every vertex is adjacent to one of a_0, b_0, and so for $1 \le i \le 8$ no vertex is adjacent to both v_i, v_{i+1}, and $G \in \mathcal{F}_1$ by 7.1. In view of (7) and (8), this proves 7.2. ■

8 Parallel-square and skew-square

A pair (s, t) of vertices of G is called a *square-forcer* if s, t are nonadjacent, and the set of vertices nonadjacent to both s, t is stable. Next we study graphs containing square-forcers. We begin with a lemma.

8.1 *Let* (s, t) *be a square-forcer in a prismatic graph* G*, and suppose that* G *contains a rotator. Then there are distinct vertices* $a_0, b_0, c_0, d_0 \in V(G) \setminus \{s, t\}$*, in the core of* G*, such that*

- a_0-c_0-b_0-d_0-a_0 *is a 4-hole*
- s *is adjacent to* a_0, c_0 *and not to* b_0, d_0*, and*
- t *is adjacent to* b_0, c_0 *and not to* a_0, d_0, s*.*

Proof. Let D be the set of vertices nonadjacent to both s, t. Thus D is stable. Choose $v_1, \ldots, v_9$ as in the definition of a rotator. Suppose first that $s = v_1$. Since s, t are nonadjacent, either $t \in \{v_5, v_6, v_8, v_9\}$, or $t \notin \{v_1, \ldots, v_9\}$. In the first case, we may assume from the symmetry that $t = v_8$; and then we may take $a_0 = v_3, b_0 = v_5, c_0 = v_2, d_0 = v_9$. In the second case, since t has a neighbour in $\{v_1, v_2, v_3\}$, we may assume that t is adjacent to v_2, and so t is nonadjacent to v_5, v_8; but then v_5, v_8 are adjacent members of D, a contradiction.

Thus we may assume that $s, t \neq v_1, v_2, v_3$. Since D is stable, it contains at most one member of each triangle, and in particular at most one of $v_1, v_2, v_3 \in D$. Hence we may assume that s is adjacent to v_1 and t to v_2, and so $v_3 \in D$. Consequently, $s, t \neq v_6, v_9$, and since at most one of $v_3, v_6, v_9 \in D$, it follows that one of v_6, v_9 is adjacent to s and the other to t. From the symmetry we may assume that s is adjacent to v_9 and t to v_6. Thus $s, t \neq v_5, v_7$. Since v_5, v_7 are adjacent, they are not both in D, and from the symmetry we may assume that v_5 is adjacent to one of s, t. We claim that $t = v_8$. For suppose not. Then t is nonadjacent to v_5 since it has only one neighbour in the triangle $\{v_2, v_5, v_8\}$; and therefore s is adjacent to v_5, a contradiction since then t has no neighbour in the triangle $\{s, v_5, v_9\}$. Thus $t = v_8$. Since s has a neighbour in $\{v_2, v_5, v_8\}$, and s is nonadjacent to v_2, v_8, it follows that s, v_5 are adjacent. But then we may take $a_0 = v_9, b_0 = v_2, c_0 = v_5, d_0 = v_3$. This proves 8.1. ■

The main result of this section is the following.

8.2 *Let G be prismatic and rigid, containing a square-forcer and containing a rotator, and with no wicket. Then G is of parallel-square type, or of skew-square type.*

Proof. Let (s, t) be a square-forcer in G. Thus s, t are nonadjacent. Let A, B, C, D be the set of all $v \in V(G) \setminus \{s, t\}$ adjacent respectively to s and not to t, to t and not to s, to both s, t, and to neither of s, t. By hypothesis, D is stable. Let W be the core of G. Choose a_0, b_0, c_0, d_0 as in 8.1; then a_0, b_0, c_0, d_0 are all in W, and belong respectively to A, B, C, D.

(1) *A, B, C, D are stable.*

For D is stable by hypothesis. If $a_1, a_2 \in A$ are adjacent, then t has no neighbour in the triangle $\{s, a_1, a_2\}$, a contradiction. Thus A is stable, and similarly so is B. If $c_1, c_2 \in C$ are adjacent then t has two neighbours in the triangle $\{s, c_1, c_2\}$, a contradiction. This proves (1).

(2) *Every triangle in G containing s consists of s, some vertex of A, and some vertex of C. A similar statement holds for t. Every triangle containing neither of s, t consists of a vertex of A, a vertex of B, and a vertex of D.*

The first assertion follows from (1). Suppose then that T is a triangle with $s, t \notin T$. By (1), $T \not\subseteq C \cup D$, and so we may assume that $T \cap A$ is nonempty. Since s has only one neighbour in T it follows that $T \cap C = \emptyset$. By (1), this proves (2).

(3) *Every vertex in $A \cup B$ has at most one neighbour in C. Every vertex in C has at most one neighbour in A and at most one in B, and every vertex in $C \cap W$ has at least one neighbour in $A \cup B$. If $a \in A$ has a neighbour $c \in C$, then every member of B is adjacent to a except for the (at most one) member of B that is adjacent to c.*

For if say $a \in A$ has two neighbours $c_1, c_2 \in C$ then c_2 has two neighbours in the triangle $\{s, a, c_1\}$, a contradiction. Thus every member of $A \cup B$ has at most one neighbour in C, and similarly every vertex in C has at most one neighbour in A and at most one in B. If $c \in C \cap W$, then since c is in a triangle, it follows from (2) that c has a neighbour in $A \cup B$. For the final claim, let $a \in A$ and $c \in C$ be adjacent, and let $b \in B$. Since b has a unique neighbour in the triangle $\{s, a, c\}$, it follows that b

is adjacent to exactly one of a, c; and there is at most one choice of b adjacent to c, as we already saw. This proves (3).

(4) *If $a \in A$ and $c \in C$ are adjacent, then every vertex in D is adjacent to exactly one of them. If $a \in A$ and $b \in B$ are nonadjacent, then they have the same set of neighbours in C. Finally, if a, b have a common neighbour in C then they are nonadjacent and have the same set of neighbours in D.*

For suppose first that $a \in A$ and $c \in C$ are adjacent. Since every vertex in D has a unique neighbour in the triangle $\{s, a, c\}$, the first claim follows. For the second claim, suppose that $a \in A$ and $b \in B$ are nonadjacent, and that a is adjacent to some $c \in C$ say. Since b has a neighbour in the triangle $\{s, a, c\}$, it follows that b, c are adjacent. This proves the second claim. For the third, suppose that $a \in A$ and $b \in B$ have a common neighbour $c \in C$. Since b has a unique neighbour in the triangle $\{s, a, c\}$, it follows that a, b are nonadjacent. Every $d \in D$ is adjacent to exactly one of a, c and to exactly one of c, b, and therefore to both or neither of a, b. This proves the third claim, and therefore proves (4).

(5) *If $a \in A \cap W$ and $b \in B$ are nonadjacent, then every vertex in D adjacent to a is also adjacent to b.*

For suppose that $d \in D$ is adjacent to a and not to b. By (4), both a, b have no neighbours in C, and in particular, $a \neq a_0$ and $b \neq b_0$. By the final assertion of (3), a is adjacent to b_0, and b to a_0. It follows that d, b_0 are nonadjacent, for otherwise b would have no neighbour in the triangle $\{a, b_0, d\}$. Consequently, $d \neq d_0$. By (4), d is adjacent to c_0, and hence not to a_0. Now $a \in W$, and since a has no neighbour in C, (2) implies that there exist $d_1 \in D$ and $b_1 \in B$ such that $\{a, b_1, d_1\}$ is a triangle. In particular, $b_1 \neq b$, and since b has a neighbour in this triangle, it follows that d_1, b are adjacent. Hence $d_1 \neq d$. Now d has no neighbour in $\{a_0, b, d_1\}$, and so d_1, a_0 are nonadjacent. In particular, $d_0 \neq d_1$. Since d has no neighbour in $\{a_0, b, d_0\}$, it follows that b, d_0 are nonadjacent. Since b has no neighbour in $\{a, b_0, d_0\}$, we deduce that a, d_0 are nonadjacent. Since d_0 has a neighbour in the triangle $\{a, b_1, d_1\}$, we deduce that b_1, d_0 are adjacent. By (3), a_0 is adjacent to b_1. Since d has a neighbour in the triangle $\{a_0, b_1, d_0\}$, it follows that d, b_1 are adjacent; but then d has two neighbours in the triangle $\{a, b_1, d_1\}$, a contradiction. This proves (5).

(6) *Every vertex in $A \cap W$ has at most one nonneighbour in $B \cap W$, and vice versa.*

For suppose that some $a_1 \in A \cap W$ has two nonneighbours $b_1, b_2 \in B \cap W$. By (4), a_1, b_1, b_2 all have the same neighbours in C, and since every vertex in C has at most one neighbour in B by (3), it follows that a_1, b_1, b_2 have no neighbours in C. Since b_2 is in a triangle, and has no neighbour in C, it follows from (2) that there is a triangle $\{a_2, b_2, d_2\}$ for some $a_2 \in A$ and $d_2 \in D$. Since a_1, b_1, b_2 have the same neighbours in D by (5), it follows that b_1 is adjacent to d_2 and therefore nonadjacent to a_2. Consequently a_1, a_2, b_1, b_2 all have the same neighbours in D by (5), and in particular d_0 is adjacent to all or none of them. If d_0 is adjacent to all of them, then since b_1, b_2 are both adjacent to a_0, the edge a_0d_0 is in two triangles, a contradiction. If d_0 is adjacent to none of a_1, a_2, b_1, b_2, then d_0 has no neighbour in the triangle $\{a_2, b_2, d_2\}$, again a contradiction. This proves (6).

Let H be the graph with $V(H) = (A \cup B \cup C) \cap W$, in which $u, v \in V(H)$ are adjacent in H if and only if either

- u, v are adjacent in G and exactly one of u, v belongs to C, or
- u, v are nonadjacent in G and one of u, v is in A and the other is in B.

By (3), (4) and (6), every component of H is a clique (in H), and contains at most one vertex of each of A, B, C. Moreover, since every member of $C \cap W$ belongs to a triangle and therefore (by (2)) has a neighbour in $(A \cup B) \cap W$, it follows that every component of H contains a vertex of $A \cup B$. For $d \in D$, let Z_d be the set of all $v \in W$ such that either $v \in A \cup B$ and v is adjacent to d, or $v \in C$ and v is nonadjacent to d.

(7) *For every $d \in D$, Z_d is the union of some set of components of H. If $\{a, b, d\}$ is a triangle with $a \in A, b \in B$ and $d \in D$, and X, Y denote the components of H containing a, b respectively, then X, Y are distinct and $Z_d = X \cup Y$. Consequently, if $d \in D \cap W$ then Z_d is the union of exactly two components of H.*

For let $d \in D$. By (4) and (5), if u, v are adjacent in H then both or neither of them belong to Z_d, and so Z_d is a union of components of H. This proves the first assertion. For the second, let a, b, d, X, Y be as given. Since X, Y are cliques in H, and a, b are nonadjacent in H, it follows that $X \neq Y$. We must show that $X \cup Y = Z_d$. For we have seen that $X \cup Y \subseteq Z_d$; let $v \in Z_d \setminus \{a, b\}$. If $v \in A$, then v, d are adjacent in G; so v, b are nonadjacent in G since v has a unique neighbour in the triangle $\{a, b, d\}$; therefore v, b are adjacent in H; and so $v \in Y$. Similarly if $v \in B$ then $v \in X$. Finally, if $v \in C$, then v, d are nonadjacent in G; hence v is adjacent to one of a, b in G, since v has a neighbour in the triangle $\{a, b, d\}$; and therefore v is adjacent to one of a, b in H, and so again $v \in X \cup Y$. This proves that $X \cup Y = Z_d$, and therefore proves the second assertion. The third follows from (2). This proves (7).

(8) *If $d, d' \in D$ are distinct with $d \in W$, then $Z_d \cap Z_{d'}$ is a component of H.*

For since $d \in W$, there exist $a \in A$ and $b \in B$ such that $\{a, b, d\}$ is a triangle; choose X, Y as in (7), and then $X \neq Y$ and $Z_d = X \cup Y$. Since d' has a unique neighbour in the triangle $\{a, b, d\}$, it is adjacent to exactly one of a, b, and so $Z_d \cap Z_{d'} \neq \emptyset$ and $Z_d \neq Z_{d'}$. The claim follows from (7).

Let $X_0 = \{a_0, b_0, c_0\}$; thus, X_0 is a component of H, and $X_0 \subseteq Z_{d_0}$.

(9) *No member of A is complete to $(B \cap W) \cup D$ in G.*

For suppose that $a \in A$ is complete to $(B \cap W) \cup D$ in G. We claim that (t, s, a) is a wicket. For certainly a, s are adjacent, and t is nonadjacent to them both. Suppose that there is a triangle T containing none of t, s, a. By (2), $T = \{a_1, b_1, d_1\}$ for some $d_1 \in D$ and some $a_1, b_2 \in Z_d$ with $a_1 \in A$ and $b_2 \in B$. Let X_1, X_2 be the components of H containing a_1, b_2 respectively. Then $X_1 \neq X_2$ and $Z_d = X_1 \cup X_2$ by (7). Since $a \notin T$ it follows that $a \neq a_1$. Since $a \in Z_d$ (because a is complete to D), we deduce that $a \in X_2$, and therefore a, b_2 are nonadjacent. But $b_2 \in W$ since $b_2 \in T$, and yet a is complete to $B \cap W$, a contradiction. This proves that every triangle contains one of t, s, a. But certainly the set of vertices nonadjacent to both a, t is stable, because this set equals $A \setminus \{a\}$ since a is complete to D; and the set of vertices nonadjacent to both s, t is stable since (s, t) is a square-forcer. This proves that (t, s, a) is a wicket, a contradiction. This proves (9).

(10) *One of the following holds:*

- *There is a component* X_1 *of* H *such that* $X_1 \subseteq Z_d$ *for every* $d \in D$*; and then* $X_1 \cap A, X_1 \cap B$ *are both nonempty.*
- $D \subseteq W$*, and* $|D| = 3$*,* $D = \{d_1, d_2, d_3\}$ *say; and there are three components* X_1, X_2, X_3 *of* H *such that* $Z_{d_1} = X_2 \cup X_3$*, and* $Z_{d_2} = X_3 \cup X_1$*, and* $Z_{d_3} = X_1 \cup X_2$*.*
- $D \cap W = \{d_0\}$*, and* $Z_{d_0} = X_1 \cup X_2$*, where* $X_i \cap A, X_i \cap B \neq \emptyset$ *for* $i = 1, 2$*, and one of* X_1, X_2 *equals* X_0*. Moreover, for every* $d \in D \setminus W$*, either* $Z_d = X_1$ *or* $Z_d = X_2$*, and there exist* $d, d' \in D \setminus W$ *with* $Z_d = X_1$ *and* $Z_{d'} = X_2$*.*

For suppose first that there is a component X_1 of H such that $X_1 \subseteq Z_d$ for every $d \in D$ If $X_1 \cap A, X_1 \cap B$ are both nonempty then the first outcome of the theorem holds. Thus we may assume that $X_1 \cap B = \emptyset$. Since every component of H meets $A \cup B$, there exists $a_1 \in A \cap X_1$, and a_1 is complete to D in G (since $X_1 \subseteq Z_d$ for every $d \in D$), and to $B \cap W$ (since $X_1 \cap B = \emptyset$), contrary to (9). Thus we may assume there is no such X_1.

Now suppose there is a component X_1 of H such that $X_1 \subseteq Z_d$ for every $d \in D \cap W$. If possible, choose such a component X_1 such that $X_1 \cap A, X_1 \cap B$ are both nonempty. From what we just proved, there exists $d_1 \in D \setminus W$ with $X_1 \cap Z_{d_1} = \emptyset$. Suppose that d_1 has a neighbour $a_2 \in A \cap W$, and a neighbour $b_2 \in B \cap W$. Since $d_1 \notin W$, it follows that a_2, b_2 are nonadjacent; and so by (6), a_2, b_2 are the only neighbours of d_1 in $(A \cup B) \cap W$. Since every component of H meets $A \cup B$, it follows that $Z_{d_1} = X_2$ where X_2 is the component of H containing a_2, b_2. By (8), Z_d includes X_2 for every $d \in D \cap W$, and also by hypotheses Z_d includes X_1 for every $d \in D \cap W$. From the choice of X_1 it follows that $X_1 \cap A, X_1 \cap B \neq \emptyset$. From (8), $D = \{d_0\}$, and so one of X_1, X_2 is X_0. Also from (8), for all $d \in D \setminus W$, Z_d includes one of X_1, X_2 and hence equals one of X_1, X_2. By our assumption of the previous paragraph, there exists $d \in D \setminus W$ with $X_2 \not\subseteq Z_d$, and there exists $d' \in D \setminus W$ with $X_1 \not\subseteq Z_{d'}$; and so the third outcome of the claim holds.

Thus we may assume that d_1 does not have neighbours in both of A, B, and so we may assume that d_1 has no neighbour in $B \cap W$. It follows that d_1 is nonadjacent to b_0, and therefore has no neighbour in $\{d_0, a_1, b_0\}$. Consequently $\{d_0, a_1, b_0\}$ is not a triangle. But d_0 is adjacent to b_0 by definition, and d_0 is adjacent to a_1 since $d_0 \in W$, and therefore $X_1 \subseteq Z_{d_0}$. Consequently a_1, b_0 are nonadjacent, and so $a_1 = a_0$, that is, $X_1 = X_0$. Suppose that there is a triangle containing none of s, t, b_0, say $\{a_2, b_2, d_2\}$ where $a_2 \in A, b_2 \in B$ and $d_2 \in D$. Since $d_2 \in W$, it follows that $X_1 \subseteq Z_{d_2}$, and so d_2 is adjacent to a_0, b_0. Hence $\{a_0, b_2, d_2\}$ is a triangle, and d_1 has no neighbour in it (for d_1 has no neighbour in $B \cap W$ and $a_0 \notin Z_{d_1}$), a contradiction. Hence there is no such triangle, and so every triangle contains one of s, t, b_0. Let $Z_{d_0} = X_0 \cup X_2$; then $X_2 \cap B = \emptyset$. Since (s, t, b_0) is not a wicket, it follows that the set of vertices nonadjacent to both s, b_0 is not stable, and therefore there exists $d_2 \in D$ with a neighbour $b_2 \in B$ and yet nonadjacent to b_0. Let $a_2 \in X_2 \cap A$. By (8), $X_2 \subseteq Z_{d_2}$, and so a_2, d_2 are adjacent. But $d_2 \notin W$, since b_0 is complete to $D \cap W$, and therefore $\{d_2, a_2, b_2\}$ is not a triangle; and hence $b_2 \in X_2$, contradicting that $X_2 \cap B = \emptyset$.

Thus the claim holds when there is a component X_1 of H such that $X_1 \subseteq Z_d$ for every $d \in D \cap W$. We may therefore assume that there is no such component X_1. By (7) and (8), there exist distinct components X_1, X_2, X_3 of H, and $d_1, d_2, d_3 \in D \cap W$, such that $Z_{d_1} = X_2 \cup X_3$, and $Z_{d_2} = X_3 \cup X_1$, and $Z_{d_3} = X_1 \cup X_2$. For any fourth member $d \in D$ different from d_1, d_2, d_3, (8) implies that Z_d meets each of Z_{d_i} for $i = 1, 2, 3$ and so Z_d includes some Z_{d_i} with $1 \le i \le 3$. But then d has two neighbours in each triangle containing d_i, a contradiction. Hence $D = \{d_1, d_2, d_3\}$, and the claim holds. This proves (10).

(11) *$A \setminus W$ is anticomplete to D and complete to $B \cap W$; and $B \setminus W$ is anticomplete to D and complete to $A \cap W$.*

For let $a \in A \setminus W$. Then a is anticomplete to $(A \setminus \{a\}) \cup C$, since $a \notin W$. We must show that A is anticomplete to D and complete to $B \cap W$. Suppose first that a is anticomplete to D. For $b \in B \cap W$, since there is a triangle T containing b with $T \setminus \{b\}$ a subset of $\{t\} \cup A \cup C \cup D$, and a is anticomplete to $\{t\} \cup (A \setminus \{a\}) \cup C \cup D$, it follows that a, b are adjacent, and so a is complete to $B \cap W$ and the claim holds. We may therefore assume that a has a neighbour in D.

Suppose that the first outcome of (10) holds, and let X_1 be a component of H included in Z_d for every $d \in D$, and let $a_1 \in A \cap X_1$ and $b_1 \in B \cap X_1$. Since $a \notin W$ and $a_1 \in W$, it follows that $a \neq a_1$. Suppose that a is not adjacent to b_1. Now a has a neighbour in the triangle $\{t, b_0, c_0\}$, and since a is nonadjacent to t, c_0, we deduce that a, b_0 are adjacent. In particular, $b_0 \neq b_1$, and so a_0, b_1 are adjacent. Moreover a, d_0 are nonadjacent (since $a \notin W$), and d_0, b_1 are adjacent (since $X_1 \subseteq Z_{d_0}$), and therefore a has no neighbour in the triangle $\{a_0, b_1, d_0\}$, a contradiction. This proves that a is adjacent to b_1. But since a has a neighbour in D, and every member of D is adjacent to b_1, this proves that a is in a triangle, a contradiction.

Next, suppose that the second outcome of (10) holds. Let $d_1, d_2, d_3, X_1, X_2, X_3$ be as in that statement. Since $X_0 \subseteq Z_{d_0}$, we may assume from the symmetry that $d_1 = d_0$ and $X_3 = X_0$. Since $d_3 \in W$, we may assume (by exchanging X_1, X_2 if necessary) that there exist $a_1 \in X_1 \cap A$ and $b_2 \in X_2 \cap B$. Since a has a neighbour in the triangle $\{t, b_0, c_0\}$, it follows that a, b_0 are adjacent, and so a is nonadjacent to d_1, d_2 (because $a \notin W$). Since a has a neighbour in D, it follows that a, d_3 are adjacent. Since $a \notin W$, we deduce that a, b_2 are nonadjacent; but then a has no neighbour in the triangle $\{a_0, b_2, d_1\}$, a contradiction.

Next, suppose that the third outcome of (10) holds. Thus $D \cap W = \{d_0\}$, and $Z_{d_0} = X_1 \cup X_2$, where $X_i \cap A, X_i \cap B \neq \emptyset$ for $i = 1, 2$. We may assume that $X_2 = X_0$. For $i = 1, 2$ let D_i be the set of vertices in $D \setminus W$ with $Z_d = X_i$; then $D_1, D_2 \neq \emptyset$, and $D_1 \cup D_2 = D \setminus W$. Let $X_1 \cap A = \{a_1\}$ and $X_1 \cap B = \{b_1\}$. Now $a \notin W$, and so $a \notin X_1 \cup X_2$; hence a is nonadjacent to c_0, and since a has a neighbour in the triangle $\{t, b_0, c_0\}$, it follows that a is adjacent to b_0. Since $a \notin W$, $\{d_0, a, b_0\}$ is not a triangle, and so a, d_0 are nonadjacent. Since a has a neighbour in both triangles $\{d_0, a_1, b_0\}$ and $d_0, a_0, b_1\}$, it follows that a is adjacent to both b_0, b_1. Since every vertex in D is adjacent to at least one of a_0, a_1, and $a \notin W$, and a has a neighbour in D, it follows that a is in a triangle and so $a \in W$, a contradiction.

From (10), this completes the proof of (11).

(12) *$A \setminus W$ is complete to B, and $B \setminus W$ is complete to A.*

For from (11), it suffices to show that if $a \in A \setminus W$ and $b \in B \setminus W$ then a, b are adjacent. Suppose not; then from rigidity, a, b have a common neighbour in W. But they are both anticomplete to C since they are not in the core, and anticomplete to D by (11); a is anticomplete to $(A \cup \{t\}) \cap W$, and b is anticomplete to $(B \cup \{s\}) \cap W$, a contradiction. This proves (12).

(13) *$C \setminus W$ is anticomplete to $A \cup B$, and complete to D.*

For let $c \in C \setminus W$. Since $c \notin W$, it is anticomplete to $A \cup B$, and by (1) anticomplete to $C \setminus \{c\}$. Let $d \in D$, and suppose that c, d are nonadjacent. Then c, d have a common neighbour in W (for if $d \in W$ this is trivial, and otherwise it follows from rigidity). But all neighbours of c are in $(D \setminus \{d\}) \cup \{s, t\}$, and d has no neighbour in this set, a contradiction. This proves (13).

It follows from (11)–(13) that if the second or third outcome of (10) holds then G is of skew-square type. We therefore assume henceforth that the first outcome of (10) holds. Let X_1 be a component of H such that $X_1 \subseteq Z_d$ for every $d \in D$, and let $a_1 \in X_1 \cap A$ and $b_1 \in X_1 \cap B$.

(14) *$|D \setminus W| \leq 1$, and if $d \in D \setminus W$ then $Z_d = X_1$.*

For suppose that $d \in D \setminus W$. Since $X_1 \subseteq Z_d$, it follows that d is adjacent to a_1, b_1. Since d is in no triangles, and a_1 is complete to $(B \cap W) \setminus \{b_1\}$, it follows that d has no neighbours in $B \cap W$ except b_1, and similarly none in $A \cap W$ except a_1. Since every component of H contains a vertex of $A \cup B$, it follows that $Z_d = X_1$. Consequently, every two members of $D \setminus W$ have the same set of neighbours in W, and therefore rigidity implies that $|D \setminus W| \leq 1$. This proves (14).

From (11)–(14), we deduce that G is a graph of parallel-square type. This proves 8.2. ■

9 A triangle meeting all triangles

It is helpful to handle one other special case separately, as follows.

9.1 *Let G be a rigid, non-orientable prismatic graph containing a rotator. If there is a triangle Z such that every triangle contains a vertex of Z, then $G \in \mathcal{F}_8$.*

Proof. Let $Z = \{v_1, v_2, v_3\}$ be a triangle that meets all triangles. For $i = 1, 2, 3$, let N_i be the set of vertices in $V(G) \setminus Z$ that are adjacent to v_i. Since G is prismatic, the sets N_1, N_2, N_3 are pairwise disjoint and have union $V(G) \setminus Z$. Since G contains a rotator, no two vertices meet all triangles of G; so for $i = 1, 2, 3$, there is a triangle disjoint from $Z \setminus \{v_i\}$, and therefore it contains v_i; choose such a triangle T_i say.

Let $T_3 = \{a_3, b_3, v_3\}$. For $i = 1, 2$, every vertex in N_i is adjacent to exactly one of a_3, b_3, since it has a unique neighbour in T_3; let B_i be the set of vertices in N_i adjacent to a_3, and $A_i = N_i \setminus B_i$ (so A_i is the set adjacent to b_3). We may write $T_i = \{a_i, b_i, v_i\}$ where $a_i \in A_i$ and $b_i \in B_i$ for $i = 1, 2$. Since a_3 is complete to $B_1 \cup B_2$ and every triangle meets Z, it follows that $B_1 \cup B_2$ is stable, and similarly $A_1 \cup A_2$ is stable. But every vertex in N_1 has a neighbour in T_2, and so b_2 is complete to A_1 and a_2 to B_1. Similarly b_1 is complete to A_2 and a_1 to B_2.

Suppose that some vertex v has a neighbour $b \in B_1$ and a neighbour $a \in A_2$. Then $v \notin Z$. Since v has a neighbour in B_1, it follows that $v \notin B_1 \cup B_2$, and similarly $v \notin A_1 \cup A_2$. Hence $v \in N_3$. Since a_1, b_2 are adjacent and every triangle meets Z, v is nonadjacent to at least one of a_1, b_2, and from the symmetry we may assume v is nonadjacent to b_2. Hence v is adjacent to a_2 since it has a neighbour in T_2. Also, b is adjacent to a_2, since b has a neighbour in T_2 and is nonadjacent to b_2 (for $B_1 \cup B_2$ is stable). But then $\{v, b, a_2\}$ is a triangle disjoint from Z, a contradiction. This proves that there is no such vertex v. Since G is rigid, it follows that B_1 is complete to A_2, and similarly A_1 is complete to B_2.

We defined A_2 to be the set of vertices in N_2 adjacent to b_3, and we just showed that it is also the set of vertices in N_2 adjacent to b_1. Let A_3, B_3 be the sets of vertices in N_3 adjacent to a_1, b_1 respectively; then it follows that $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$ are stable, and A_i is complete to B_j for all distinct $i, j \in \{1, 2, 3\}$. In particular $\{a_1, a_2, a_3\}$ is complete to $\{b_1, b_2, b_3\}$. But then $G \in \mathcal{F}_8$ (to see this, take $v_4, \ldots, v_9$ to be $a_1, a_2, a_3, b_1, b_2, b_3$ respectively). This proves 9.1. ∎

10 Schläfli-prismatic graphs

In this section we prove the following.

10.1 *Let G be a rigid prismatic graph containing a rotator and with no square-forcer. Then G is in the menagerie.*

The proof is in several steps, and we first set up some notation that we use throughout this section. A set $X \subseteq V(G)$ is a *homogeneous stable set* if it is stable and all its members have the same set of neighbours. In a rigid graph, every homogeneous stable set has at most one member.

Let G satisfy the hypotheses of 10.1. The complement of the Schläfli graph has an induced subgraph which is a rotator, for instance the subgraph induced on the vertex set

$$\{r_1^3, r_2^3, r_3^3, t_3^1, t_3^2, t_3^3\} \cup \{s_i^i : (i, i) \in I\},$$

where $I = \{(1,1), (2,2), (3,3)\}$. (The reason for this puzzling notation will appear in a moment.) Since G also contains a rotator, we have therefore found an induced subgraph of G that is isomorphic to an induced subgraph of the complement of the Schläfli graph, but it might not be maximal. If possible, let us enlarge it, adding to it any further vertices that can be labelled s_j^i for some i, j with adjacency consistent with the complement of the Schläfli graph, and adding these new pairs (i, j) to the set I. Moreover, it is helpful for purposes of symmetry to replace the hypothesis that $(i, i) \in I$ $(1 \leq i \leq 3)$ by the weaker hypothesis that I "includes a permutation".

Let us state this more precisely. We choose a subset $I \subseteq \{(i, j) : 1 \leq i, j \leq 3\}$, maximal such that there are distinct vertices $r_1^3, r_2^3, r_3^3, t_3^1, t_3^2, t_3^3$ and s_j^i $((i, j) \in I)$ satisfying the following:

- $\{r_1^3, r_2^3, r_3^3\}$ and $\{t_3^1, t_3^2, t_3^3\}$ are each stable, and complete to each other;
- there exist $(i, j), (i', j'), (i'', j'') \in I$ such that $\{i, i', i''\} = \{j, j', j''\} = \{1, 2, 3\}$ (briefly, we say "I includes a permutation");
- for distinct $(i, j), (i', j') \in I$, s_j^i and $s_{j'}^{i'}$ are adjacent if and only if $i \neq i'$ and $j \neq j'$;
- for $(i, j) \in I$ and $k = 1, 2, 3$, r_k^3 and s_j^i are adjacent if and only if $k = i$;
- for $(i, j) \in I$ and $k = 1, 2, 3$, t_3^k and s_j^i are adjacent if and only if $k = j$.

Let $S = \{s^i_j : (i,j) \in I\}$, let $R^3 = \{r^3_1, r^3_2, r^3_3\}$, and let $T_3 = \{t^1_3, t^2_3, t^3_3\}$. For $i = 1,2,3$ let $S^i = \{s^i_j : 1 \leq j \leq 3$ and $(i,j) \in I\}$, and for $j = 1,2,3$ let $S_j = \{s^i_j : 1 \leq i \leq 3$ and $(i,j) \in I\}$. Let $Z = R^3 \cup S \cup T_3$. For $i = 1,2,3$ let R_i denote the set of all vertices in $V(G) \setminus Z$ that are complete to $S^i \cup (R^3 \setminus \{r^3_i\})$ and anticomplete to $\{r^3_i\} \cup (S \setminus S^i) \cup T_3$; and let $R = R_1 \cup R_2 \cup R_3$. For $j = 1,2,3$ let T^j denote the set of all vertices in $V(G) \setminus Z$ that are complete to $S_j \cup (T_3 \setminus \{t^j_3\})$ and anticomplete to $R^3 \cup (S \setminus S_j) \cup \{t^j_3\}$; and let $T = T^1 \cup T^2 \cup T^3$.

10.2 *The sets $R_1, R_2, R_3, R^3, S, T^1, T^2, T^3, T_3$ are pairwise disjoint and have union $V(G)$.*

Proof. Clearly they are pairwise disjoint. Let $v \in V(G) \setminus Z$, and let N be the set of neighbours of v in Z. Since I includes a permutation, we may assume from the symmetry that $(1,1),(2,2),(3,3) \in I$. Since $\{s^1_1, s^2_2, s^3_3\}$ is a triangle and v has a unique neighbour in it, we may assume that $s^1_1 \in N$ and $s^2_2, s^3_3 \notin N$. For $i = 1,2,3$, $\{r^3_i, s^i_i, t^i_3\}$ is a triangle, and it follows that $r^3_1, t^1_3 \notin N$, and for $i = 2,3$ exactly one of $r^3_i, t^i_3 \in N$. From the symmetry we may assume that $r^3_2 \in N$ and $t^2_3 \notin N$. Suppose first that $r^3_3 \notin N$, and $t^3_3 \in N$. Let $(i,j) \in I$. Since $\{r^3_i, s^i_j, t^j_3\}$ is a triangle, it follows that $s^i_j \in N$ if and only if both $r^3_i, t^j_3 \notin N$, that is, if and only if $i \neq 2$ and $j \neq 3$. Moreover, since v has a unique neighbour in this triangle, it follows that not both $r^3_i, t^j_3 \in N$, that is, $(i,j) \neq (2,3)$, and so $(2,3) \notin I$. But then we could set $s^2_3 = v$ and add the pair $(2,3)$ to I, contrary to the maximality of I. This proves that $r^3_3 \in N$ and $t^3_3 \notin N$. Let $(i,j) \in I$; then $\{r^3_i, s^i_j, t^j_3\}$ is a triangle, and therefore $s^i_j \in N$ if and only if both $r^3_i, t^j_3 \notin N$, that is, if and only if $i = 1$. Consequently $v \in R_1$ as required. This proves 10.2. ■

Let $\{i,j,k\} = \{1,2,3\}$, and let $r_i \in R_i$ and $r_j \in R_j$ be adjacent. Then $\{r_i, r_j, r^3_k\}$ is a triangle, and we call such a triangle an *R-triangle*. We define T-triangles similarly. If $(i,j) \in I$ and $r_i \in R_i$ is adjacent to $t^j \in T^j$, then $\{r_i, s^i_j, t^j\}$ is a triangle, and we call triangles of this form *diagonal triangles*. Finally, if $(i,j) \in I$ then $\{r^3_i, s^i_j, t^j_3\}$ is a triangle, and we call triangles of this form *marginal triangles*.

10.3 *The sets $R_i \cup \{r^3_i\}$ $(i = 1,2,3)$ and the sets $T^j \cup \{t^j_3\}$ $(j = 1,2,3)$ are stable. Moreover, every triangle in G is either a subset of S, or an R-triangle, or a T-triangle, or a diagonal triangle, or a marginal triangle.*

Proof. For the first claim, suppose that $u, v \in R_i \cup \{r^3_i\}$ are adjacent, say. Choose $j \in \{1,2,3\}$ such that $(i,j) \in I$. Thus $\{s^i_j, u, v\}$ is a triangle D say. Since t^j_3 has a unique neighbour in D, and t^j_3 is adjacent to s^i_j, r^3_i, it follows that $u, v \neq r^3_i$. Choose $k \in \{1,2,3\}$ with $k \neq i$. Then r^3_k is adjacent to both u, v, and therefore has two neighbours in D, a contradiction. This proves the first claim.

For the second, let $D = \{u,v,w\}$ be a triangle. Suppose that $v \in S$. We may assume that $D \not\subseteq S$, and so we may assume that $u \in R_i \cup \{r^3_i\}$ say. Then $v \in S^i$ since u, v are adjacent, say $v = s^i_j$, where $(i,j) \in I$. Now v has no neighbour in S^i, and u has no neighbour in $S \setminus S^i$, and therefore $w \notin S$. Since u has no neighbour in R_i and v has none in $R \setminus R^i$ it follows that $w \notin R$. For $j = 1,2,3$, u is nonadjacent to r^3_j or equal to it if $j = i$, and v is nonadjacent to r^3_j if $j \neq i$, and so $w \notin R^3$. Thus $w \in T \cup T_3$. If $u \in R_i$ then u is anticomplete to T_3 and v is anticomplete to $T \setminus T^j$, and so $w \in T_j$ and D is a diagonal triangle. If $u = r^3_i$ then u is anticomplete to T and v is anticomplete to $T_3 \setminus \{t^j_3\}$, and so $w = t^j_3$ and D is a marginal triangle. Hence the result holds if $v \in S$.

Consequently we may assume that $D \cap S = \emptyset$. Now suppose that $|D \cap R| \geq 2$. Then from the symmetry we may assume that $|D \cap (R_1 \cup R_2)| \geq 2$, and therefore r^3_3 has two neighbours in D. It follows that $r_3^3 \in D$, and so D is an R-triangle. We may assume therefore that $|D \cap R| \leq 1$, and similarly that $|D \cap T| \leq 1$. But also $|D \cap R^3|, |D \cap T_3| \leq 1$, since R^3, T_3 are stable. Hence D has nonempty intersection with three of the sets R, T, R^3, T_3, which is impossible since R is anticomplete to T_3 and T is anticomplete to R^3. This proves 10.3. ■

The next lemma is a useful assortment of easy facts.

10.4 *The following hold.*

1. *If $u, v \in R$ are adjacent, then every vertex in T is adjacent to exactly one of u, v.*
2. *If $\{i, j, k\} = \{1, 2, 3\}$ and $u \in R_i$ and $v \in R_j$ are adjacent, then every vertex in R_k is adjacent to exactly one of u, v.*
3. *Let $i, j \in \{1, 2, 3\}$ be distinct; then every vertex in R_i has at most one neighbour in R_j.*
4. *If $(i, j) \in I$, then every vertex in R_i has at most one neighbour in T^j.*
5. *If $(i, j) \in I$, and $u \in R_i$ and $v \in T^j$ are adjacent, then every vertex in $R \setminus R_i$ is adjacent to exactly one of u, v.*

The analogous statements with R, T exchanged also hold.

Proof. For the first and second statements, let $u \in R_1, v \in R_2$ say; then $\{u, v, r_3^3\}$ is an R-triangle. Every vertex in $T \cup R_3$ has a unique neighbour in this triangle, and is nonadjacent to r_3^3, and therefore is adjacent to exactly one of u, v. This proves the first two statements.

For the third, suppose that $u \in R_i$ has two neighbours $v, v' \in R_j$. Then v' has two neighbours in the R-triangle containing u, v, a contradiction. For the fourth, suppose that $u \in R_i$ has two neighbours $v, v' \in T^j$. Then v' has two neighbours in the triangle $\{u, v, s_j^i\}$, a contradiction. Finally, for the fifth statement, if $(i, j) \in I$, and $u \in R_i$ and $v \in T^j$ are adjacent, then every vertex in $R \setminus R_i$ has a unique neighbour in the triangle $\{u, v, s_j^i\}$ and is nonadjacent to s_j^i, and so is adjacent to exactly one of u, v. This proves 10.4. ■

10.5 *Every component of $G|R$ is either a path with at most five vertices, or a cycle of length six. Moreover, one of the following holds:*

- *R is stable*
- *exactly two of R_1, R_2, R_3 are nonempty, and each component of $G|R$ has at most two vertices*
- *R_1, R_2, R_3 are nonempty, and $G|R$ is connected*
- *R_1, R_2, R_3 are nonempty; and two of R_1, R_2, R_3 have only one member, say R_1, R_2; the third (R_3) has at least two members; one of its members is complete to $R_1 \cup R_2$, and all the other members are anticomplete to $R_1 \cup R_2$.*

Proof. Every component of $G|R$ is either a path or a cycle by 10.4.3. Suppose that $v_1\text{-}\cdots\text{-}v_6$ is an induced path in $G|R$. We may assume that $v_1 \in R_1$ and $v_2 \in R_2$. By 10.4.3, it follows that $v_3 \in R_3, v_4 \in R_1, v_5 \in R_2$ and $v_6 \in R_3$. But then v_6 is nonadjacent to v_1, v_2 contrary to 10.4.2. Thus every component of G has at most five vertices unless it is a cycle of length six. This proves the first claim.

For the second claim, if two of R_1, R_2, R_3 are empty the first outcome holds, and if one of them is empty then the second holds, by 10.4.3. Thus we may assume that R_1, R_2, R_3 are all nonempty, and $r_1 \in R_1$ is adjacent to $r_3 \in R_3$. Every member of R_2 is adjacent to one of r_1, r_3, by 10.4.2, and so we may assume that there exists $r_2 \in R_2$ adjacent to r_3. Let C be the component of $G|R$ containing r_1, r_2, r_3. We may assume that $G|R$ is not connected, and so there exists $v \in R$ that does not belong to C. Every vertex in R_1 is adjacent to one of r_2, r_3 by 10.4.2, and therefore belongs to C, and similarly $R_2 \subseteq C$, and so $v \in R_3 \setminus \{r_3\}$. If there exists $u \in R_1 \setminus \{r_1\}$, then u is not adjacent to r_3 by 10.4.3, and so u is adjacent to r_2 by 10.4.2; and then v is adjacent to one of u, r_2 by 10.4.2, contradicting that $v \notin C$. So $R_1 = \{r_1\}$ and similarly $R_2 = \{r_2\}$. Hence all members of $R_3 \setminus \{r_3\}$ are anticomplete to $R_1 \cup R_2$, and the fourth outcome holds. This proves 10.5. ■

Let us say $u, v \in R$ are *collinear* if $u, v \in R_i$ for some $i \in \{1, 2, 3\}$; and similarly, $u, v \in T$ are *collinear* if $u, v \in T^j$ for some $j \in \{1, 2, 3\}$. If $u, v \in T$, we say that they are *R-equal* if they have the same set of neighbours in R, and *R-opposite* if every vertex in R is adjacent to exactly one of them. They are *R-consistent* if either they are R-equal or R-opposite. (Being T-equal, T-opposite and T-consistent is defined analogously.) Here is a convenient corollary of 10.5.

10.6 *If $u, v \in R$ are adjacent, then they are T-opposite. If $u, v \in R$ are nonadjacent and noncollinear, and in the same component of $G|R$, then they are T-equal.*

Proof. The first claim follows from 10.4.1. For the second, let $u, v \in R$ be nonadjacent and noncollinear, and let P be a path of $G|R$ between u, v. From 10.4.3 and 10.5 it follows that P has even length. Since every two consecutive vertices of P are T-opposite, it follows that u, v are T-equal. This proves 10.6. ■

Let H, J be the graphs with vertex set R, T respectively, in which distinct vertices u, v are adjacent if they are noncollinear and nonadjacent in G.

10.7 *The following hold:*

1. *If at least two of R_1, R_2, R_3 are nonempty, then H has at most two components.*
2. *If at least two of R_1, R_2, R_3 are nonempty, then no vertex in T is complete in G to more than one component of H, and no vertex in T is anticomplete in G to more than one component of H.*
3. *If all of R_1, R_2, R_3 are nonempty, then every component of H is stable in G.*

The analogous statements with R, T exchanged also hold.

Proof. For the first statement, suppose that at least two of R_1, R_2, R_3 are nonempty, and that C_1, C_2, C_3 are distinct components of H. Hence (from the symmetry, and by choosing C_1, C_2, C_3 appropriately) we may assume that there exist $v_1 \in C_1 \cap R_1$ and $v_2 \in C_2 \cap R_2$. Consequently v_1, v_2

are adjacent in G (because they are nonadjacent in H). Choose $v_3 \in C_3$. By 10.3, v_3 is adjacent in G to at most one of v_1, v_2, and therefore we may assume that v_3 is nonadjacent to v_1 in G. Since v_1, v_3 belong to different components of H, it follows that v_1, v_3 are collinear and so $v_3 \in R_1$; but then v_2 has two neighbours in R_1, contrary to 10.4.3. This proves the first statement.

For the second, let $t \in T$. From the first statement, we may assume that H has exactly two components C_1, C_2 say. Since at least two of R_1, R_2, R_3 are nonempty, we may assume that there exist $v_1 \in C_1 \cap R_1$ and $v_2 \in C_2 \cap R_2$. Since v_1, v_2 are nonadjacent in H, they are adjacent in G. By 10.4.1, t is adjacent to exactly one of v_1, v_2, and therefore complete to at most one of C_1, C_2, and anticomplete to at most one of C_1, C_2. This proves the second statement. The third statement follows immediately from 10.5. This completes the proof of 10.7. ■

One reason for the usefulness of 10.7 is that it can often be combined with the following to deduce that G is Schläfli-prismatic.

10.8 *If all the following hold, and so do the analogous statements with R, T exchanged, then G is Schläfli-prismatic:*

- *H has at most two components;*
- *every component of H is stable in G;*
- *every two nonadjacent noncollinear vertices in T are R-equal;*
- *every vertex in T is complete in G to at most one component of H, and anticomplete in G to at most one component of H.*

Proof. Suppose the bulleted conditions hold. For each component C of H, $C \cap R_i$ is a homogeneous stable set for $i = 1, 2, 3$. Thus $|C \cap R_i| \leq 1$ for $i = 1, 2, 3$, and the same holds in J, and it follows easily that G is Schläfli-prismatic. This proves 10.8. ■

Let us apply these lemmas to dispose of some of the easier cases.

10.9 *If either $R = \emptyset$, or $T = \emptyset$, or at least four of $R_1, R_2, R_3, T^1, T^2, T^3$ are empty then G is either Schläfli-prismatic or fuzzily Schläfli-prismatic.*

Proof. Suppose first that $T = \emptyset$. If also $R_3 = \emptyset$, then G is fuzzily Schläfli-prismatic at (R_1, R_2, r_3^3), and the theorem holds. So we may assume that R_1, R_2, R_3 are all nonempty. By 10.7 and 10.8, G is Schläfli-prismatic and again the theorem holds.

We may therefore assume that $R_1, T^1 \neq \emptyset$, and therefore $R_2 = R_3 = T^2 = T^3 = \emptyset$. If R_1 is complete to T^1 then R_1, T^1 are homogeneous stable sets, and so $|R_1|, |T^1| = 1$; but then the theorem holds by 10.8. We therefore assume that R_1 is not complete to T^1. Since G is rigid, there is a vertex with a neighbour in R_1 and a neighbour in T^1, and so $(1, 1) \in I$; but then G is fuzzily Schläfli-prismatic at (R_1, T^1, s_1^1) and the theorem holds. This proves 10.9. ■

Henceforth, therefore, we assume that $R, T \neq \emptyset$ and at least three of $R_1, R_2, R_3, T^1, T^2, T^3$ are nonempty. Now we can bootstrap 10.8 to a stronger version. Let us say that $u, v \in R$ are *distant* if they are nonadjacent, noncollinear, and belong to different components of $G|R$. (We define when $u, v \in T$ are distant analogously.)

10.10 *If every two distant vertices in T are R-equal, and every two distant vertices in R are T-equal, then G is Schläfli-prismatic.*

Proof.

(1) *Every two nonadjacent noncollinear vertices in R are T-equal, and every two nonadjacent noncollinear vertices in T are R-equal.*

For let $u, v \in R$ be nonadjacent and noncollinear. If they belong to different components of $G|R$ then they are distant and therefore T-equal by hypothesis. If they belong to the same component of $G|R$, they are T-equal by 10.6. This proves (1).

(2) *Every two members of R are T-consistent, and every two members of T are R-consistent.*

For from the symmetry between R and T, we may assume that at least two of R_1, R_2, R_3 are nonempty. We claim that every two members of R are T-consistent. For let $u, v \in R$. If they are adjacent in G, they are T-opposite and therefore T-consistent by 10.4.1; if u, v are noncollinear and nonadjacent, they are T-equal and therefore T-consistent by (1); and if u, v are collinear, then since at least two of R_1, R_2, R_3 are nonempty, there exists $z \in R$ noncollinear with both u, v and therefore T-consistent with both u, v, and again it follows that u, v are T-consistent. This proves that every two members of R are T-consistent. Consequently there is a subset $Y \subseteq T$, such that for every $r \in R$, its set of neighbours in T is either Y or $T \setminus Y$. Let X be the set of vertices in R with neighbour set Y. Then for every $t \in T$, if $t \in Y$ then the neighbour set of t in R is X, and otherwise it is $R \setminus X$. Consequently every two members of T are R-consistent. This proves (2).

Now we verify that the four hypotheses of 10.8 hold. First, we must show that H has at most two components. This follows from 10.7.1 if at least two of R_1, R_2, R_3 are nonempty; so we may assume that $R_2 = R_3 = \emptyset$. By (2), every two members of R are T-consistent. But no two members of R are T-equal, since $R_2 = R_3 = \emptyset$ and G is rigid; and so $|R| \leq 2$, and therefore H has at most two components. Thus the first hypothesis of 10.8 holds.

Moreover, by (1), if $t \in T$ then t is either complete or anticomplete in G to every component of H. By 10.4.1, $t \in T$ is neither complete nor anticomplete to any subset of R that is not stable, and so (since there exists $t \in T$), every component of H is stable in G. Hence the second hypothesis of 10.8 holds. The third obviously holds; and the fourth follows from 10.7.2. Thus the claim follows from 10.8. This proves 10.10. ■

Let us tackle the next easiest case.

10.11 *If exactly three of $R_1, R_2, R_3, T^1, T^2, T^3$ are empty then G is either Schläfli-prismatic or fuzzily Schläfli-prismatic, or $G \in \mathcal{F}_4$.*

Proof. By 10.9, we may assume that $R_2 = R_3 = T^3 = \emptyset$, and R_1, T^1, T^2 are nonempty. From 10.10 we may assume that there exist $t^1 \in T^1$ and $t^2 \in T^2$, nonadjacent and not R-equal. Choose $r \in R_1$ adjacent to exactly one of them; say r is adjacent to t^2 and not to t^1. Thus $(1, 2) \notin I$, by 10.4.5. If $(1, 1) \in I$, then (s_1^1, t_3^3) is a square-forcer, a contradiction.

So $(1,1) \notin I$, and similarly $(1,2) \notin I$, and therefore $(1,3) \in I$. If $(2,1) \notin I$, then $(2,2) \in I$, since I includes a permutation, and then (s_2^2, t_3^3) is a square-forcer, a contradiction. So $(2,1) \in I$, and similarly $(2,2),(3,1),(3,2) \in I$. If a vertex in T^1 and a vertex in R_1 are nonadjacent, then they have a common neighbour since G is rigid, and this must be in T^2. Consequently any member of T^1 with no neighbour in T^2 is complete to R_1. Similarly any member of T^2 with no neighbour in T^1 is complete to R_1. But then $G \in \mathcal{F}_4$ (there is an isomorphism from G to the graph, G' say, described in the definition of $\mathcal{F}_4$, mapping the vertices

$$t_3^2, t_3^1, s_3^1, s_1^2, s_2^2, r_3^3, s_1^3, s_2^3, r_3^2, r_3^1, t_3^3$$

of G to the vertices of G' called $s_1^1, s_2^1, \ldots, s_3^3, r_1^3, t_3^3$ in the definition of $\mathcal{F}_4$, and mapping s_3^3 to r_2^3 and s_3^2 to r_3^3 if they exist.) This proves 10.11. ■

Another special case:

10.12 *Suppose that R, T are both stable and there is no diagonal triangle. Then $G \in \mathcal{F}_7$.*

Proof. For then the only triangles in G are either included in S or are marginal triangles. For $1 \le i,j \le 3$, if $(i,j) \in I$ then R_i is anticomplete to T^j, because there are no diagonal triangles; and if $(i,j) \notin I$ then R_i is complete to T^j, since no vertex has a neighbour in R_i and in T^j and G is rigid. Hence each of the sets $R_1, R_2, R_3, T^1, T^2, T^3$ is a homogeneous stable set, and therefore has at most one member. The subgraph of G induced on

$$\{r_1^3, r_2^3, r_3^3\} \cup \{s_j^i : (i,j) \in I\} \cup \{t_3^1, t_3^2, t_3^3\}$$

is the complement of the line graph of some graph K with six vertices (for instance, if $|I| = 9$ this graph is the complement of the line graph of K_6). Since I includes a permutation, there exists $I' \subseteq I$ with $|I'| = 3$ such that the subgraph of K formed by the edges

$$\{r_1^3, r_2^3, r_3^3\} \cup \{s_j^i \ : \ (i,j) \in I'\} \cup \{t_3^1, t_3^2, t_3^3\}$$

is the six-vertex prism; for each member of $R \cup T$, its set of neighbours in $E(K)$ is the set of edges of K incident in K with a vertex of K; and two such members of $R \cup T$ are adjacent if and only if the corresponding vertices of K are nonadjacent in K. It follows that $G \in \mathcal{F}_7$, and the theorem holds. This proves 10.12. ■

We find that if there is a stable 3-vertex set that meets every triangle of G, then G can admit structures that do not otherwise show up, so it is helpful to handle this case separately. That is the objective of the next result.

10.13 *Suppose that $(1,1) \in I$, and every triangle of G contains one of r_2^3, r_3^3, s_1^1. Then G is in the menagerie.*

Proof. Let $Z = \{r_2^3, r_3^3, s_1^1\}$. Since every triangle meets Z, it follows that $(1,2),(1,3) \notin I$, and T is stable. Since I includes a permutation, we may assume that $(2,2),(3,3) \in I$. Since every triangle meets Z, it follows that R_i is anticomplete to T^i for $i = 2,3$.

(1) *Either $R_2 \neq \emptyset$ or $(3,1) \in I$; and either $R_3 \neq \emptyset$ or $(2,1) \in I$.*

For since (s_2^1, r_2^2) is not a square-forcer, it follows that either $R_2 = \emptyset$ or $(3,1) \in I$, proving the first claim, and the second follows similarly. This proves (1).

(2) *If $R_2 = R_3 = \emptyset$, then G is in the menagerie.*

For then R, T are stable, and by rigidity R_1 is complete to $T^2 \cup T^3$. But then G is fuzzily Schläfli-prismatic at (R_1, T^1, s_1^1). This proves (2).

(3) *If $R_3 = \emptyset$ and $R_2 \neq \emptyset$ then G is in the menagerie.*

For then $(2,1) \in I$ by (1), and so R_2 is anticomplete to T^1 since Z meets every triangle. By rigidity, R_1 is complete to T^2, and consequently T^2 is a homogeneous stable set and so $|T^2| \leq 1$.

Suppose that $T^1 = \emptyset$. Since (r_3^3, s_2^2) is not a square-forcer, it follows that $(2,3) \in I$; then there is symmetry between T^2 and T^3, so similarly T^3 is complete to R_1 and anticomplete to R_2, and therefore G is fuzzily Schläfli-prismatic at (R_1, R_2, r_3^3). We may therefore assume that $T^1 \neq \emptyset$. By 10.11 we may also assume that $T^2 \cup T^3 \neq \emptyset$.

We claim that R_1 is complete to T^3; for suppose that $r_1 \in R_1$ and $t^3 \in T^3$ are nonadjacent. By rigidity, they have a common neighbour, and this must belong to R_2, say $r_2 \in R_2$ is adjacent to both r_1, t^3. Choose $t^1 \in T^1$. Now t^3 has no neighbour in $\{r_1, s_1^1, t^1\}$, and Z is disjoint from $\{r_2, s_1^2, t^1\}$, and so neither of these are triangles; and hence t^1 is nonadjacent to both r_1, r_2, contrary to 10.4.2. This proves that R_1 is complete to T^3.

Let R_1' be the set of vertices in R_1 with a neighbour in $R_2 \cup T^1$. By 10.12 we may assume that either R is not stable or there is a diagonal triangle, and in either case it follows that $R_1' \neq \emptyset$. Since R_2 is anticomplete to T^1, 10.4.1 and 10.4.5 imply that every vertex in R_1 with a neighbour in R_2 is complete to T^1, and every vertex in R_1 with a neighbour in T^1 is complete to R_2. Since $R_2, T^1 \neq \emptyset$, it follows that every vertex in R_1' is complete to $R_2 \cup T_1$. In particular, 10.4.3 and 10.4.4 imply that $|R_1'| = |R_2| = |T^1| = 1$. By 10.4.1, R_2 is anticomplete to T^3. Since T^3 and $R_1 \setminus R_1'$ are homogeneous stable sets, they have at most one member. If $R_1 = R_1'$, then G is Schläfli-prismatic; and if there exists $z \in R_1 \setminus R_1'$, then $G \in \mathcal{F}_9$ (to see this, let $R_1' = \{r_1^2\}$, let $R_2 = \{r_2^1\}$, and let $T^j = \{t_2^j\}$ if $T^j \neq \emptyset$ for $j = 1, 2, 3$). This proves (3).

(4) *If R_2, R_3 are both nonempty and R is stable, then G is in the menagerie.*

Suppose that R is stable. Then by 10.12 we may assume there is a diagonal triangle, and since Z meets all triangles, there is an edge between some $r_1 \in R_1$ and some $t^1 \in T^1$. By 10.4.5, every vertex in $R_2 \cup R_3$ is adjacent to t^1. Since $R_2, R_3 \neq \emptyset$ and Z meets all triangles, it follows that $(2,1), (3,1) \notin I$. If also $(2,3), (3,2) \notin I$ then every triangle contains one of s_1^1, s_2^2, s_3^3 and so $G \in \mathcal{F}_8$ by 9.1; so from the symmetry we may assume that $(2,3) \in I$. Since Z meets all triangles, R_2 is anticomplete to $T^2 \cup T^3$. If $(3,2) \in I$, then R_3 is also anticomplete to $T^2 \cup T^3$, and then G is fuzzily Schläfli-prismatic at $(R_1 \cup \{t_3^1\}, T^1 \cup \{r_1^3\}, s_1^1)$. Thus we assume that $(3,2) \notin I$. By rigidity, R_3 is complete to T^2. Hence R_2, R_3, T^2, T^3 all have cardinality at most one since they are homogeneous stable sets. Then G can be obtained from the subgraph $G \setminus (R_1 \cup T^1)$ by multiplying $\{r_1^3, t_3^1\}$; and so $G \in \mathcal{F}_6$. This proves (4).

(5) *If R_2, R_3 are both nonempty and R is not stable, then G is in the menagerie.*

For R_1' be the set of vertices in R_1 that have a neighbour in $R_2 \cup R_3$. Since $R_2 \cup R_3$ is stable, and R is not stable, it follows that $R_1' \neq \emptyset$. Since R_2, R_3 are nonempty, 10.4.2 implies that R_1' is complete to $R_2 \cup R_3$. Since $R_2 \neq \emptyset$, 10.4.3 implies that $|R_1'| = 1$, say $R_1' = \{r_1'\}$. For $t^2 \in T^2$, since t^2 has no neighbour in R_2, 10.4.1 implies that t^2 is adjacent to r_1', and similarly for $t^3 \in T^3$; so r_1' is complete to $T^2 \cup T^3$. Also $R_1 \setminus R_1'$ is complete to $T^2 \cup T^3$ by rigidity, so R_1 is complete to $T^2 \cup T^3$. Let $T^{1\prime}$ be the set of vertices in T^1 adjacent to r_1'. By 10.4.1, $T^{1\prime}$ is anticomplete to $R_2 \cup R_3$. Also, 10.4.4 implies that $T^{1\prime}$ is anticomplete to $R_1 \setminus \{r_1{}'\}$, and 10.4.1 implies that $T^1 \setminus T^{1\prime}$ is complete to $R_2 \cup R_3$. Since r_1' is complete to $R_2 \cup R_3 \cup T^2 \cup T^3$, 10.3 implies that $R_2 \cup R_3$ is anticomplete to $T^2 \cup T^3$. Since $R_2, R_3, T^{1\prime}, T^2, T^3$ are homogeneous stable sets, they each have at most one member. Suppose that $T^{1\prime} = T^1$. Then $R_1 \setminus R_1'$ is also a homogeneous stable set and so has cardinality at most one. If $R_1 = R_1'$, then G is Schläfli-prismatic, so we may assume that there exists $z \in R_1 \setminus R_1'$. The set of neighbours of z is precisely Z together with all vertices that are anticomplete to Z, and so $G \in \mathcal{F}_9$. Hence we may assume that $T^{1\prime} \neq T^1$. Since $T^1 \setminus T^{1\prime}$ is complete to $R_2 \cup R_3$, and R_2, R_3 are nonempty, and Z meets every triangle, it follows that $(2,1), (3,1) \notin I$. But then G is fuzzily Schläfli-prismatic at $((R_1 \setminus R_1') \cup \{t_3^1\}, (T^1 \setminus T^{1\prime}) \cup \{r_1^3\}, s_1^1)$. This proves (5).

From (2)–(5), this proves 10.13. ■

10.14 *If R, T are both stable, then G is in the menagerie.*

Proof. If R is complete to T, then G is Schläfli-prismatic by 10.10, so we assume that R is not complete to T. By 10.12, we may assume that $(1,1) \in I$ and R_1 is not anticomplete to T^1. Let R_1' be the set of vertices in R_1 with a neighbour in T^1, and let $T^{1\prime}$ be the set of vertices in T^1 with a neighbour in R_1; thus, $R_1', T^{1\prime} \neq \emptyset$. By 10.4.5, R_1' is complete to $T^2 \cup T^3$, and $T^{1\prime}$ is complete to $R_2 \cup R_3$.

(1) *If $R_2 = R_3 = \emptyset$ then G is in the menagerie.*

For if $(1,2), (1,3) \notin I$, then every triangle contains one of s_1^1, r_2^3, r_3^3, and the result follows from 10.13. Thus from the symmetry we may assume that $(1,2) \in I$. By 10.4.5, R_1' is complete to $T^2 \cup T^3$. By the same argument with T^1, T^2 exchanged, every vertex in R_1 with a neighbour in R_2 is complete to T^1, and therefore belongs to R_1'. Hence R_1' is complete to T, and $R_1 \setminus R_1'$ is anticomplete to $T^1 \cup T^2$. By 10.4.4, $|R_1'| = 1$, and $|T^1| = |T^2| = 1$. If in addition $(1,3) \in I$, then similarly $R_1 \setminus R_1'$ is anticomplete to T^3, and so $|R_1 \setminus R_1'| \leq 1$ (since $R_1 \setminus X$ is a homogeneous stable set), and G is Schläfli-prismatic, by 10.10. Thus we may assume that $(1,3) \notin I$. Then no vertex has a neighbour in R_1 and a neighbour in T^3, and so since G is rigid it follows that R_1 is complete to T^3. Consequently the sets $R_1', R_1 \setminus R_1', T^3$ each have cardinality at most 1, since they are all homogeneous stable sets. But then $G \in \mathcal{F}_5$. This proves (1).

In view of (1) we assume henceforth that least one of R_2, R_3 is nonempty, and similarly at least one of T^2, T^3 is nonempty.

(2) *Not both* $(2,1),(3,1) \in I$.

For suppose that $(2,1),(3,1) \in I$. Then since $T^{1\prime}$ is complete to $R_2 \cup R_3$, 10.4.4 implies that $|R_2|, |R_3| \leq 1$ and $|T^{1\prime}| = 1$, say $T^{1\prime} = \{t^1\}$. We claim that $|R_1| = 1$ and R is complete to $T^2 \cup T^3$. For at least one of R_2, R_3 is nonempty, say R_i; choose $r_i \in R_i$. Since $\{r_i, s^i_1, t^1\}$ is a diagonal triangle, there is symmetry between R_1, R_i, and so by exchanging R_1, R_i it follows that $|R_1| = 1$, say $R_1 = \{r_1\}$, and R_i is complete to $T^2 \cup T^3$. Thus R is complete to $T^2 \cup T^3$. This proves our claim.

Since R is not complete to T, it follows that there exists $t \in T^1 \setminus T^{1\prime}$. Thus t is anticomplete to R. At least one of T^2, T^3 is nonempty, so we may assume that there exists $t^2 \in T^2$; choose j such that $(j,2) \in I$. Now t^3 is complete to R_j, and if there exists $r \in R_j$ then t is nonadjacent to both r, t^j, contrary to 10.4.5. Thus $R_j = \emptyset$. Consequently $j \neq 1$, so $(1,2) \notin I$; and we may assume that $j = 3$; so $(3,2) \in I$ and $R_3 = \emptyset$. Hence $R_2 \neq \emptyset$, and therefore $(2,2) \notin I$. By the same argument, if $T^3 \neq \emptyset$ then $(3,1),(3,2) \notin I$, which is impossible since I includes a permutation. Thus $R_3 = \emptyset$. But then (s^3_2, t^3_3) is a square-forcer, a contradiction. This proves (2).

(3) *If* $R_2 \neq \emptyset$ *and* $(2,1) \in I$ *then* G *is in the menagerie.*

For then by (2) we may assume that $(3,1) \notin I$. Suppose that there is no pair $(i,j) \in I$ with $i \in \{1,2\}$ and $j \in \{2,3\}$ such that $T^j \neq \emptyset$. Since one of T^2, T^3 is nonempty, say T^2, it follows that $(1,2),(2,2) \notin I$, and therefore $(3,2) \in I$; and since one of $(1,3),(2,3) \in I$ (because I includes a permutation), it follows that $T^3 = \emptyset$. But then (s^3_2, t^3_3) is a square-forcer, a contradiction. Thus there is a pair (i,j), and we may therefore assume that $(i,j) = (1,2)$, that is, $(1,2) \in I$ and $T^2 \neq \emptyset$. By (2) (with R,T exchanged), $(1,3) \notin I$. Moreover, $|R_1| = |R_2| = |T^1| = |T^2| = 1$; and $R_1 \cup R_2$ is complete to T, and $T^1 \cup T^2$ is complete to R. Since R is not complete to T, there exists $r_3 \in R_3$ and $t^3 \in T^3$, nonadjacent. Since G is rigid, there is a vertex adjacent to both r_3, t^3 and so $(3,3) \in I$. Then G is fuzzily Schläfli-prismatic at (R_3, T^3, s^3_3). This proves (3).

Since at least one of R_2, R_3 is nonempty, we may assume that $R_2 \neq \emptyset$, and so by (3) we may assume that $(2,1) \notin I$. Similarly we may assume that $T^2 \neq \emptyset$ and $(1,2) \notin I$. Suppose that $R_3 = \emptyset$. By (3) we may assume that not both $(1,3) \in I$ and $T^3 \neq \emptyset$; but then (r^3_3, s^1_1) is a square-forcer, a contradiction. Thus $R_3 \neq \emptyset$, and similarly $T^3 \neq \emptyset$. By (3) we may assume that $(1,3),(3,1) \notin I$. We have shown then that if $(i,j) \in I$ and s^i_j belongs to a diagonal triangle, then $(i,j') \notin I$ for all $j' \neq j$, and $(i',j) \notin I$ for all $i' \neq i$. If there is no $(i,j) \neq (1,1)$ such that s^i_j belongs to a diagonal triangle, then every triangle contains one of r^3_2, r^3_3, s^1_1 and the result follows from 10.13. Thus we may assume that $(2,2) \in I$ and s^2_2 belongs to a diagonal triangle. Consequently $(2,3),(3,2) \notin I$, and so $I = \{(1,1),(2,2),(3,3)\}$; but then every triangle contains one of s^1_1, s^2_2, s^3_3 and the result follows from 9.1. This proves 10.14. ■

10.15 *If exactly two of* R_1, R_2, R_3 *are empty and* T^1, T^2, T^3 *are all nonempty, or exactly two of* T^1, T^2, T^3 *are empty and* R_1, R_2, R_3 *are all nonempty, then* G *is in the menagerie.*

Proof. We may assume that R_2, R_3 are empty, and R_1, T^1, T^2, T^3 are nonempty. By 10.10 we may assume that $G|T$ is not connected, and by 10.14 we may assume that T is not stable. By 10.5 we may assume that $t^1 \in T^1$ is adjacent to $t^2 \in T^2$ and to $t^3 \in T^3$, and $|T^2| = |T^3| = 1$, and $|T^1| > 1$. Let X be the set of neighbours of t^1 in R_1, and $Y = R_1 \setminus X$. By 10.4.1, Y is the set of neighbours in R_1 of t^2, and also of t^3. Suppose first that $(1,1) \in I$. Then by 10.4.4, $|X| \leq 1$. Then G is fuzzily Schläfli-prismatic at $(Y, T^1 \setminus \{t^1\}, s_1^1)$.

Thus we may assume that $(1,1) \notin I$, and since I includes a permutation, we may assume from the symmetry that $(1,2),(2,1),(3,3) \in I$. Since Y is complete to t^2, 10.4.4 implies that $|Y| \leq 1$. If $r_1 \in R_1$ and $t \in T^1 \setminus \{t^1\}$, then r_1, t have no common neighbour, and therefore they are adjacent since G is rigid. Thus R_1 is complete to $T^1 \setminus \{t^1\}$. Then $X, T^1 \setminus \{t^1\}$ are homogeneous stable sets, so $|X|, |T^1 \setminus \{t^1\}| \leq 1$. If $X = \emptyset$ or $T^1 \setminus \{t^1\} = \emptyset$ then G is Schläfli-prismatic, by 10.10; so we may assume that $|X| = |T^1 \setminus \{t^1\}| = 1$. But then $G \in \mathcal{F}_5$. This proves 10.15. ■

10.16 *If exactly one R_1, R_2, R_3 is empty and exactly one of T^1, T^2, T^3 is empty, then G is in the menagerie.*

Proof. We may assume that $R_3 = T^3 = \emptyset$, and R_1, R_2, T^1, T^2 are nonempty.

(1) *If $(1,1) \in I$, then one of $(1,2),(2,1) \in I$.*

This follows since (r_3^3, s_1^1) is not a square-forcer.

(2) *If there is no vertex of R that belongs to a diagonal triangle and has a neighbour in R, then G is in the menagerie.*

For let Q_1, Q_2 be the sets of vertices in R_1, R_2 respectively that do not belong to diagonal triangles. We may assume by 10.14 that there is an edge $r_1 r_2$ with both ends in R, and so from the hypothesis of (2), we may assume that $r_1 \in Q_1$ and $r_2 \in Q_2$. Let X be the set of neighbours of $r_1 \in T$. Thus $T \setminus X$ is the set of neighbours of r_2 in T, and from the symmetry we may assume that $X \cap T^1 \neq \emptyset$. Since r_1 belongs to no diagonal triangle, $(1,1) \notin I$.

Suppose that $T^2 \setminus X \neq \emptyset$. Then $(2,2) \notin I$, and therefore one of $(1,2),(2,1) \in I$ since I includes a permutation, contrary to (1). So $T^2 \subseteq X$, and since $T^2 \neq \emptyset$, we deduce that $X \cap T^2 \neq \emptyset$. By exchanging T^1 and T^2, it follows that $(1,2) \notin I$, and $T^1 \subseteq X$, and so $X = T$. Hence $(1,3) \in I$. Also no vertex of R_1 is in a diagonal triangle, and so $Q_1 = R_1$. At least one of $(2,1),(2,2) \in I$, and hence both by (1). Since r_1 is complete to T, 10.4.1 implies that T is stable; and from the definition of Q_2, Q_2 is anticomplete to T.

By 10.13, we may assume that not every triangle contains one of r_2^3, r_3^3, s_3^1, and so there is a diagonal triangle, that is, $Q_2 \neq R_2$. Choose $r_2' \in R_2 \setminus Q_2$. It belongs to a diagonal triangle, say $\{r_2', s_1^2, t^1\}$ with $t^1 \in T^1$ without loss of generality. Every vertex of T^2 has a neighbour in this triangle, and since T is stable, it follows that r_2' is complete to T. In particular, there is a diagonal triangle $\{r_2', s_2^2, t^2\}$ with $t^2 \in T^2$, and so by the same argument with T^1, T^2 exchanged, r_2' is complete to T^1. Hence $R_2 \setminus Q_2$ is complete to T. From the hypothesis of (2), $R_2 \setminus Q_2$ is anticomplete to $Q_1 = R_1$, and so $R_2 \setminus Q_2$ is a homogeneous stable set; and hence $R_2 \setminus Q_2 = \{r_2'\}$. By 10.4.4, $|T^i| = 1$ for $i = 1,2$; let $T^i = \{t^i\}$ for $i = 1,2$. Since 10.4.5 implies that every vertex in R_1 is adjacent to one of r_2', t^i for $i = 1,2$, and is nonadjacent to r_2', it follows that R_1 is complete to T. But then G is fuzzily Schläfli-prismatic at (R_1, Q_2, r_3^3). This proves (2).

From (2), we may assume that $(1,1) \in I$, and $\{r_1, s_1^1, t^1\}$ is a diagonal triangle, with $r_1 \in R_1$ and $t^1 \in T^1$ say; and r_1 is adjacent to a vertex $r_2 \in R_2$. Let X be the set of neighbours of r_1 in T; thus by 10.4.1, $T \setminus X$ is the set of neighbours of r_2 in T. We need to determine the adjacencies between the eight sets

$$\{r_1\}, R_1 \setminus \{r_1\}, \{r_2\}, R_2 \setminus \{r_2\}, X \cap T^1, T^1 \setminus X, X \cap T^2, T^2 \setminus X.$$

From 10.4.3 and the definition of X, $\{r_1\}$ is complete to $\{r_2\}, X \cap T^1, X \cap T^2$ and anticomplete to the other four sets. Also $\{r_2\}$ is complete to $T^1 \setminus X, T^2 \setminus X$, and anticomplete to $R_1 \setminus \{r_1\}, R_2 \setminus \{r_2\}$, $X \cap T^1, X \cap T^2$. By 10.4.4, $X \cap T^1 = \{t^1\}$, and it is anticomplete to $T^1 \setminus X$. Next, we determine the adjacencies between t^1 and the other sets.

(3) *The adjacencies between $X \cap T^1 = \{t^1\}$ and the other sets are as follows:*

- *t^1 is complete to $R_2 \setminus \{r_2\}$ and to $T^2 \setminus X$*
- *t^1 is anticomplete to $X \cap T^2$ and to $R_1 \setminus \{r_1\}$.*

These are consequences of 10.4.5. t^1 is complete to $R_2 \setminus \{r_2\}$ because r_2 is the only neighbour of r_1 in R_2 by 10.4.3; t^1 is complete to $T^2 \setminus X$ because vertices in $T^2 \setminus X$ are nonadjacent to r_2; t^1 is anticomplete to $X \cap T^2$ by 10.4.1; and t^1 is anticomplete to $R_1 \setminus \{r_1\}$ by 10.4.4. This proves (3).

(4) $|T^2 \setminus X| \leq 1$, *and the adjacencies between $T^2 \setminus X$ and the remaining sets are as follows:*

- *$T^2 \setminus X$ is anticomplete to $X \cap T^2$, to $T^1 \setminus X$, and to $R_2 \setminus \{r_2\}$, and*
- *$T^2 \setminus X$ is complete to $R_1 \setminus \{r_1\}$.*

For $T^2 \setminus X$ is certainly anticomplete to $X \cap T^2$, and by 10.4.1 $T^2 \setminus X$ is anticomplete to $T^1 \setminus X$. From (3) and 10.3, it follows that $T^2 \setminus X$ is anticomplete to $R_2 \setminus \{r_2\}$. If $r_1' \in R_1 \setminus \{r_1\}$ and $t^2 \in T^2 \setminus X$, then they are adjacent by 10.4.1, since t^1, t^2 are adjacent and r_1', t^1 are nonadjacent. Hence $T^2 \setminus X$ is complete to $R_1 \setminus \{r_1\}$. Consequently $T^2 \setminus X$ is a homogeneous stable set, and so $|T^2 \setminus X| \leq 1$. This proves (4).

Thus the only adjacencies between the eight sets that are still undetermined are those between the four sets

$$R_1 \setminus \{r_1\}, R_2 \setminus \{r_2\}, T^1 \setminus X, X \cap T^2.$$

There are a couple more that we can determine.

(5) *$X \cap T^2$ is complete to $T^1 \setminus X$.*

For let $t^2 \in X \cap T^2$ and $t \in X \cap T^1$. If $(1,2) \in I$ then t, t^2 are adjacent by 10.4.5, and if $(1,2) \notin I$ then by (1), $(2,1) \in I$ and t^2 is adjacent to one of r_2, t (and hence to t) by 10.4.5. In either case t, t^2 are adjacent. This proves (5).

(6) *$X \cap T_2$ is complete to $R_2 \setminus \{r_2\}$.*

For let $t^2 \in X \cap T^2$, and $r_2' \in R_2 \setminus \{r_2\}$. If $(2,1) \in I$ then t^2 has a neighbour in the triangle $\{r_2', s_1^2, t^1\}$, and if $(2,1) \notin I$ then by (1), $(1,2) \in I$, and r_2' has a neighbour in the triangle $\{r_1, s_2^1, t^2\}$. In either case it follows that r_2', t^2 are adjacent. This proves (6).

(7) *If $(2,1) \in I$ then G is in the menagerie.*

For let $(2,1) \in I$. For all $r_1' \in R_1 \setminus \{r_1\}$ and $r_2' \in R_2 \setminus \{r_2\}$, r_1' has a neighbour in the triangle $\{r_2', s_1^2, t^1\}$, and therefore is adjacent to r_2'; and hence $R_1 \setminus \{r_1\}$ is complete to $R_2 \setminus \{r_2\}$. Secondly, for all $t \in T^1 \setminus X$ and $r_2' \in R_2 \setminus \{r_2\}$, since r_1 has no neighbour in $\{r_2', s_1^2, t\}$, it follows that the latter is not a triangle and so r_2', t are nonadjacent; and hence $T^1 \setminus X$ is anticomplete to $R_2 \setminus \{r_2\}$. Third, for all $r_1' \in R_1 \setminus \{r_1\}$ and $t \in T^1 \setminus X$, since 10.4.5 implies that r_1' is adjacent to one of r_2, t, it follows that r_1', t are adjacent, and hence $R_1 \setminus \{r_1\}$ is complete to $T^1 \setminus X$. Thus in this case the only undecided adjacency is between $X \cap T^2$ and $R_1 \setminus \{r_1\}$. If $X \cap T^2$ is anticomplete to $R_1 \setminus \{r_1\}$, then G is Schläfli-prismatic by 10.10. So we may assume that there exist $t^2 \in X \cap T^2$ and $r_1' \in R_1 \setminus \{r_1\}$ that are adjacent. Since r_1' is complete to $T^1 \setminus X$, 10.4.1 implies that $T^1 \setminus X = \emptyset$. Since any vertex in $R_2 \setminus \{r_2\}$ is adjacent to both ends of the edge $r_1't^2$, 10.4.1 implies that $R_2 \setminus \{r_2\} = \emptyset$. Also, since r_1 would have no neighbour in a triangle $\{r_1', s_2^1, t^2\}$, it follows that $(1,2) \notin I$. Since no vertex has a neighbour in $R_1 \setminus \{r_1\}$ and a neighbour in $T^2 \setminus \{t^2\}$, these two sets are complete to each other, and therefore are homogeneous stable sets and hence have cardinality one. But then $G \in \mathcal{F}_5$ and the claim holds. This proves (7).

Hence we may assume that $(2,1) \notin I$, and so $(1,2) \in I$ by (1). For $r_1' \in R_1 \setminus \{r_1\}$ and $t^2 \in X \cap T^2$, 10.4.5 implies that r_1' is adjacent to only one of r_1, t^2, and so r_1', t^2 are nonadjacent; and hence $R_1 \setminus \{r_1\}$ is anticomplete to $X \cap T^2$. Consequently $X \cap T^2$ is a homogeneous stable set, and therefore $|X \cap T^2| \leq 1$.

(8) *If $X \cap T^2 \neq \emptyset$ then G is in the menagerie.*

For let $t^2 \in X \cap T^2$. If $t \in T^1 \setminus X$ and $r_1' \in R_1 \setminus \{r_1\}$, 10.4.1 implies that r_1' is adjacent to t, since it is not adjacent to t^2; and hence $T^1 \setminus X$ is complete to $R_1 \setminus \{r_1\}$. Moreover, for $t \in T^1 \setminus X$ and $r_2' \in R_2 \setminus \{r_2\}$, 10.4.1 implies that r_2' is nonadjacent to t, since it is adjacent to t^2; and hence $T^1 \setminus X$ is anticomplete to $R_2 \setminus \{r_2\}$. If $R_1 \setminus \{r_1\}$ is complete to $R_2 \setminus \{r_2\}$ then G is Schläfli-prismatic by 10.10, so we may assume that there exist $r_1' \in R_1 \setminus \{r_1\}$ and $r_2' \in R_2 \setminus \{r_2\}$ that are nonadjacent. Since r_2' would have no neighbour in the triangle $\{r_1', s_1^1, t\}$ for $t \in T^1 \setminus \{t^1\}$, it follows that $T^1 = \{t^1\}$; and since r_2' would have no neighbour in a triangle $\{r_1', s_2^1, t\}$ for $t \in T^2 \setminus X$, it follows that $T^2 \setminus X = \emptyset$. Moreover, r_1' is nonadjacent to r_2', t^2, and so 10.4.5 implies that $(2,2) \notin I$. Then G is fuzzily Schläfli-prismatic at $(R_1 \setminus \{r_1\}, R_2 \setminus \{r_2\}, r_3^3)$. This proves (8).

Thus we may assume that $X \cap T^2 = \emptyset$, and therefore $T_2 \setminus X \neq \emptyset$. Let $t^2 \in T^2 \setminus X$. For $r_1' \in R_1 \setminus \{r_1\}$ and $t \in T^1 \setminus X$, 10.4.5 implies that t is adjacent to one of r_1', t^2, and therefore r_1', t are adjacent; and hence $R_1 \setminus \{r_1\}$ is complete to $T^1 \setminus X$. Moreover, for $r_1' \in R_1 \setminus \{r_1\}$ and $r_2' \in R_2 \setminus \{r_2\}$, 10.4.5 implies that r_2' is adjacent to one of r_1', t^2, and so r_1', r_2' are adjacent; and hence $R_1 \setminus \{r_1\}$ is complete to $R_2 \setminus \{r_2\}$. Consequently $R_1 \setminus \{r_1\}$ is a homogeneous stable set, and so $|R_1 \setminus \{r_1\}| \leq 1$. If $T^1 \setminus X$ is anticomplete to $R_2 \setminus \{r_2\}$, then both these sets have cardinality at most 1 (since they are both homogeneous stable sets) and G is Schläfli-prismatic and the claim holds. So we may assume that there exist $t \in T^1 \setminus X$ and $r_2' \in R_2 \setminus \{r_2\}$, adjacent. By 10.4.1, $R_1 \setminus \{r_1\} = \emptyset$, since any member of $R_1 \setminus \{r_1\}$ would form a triangle with t and r_2', contrary to 10.3. No vertex has a neighbour in $R_2 \setminus \{r_2\}$ and a neighbour in $T^1 \setminus X$, and since G is rigid it follows that $R_2 \setminus \{r_2\}$ is complete to

$T^1 \setminus X$; and so $R_2 \setminus \{r_2\}, T^1 \setminus X$ are homogeneous stable sets, and hence $|R_2 \setminus \{r_2\}| = |T^1 \setminus X| = 1$, and therefore $G \in \mathcal{F}_5$. This proves 10.16. ■

10.17 *If exactly one of $R_1, R_2, R_3, T^1, T^2, T^3$ is empty, then G is in the menagerie.*

Proof. We may assume that $R_3 = \emptyset$, and the other five sets are nonempty.

(1) *If every two distant vertices in T are R-equal, then the theorem holds.*

For then every two vertices in T are R-equal or R-opposite. Choose $C_1 \subseteq R$ such that for every vertex in T, its neighbour set in R is either C_1 or C_2, where $C_2 = R \setminus C_1$. For $i = 1, 2$ let D_i be the set of vertices in T whose neighbour set in R is C_i. Thus D_1, D_2 are disjoint and have union T. By 10.10, we may assume that there are two distant vertices in R that are not T-equal; and therefore one belongs to C_1 and the other to C_2, say $r_1 \in R_1 \cap C_1$ and $r_2 \in R_2 \cap C_2$. Since r_i is complete to D_i, it follows from 10.4.1 that D_i is stable for $i = 1, 2$. Since every two distant vertices in T are R-equal, no two such vertices belong to different sets D_1, D_2; and therefore there is no edge of J between D_1 and D_2. So each of the six sets $D_i \cap T^j :\ i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ is a homogeneous stable set and so has cardinality at most one. From the symmetry we may assume that $(1, 1), (2, 2), (3, 3) \in I$. Since r_2 would have no neighbour in a triangle $\{r_1, s_1^1, t^1\}$ where $t^1 \in D_1 \cap T^1$, it follows that $D_1 \cap T^1 = \emptyset$, and similarly $D_2 \cap T^2 = \emptyset$. Since T^1, T^2 are nonempty and $|D_2 \cap T_1|, |D_1 \cap T_2| \le 1$, it follows that there are vertices t^1, t^2 so that $T^i = \{t^i\}$ for $i = 1, 2$, and $t^1 \in D_2, t^2 \in D_1$. Since $T^3 \neq \emptyset$, we may assume that there exists $t^3 \in D_1 \cap T^3$ from the symmetry. For $j = 2, 3$, since $t^j \in D_1 \cap T^j$ and r_2 is nonadjacent to both r_1, t^j, 10.4.5 implies that $(1, j) \notin I$; that is, $(1, 2), (1, 3) \notin I$. Similarly $(2, 1) \notin I$, and if $D_2 \cap T^3 \neq \emptyset$ then $(2, 3) \notin I$. Since (r_3^3, s_1^1) is not a square-forcer, it follows that $T^2 \cup T^3$ is not stable, and so $D_2 \cap T^3 \neq \emptyset$; and therefore $(2, 3) \notin I$. Since t^1 is complete to C_2 it follows that C_2 is stable, and similarly C_1 is stable. Hence $C_2 \cap R_1, C_1 \cap R_2$ are both homogeneous stable sets, and therefore have cardinality at most one. Consequently G is fuzzily Schläfli-prismatic at $(C_1 \cap R_1, C_2 \cap R_2, r_3^3)$ and the theorem holds. This proves (1).

In the case when T is stable, we may assume that R is not stable, by 10.14. In this case let A, B, C be respectively the sets of vertices in R that are complete to T, anticomplete to T, and neither complete nor anticomplete to T.

(2) *If T is stable, then $A, B \neq \emptyset$, and every edge with both ends in R is between A and B.*

For let $r_1 \in R_1, r_2 \in R_2$ be adjacent, say. If one of r_1, r_2 is in $A \cup B$, then by 10.4.1 the edge is between A and B. So suppose they are both in C. Let X be the set of neighbours of r_1 in T; then $T \setminus X$ is the set of neighbours of r_2, and both $X, T \setminus X$ are nonempty. Suppose that r_1 belongs to a diagonal triangle, say $\{r_1, s_1^1, t^1\}$, where $(1, 1) \in I$ and $t^1 \in T^1$. Since every vertex in $T^2 \cup T^3$ has a neighbour in this triangle, it follows that $T^2, T^3 \subseteq X$. Hence $T^1 \not\subseteq X$. But then, since a vertex in $T^1 \setminus X$ is nonadjacent to both r^1, t^2, where $t^2 \in T^2$, 10.4.5 implies that $(1, 2) \notin I$, and similarly $(1, 3) \notin I$. Also, since r_1 is nonadjacent to r_2, t, where $t \in T^1 \setminus X$, 10.4.5 implies that $(2, 1) \notin I$. But then (r_3^3, s_1^1) is a square-forcer, a contradiction.

This proves that r_1 is not in a diagonal triangle, and similarly neither is r_2. From the symmetry we may assume that there exists $t^1 \in T^1 \cap X$ and $t^3 \in T^3 \cap X$, and $t^2 \in T^2 \setminus X$. Since r_1, r_2 are not in diagonal triangles, $(1,1),(1,3),(2,2) \notin I$. Consequently $(1,2) \in I$, and (r_3^3, s_2^1) is a square-forcer, a contradiction. This proves (2).

(3) *If T is stable, then every vertex of R in a diagonal triangle belongs to A.*

For let $\{r_1, s_1^1, t^1\}$ be a diagonal triangle, where $(1,1) \in I$ and $r_1 \in R_1$ and $t^1 \in T^1$, and suppose that $r_1 \notin A$. Every vertex in $T \setminus T^1$ has a neighbour in this triangle and therefore is adjacent to r_1, so r_1 is complete to $T^2 \cup T^3$. Consequently r_1 has a nonneighbour in $T^1 \setminus \{t^1\}$; so $|T^1| > 1$, and $(1,2),(1,3) \notin I$. Also, r_1 is the only vertex of R_1 adjacent to t^1 by 10.4.4, so $A \cap R_1 = \emptyset$. But $A \neq \emptyset$ by (2). Choose $r_2 \in A$; then $r_2 \in R_2$. Since r_2 is complete to T^1 and $|T^1| > 1$ it follows that $(2,1) \notin I$. But then (r_3^3, s_1^1) is a square-forcer, a contradiction. This proves (3).

(4) *If T is stable then G is in the menagerie.*

For we may assume that $(1,1),(2,2) \in I$. By (3), $C \cap R_i$ is anticomplete to T^i for $i = 1,2$. Since R is not stable, (2) implies that $A \neq \emptyset$.

Suppose that $(1,3),(2,3) \in I$. Then by (3), C is anticomplete to T^3. From the symmetry we may assume that there exists $r_1 \in A \cap R_1$. Any vertex in $C \cap R_2$ is nonadjacent to both r_1, t^3 where $t^3 \in T_3$; so 10.4.5 implies that $C \cap R_2 = \emptyset$, and therefore $C \cap R_1 \neq \emptyset$. By the same argument, it follows that $A \cap R_2 = \emptyset$. Choose $c \in C \cap R_1$. Since c has a neighbour in T, necessarily in T^2, (3) implies that $(1,2) \notin A$. But then no vertex has a neighbour in $R_1 \setminus A$ and a neighbour in T^2, and since G is rigid it follows that $R_1 \setminus A$ is complete to T^2. Then $R_1 \setminus A, T^2$ are both homogeneous stable sets, and so $|R_1 \setminus A| \leq 1$ and $|T_2| = 1$; since $\emptyset \neq C \subseteq R_1$, it follows that $B \cap R_1 = \emptyset$ and $|C| = 1$; also $A \cap R_1, B \cap R_2, T^1, T^3$ are all nonempty homogeneous stable sets, and so all have cardinality one; and then $G \in \mathcal{F}_5$.

Thus we may assume that not both $(1,3),(2,3) \in I$. If $(1,3) \notin I$, then (r_3^3, s_1^1) is not a square-forcer, and if $(2,3) \notin I$ then (r_3^3, s_2^2) is not a square-forcer, and in either case it follows that one of $(1,2),(2,1) \in I$. We claim that not both of $A \cap R_1, C \cap R_2$ are nonempty. For suppose that there exist $r_1 \in A \cap R_1$ and $r_2 \in C \cap R_2$. By (2) (or 10.4.1) they are nonadjacent. If $(2,1) \in I$, then r_2 is anticomplete to T^1 by (3), and so has no neighbour in the triangle $\{r_1, s_1^1, t^1\}$ where $t^1 \in T^1$, a contradiction. If $(1,2) \in I$, then r_2 has no neighbour in the triangle $\{r_1, s_2^1, t^2\}$ where $t^2 \in T^2$, again a contradiction. This proves our claim that not both of $A \cap R_1, C \cap R_2$ are nonempty. Similarly not both $A \cap R_2, C \cap R_1$ are nonempty. Since $C \neq \emptyset$, from the symmetry we may assume that $C \cap R_1 \neq \emptyset$. Therefore $A \subseteq R_1$, and in particular $A \cap R_1 \neq \emptyset$; and so $C \subseteq R_1$. Hence $R_2 \subseteq B$. Every vertex in $B \cap R_2$ has a neighbour in each of the diagonal triangles that contain a vertex of $A \cap R_1$, and so $B \cap R_2$ is complete to $A \cap R_1$. Since $C \subseteq R_1$ is nonempty and a vertex in C has a neighbour in T and is not in a diagonal triangle, it follows that not both $(1,2),(1,3) \in I$. If $(1,2),(1,3) \notin I$, then every triangle contains one of r_2^3, r_3^3, s_1^1, and the claim follows from 10.13. Thus we may assume that exactly one of $(1,2),(1,3)$ belongs to I. There is not quite symmetry between $(1,2),(1,3)$, since we assumed that $(2,2) \in I$; to restore the symmetry, let us drop that assumption, and then we may assume that $(1,2) \in I$ and $(1,3) \notin I$. Since $(1,2) \in I$, C is anticomplete to T^2. No vertex has a neighbour in $R_1 \setminus A$ and a neighbour in T^3, and so $R_1 \setminus A$ is complete to T^3 since G is rigid. Consequently $B \cap R_1 = \emptyset$ and $|C| = |T^3| = 1$. Also, $T^1, T^2, A \cap R_1, B \cap R_2$ are all nonempty homogeneous stable sets, and so they all have cardinality one. But then $G \in \mathcal{F}_5$ and the theorem holds. This proves (4).

In view of (4), we may assume that T is not stable. By 10.7.3, each component of J is stable in G, so J has at least two components; and by 10.7.1 it has exactly two, say D, D'. By (1), we may assume that there are distant vertices in T that are not R-equal; say $t^1 \in T^1$ and $t^3 \in T^3$ are distant and not R-equal. Hence $G|T$ is not connected, so by 10.5 we may assume that $|T^1| = |T^2| = 1$, and there exists $d \in T^3 \setminus \{t^3\}$ complete to $T^1 \cup T^2$, and $T_3 \setminus \{t^3\}$ is anticomplete to $T^1 \cup T^2$. Then J has two components, $D = \{d\}$ and $D' = T \setminus \{d\}$. Let $T^i = \{t^i\}$ for $i = 1, 2$. Let X be the set of neighbours of d in R. Thus by 10.4.1, X and $R \setminus X$ are stable. Since d is adjacent to t^1, t^2, it follows that $R \setminus X$ is the set of neighbours of t^1, t^2 in R.

(5) *If $\{i, i'\} = 1, 2$ and $(i, 3) \in I$, then $|R_i \cap X| \leq 1$, $R_i \cap X$ is complete to $R_{i'} \setminus X$, and $R_i \cap X$ is anticomplete to $T^3 \setminus \{d\}$.*

For $|R_i \cap X| \leq 1$ by 10.4.4; $R_i \cap X$ is complete to $R_{i'} \setminus X$ since 10.4.5 implies that for $r_i \in R_i \cap X$, every vertex in $R_{i'} \setminus X$ is adjacent to one of r_i, d; and $R_i \cap X$ is anticomplete to $T^3 \setminus \{d\}$ by 10.4.4. This proves (5).

(6) *If $\{i, i'\} = \{1, 2\}$ and $(i, j) \in I$ for some $j \in \{1, 2\}$, then $|R_i \setminus X| \leq 1$ and $R_i \setminus X$ is complete to $(R_{i'} \cap X) \cup (T^3 \setminus \{d\})$.*

For $|R_i \setminus X| \leq 1$ by 10.4.4, since t^j is complete to $R_i \setminus X$; and $R_i \setminus X$ is complete to $(R_{i'} \cap X) \cup (T^3 \setminus \{d\})$ since 10.4.5 implies that for $r_i \in R_i \setminus X$, every vertex in $(R_{i'} \cap X) \cup (T^3 \setminus \{d\})$ is adjacent to one of r_i, t^j. This proves (6).

(7) *If $(1, 1), (2, 3) \in I$ then G is in the menagerie.*

For suppose that $(1, 1), (2, 3) \in I$. Then by (5) and (6), $|R_1 \setminus X|, |R_2 \cap X| \leq 1$, $(R_2 \cap X) \cup (T^3 \setminus \{d\})$ is complete to $R_1 \setminus X$, and $R_2 \cap X$ is anticomplete to $T^3 \setminus \{d\}$. Now suppose in addition that $R_1 \cap X$ is complete to $R_2 \setminus X$ and anticomplete to $T^3 \setminus \{d\}$. Since t^1, t^3 are not R-equal, it follows that t^3 has a nonneighbour $r_2 \in R_2 \setminus X$. But $R_1 \cap X$ is a homogeneous stable set, and so $|R_1 \cap X| \leq 1$; and then G is fuzzily Schläfli-prismatic at $(R_2 \setminus X, T_3 \setminus \{d\}, s_3^2)$. Hence we may assume that $R_1 \cap X$ is not both complete to $R_2 \setminus X$ and anticomplete to $T^3 \setminus \{d\}$. From (5), $(1, 3) \notin I$; and since (r_3^3, s_3^2) is not a square-forcer, it follows that I contains one of $(2, 1), (2, 2)$, say $(2, j)$. From (6) (applied to $(2, j)$), $|R_2 \setminus X| \leq 1$ and $R_2 \setminus X$ is complete to $(R_1 \cap X) \cup (T^3 \setminus \{d\})$. Since t^1, t^3 are not R-equal, t^3 has a neighbour $r_1 \in R_1 \cap X$. Since r_1, t^3 are complete to $R_2 \setminus X$, 10.3 implies that $R_2 \setminus X = \emptyset$. But then $G \in \mathcal{F}_5$. This proves (7).

Thus we may assume that if $(2, 3) \in I$, then $(1, 1) \notin I$ and similarly $(1, 2) \notin I$, and consequently $(1, 3) \in I$ since I includes a permutation. Consequently if I contains either of $(1, 3), (2, 3)$, then it contains both, and contains none of $(1, 1), (1, 2), (2, 1), (2, 2)$, which is impossible since I includes a permutation. Thus $(1, 3), (2, 3) \in I$.

Since I includes a permutation, we may assume that $(1, 1), (2, 2), (3, 3) \in I$. By two applications of (6), we deduce that $|R_1 \setminus X|, |R_2 \setminus X| \leq 1$, and $R_1 \setminus X$ is complete to $(R_2 \cap X) \cup (T^3 \setminus \{d\})$, and $R_2 \setminus X$ is complete to $(R_1 \cap X) \cup (T^3 \setminus \{d\})$. Since t^1, t^3 are not R-equal, t^3 has a neighbour in X, and from the symmetry we may assume that t^3 is adjacent to $r_1 \in R_1 \cap X$. Since t^3, r_1 are complete to $R_2 \setminus X$, it follows from 10.3 that $R_2 \setminus X = \emptyset$. By 10.13, we may assume that, there is a triangle containing none of t_3^1, t_3^2, s_3^3, and therefore $R_1 \setminus X \neq \emptyset$. Since $R_1 \setminus X$ is complete to $T^3 \setminus \{d\}$ and to $R_2 \cap X$, 10.3 implies that $T^3 \setminus \{d\}$ is anticomplete to $R_2 \cap X$. No vertex has a neighbour in $T^3 \setminus \{d\}$ and a neighbour in $R_1 \cap X$, and so $T^3 \setminus \{d\}$ is complete to $R_1 \cap X$ since G is rigid; and therefore both these sets are homogeneous stable sets, and so $T^3 = \{t^3, d\}$ and $R_1 \cap X = \{r_1\}$. But then $G \in \mathcal{F}_5$ and the theorem holds. This proves 10.17. ■

The last case is:

10.18 *If $R_1, R_2, R_3, T^1, T^2, T^3$ are all nonempty, then G is in the menagerie.*

Proof. By 10.7.1, H and J both have at most two components.

(1) *We may assume that if R is not stable, then H has exactly two components C, C', where $C = \{c\}$ and $c \in R_i$ say, and $|R_j| = 1$ and c is complete to R_j for all $j \in \{1, 2, 3\} \setminus \{i\}$.*

For if $G|R$ is not connected, this follows from 10.5. We assume therefore that $G|R$ is connected. By 10.5, $G|R$ is a path or cycle, with at least three vertices since R_1, R_2, R_3 are nonempty. If it is a path with exactly three vertices, then the claim holds, so we may assume that it has at least four vertices. Hence H has exactly two components, say C_1, C_2, and $|C_1|, |C_2| > 1$, and they are stable in G. By 10.6, every two nonadjacent noncollinear vertices in R are T-equal. Consequently, for $i = 1, 2$ all vertices in C_i have the same set of neighbours in T; call this set X_i.

By 10.10 we may assume that there exist distant vertices in T that are not R-equal; say $t^1 \in T^1$ and $t^2 \in T^2$. Since t^1, t^2 are not R-equal, we may assume that $t^1 \in X_1$ and $t^2 \notin X_1$. Since $|C_1|, |C_2| > 1$, there exists $i \in \{1, 2, 3\}$ such that $C_1 \cap R_i, C_2 \cap R_i$ are both nonempty (for if say $R_1 \cup R_2 = C_1$ and $R_3 = C_2$ then C_2 is a homogeneous stable set, contradicting that $|C_2| > 1$). Hence we may assume that $C_1 \cap R_1, C_2 \cap R_1 \neq \emptyset$. Choose $c_i \in C_i \cap R_1$ for $i = 1, 2$. Since there exists $r_2 \in R_2$, and r_2 belongs to one of C_1, C_2, it follows that r_2 is T-equal to one of c_1, c_2, and adjacent (and therefore T-opposite) to the other. Consequently c_1, c_2 are T-opposite; that is, $X_1 \cup X_2 = T$ and $X_1 \cap X_2 = \emptyset$. Since t^2 is nonadjacent to both c_1, t^1, 10.4.5 implies that $(1, 1) \notin I$, and similarly $(1, 2) \notin I$, and therefore $(1, 3) \in I$. Choose $t^3 \in T^3$. Since $X_1 \cup X_2 = T$, we may assume that $t^3 \in X_1$ without loss of generality. Since 10.4.5 implies that t^2 is adjacent to one of r_1, t^3, it follows that t^2, t^3 are adjacent; since 10.4.2 implies that t^1 is adjacent to one of t^2, t^3, it follows that t^1, t^3 are adjacent; and so $\{c_1, t^1, t^3\}$ is a triangle violating 10.3, a contradiction. This proves (1).

By 10.14 we may assume that T is not stable. Hence by (1) (with R, T exchanged), we may assume that J has two components D, D', where $D = \{d\}$ and $d \in T^3$, and $|T^1| = |T^2| = 1$, and $d \in T^3$ is complete to $T^1 \cup T^2$. Let X be the set of neighbours of d in R, and let $T^i = \{t^i\}$ for $i = 1, 2$. By 10.6, $R \setminus X$ is the set of neighbours in R of both t^1, t^2.

(2) *We may assume that X is a union of components of H.*

For suppose not; then there is a component of H meeting both X and $R \setminus X$, and therefore there exist two nonadjacent noncollinear vertices in R, exactly one of which is in X. Thus we may assume that $r_1 \in R_1 \cap X$ and $r_3 \in R_3 \setminus X$ are nonadjacent. Since r_3 is nonadjacent to both r_1, d, 10.4.5

implies that $(1,3) \notin I$; and since r_1 is nonadjacent to both r_3, t^i, 10.4.5 implies that $(3,i) \notin I$ for $i = 1,2$. Hence $(3,3) \in I$. Choose $r_2 \in R_2$. Since r_1, r_3 are not T-equal, they are not both adjacent to r_2 by 10.4.1, so all three belong to the same component of H, and therefore r_2 is nonadjacent to both r_1, r_3 by 10.7.3. Thus R_2 is anticomplete to r_1, r_3. Now I contains at least one of $(2,1), (2,2)$ since it includes a permutation, say $(2,i)$. Since r_1 has no neighbour in $\{r_2, s_i^2, t^i\}$, the latter is not a triangle and so $r_2 \in X$. Hence $R_2 \subseteq X$. From the symmetry between r_1 and r_2, it follows that $(2,3) \notin I$ and R_1 is anticomplete to r_2, r_3, and $R_1 \subseteq X$. But then every triangle contains one of s_3^3, t_3^1, t_3^2, and the theorem holds by 10.13. This proves (2).

(3) *If R is not stable then G is in the menagerie.*

For then by (1) we may assume that H has two components C, C', where $C = \{c\}$ and $c \in R_3$ say, and $|R_i| = 1$ for $i = 1,2$, and c is complete to $R_1 \cup R_2$. Let $R_i = \{r_i\}$ say $(i = 1,2)$. Suppose first that c, d are adjacent. Since c has a neighbour in C', it follows that d is not complete to C', and therefore d is anticomplete to C' by (2), and similarly c is anticomplete to D'. Thus $X = \{c\}$; and so $R \setminus \{c\}$ is the set of neighbours of both r_1, r_2 in R. Similarly $T \setminus \{d\}$ is the set of neighbours of both r_1, r_2 in T. If R_3 is complete to T^3 then G is Schläfli-prismatic, so we may assume that there exist nonadjacent $r_3 \in R_3$ and $t^3 \in T^3$. Since G is rigid, there is a vertex adjacent to both r_3, t^3, and so $(3,3) \in I$; and then G is fuzzily Schläfli-prismatic at (R_3, T^3, r_3^3), and the claim holds. Now suppose that c, d are nonadjacent. By 10.4.1, d is adjacent to r_1, r_2, and therefore d is complete to C', since X is a union of components of H. Hence $R \setminus \{c\}$ is the set of neighbours of d in R, and similarly $T \setminus \{d\}$ is the set of neighbours of c in T. It follows that c is the unique neighbour in R of t^1, and of t^2, and d is the unique neighbour in T of r_1, r_2. If R_3 is anticomplete to T^3 then G is Schläfli-prismatic, so we assume there is an edge r_3t^3 with $r_3 \in R_3$ and $t^3 \in T^3$. By 10.4.4, $(3,3) \notin I$. Hence no vertex has a neighbour in R_3 and a neighbour in T^3, and therefore R_3 is complete to T^3 since G is rigid. Thus R_3, T^3 are both homogeneous stable sets, so $R_3 = \{r_3\}, T^3 = \{t^3\}$. Then $G \in \mathcal{F}_5$, and the claim holds. This proves (3).

We may therefore assume that R is stable. Hence H has only one component, and so $X = \emptyset$ or $X = R$. Suppose first that $X = \emptyset$, and so d has no neighbours in R. Hence t_1, t_2 are complete to R. If $T^3 \setminus \{d\}$ is also complete to R, then by 10.10 the theorem holds; so we may assume that there exists $t^3 \in T^3 \setminus \{d\}$ and say $r_3 \in R_3$ that are nonadjacent. For $j = 1,2$, t^3 is nonadjacent to both r_3, t^j, and so 10.4.5 implies that $(3,1), (3,2) \notin I$; and therefore $(3,3) \in I$, and we may assume that $(1,1), (2,2) \in I$. For $i = 1,2$ and every $r_i \in R_i$, 10.4.5 implies that every vertex in $T^3 \setminus \{d\}$ is adjacent to one of r_i, t^i, and so $T^3 \setminus \{d\}$ is complete to R_1, R_2. For $i = 1,2$, $|R_i| = 1$ since R_i is a homogeneous stable set. But then G is fuzzily Schläfli-prismatic at $(R_3, T^3 \setminus \{d\}, s_3^3)$, and the theorem holds.

Finally suppose that $X = R$; so d is complete to R, and t^1, t^2 are anticomplete to R. If also $T^3 \setminus \{d\}$ is anticomplete to R, then G is Schläfli-prismatic and the theorem holds; so we assume that there exist $t^3 \in T^3 \setminus \{d\}$ and say $r_3 \in R_3$ that are adjacent. Thus r_3 has two neighbours in T^3, and so $(3,3) \notin I$ by 10.4.4. Hence we may assume that $(1,3) \in I$. By the same argument it follows that $T^3 \setminus \{d\}$ is anticomplete to R_1. If $(2,3) \notin I$, then every triangle contains one of t_3^1, t_3^2, s_3^3, and the theorem follows from 10.13. Thus we may assume that $(2,3) \in I$, and therefore $T^3 \setminus \{d\}$ is anticomplete to R_2. No vertex has a neighbour in $T^3 \setminus \{d\}$ and a neighbour in R_3, and so $T^3 \setminus \{d\}$ is complete to R_3 since G is rigid. Both these sets are therefore homogeneous stable sets, so

$T^3 \setminus \{d\} = \{t^3\}$ and $R_3 = \{r_3\}$, and $G \in \mathcal{F}_5$ and the theorem holds. This completes the proof. ■

Now we can prove 4.1, which we restate.

10.19 *Every rigid non-orientable prismatic graph is in the menagerie.*

Proof. Let G be a rigid non-orientable prismatic graph. By 6.1, there is an induced subgraph of G which is either a twister or a rotator. If no induced subgraph is a rotator, then the theorem holds by 7.2. Thus we may assume that some induced subgraph is a rotator. If G contains a square-forcer, then by 8.2, G is of parallel-square or skew-square type. If it does not contain a square-forcer, then the theorem holds by 10.1. This proves 10.19. ■

11 Changeable edges

Let us say an edge $e = uv$ of a prismatic graph G is *changeable* if the graph $G \setminus e$ is also prismatic; that is (by a theorem of [2]) if either u, v are both in no triangle, or there is a leaf triangle $\{u, v, w\}$ at some w. In a future paper, we claim to have an explicit description of all claw-free "trigraphs", and for part of that description, we need to describe all the pairs (G, X) where G is a prismatic graph and $X \subseteq E(G)$ is a set of changeable edges forming a matching. For the orientable case we accomplished this in [2], and for the nonorientable case it suffices to list all leaf triangles in nonorientable prismatic graphs G; and we may assume that G is rigid, and so we just have to go through all the classes of graphs in the menagerie, and figure out which of their triangles can be leaf triangles. This is mechanical, but tedious and we leave it to the reader. Here are two observations that might help.

First, if $\{a, b, c\}$ is a leaf triangle at c of some prismatic graph H, and G is obtained from H either by multiplying or by exponentiating this leaf triangle, with new sets A, B corresponding to a, b respectively, then any edge of G between A and B is changeable. Since many of our basic classes of prismatic graphs are defined by multiplying or exponentiating some leaf triangle in some other prismatic graph, they have changeable edges in the corresponding positions. Let us call such leaf triangles "expected".

There are other leaf triangles as well; but we can limit where we need to look, to cut down the case analysis, by the following device. Suppose that $T = \{u, v, w\}$ is a leaf triangle at w in some nonorientable prismatic graph G. Let G' be obtained from G by multiplying the leaf triangle T, replacing u by a set A' and v with a set B' where $|A'| = |B'|$ and every vertex in A' has a neighbour in B' and vice versa, with $|A'| = |B'| = 101$. Moreover, we may assume that $u \in A'$ and $v \in B'$. Now we apply 4.1 to G' rather than to G; and we are able to eliminate most possibilities for u, v very quickly. For instance, because $|V(G')| > 27$, G' is not Schläfli-prismatic; suppose it is fuzzily Schläfli-prismatic. Let $\{a, b, c\}$ be a leaf triangle on c in a Schläfli-prismatic graph H, so that G' can be obtained from H by multiplying $\{a, b\}$, and A, B are the two sets of new vertices corresponding to a, b respectively. Now $|A' \cap (A \cup B)| \geq 74$, so we may assume that $|A \cap A'| \geq 37$. Since A is stable and there are at least 37 vertices in B' with a neighbour in $A \cap A'$, it follows that $|B \cap B'| \geq 20$. Thus $w = c$, and we may assume that $u \in A$ and $v \in B$. Since G' is obtained from H by multiplying $\{a, b, c\}$, it follows that G is also obtained from an induced subgraph of H by multiplying the same leaf triangle, and so G is fuzzily Schläfli-prismatic, and the leaf triangle T is is one of those that are expected when we multiply $\{a, b, c\}$ to produce G. A similar argument can be applied in all the other cases of 4.1 (although in some there are two or three different possible positions for leaf triangles to be listed). We omit the details.

12 Colouring

In another future paper, we need the following:

12.1 *Let G be prismatic, with at least one triangle. Then one of the following holds:*

- *there is a rigid, Schläfli-prismatic, non-orientable prismatic graph G_0 with no changeable edges, such that G can be obtained from G_0 by replicating vertices not in the core, or*
- *for some k with $1 \leq k \leq 3$, there is a list of $4k$ stable sets of G such that every vertex belongs to exactly k of them.*

The main part of the proof of 12.1 is the following lemma.

12.2 *Let G be prismatic, non-orientable, and rigid. Then either*

- *G is Schläfli-prismatic and has no changeable edges, or*
- *for some with $1 \leq k \leq 3$, there is a list of $4k$ stable sets of G such that every vertex belongs to exactly k of them.*

Proof. We may assume that

(1) *For $k = 1, 2, 3$, there is no list of $4k$ stable sets of G such that every vertex belongs to k of them.*

Since G is rigid and non-orientable, we may apply 4.1, and deduce that G is in the menagerie. Thus either G is of parallel-square or skew-square type, or is Schläfli-prismatic or fuzzily Schläfli-prismatic, or $G \in \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_9$. We treat these cases separately.

(2) *G cannot be obtained from any graph by multiplying or exponentiating a leaf triangle.*

For suppose that $T = \{a, b, c\}$ is a leaf triangle at c in a prismatic graph H. For $v = a, b, c$, let N_v be the set of vertices in $V(H) \setminus T$ adjacent to v; thus the sets N_a, N_b, N_c are pairwise disjoint and have union $V(G) \setminus T$. Moreover, N_a, N_b are stable, since T is a leaf triangle. Also, since every vertex in N_c has at most one neighbour in N_c, it follows that N_c is the union of two stable sets P, Q say. Suppose first that G is obtained from H by multiplying $\{a, b\}$, and let A, B be the two corresponding sets of new vertices of G. Then $A \cup P, B \cup Q$ are stable in G, and so the four sets Then $N_a \cup \{c\}, N_b, A \cup P, B \cup Q$ are four stable sets of G and every vertex belongs to one of them, contrary to (1). Suppose now that G is obtained from H by exponentiating T, and let A, B, C, D_1, D_2, D_3 be as in the definition of exponentiating a triangle. Then the four sets $N_a \cup C \cup \{c\}, N_b, A \cup P, B \cup Q$ are stable sets of G and every vertex belongs to one of them, contrary to (1). This proves (2).

From (2) we deduce that G is not fuzzily Schläfli-prismatic, and not in $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_6, \mathcal{F}_8$. Since there do not exist four stable sets with union $V(G)$, it follows that G is not of parallel-square or skew-square type, and $G \notin \mathcal{F}_1$. (For the same reason, $G \notin \mathcal{F}_9$, but we show that explicitly below.) If $G \in \mathcal{F}_5$, let r^i_j, s^i_j, t^i_j ($1 \leq i, j \leq 3$) be as in the definition of $\mathcal{F}_5$; then the twelve sets

$$\{r_1^1, r_1^2, s_2^1, s_3^1, t_3^2, t_3^3\}, \{r_2^3, r_3^3, s_1^2, s_1^3, t_1^1, t_2^1\}, \{r_1^1, r_1^2, r_1^3, s_3^1, s_3^2, s_3^3\},$$
$$\{r_1^1, r_2^1, s_3^1, s_3^2, t_2^3, t_3^3\}, \{r_1^1, r_2^1, r_3^1, t_3^1, t_3^2, t_3^3\}, \{r_1^1, r_3^1, s_2^1, s_2^3, t_2^2, t_3^2\},$$
$$\{r_1^1, r_1^2, r_1^3, s_2^1, s_2^2, s_2^3\}, \{s_1^2, s_2^2, s_3^2, t_1^1, t_2^1, t_3^1\}, \{r_3^1, r_3^3, s_1^2, s_2^2, t_2^1, t_2^3\},$$
$$\{r_1^3, r_2^3, r_3^3, t_2^1, t_2^2, t_2^3\}, \{r_2^1, r_2^3, s_1^3, s_3^3, t_2^1, t_2^2\}, \{s_1^3, s_2^3, s_3^3, t_1^1, t_2^1, t_3^1\},$$

are stable sets of the complement of the Schläfli graph, containing every vertex of G at least three times, and none containing both r_1^1, t_2^1; so there are twelve stable sets of G containing every vertex three times, contrary to (1).

Next suppose that $G \in \mathcal{F}_7$, and let K be as in the definition of $\mathcal{F}_7$. For each $v \in V(K)$, let M_v be the set of vertices of K different from and nonadjacent to v, and let D_v be the set of edges of K incident with v. Then D_v is a stable set of G; M_v is anticomplete in G to D_v; and $M_v \cap V(G)$ is a stable set of G (because K has no stable set of size three). Thus $D_v \cup (M_v \cap V(G))$ is stable in G, and this gives a list of six stable sets of G such that every edge of K belongs to two of them, and each $v \in V(K) \cap V(G)$ belongs to $|M_v|$ of them. Let X_i be the set of all $v \in V(K) \cap V(G)$ with $|M_v| = i$, for $i = 0, 1$. Then X_0 is a clique of K, and complete in K to X_1, and every vertex in X_1 is nonadjacent in K to at most one other vertex of X_1. Consequently there are two cliques of K (and hence stable sets of G) covering every vertex in X_0 twice, and every vertex in X_1 once. Combined with the other six, we have eight stable sets of G containing every vertex in G twice, contrary to (1).

Next suppose that $G \in \mathcal{F}_9$. Let r_j^i, s_j^i, t_j^i $(1 \le i, j \le 3)$ be as in the definition of $\mathcal{F}_9$, and let z be the new vertex. Then

$$\{r_2^3, r_3^3, s_1^1, t_2^2, t_2^3\}$$
$$\{r_1^2, r_1^3, s_1^2, s_1^3, t_3^2, t_3^3\}$$
$$\{r_2^1, s_2^3, s_3^3, t_2^1, t_3^1\}$$
$$\{r_3^1, s_2^2, s_3^2, z\}$$

is a list of four stable sets of G with union $V(G)$, contrary to (1).

We deduce from 4.1 that G is Schläfli-prismatic. Suppose that uv is a changeable edge. Since G has no leaf triangle by (2), it follows that u, v are not in the core, and so have no common neighbour. Let N_u, N_v be respectively the sets of vertices in $V(G) \setminus \{u, v\}$ adjacent to u and to v. Since u is not in the core, N_u is stable, and so is N_v. But for every edge of the complement of the Schläfli graph, the set of vertices nonadjacent to both ends of the edge can be partitioned into two stable sets (for instance, if we take the edge $s_1^1 s_2^2$ in the usual notation, then the set of vertices nonadjacent to both s_1^1, s_2^2 is the union of the stable sets $\{t_3^1, t_3^2, t_3^3, s_1^2\}, \{r_1^3, r_2^3, r_3^3, s_2^1\}$). Consequently $V(G) \setminus (N_u \cup N_v \cup \{u, v\})$ can be partitioned into two sets P, Q both stable in G, and so $V(G)$ is the union of four stable sets $N_u \cup \{v\}, N_v \cup \{u\}, P, Q$, contrary to (1). Thus G has no changeable edge and so the theorem holds. This proves 12.2. ∎

Proof of 12.1. If G is orientable, the result follows from a theorem proved in [2], so we assume that G is nonorientable. By 2.2, G can be obtained from a rigid non-orientable prismatic graph G_0 by replicating vertices not in the core, and then deleting edges between vertices not in the core. If for some $k \in \{1, 2, 3\}$ there is a list of $4k$ sets, stable in G_0, such that every vertex of $V(G_0)$ is in k of them, then the same $4k$ sets are also stable in G and therefore G satisfies the theorem. We assume therefore that there is no such list. By 12.2, G_0 is Schläfli-prismatic and has no changeable edges. In particular, since G_0 has no changeable edges, it follows that no two vertices of G_0 not in the core are adjacent, and so G is obtained from G_0 just by replicating vertices not in the core. But then the theorem holds. This proves 12.1. ■

References

[1] Peter Cameron, "Strongly regular graphs", in *Topics in Algebraic Graph Theory* (ed. L. W. Beineke and R. J. Wilson), Cambridge Univ. Press, Cambridge, 2004, 203-221.

[2] Maria Chudnovsky and Paul Seymour, "Claw-free Graphs. I. Orientable prismatic graphs", *J. Combinatorial Theory, Ser. B*, to appear (manuscript February 2004).