Even-hole-free graphs still have bisimplicial vertices

Maria Chudnovsky[1] and Paul Seymour[2]
Princeton University, Princeton, NJ 08544

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Abstract

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. A hole is called *even* if it has an even number of vertices. An *even-hole-free* graph is a graph with no even holes. A vertex of a graph is *bisimplicial* if the set of its neighbours is the union of two cliques.

In an earlier paper [1], Addario-Berry, Havet and Reed, with the authors, claimed to prove a conjecture of Reed, that every even-hole-free graph has a bisimplicial vertex, but we have recently been shown that the "proof" has a serious error. Here we give a proof using a different approach.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. A *clique* in G is a set of pairwise adjacent vertices. A vertex is *bisimplicial (in G)* if its neighbourhood is the union of two cliques. A *hole* in a graph is an induced subgraph that is a cycle of length at least four. A hole is *even* if it has even length and *odd* otherwise. A graph is *even-hole-free* if it contains no even hole. The following was conjectured in [4]:

1.1 *Every non-null even-hole-free graph has a bisimplicial vertex.*

Louigi Addario-Berry, Frédéric Havet and Bruce Reed, with the authors, published a "proof" in [1]. However, there is a major error in this proof, pointed out to us recently by Rong Wu. The flawed proof is for a result (theorem 3.1 of that paper) that is fundamental to much of the remainder of the paper, and we have not been able to fix the error (although we still believe 3.1 of that paper to be true). The error in [1] is in the very last line of the proof of theorem 3.1 of that paper: we say "it follows that $N_G(v) = N_{G'}(v)$, and so v is bisimplicial in G"; and this is not correct, since cliques of G' may not be cliques of G.

In this paper we give a different proof of 1.1. For inductive purposes we prove something a little stronger, namely:

1.2 *Let G be even-hole-free, and let K be a clique of G with $|K| \leq 2$. Let M be the set of vertices in $V(G) \setminus V(K)$ with no neighbour in $V(K)$. If $M \neq \emptyset$, then some vertex in M is bisimplicial in G.*

The proof is via two decomposition theorems for even-hole-free graphs. Most of the paper is concerned with proving these decomposition theorems, and at the end we give the application to finding a bisimplicial vertex.

2 Preliminaries, and a sketch of the proof

Before we can outline the proof we need more definitions. Let S be a subset of $V(G)$. We denote by $G[S]$ the subgraph of G induced on S, and by $G \setminus S$ the subgraph of G induced on $V(G) \setminus S$. We say $S \subseteq V(G)$ is *connected* if $G[S]$ is connected. The *neighbourhood* of S, denoted by $N_G(S)$ (or $N(S)$ when there is no risk of confusion), is the set of all vertices of $V(G) \setminus S$ with a neighbour in S, and $N[S]$ means $N(S) \cup S$. If $S = \{v\}$, we write $N_G(v)$ instead of $N_G(\{v\})$; for an induced subgraph H of G, we define $N(H)$ to be $N(V(H))$, and so on. A subgraph S is *dominating* in G if $N[S] = V(G)$, and *non-dominating* otherwise.

Two disjoint subsets A, B of $V(G)$ are *complete* to each other if every vertex of A is adjacent to every vertex of B, and *anticomplete* to each other if no vertex of A is adjacent to any vertex of B. If $A = \{a\}$, we write "a is complete (anticomplete) to B" instead of "$\{a\}$ is complete (anticomplete) to B".

The *length* of a path is the number of edges in it. A path is called *even* if its length is even, and *odd* otherwise. Let the vertices of P be $p_1, \dots, p_k$ in order. Then p_1, p_k are called the *ends* of P (sometimes we say P is *from p_1 to p_k* or *between p_1 and p_k*), and the set $V(P) \setminus \{p_1, p_k\}$ is the *interior* of P and is denoted by P^*. For $1 \leq i < j \leq k$ we will write p_i-P-p_j or p_j-P-p_i to mean the subpath of P between p_i and p_j. More generally, if S is an induced subgraph of a graph G, and u, v both have neighbours in $V(S)$, we denote by u-S-v some induced path between u, v with interior in $V(S)$. (Here u, v might or might not belong to $V(S)$.) If H is a cycle, and a, b and c are three vertices of H such that a is adjacent to b, then a-b-H-c is a path, consisting of a, and the subpath of $H \setminus \{a\}$ between b and c. A *triangle* is a set of three vertices, pairwise adjacent, and we use the same word for the subgraph induced on a triangle.

Here are some types of graph that we will need:

- A *theta* with *ends* s, t is a graph that is the union of three paths R_1, R_2, R_3, each with the same pair of ends s, t, each of length more than one, and pairwise vertex-disjoint except for their ends.

Figure 1: A theta and a pyramid (dashed lines mean paths of arbitrary positive length)

- A *pyramid* with *apex* a and *base* $\{b_1, b_2, b_3\}$ is a graph P in which
  - a, b_1, b_2, b_3 are distinct vertices, and $\{b_1, b_2, b_3\}$ is a triangle,
  - P is the union of this triangle and three paths R_1, R_2, R_3, where R_i has ends a, b_i for $i = 1, 2, 3$, and
  - R_1, R_2, R_3 are pairwise vertex-disjoint except for their common end, and at least two of R_1, R_2, R_3 have length at least two.
- A *near-prism* with *bases* $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ is a graph P in which
  - $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles, and $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \{a_3\} \cap \{b_3\}$ (that is, the triangles are disjoint except that possibly $a_3 = b_3$).
  - P is the union of these two triangles and three paths R_1, R_2, R_3, where R_i has ends a_i, b_i for $i = 1, 2, 3$ (and so R_3 has length zero if $a_3 = b_3$).
  - R_1, R_2, R_3 are pairwise vertex-disjoint.

  If $a_3 \neq b_3$, P is also called a *prism*.

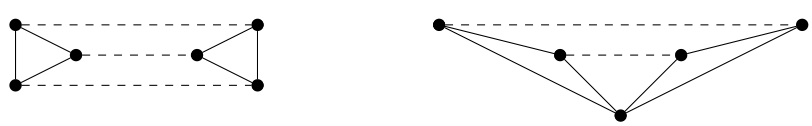

Figure 2: Near-prisms

- A *wheel* is a graph consisting of a hole H and a vertex $v \notin V(H)$ with at least three neighbours in $V(H)$, and if it has exactly three neighbours in $V(H)$ then no two of them are adjacent. We call v its *centre* and H its *hole*. If v has k neighbours in H we also call it a *k-wheel*. If k is even we call it an *even wheel*.

For a theta, pyramid or near-prism, we call R_1, R_2, R_3 its *constituent paths*. It is easy to see that:

2.1 *No even-hole-free graph contains a theta, a near-prism or an even wheel as an induced subgraph.*

Even-hole-free graphs can contain pyramids, however. A pyramid is *short* if one of the three constituent paths has length one.

An *extended near-prism* is a graph obtained from a near-prism by adding one extra edge, as follows. Let R_1, R_2, R_3 be as in the definition of a near-prism, and let $a \in R_1^*$ and $b \in R_2^*$; and add an edge ab. (It is important that a, b do not belong to the triangles.) We call ab the *cross-edge* of the extended near-prism.

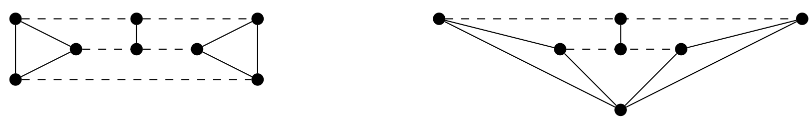

Figure 3: Extended near-prisms

A vertex $a \in V(G)$ is *splendid* if

- $V(G) \setminus N[a]$ is connected;
- every vertex in $N(a)$ has a neighbour in $V(G) \setminus N[a]$; and
- there is no short pyramid with apex a in G.

Now we can sketch the idea of the proof. In order to prove 1.2, we use induction on $|V(G)|$. From a result of [1] (that did not depend on theorem 3.1 of that paper, and so is still valid), we may assume that G admits no "full star cutset" (defined later). It follows that, with K as in 1.2, there is a splendid vertex $a \in V(G) \setminus N[K]$. We can assume that a is not bisimplicial. Now there are two possibilities:

- there is an extended near-prism in which a belongs to the cross-edge;
- there is a pyramid with apex a, but there is no extended near-prism in which a belongs to the cross-edge.

In both cases we use a decomposition theorem to find a smaller subgraph to which we can apply the inductive hypthesis and win. There are two different decompositions theorems. The first gives a decomposition of G relative to an extended near-prism, and is fully general (that is, it does not require any vertex to be splendid), and so may be useful in other applications. The second is more tailored to our application, in that it needs a to be splendid.

To apply these to find bisimplicial vertices, we use that both theorems provide a choice of subgraphs (two in the first case, three in the second) that are separated from the remainder of the graph in a convenient way, and we can prove inductively that all these subgraphs contain bisimplicial vertices of G; and in both cases these subgraphs are sufficiently widely separated that at least one of these bisimplicial vertices has no neighbours in K.

The main part of the paper concerns proving the two decomposition theorems, and we use them to prove 1.2 in the final section.

3 Some results from [1].

We will need to use some results of [1] that did not depend on the flawed theorem 3.1 of that paper. A *cutset* in G is a subset C of $V(G)$ such that $V(G)\setminus C$ is the union of two non-empty sets, anticomplete to each other. A *star cutset* is a cutset consisting of a vertex and some of its neighbours. If v together with a subset of $N(v)$ is a cutset, we say that v is a *centre* of this star cutset. A star cutset C is called *full* if it consists of a vertex and all its neighbours. We need the following, theorem 4.2 of [1]:

3.1 *Let G be an even-hole-free graph such that, for every even-hole-free graph H with fewer vertices than G, and every non-dominating clique J of H with $|J| \le 2$, there is a bisimplicial vertex of H in $V(H) \setminus N_H(J)$. Assume that there exists a non-dominating clique K with $|K| \le 2$ such that no vertex of $V(G) \setminus N_G(K)$ is bisimplicial in G. Then G does not admit a full star cutset.*

(Actually, theorem 4.2 in [1] takes a stronger hypothesis than we give here, requiring that the dubious theorem 3.1 of that paper holds for all graphs with fewer vertices than G; but fortunately its proof in that paper does not use the extra hypothesis, so we can legitimately omit it.) We will also need the following consequence of theorem 4.5 of [1]:

3.2 *Let G be even-hole-free, let H be a hole in G, and let $a \notin V(H)$. If G admits no full star cutset with centre a, then either*

- *a is complete or anticomplete to $V(H)$; or*
- *$H[V(H) \cap N(a)]$ is a path; or*
- *a has exactly three neighbours in H, and two of them are adjacent.*

4 Tree strip systems

In this section and the next, we state and prove the decomposition theorem for even-hole-free graphs that contain an extended near-prism.

Here is an example of an even-hole-free graph, due to Conforti, Cornuéjols and Vušković [2], and see also [3]. Start with a tree T with $|V(T)| \ge 3$. (A *leaf* of T means a vertex of degree exactly one, and a *leaf-edge* is an edge incident with a leaf.) Let (A', B') be a bipartition of T. Since $|V(T)| \ge 3$, each leaf-edge is incident with only one leaf; let A be the set of leaf-edges incident with a leaf in A', and define B similarly. Let $L(T)$ be the line-graph of T. Thus the vertex set of $L(T)$ is the edge set of T, and A, B are disjoint subsets of $V(L(T))$. Add to $L(T)$ two more vertices a, b and the edge ab, and make a complete to A and b complete to B, forming a graph $H(T)$ say. Thus $H(T)$ has vertex set $E(T) \cup \{a, b\}$. This graph $H(T)$ is even-hole-free, but it is helpful for our purposes to impose additional conditions on T. We will assume that T has at least three leaves, and for every $v \in V(T)$, there is at most one component C of $V(T) \setminus v$ such that $A' \cap V(C) = \emptyset$, and at most one such that $B' \cap V(C) = \emptyset$. (Note that every component C of $V(T) \setminus v$ contains a leaf of T and therefore meets at least one of A', B'.) If this additional condition is satisfied, we say that $H(T)$ is an *extended tree line-graph*, and ab is its *cross-edge*.

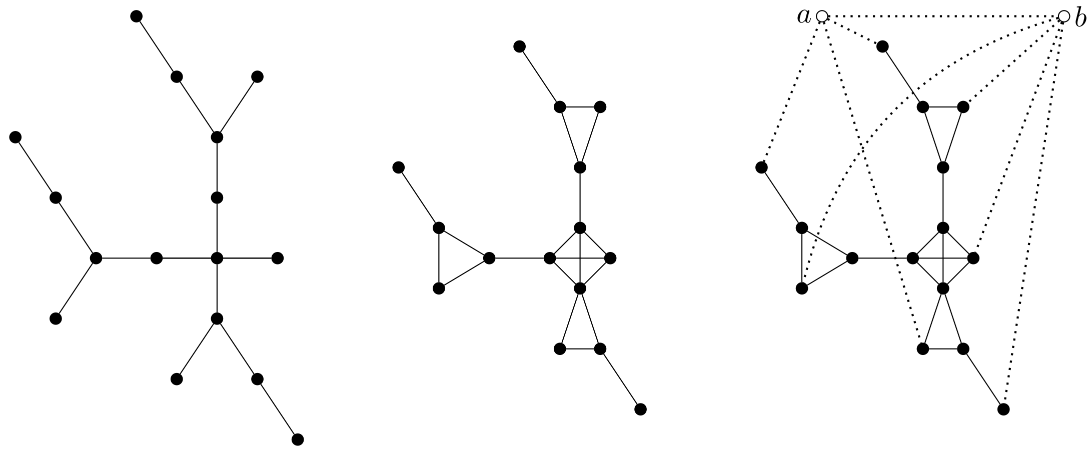

Figure 4: A tree T, $L(T)$, and $H(T)$. (The dotted lines are just edges.)

Every extended near-prism is an extended tree line-graph, where the corresponding tree has four leaves and exactly two vertices of degree three. In the next few sections we will be working with even-hole-free graphs G that contain extended near-prisms, and therefore the graph also contains an extended tree line-graph that is maximal (subject to keeping the cross-edge fixed); and examining how the remainder of the graph attaches to this subgraph will lead us to the decomposition.

Sometimes we have different graphs with the same vertex set or edge set, and we say *G-incident* to mean incident in G, and *G-adjacent* to mean adjacent in G, and so on. A *branch-vertex* of a tree means a vertex of degree different from two (thus, leaves are branch-vertices). A *branch* of a tree T means a path P of T with distinct ends u, v, both branch-vertices, such that every vertex of P^* has degree two in T. Every edge of T belongs to a unique branch. A *leaf-branch* is a branch such that one of its ends is a leaf of T. In general, a *leaf-path* of T means a path of T with one end a leaf of T and the other end a vertex of T that is not a leaf.

Let T be a tree, and let U be the set of branch-vertices of T; and make a new tree J with vertex set U by making $u, v \in U$ J-adjacent if there is a branch of T with ends u, v. We call J the *shape* of T. Thus J has no vertices of degree two; and T is obtained from J by replacing each edge by a path of positive length.

Let A, B, C be subsets of $V(G)$, with $A, B \neq \emptyset$ and disjoint from C, and let $S = (A, B, C)$. A rung of S, or an *S-rung*, is an induced path $p_1\text{-}\cdots\text{-}p_k$ of $G[A \cup B \cup C]$ such that $p_1 \in A$, $p_k \in B$ and $p_2, \ldots, p_{k-1} \in C$, and if $k > 0$ then $p_1 \notin B$ and $p_k \notin A$. (If $A \cap B \neq \emptyset$ then perhaps $k = 0$.) If every vertex in $A \cup B \cup C$ belongs to an S-rung we call S a *strip*. We denote $A \cup B \cup C$ by $V(S)$. In the later part of the paper, concerned with "pyramid strip systems", we will only need strips (A, B, C) with $A \cap B = \emptyset$, but earlier when we look at "tree strip systems" we need to allow A, B to intersect. A strip (A, B, C) is *proper* if $A \cap B = \emptyset$.

Let J be a tree with at least three vertices. A *J-strip system* M in a graph G means:

- for each edge $e = uv$ of J, a subset $M_{uv} = M_{vu} = M_e$ of $V(G)$
- for each $v \in V(J)$, a subset M_v of $V(G)$

satisfying the following conditions:

- the sets M_e ($e \in E(J)$) are pairwise disjoint;
- for each $u \in V(J)$, $M_u \subseteq \bigcup(M_{uv} : v \in V(J)$ adjacent to $u)$;
- for each $uv \in E(J)$, $(M_{uv} \cap M_u, M_{uv} \cap M_v, M_{uv} \setminus (M_u \cup M_v))$ is a strip (not necessarily proper);
- if $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between M_{uv} and M_{wx};
- if $uv, uw \in E(J)$ with $v \neq w$, then $M_u \cap M_{uv}$ is complete to $M_u \cap M_{uw}$, and there are no other edges between M_{uv} and M_{uw}.

A rung of the strip $(M_{uv} \cap M_u, M_{uv} \cap M_v, M_{uv} \setminus (M_u \cup M_v))$ will be called an *e-rung* or *uv-rung*. (We leave the dependence on M and J to be understood, for the sake of brevity.) Let $V(M)$ denote the union of the sets M_e ($e \in E(J)$).

Let J be a tree, let M be a J-strip system in G, and let (α, β) be a partition of the set of leaves of J. We say an edge ab of G is a *cross-edge* for M with *partition* (α, β) if:

- J has no vertex of degree two, and at least three vertices;
- for every vertex $s \in V(J)$, s has at most one neighbour in α, and at most one in β;
- for all $e \in E(J)$, $a, b \notin M_e$;
- a is complete to $\bigcup_{u \in \alpha} M_u$, and a has no other neighbours in $V(M)$; b is complete to $\bigcup_{u \in \beta} M_u$, and b has no other neighbours in $V(M)$.

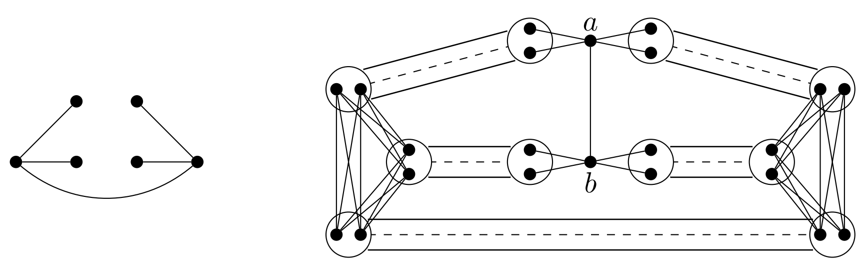

Figure 5: The smallest possible J, and a J-strip system with cross-edge

If we are given J, M and ab then we can reconstruct α, β, so we call (α, β) the *corresponding partition*. If G is an extended tree line-graph $H(T)$ with cross-edge ab, where T is a tree, and J is the shape of T, then there is a J-strip system in G with the same cross-edge ab, defined as follows. Let (A', B') be a bipartition of T, as in the definition of $H(T)$, and let $\alpha = V(J) \cap A'$, and $\beta = V(J) \cap B'$. For each edge e of J, define M_e to be the edge-set of the corresponding branch of T; and for each $u \in V(J)$, let M_u be the set of edges of T incident with u. This defines a J-strip system. (Note that some strips might not be proper; if some branch of T has length one then the J-strip system is not proper.)

Let M be a J-strip system in G with cross-edge ab. If D is a subtree of J, and we choose an e-rung R_e for each $e \in E(J)$, then the subgraph of G induced on $\bigcup_{e \in E(D)} V(R_e)$, denoted by R_D, is the line graph of some tree that has the same shape as D. Thus, R_D depends on the choices of the individual e-rungs R_e, but we leave this dependence implicit.

Let M be a J-strip system in G with cross-edge ab and partition (α, β). We say $X \subseteq V(M) \cup \{a, b\}$ is *local* if either:

- $X \subseteq M_e$ for some $e \in E(J)$; or
- $X \subseteq M_u$ for some $u \in V(J)$; or
- X contains a and not b, and $X \setminus \{a\} \subseteq M_u$ for some leaf $u \in \alpha$; or X contains b and not a, and $X \setminus \{a\} \subseteq M_u$ for some leaf $u \in \beta$.

We need a lemma:

4.1 *If $X \subseteq V(M) \cup \{a, b\}$ is not local, and $\{a, b\} \not\subseteq X$, then there exist $x, y \in X$ such that $\{x, y\}$ is not local.*

Proof. Suppose first that $a, b \notin X$. Choose $x \in X$, and choose $uv \in E(J)$ such that $x \in M_{uv}$. There exists $y \in X \setminus M_{uv}$, and we may assume that $\{x, y\}$ is local; so we may assume that $x, y \in M_u$. There exists $z \in X \setminus M_u$; and we may assume that $\{x, z\}$ is local, and so either $z \in M_{uv}$, or $x, z \in M_v$. In either case $\{y, z\}$ is not local, since J is a tree.

Thus we may assume that $a \in X$, and $b \notin X$. Also there exists $x \in X \setminus \{a, b\}$; and we may assume that $\{a, x\}$ is local, and so $x \in M_u$ for some $u \in \alpha$. There exists $y \in X \setminus (M_u \cup \{a\})$. Since we may assume that $\{a, y\}$ is local, $y \in M_v$ for some $v \in \alpha$, and so $v \neq u$. From the definition of cross-edge, u, v have no common neighbour in J, and so $\{x, y\}$ is not local. This proves 4.1. ∎

We will need two maximizations:

- We start with an even-hole-free graph G, and an edge ab of G, such that there is an extended tree line-graph $H(T)$ that is an induced subgraph of G, with cross-edge ab. Subject to this we choose T with as many branches as possible, that is, such that its shape J has $|E(J)|$ maximum.
- Then we choose a J-strip system M in G with the same cross-edge ab, with $V(M)$ maximal.

In these circumstances we say that (J, M) is *optimal for ab*. Our first goal is to prove:

4.2 *Let ab be an edge of an even-hole-free graph G, and let (J, M) be optimal for ab. Let Z be the set of vertices of G adjacent to both a, b. Then for every connected induced subgraph F of $G \setminus (Z \cup V(M))$:*

- *if not both a, b have neighbours in $V(F)$, then the set of vertices in $V(M)$ with a neighbour in $V(F)$ is local;*
- *if both a, b have neighbours in $V(F)$, then there exists a leaf t of J such that every vertex of $V(M)$ with a neighbour in $V(F)$ belongs to M_t.*

We break the proof into three steps, 4.3, 4.4, and 4.5 below, depending on the number of $a, b \in N(F)$.

Under the hypotheses of 4.2, let (α, β) be the corresponding partition. Let us say that a subgraph F is *small* if F is connected and F is an induced subgraph of $G \setminus (Z \cup V(M))$; and a *small component* is a component of $G \setminus (V(M) \setminus Z)$. A small subgraph F is *α-peripheral* if $X(F) \subseteq M_t$ for some $t \in \alpha$. We define *β-peripheral* similarly; and F is *peripheral* if it is either α- or β-peripheral. If F is small, the set of vertices in $V(M)$ with a neighbour in $V(F)$ is denoted by $X(F)$. We begin with:

4.3 *Under the hypotheses of 4.2, if F is small, and $a, b \notin N(F)$, then $X(F)$ is local.*

Proof. Suppose the theorem is false, and choose a small subgraph F not satisfying the theorem, with F minimal. By 4.1, there exist $x, y \in X(F)$ such that $\{x, y\}$ is not local, and so F is a path joining these two vertices. Let F have ends f_1, f_2.

For $x_1, x_2 \in V(M)$, let us say $s \in V(J)$ *separates* x_1, x_2 if $x_1, x_2 \notin M_s$, and s lies on the path of J between e_1, e_2, where $x_i \in M_{e_i}$ $(i = 1, 2)$.

(1) *If $x_1, x_2 \in X(F)$, there is no $s \in V(J)$ that separates x_1, x_2.*

Let $x_i \in M_{e_i}$ $(i = 1, 2)$, and suppose that $s \in V(J)$ separates x_1, x_2. Then $\{x_1, x_2\}$ is not local, and so we may assume that f_1x_1 and f_2x_2 are edges. Choose three leaf-paths S_1, S_2, S_3 of J, each with one end s and otherwise pairwise vertex-disjoint, with $e_i \in E(S_i)$ for $i = 1, 2$. For $i = 1, 2, 3$ let s_i be the edge of S_i incident with s. For $i = 1, 2, 3$ and each $e \in E(S_i)$, choose an e-rung R_e, such that $x_i \in V(R_{e_i})$ for $i = 1, 2$. For $i = 1, 2, 3$, let u_i be the end of R_{s_i} in M_s. Then R_{S_i} is an induced path of G from u_i to some $p_i \in N(\{a, b\})$. We may assume that x_1, x_2 have been chosen such that for $i = 1, 2$ the subpath of R_{S_i} between x_i, p_i is minimal.

Suppose that there exists $x_3 \in X(F)$, in $V(R_{S_1} \cup R_{S_2} \cup R_{S_3})$ and different from and nonadjacent to x_1, x_2. Choose x_3 such that the subpath of R_{S_3} between x_3 and p_3 is minimal. We claim that $|V(F)| = 1$. For if not, we may assume that x_3 has a neighbour in $V(F \setminus f_2)$, and since $X(F \setminus f_2)$ is local (from the minimality of F) and contains x_1, x_3, and x_1, x_3 are nonadjacent, it follows that $X(F \setminus f_2) \subseteq M_{e_1}$, and in particular x_3 belongs to R_{e_1}. But then there is an induced path between the ends of R_{e_1} and contained in $G[V(R_{e_1} \cup (F \setminus f_2))]$, that contains at least one vertex of $F \setminus f_2$, and the vertices of this path can be added to M_{e_1}, contrary to the maximality of $V(M)$. This proves that $|V(F)| = 1$.

If $p_1, p_2, p_3 \in N(a)$, there is a theta with ends f_1, a and constituent paths f_1-x_i-R_{S_i}-p_i-a for $i = 1, 2, 3$; and similarly not all $p_1, p_2, p_3 \in N(b)$. By exchanging a, b if necessary, we may assume that two of $p_1, p_2, p_3 \in N(a)$; then there is a theta with ends f_1, a with constituent paths f_1-x_i-R_{S_i}-p_i-a for the two values of i with $p_i \in N(a)$, and f_1-x_i-R_{S_i}-p_i-b-a for the value of i with $p_i \in N(b)$.

This proves that $X(F) \cap V(R_{S_3}) = \emptyset$, and every vertex of $X(F) \cap V(R_{S_1} \cup R_{S_2})$ is equal or adjacent to one of x_1, x_2. For $i = 1, 2$, let y_i be the neighbour of x_i in R_{S_i} between x_i and u_i (this exists, since $x_i \notin M_s$.) The path $R_{S_1 \cup S_2}$ can be completed to a hole H by adding a or b or both. From the minimality of F, $X(F \setminus \{f_1, f_2\}) = \emptyset$. We claim that the only edges between $V(R_{S_1} \cup R_{S_2} \cup R_{S_3})$ and $V(F)$ are the edges f_1x_1, f_2x_2 and exactly one of f_1y_1, f_2y_2. If $|V(F)| = 1$ this is true since f_1 cannot have two nonadjacent neighbours in H, or four neighbours in H. If $f_1 \neq f_2$ then from the minimality of F, f_1 is nonadjacent to y_2, x_2, and f_2 is nonadjacent to x_1, y_1; at least one of the pairs f_1y_1, f_2y_2 is an edge since otherwise the subgraph induced on $V(H) \cup V(F)$ is a theta, and not both since otherwise the same subgraph is a prism. (Note that $y_1 \neq y_2$ since $x_1, x_2 \notin M_s$.) Thus we may assume that f_1x_1, f_1y_1, f_2x_2 are edges, and there are no other edges between $V(R_{S_1} \cup R_{S_2} \cup R_{S_3})$ and $V(F)$. If $x_2 \notin N(\{a, b\})$, we may assume that at least two of p_1, p_2, p_3 are adjacent to a, and then there is a theta between x_2 and a with constituent paths

$$x_2\text{-}R_{S_2}\text{-}u_2\text{-}u_3\text{-}R_{S_3}\text{-}p_3\text{-}a,$$

$$x_2\text{-}f_2\text{-}F\text{-}f_1\text{-}x_1\text{-}R_{S_1}\text{-}p_1\text{-}a,$$

$$x_2\text{-}R_{S_2}\text{-}p_2\text{-}a,$$

inserting b before a in one of these paths if necessary. Thus e_2 is a leaf-edge of J, and $x_2 = p_2 \in N(\{a,b\})$, and we may assume that $x_2 \in N(b)$. We can choose S_3 such that it has an end in α (from the definition of a crossedge for a tree strip system), and hence we may assume that $p_3 \in N(a)$. If $p_1 \in N(a)$ then the same argument gives a theta, which is impossible; so we may assume that every choice of S_1 has an end in β, and so e_1 is also a leaf-edge of J. Let r be the end of e_1 that is not a leaf of J, and let t be the end of e_2 that is not a leaf. From the definition of a crossedge, $r \neq t$. Exactly two vertices of $R_{S_1 \cup S_2}$ belong to M_r, and they are adjacent, say r_1, r_2; and define t_1, t_2 similarly, where r_1, r_2, t_1, t_2 are in order in $R_{S_1 \cup S_2}$ (possibly $r_2 = t_1$). By choosing a leaf-path of J with one end r that is edge-disjoint from S_1, S_2, and has an end in β, and choosing a rung for each of its edges, we find a path R say of $G[V(M)]$ with ends r_3, r_0 say, where r_3 is adjacent to r_1, r_2, and $r_0 \in N(b)$ and there are no other edges between $V(R)$ and $V(R_1 \cup R_2)$, Define a path T with ends t_3, t_0 similarly, where t_3 is adjacent to t_1, t_2 and $t_0 \in N(b)$, and there are no edges between $V(R)$ and $V(T)$. There is a near-prism with bases $\{r_1, r_2, r_3\}$, $\{t_1, t_2, t_3\}$ and constituent paths

$$r_1\text{-}R_{S_1}\text{-}y_1\text{-}f_1\text{-}F\text{-}f_2\text{-}x_2\text{-}R_{S_2}\text{-}t_2,$$

$$r_3\text{-}R\text{-}r_0\text{-}b\text{-}t_0\text{-}T\text{-}t_3,$$

$$r_2\text{-}R_{S_1 \cup S_2}\text{-}t_1,$$

contrary to 2.1 (note that possibly $r_2 = t_1$). This proves (1).

(2) *There is an edge uv of J such that $X(F) \subseteq M_u \cup M_v \cup M_{uv}$.*

Suppose first that for some $uv \in E(J)$, there exists $x \in X(F) \in M_{uv} \setminus (M_u \cup M_v)$. Then for each $y \in X(F)$, (1) implies that no vertex of S separates x, y, and so $y \in M_u \cup M_v \cup M_{uv}$ as required. Thus we may assume that $X(F) \subseteq \bigcup_{v \in V(J)} M_v$. Suppose next that some $x \in X$ belongs to M_v for only one value of $v \in V(J)$. Let $y \in X(F) \setminus M_v$, and choose $u \in V(J)$ with $y \in M_u$. Let w be the neighbour of v in J on the path of J between $v, u(y)$. Since w does not separate x, y, and $x \notin M_w$ from the choice of x, it follows that $y \in M_w$; define $w(y) = w$. If there exist $y_1, y_2 \in X(F) \setminus M_v$ with $w(y_1) \neq w(y_1)$, then v separates y_1, y_2, contrary to (1). So there exists a neighbour w of v in J, such that $y \in M_w$ for all $y \in X(F) \setminus M_v$; and so the claim holds.

We may therefore assume that every vertex in $X(F)$ belongs to M_v for two vertices $v \in V(J)$. For each $x \in X(F)$, let $e(x)$ be the edge uv of J such that $x \in M_u \cap M_v$. If all the edges $e(x)$ ($x \in X(F)$) have a common end, then the claim holds; so we may assume that there exist $x_1, x_2 \in X(F)$ such that $e(x_1), e(x_2)$ have no common end. Let $e(x_i) = u_i v_i$ for $i = 1, 2$; thus u_1, v_1, u_2, v_2 are distinct vertices of J. Since no vertex of J separates x_1, x_2 by (1), it follows that one of u_1, v_1 is J-adjacent to one of u_2, v_2; say v_1, v_2 are J-adjacent, and so $u_1\text{-}v_1\text{-}v_2\text{-}u_2$ is a path of J. Suppose there exists $x_3 \in X(F)$ such that $x_3 \notin M_{v_1} \cup M_{v_2}$. Let $e(x_3) = u_3 v_3$ say. Thus $u_3, v_3 \neq v_1, v_2$; and we may assume that v_1 lies on the path of J between v_2 and u_3, by exchanging x_1, x_2 if necessary. But then v_1 separate x_2, x_3, contrary to (1). This proves (2).

(3) *There is an edge uv of J such that $X(F) \subseteq M_u \cup M_{uv}$.*

Suppose not. Choose uv as in (2); then there exist $x_1, x_2 \in X(F)$ with $x_1 \in M_u \setminus M_{uv}$ and

$x_2 \in M_v \setminus M_{uv}$. We may assume that f_1x_1 and f_2x_2 are edges. From the minimality of F, there are no edges between $V(F\setminus f_1)$ and $(M_u\cup M_{uv})\setminus M_v$, and no edges between $V(F\setminus f_2)$ and $(M_v\cup M_{uv})\setminus M_u$.

Let $c_1, \ldots, c_k$ be the edges of J incident with u, and different from uv; and let $d_1, \ldots, d_\ell$ be those incident with v and different from uv. Thus $k, \ell \geq 2$. If f_1 is complete to $M_u \setminus M_{uv}$ and f_2 is complete to $M_v \setminus M_{uv}$, we can add f_1 to M_u, add f_2 to M_v, and add $V(F)$ to M_{uv}, contrary to the maximality of $V(M)$. Thus we may assume that f_1 has a non-neighbour in $M_u \setminus M_{uv}$; and since $x_1 \in M_u \setminus M_{uv}$ and $k \geq 2$, we may assume that $x_1 \in M_{c_1} \cap M_u$ and $y_1 \in M_{c_2} \cap M_u$, and f_1, y_1 are nonadjacent. Also we may assume $x_2 \in M_{d_1} \setminus M_{uv}$. For $1 \leq i \leq k$ choose a leaf-path C_i of J from u and using c_i; and for $1 \leq i \leq \ell$ define D_i similarly; and choose an e-rung R_e for each of their edges e, containing x_1, x_2, y_1. If $x_1 \notin N(\{a, b\})$, we may assume that at least two of C_1, C_2, D_1 have an end in α; and then there is a theta in G with ends x_1, a and constituent paths

$$x_1\text{-}R_{C_1}\text{-}a,$$

$$x_1\text{-}y_1\text{-}R_{C_2}\text{-}a,$$

$$x_1\text{-}f_1\text{-}F\text{-}f_2\text{-}x_2\text{-}R_{D_1}\text{-}a,$$

inserting b into one of these if necessary. Thus we may assume that $x_1 \in N(b)$ say. Consequently c_1 has an end in β; and so C_2 can be chosen with an end in α. If also D_1 can be chosen with an end in α then the same construction still gives a theta; so the leaf of D_1 is in β. Hence the leaf of D_2 is not in β, so f_1 has no neighbour in M_{d_2}. This restores the symmetry between u, v. Let R_{uv} be a uv-rung, with ends $r_1 \in M_u \cap M_{uv}$ and $r_2 \in M_v \cap M_{uv}$. Since b is adjacent to both x_1, x_2, it follows that $f_1 \neq f_2$, and for the same reason, $r_1 \neq r_2$. From the minimality of F, the only edges between $V(F)$ and $V(R_{uv})$ are either f_1r_1 or f_2r_2; since $x_1\text{-}r_1\text{-}R_{uv}\text{-}r_2\text{-}x_1$ and $x_1\text{-}f_1\text{-}F\text{-}f_2\text{-}x_2$ are both odd, exactly one of these two edges is present, say f_1r_1 (without loss of generality, since the symmetry between u, v was restored). But then there is a theta with ends r_1, x_2 and constituent paths

$$r_1\text{-}R_{uv}\text{-}r_2\text{-}x_2,$$

$$r_1\text{-}f_1\text{-}F\text{-}f_2\text{-}x_2,$$

$$r_1\text{-}y_1\text{-}R_{C_2}\text{-}a\text{-}b\text{-}x_2,$$

contrary to 2.1. This proves (3).

Choose uv as in (3). Let $c_1, \ldots, c_k$ be the edges of J incident with u, and different from uv. Since $X(F)$ is not local, there exists $x_1 \in M_u \setminus M_{uv}$ and $x_2 \in M_{uv} \setminus M_u$. We may assume that f_1x_1 and f_2x_2 are edges. From the minimality of F, there are no edges between $V(F \setminus f_2)$ and $M_{uv} \setminus M_u$, and none between $V(F \setminus f_1)$ and $M_u \setminus M_{uv}$. If f_1 is complete to $M_u \setminus M_{uv}$, we can add f_1 to M_u and $V(F)$ to M_{uv}, contrary to the maximality of $V(M)$; so we may assume that $x_1 \in M_{c_1} \cap M_u$ and $y_1 \in M_{c_2} \cap M_u$, and f_1, y_1 are nonadjacent. For $1 \leq i \leq k$ choose a leaf-path C_i of J from u and using c_i. Choose a leaf-path D of J from u and using uv. (Possibly D has length one.) For each edge e of $C_1, \ldots, C_k, D$ choose an e-rung R_e, where R_{c_1} contains x_1, R_{c_2} contains y_1, and R_{uv} contains x_2.

Suppose that $x_1 \notin N(\{a, b\})$; then we may assume that at least two of C_1, C_2, D have an end in α; and then there is a theta in G with ends x_1, a and constituent paths

$$x_1\text{-}R_{C_1}\text{-}a,$$

$$x_1\text{-}y_1\text{-}R_{C_2}\text{-}a,$$

$$x_1\text{-}f_1\text{-}F\text{-}f_2\text{-}x_2\text{-}R_D\text{-}a,$$

inserting b into one of these if necessary. Thus we may assume that $x_1 \in N(b)$ say. Hence C_2 and D can be chosen to have an end in α, and the same construction still serves to find a theta, a contradiction. This proves 4.3. ∎

4.4 *Under the hypotheses of 4.2, if F is small, and $a \in N(F)$ and $b \notin N(F)$, then F is α-peripheral.*

Proof. Suppose the theorem is false, and choose a small subgraph F not satisfying the theorem, with F minimal. By 4.1, there exist $x, y \in X(F) \cup \{a\}$ such that $\{x, y\}$ is not local, and so F contains a path joining these two vertices; and a has a neighbour in this path, by 4.3, and so F is this path, from the minimality of F. Let F have ends f_1, f_2.

(1) *Let D be a path of J with distinct ends both in β, and for each $e \in E(D)$ choose an e-rung R_e. Then either $X(F)$ contains no vertices of R_D, or it contains exactly two and they are adjacent.*

Let the ends of D be $t_1, t_2 \in \beta$. Since R_d has both ends in $N(b)$, it follows that a has no G-neighbours in $V(R_d)$; and by adding b to R_D we obtain a hole H, and so a has a unique G-neighbour b in $V(H)$. We may assume there exists $y \in V(H) \cap X(F)$; and since $\{a, y\}$ is not local, the minimality of F implies that F is a path between a, y; say a is adjacent to f_1 and to no other vertex of F, and y is adjacent to f_2 and to no other vertex of F. For the same reason, $F \setminus f_2$ is anticomplete to $V(H)$.

If f_2 has two nonadjacent vertices in $V(H)$, there are two paths P_1, P_2 between f_2, b with interior in $V(H)$, and with union a hole; but then there is a theta with ends f_2, b and constituent paths

$$f_2\text{-}F\text{-}f_1\text{-}a\text{-}b,$$

$$f_2\text{-}P_1\text{-}b,$$

$$f_2\text{-}P_2\text{-}b,$$

a contradiction.

If f_2 has a unique neighbour in $V(H)$, say x, and x is nonadjacent to b, then $G[V(H \cup F)]$ is a theta with ends x, b, again a contradiction.

Suppose next that f_2 has a unique neighbour in $V(H)$, say x, and x is adjacent to b. Let $x \in M_{t_1}$, say, and let s_1 be the neighbour of t_1 in J. Since a-b-x-f_2-a is not a 4-hole, it follows that a, f_2 are not adjacent, and therefore $f_1 \neq f_2$, and so $a \in N(F \setminus f_2)$. From the minimality of F, $X(F \setminus f_2) \cup \{a\}$ is local. Choose $t_3 \in \alpha$ such that $X(F \setminus f_2) \cap M_e = \emptyset$ (this is possible, since $|\alpha| \geq 2$ and $X(F \setminus f_2) \cup \{a\}$ is local). Let D_3 be a path of J, edge-disjoint from D and with ends d, t_3 where $d \in V(D)$. For each $e \in E(D_3)$ choose an e-rung R_e. Let D_1, D_2 be the subpaths of D with ends d and t_1, t_2 respectively.

If F is anticomplete to R_{D_3}, there is a theta with ends x, a, and constituent paths

$$x\text{-}b\text{-}a,$$

$$x\text{-}f_2\text{-}F\text{-}f_1\text{-}a,$$

$$x\text{-}R_{D_1 \cup D_3}\text{-}a,$$

contrary to 2.1. Thus F is not anticomplete to R_{D_3}. Now F, R_{D_3} are vertex-disjoint, since $V(F)$ is disjoint from $V(M)$ and $V(R_{D_3}) \subseteq V(M)$. Let $y \in V(R_{D_3})$ with a neighbour in $V(F)$. If y has a neighbour in $V(F \setminus f_2)$, then $\{a, y\}$ is local, from the minimality of F; but then $y \in N(a)$ and so $y \in M_{t_3}$, contrary to the choice of e_3. Thus y is adjacent to f_2 and to no other vertex of F. From the minimality of F, $\{x, y\}$ is local; and so either $y \in M_{s_1 t_1}$ or $x, y \in M_{s_1}$. The first is impossible since $s_1 t_1$ is not an edge of D_3; and so $x, y \in M_{s_1}$. In particular, $d = s_1$, and y is the end of R_{D_3} in M_d. But then

$$y\text{-}R_{D_2}\text{-}b\text{-}a\text{-}f_1\text{-}F\text{-}f_2\text{-}y$$

is a hole, in which x has exactly four neighbours, making a 4-wheel, a contradiction. This proves (1).

Let X_1 be the set of $x \in X(F) \cap V(M)$ such that $x \in M_e$ for some $e \in E(J)$ not incident with any vertex in α, and let $X_2 = X(F) \setminus X_1$. From the minimality of F, there are no edges between $V(F \setminus f_2)$ and X_1.

(2) $X_1 \neq \emptyset$.

Suppose that $X_1 = \emptyset$. Consequently the only edges $e \in E(J)$ with $X(F) \cap M_e \neq \emptyset$ are those with an end in α. Suppose that there are distinct e_1, e_2, both with an end in α, such that $X(F) \cap M_{e_i} \neq \emptyset$ for $i = 1, 2$. Let $e_i = s_i t_i$ where $t_i \in \alpha$ for $i = 1, 2$. Let D be a path of J with both ends in β, containing s_1 and s_2. Let D have end-edges t_1', t_2', where t_1', s_1, s_2, t_2' are in order on D. For $i = 1, 2$ let D_i be the subpath of D between t_i', s_i; and let D_3 be the subpath between s_1, s_2. For each $e \in E(D) \cup \{e_1, e_2\}$ choose an e-rung R_e, with $V(R_{e_i}) \cap X(F)$ nonempty for $i = 1, 2$. Let the ends of $R_{e_1}, R_{D_1}, R_{D_3}$ in M_{s_1} be p_1, p_2, p_3 respectively, and let the ends of $R_{e_2}, R_{D_2}, R_{D_3}$ in M_{s_2} be q_1, q_2, q_3 respectively. Now F is anticomplete to R_D, since $X_1 = \emptyset$. Since $X(F)$ meets both R_{e_1}, R_{e_2}, there is an induced path Q between p_1, q_1 with interior in $V(R_{e_1} \cup R_{e_2} \cup F)$. There is a near-prism in G with bases $\{p_1, p_2, p_3\}, \{q_1, q_2, q_3\}$ and constituent paths

$$p_2\text{-}R_{D_1}\text{-}b\text{-}R_{D_2}\text{-}q_2,$$

$$p_1\text{-}Q\text{-}q_1,$$

$$p_3\text{-}R_{D_3}\text{-}q_3,$$

a contradiction.

Consequently there is a unique $e \in E(J)$ such that $X(F) \cap M_e \neq \emptyset$, say $e = st$ where $t \in \alpha$. Since $X(F)$ does not satisfy the theorem, it follows that $X(F) \not\subseteq M_t$; let $x \in X(F) \setminus M_t$. Since $\{a, x\}$ is not local, we may assume that $f_1 a$ and $f_2 x$ are edges. But then we can add $V(F)$ to M_e and f_1 to M_t, contrary to the maximality of $V(M)$. This proves (2).

For each edge $e \in E(J)$, choose an e-rung R_e. The subgraph induced on $\bigcup_{e \in E(J)} V(R_e)$ is the line-graph $L(T)$ of a tree T, where T has shape J, and $E(T) = \bigcup_{e \in E(J)} V(R_e)$. In particular, $E(T) = V(R_J)$, and $V(J)$ is the set of branch-vertices of T. Let us call such a tree T a *realization* of M. If P is a subgraph of T, then $E(P)$ is a set of vertices of G, and we denote $G[E(P)]$ by $L(P)$ (it is indeed the line-graph of P).

(3) *For every realization T with $E(T) \cap X_1 \neq \emptyset$, there exists $d \in V(T)$ such that $X_1 \cap E(T)$ consists of all edges of T incident with d that belong to branches of T that do not have an end-edge in $N(a)$.*

Let P be a path of T with distinct ends, and both end-edges in $N(b)$, with $E(P) \cap X_1 \neq \emptyset$. By (1) there exists $d \in V(P)$ such that $X(F) \cap E(P)$ is the set of edges of P that are T-incident with d. We will show that d satisfies the claim. Let P_1, P_2 be the two subpaths of P between d and an end of P, and let x_1, x_2 respectively be the edges of P_1, P_2 that are T-incident with d. Suppose that $x_3 \in E(T) \cap X(F)$; we will show that x_3 is incident with d in T. We may assume that $x_3 \neq E(P)$. Let $e_3 \in E(J)$ with $x_3 \in M_{e_3}$. Since $x_3 \notin X_2$, there is a path of J with both ends in β containing e_3; and hence there is a path of T containing x_3 with both end-edges in $N(b)$. Choose a path P_3 of T containing x_3 with one end-edge in $N(b)$ and the other in $V(P)$, edge-disjoint from P. Let p be the end of P_3 in $V(P)$; and let P_1', P_2' be the two subpaths of P between p and the ends of P. If $p \neq d$, then d is an internal vertex of one of P_1', P_2', say P_1'; and $X(F)$ contains two nonconsecutive edges of the path $P_1' \cup P_3$, contrary to (1). So $p = d$. From (1) applied to the path $P_1 \cup P_3$, it follows that there is a unique edge of P_3 in $X(F)$, and it is T-incident with d. This proves that all edges of T in $X(F)$ are T-incident with d.

Next we show that every edge of T that is T-incident with d, and not in a branch of T with end-edge in $N(a)$, belongs to X_1. Let y be an edge of T that is T-incident with d, and let $y \in M_e$ say, with no end in α. We must show that $y \in X(F)$. To see this, choose a path P_3 of T containing y with one end-edge in $N(b)$ and one end d, edge-disjoint from P. From (1) applied to $P_1 \cup P_3$ it follows that $y \in X(F)$. This proves (3).

(4) *Let T, d be as in (3). Then there is a branch S of T with one end d and with an end-edge in $N(a)$, such that $X_2 \cap E(T) \subseteq E(S)$. In particular $d \in V(J)$, and so $X_1 \cap E(T) \subseteq M_d$.*

If $X_2 \cap E(T) = \emptyset$, we can assume there is no branch S of T with one end d and with an end-edge in $N(a)$ (for otherwise the claim holds); and then by (3), $X(F) \cap E(T)$ consists of all edges of T incident with d, and the subgraph of G induced on $E(T) \cup V(F) \cup \{a, b\}$ is an extended tree line-graph $H(T')$ with cross-edge ab, for some tree T' whose shape has more edges than J, contrary to the choice of J. Thus we may assume that $X_2 \cap E(T) \neq \emptyset$. Let $t \in \alpha$ with J-neighbour s, such that the branch, S say, of T with ends s, t contains an edge in $X(F)$. If $s = d$ for every such choice of t, then the claim holds (because there is at most one leaf of J in α J-adjacent to d). Thus we may assume that $s \neq d$. Let P be a path of T, including the subpath of T between s, d, and with both end-edges in $N(b)$. Now P is divided into three subpaths by s, d, namely from an end of P to s, from s to d, and from d to the other end of P. We call these P_1, P_2, P_3 respectively. Let d_1, d_2, d_3 be the edges of T incident with s that belong to $E(P_1), E(P_2), E(S)$ respectively. Thus exactly one of x_1, x_2 belongs to $E(P_2)$, say x_1. Since there are edges between $V(F)$ and $V(L(S))$, there is an induced path Q between d_3, f_2 with interior in $V(L(S) \cup F)$. Then there is a near-prism in G with bases $\{d_1, d_2, d_3\}$, $\{f_2, x_1, x_2\}$ and constituent paths

$$d_1\text{-}L(P_1)\text{-}b\text{-}L(P_3)\text{-}x_2,$$

$$d_3\text{-}Q\text{-}f_2,$$

$$d_2\text{-}L(P_2)\text{-}x_1,$$

contrary to 2.1. This proves (4).

(5) *Let T, d, S be as in (4), and let S have ends s, d say; then $X_2 \subseteq M_{sd}$.*

For each $e \in E(J)$, let R_e be the e-rung used to define T. If some vertex $x \in X_2$ belongs to M_e say where $e \in E(J)$, then e has an end in α from the definition of X_2, and if $e \neq sd$, we could replace R_e with an e-rung that contains x, to obtain a realization that violates (4). This proves (5).

(6) *Let T, d, S be as in (4), and let S have ends s, d say; then $X_1 = M_d \setminus M_{sd}$.*

There are at least two edges e_1, e_2 of J, J-incident with d and with no end in α; let x_1, x_2 be the edges of the corresponding branches of T that are T-incident with d. We show first that $X_1 \subseteq M_d \setminus M_{sd}$. Let $x \in X_1$, and let $x \in M_e$ where $e \in E(J)$. Let R'_e be an e-rung containing x. Let T' be the realization of M obtained by replacing R_e by R'_e, and otherwise using all the same rungs. Since $e_1 \neq e_2$ we may assume that $e \neq e_2$; and so $x_2, x \in V(T')$. Hence by (4) applied to T', e_2, e have a common end $d' \in V(J)$, and $x_2, x \in M_{d'}$. Also either $e = e_1$ or $x_1 \in E(T')$; and so in either case e_1 is incident with d'. Consequently d' is the common end of e_1, e_2 in J, and so $d' = d$. This proves that $x \in M_d$, and so $X_1 \subseteq M_d \setminus M_{sd}$.

Next we show that $M_d \setminus M_{sd} \subseteq X_1$. To see this, let $y \in M_d \setminus M_{sd}$. Let $e \in E(J)$ with $y \in M_e$; since $y \notin M_{sd}$ it follows that e has no end in α. Let R'_e be an e-rung containing y. Since $y \in M_d$ it follows that e is J-incident with d. Let T' be the realization obtained by replacing R_e by R'_e. Since $e_1 \neq e_2$ we may assume that $e \neq e_2$. Since e has no end in α, there is a path P' of T' with $x_2, y \in E(P')$ and with both end-edges in $N(b)$; and so X_1 contains either zero or two consecutive edges in this path, by (1). Not zero, since $x_2 \in E(P')$; so a unique vertex of R'_e belongs to X_1, and that vertex is in M_d. Since y is the only vertex of R'_e in M_d, it follows that $y \in X_1$. This proves (6).

From (5) and (6) we can add f_2 to M_d and add f_1 to M_s, and add $V(F)$ to M_{sd}, contrary to the maximality of $V(M)$. This proves 4.4. ∎

4.5 *Under the hypotheses of 4.2, if F is small, and $a, b \in N(F)$, then F is peripheral.*

Proof. We claim first:

(1) $X(F) \subseteq N[\{a, b\}]$.

Suppose $x \in V(M)$ has a neighbour in $V(F)$, and $x \notin N(\{a, b\})$. Choose a minimal path P of F such that x and at least one of a, b has a neighbour in $V(P)$. Thus P has one end adjacent to x and the other to a, say. But a, b have no common neighbour in $V(F)$, since $V(F) \cap Z = \emptyset$; and so from the minimality of P, b has no neighbour in $V(P)$. But then P violates 4.4. This proves (1).

(2) *Either $X(F) \subseteq N[a]$ or $X(F) \subseteq N[b]$.*

Suppose not; then there is a vertex $c \in V(M) \cap N[a]$ and $d \in V(M) \cap N[b]$, joined by a path P with interior in $V(F)$. Choose c, d and P such that P has minimum length. Choose $u \in \alpha$ and $v \in \beta$ with $c \in M_u$ and $d \in M_v$, and let D be a path of J with ends u, v. Let p, q be the neighbours in

P of c,d respectively. Let $c_1,\dots,c_k$ be the vertices of $N(a)\cap V(P)$ in order on P, with $c_1=c$. Note that $c_1,\dots,c_k$ are not adjacent to b since $V(P)\cap Z=\emptyset$. For $1\le i<k$, let P_i be the subpath of P between c_i and c_{i+1}. Since a-c_i-P_i-c_{i+1}-a is a hole, and b is adjacent to a and not to c_i,c_{i+1}, it follows that b has an even number of neighbours in P_i. Choose $u'\in\alpha\setminus\{u\}$ and $c'\in M_{u'}$. By 4.3 and 4.4, p has no neighbour in $M_{u'}$, since p has a neighbour in M_u and $X(p)$ is local; and by the minimality of P, no vertex of P different from p has a neighbour in $M_{u'}$. In particular c_k,c' are nonadjacent. Let S' be an induced path of G between c',d with interior in $V(M)\setminus N[\{a,b\}]$. Then a-c'-S'-d-P-c_k-a is a hole (note that c_k is not adjacent to c'), and b has at least two nonadjacent neighbours in it (a and d), and so it has an odd number; and therefore b has an even number of neighbours in the subpath of P between c_k,d. Hence b has an even number of neighbours in $V(P)$ altogether. Also, d-P-c-R_D-d is a hole, and b has an even number of neighbours in it, at least one; and it has exactly two and they are adjacent. Consequently b is adjacent to q and has no other neighbours in $V(P)$ except d. Similarly a is adjacent to c,p and has no other neighbours in $V(P)$. But then the subgraph induced on $V(R_D)\cup V(P)\cup\{a,b\}$ is a prism, a contradiction. This proves (2).

From (2) we may assume that $X(F)\subseteq N[a]$. Suppose that there exist distinct $u,u'\in\alpha$ such that $X(F)\cap M_u, X(F)\cap M_{u'}\neq\emptyset$. Choose $c\in X(F)\cap M_u$ and $c'\in X(F)\cap M_{u'}$, such that there is an induced path P between c,c' with interior in $V(F)$. Both ends of P are adjacent to a; let the neighbours of a in P be $c_1,\dots,c_k$ in order on P, where $c_1=c$ and $c_k=c'$. For $1\le i<k$, let P_i be the subpath of P between c_i,c_{i+1}. For $1\le i<k$, a-c_i-P_i-c_{i+1}-a is a hole, and since b is adjacent to a and not to c_i,c_{i+1}, b has an odd number of neighbours in this hole. Hence it has an even number in P_i for each i, and so an even number in P altogether. Let D be the path of J with ends u,u', and choose an internal vertex $d\in V(D)$. Let D_1 be the subpath of D with ends d,u, and let D_2 be the subpath with ends d,u'. Let D_3 be a path of J between d,v where $v\in\beta$. For each edge g of $D_1\cup D_2\cup D_3$, choose a g-rung R_g, with $c\in V(R_e)$ and $c'\in V(R_{e'})$. For $i=1,2,3$ let d_i be the end of R_{D_i} in M_d. Then

$$c\text{-}R_{D_1}\text{-}d_1\text{-}d_2\text{-}R_{D_2}\text{-}c'\text{-}P\text{-}c$$

is a hole, and b has an even number of neighbours in it; so it has zero, or exactly two adjacent neighbours. Zero is impossible since then 4.3 and 4.4 would imply that $X(P)$ is local. Thus b has exactly two neighbours x,y in $V(P)$, and they are adjacent. Since $x\notin Z$ it follows that c,x,y,c' are all distinct. Let c,x,y,d be in order in P. Then the subgraph induced on $V(R_{D_1}\cup R_{D_2}\cup R_{D_3}\cup P)$ is a prism, with bases $\{b,x,y\},\{d_1,d_2,d_3\}$, and constituent paths

$$d_1\text{-}R_{D_1}\text{-}c\text{-}P\text{-}x,$$

$$d_2\text{-}R_{D_2}\text{-}c'\text{-}P\text{-}y,$$

$$d_3\text{-}R_{D_3}\text{-}b,$$

a contradiction. This proves 4.5. ∎

From 4.3, 4.4 and 4.5, this proves 4.2.

5 Triangles through the cross-edge

Next we prove some results about the set called Z in 4.2. We need the following lemma.

5.1 *Let G be even-hole-free, and let H be a hole of G, with vertices h_1-h_2-$\cdots$-h_n-h_1 in order. Let $a, b \in V(G) \setminus V(H)$ each have at least three neighbours in $V(H)$, and let $\{a, b\}$ be complete to $\{h_1, h_n\}$. If a, b are nonadjacent, then one of a, b is adjacent to h_{n-1}, h_n, h_1 and to no other vertices in $V(H)$, and the other is adjacent to h_n, h_1, h_2 and to no other vertices in $V(H)$.*

Proof. Let P be the path h_2-h_3-$\cdots$-h_{n-1}, and let A, B be the sets of neighbours of a, b respectively in $V(P)$. Since G has no 4-hole, it follows that $A \cap B = \emptyset$. An (A, B)-*gap* means a subpath of P with one end in A and the other in B, and with no internal vertices in $A \cup B$. If there is an (A, B)-gap containing both h_{n-1}, h_2 then the theorem holds, and so we may assume not; and hence every (A, B)-gap is anticomplete to one of h_n, h_1, and therefore has odd length (because it can be completed to a hole by adding a, b and one of h_n, h_1). It follows that no two (A, B)-gaps are anticomplete; because their union with $\{a, b\}$ would induce an even hole.

There is an (A, B)-gap, since a, b each have at least three neighbours in $V(H)$. Choose an (A, B)-gap h_i-$\cdots$-h_j with $i < j$ and i minimum, and we may assume that $h_i \in A$. Hence b is nonadjacent to $h_2, \ldots, h_{j-1}$, and so b-h_1-$\cdots$-h_j-b is a hole, and therefore j is even. Moreover, $j - i$ is odd, since h_i-$\cdots$-h_j is an (A, B)-gap; and since n is odd, it follows that $n - i = n + (j - i) - j$ is even. Consequently a-h_i-$\cdots$-h_n-a is not a hole, and so there exists $k \in \{j + 1, \ldots, n - 1\}$ minimum such that $h_k \in A$. If $B \cap \{h_i, \ldots, h_k\} = \{h_j\}$, there is a theta with ends b, h_j induced on $\{a, b, h_i, \ldots, h_k\}$, a contradiction. Thus one of $h_{j+1}, \ldots, h_{k-1}$ is in B, and since no two (A, B)-gaps are anticomplete, it follows that $h_{j+1} \in B$ and $h_{j+2}, \ldots, h_k \notin B$. Since no two (A, B)-gaps are anticomplete, b has no more neighbours in $V(P)$; but then it is the centre of a 4-wheel with hole H, a contradiction. This proves 5.1. ∎

Let G be even-hole-free, let $ab \in E(G)$, and let (J, M) be optimal for ab. Let Z be the set of common neighbours of a, b in G. It would be helpful if Z were a clique, but unfortunately this is not true, even assuming that a is splendid. It *is* true if both a, b are splendid, but that assumption is too strong for our application (to find a bisimplicial vertex, later). But here is something on those lines, good enough for the application and true without any additional hypothesis. Let us say that a vertex $y \in Z$ is *a-external* if there is a path from y to $V(M) \setminus N[a]$ containing no neighbours of a except y, and we define *b-external* similarly. Let us say a vertex y is *major* if $y \in Z$, and y is both a-external and b-external. For convenience we write $N[a, b]$ for $N[\{a, b\}]$.

5.2 *Let ab be an edge of an even-hole-free graph G, and let (J, M) be optimal for ab. Then the set of all major vertices is a clique.*

Proof. Let Z be the set of common neighbours of a, b in G, and Y the set of major vertices (thus $Y \subseteq Z$).

(1) *If $y \in Y$, then either y is complete to one of $N(a) \cap V(M), N(b) \cap V(M)$, or there is a path from y to $V(M) \setminus N[a, b]$ containing no neighbours of a or b except y.*

We may assume that $X(y) \subseteq N[a, b]$, for otherwise a path of length one satisfies the claim. Since y is b-external, there is a minimal path P with one end y, containing no neighbour of b except y, such that its other end (p say) has a neighbour in $V(M) \setminus N[b]$. It follows that $V(P) \cap V(M) = \emptyset$. Similarly, there is a minimal path Q between y and q say, containing no neighbour of a except y, where $X(q) \not\subseteq N[a]$. Thus a might have neighbours in $V(P \setminus y)$, and b might have neighbours in $V(Q \setminus y)$.

If $X(p) \not\subseteq N[a, b]$, then by 4.2, a has no neighbour in $V(P \setminus y)$ and the claim holds. Thus we may assume that $X(p) \subseteq N[a, b]$, and $X(p) \not\subseteq N[b]$ from the definition of P. We claim that $X(p) \subseteq N[a]$. Suppose not; then p has a neighbour in $V(M) \cap N[a]$ and one in $V(M) \cap N[b]$. By 4.2, p is adjacent to both a, b, and so $p = y$. Choose $t_1 \in \alpha$ such that $X(p) \cap M_{t_1} \neq \emptyset$, and $t_2 \in \beta$ such that $X(p) \cap M_{t_2} \neq \emptyset$, and let D be a path of J with ends t_1, t_2. For $i = 1, 2$ let $e_i \in E(J)$ be incident with t_i. For each $e \in E(D)$, choose an e-rung R_e, such that R_{e_1} contains a vertex in $X(p) \cap N[a]$ and R_{e_2} contains a vertex in $X(p) \cap N[b]$. Then p has exactly four neighbours in the hole a-R_D-b-a, since $X(y) \subseteq N[a] \cup N[b]$, and so G contains a 4-wheel, a contradiction. This proves that $X(p) \subseteq N[a]$. Similarly $X(q) \subseteq N[b]$.

Let $t_1 \in \alpha$ and $t_2 \in \beta$, such that $X(p) \cap M_{t_1} \neq \emptyset$, and $X(q) \cap M_{t_2} \neq \emptyset$. For $i = 1, 2$ let $e_i \in E(J)$ be J-incident with t_i. Choose $v_1 \in X(p) \cap M_{t_1}$ and $v_2 \in X(q) \cap M_{t_2}$. Let D be a path of J with ends t_1, t_2, and for each $e \in E(D)$ let R_e be an e-rung, with $v_i \in V(R_{e_i})$ for $i = 1, 2$. Then R_D is an induced path with ends v_1, v_2, and with interior anticomplete to a, b and to $V(P \cup Q)$.

By 4.2, there is no path between v_1, v_2, with interior disjoint from $V(M) \cup Z$, and so $V(P \setminus y) \cup \{v_1\}$ is disjoint from and anticomplete to $V(Q \setminus y) \cup \{v_2\}$. Consequently v_1-P-y-Q-v_2 is an induced path. Now as we saw above, $p \neq q$ and so at least one of P, Q has length at least one, say Q. Thus b has two nonadjacent neighbours in the hole

$$v_2\text{-}q\text{-}Q\text{-}y\text{-}P\text{-}p\text{-}v_1\text{-}R_D\text{-}v_2,$$

and so has an odd number, at least three. They all belong to the path v_2-q-Q-y. We may assume that y is not complete to $V(M) \cap N[a]$, so there exists $e_3 = s_3t_3$ where $t_3 \in \alpha$ and an e_3-rung R_{e_3} such that y has no neighbour in $V(R_{e_3})$ (because $X(y) \subseteq N[a, b]$). Let D be a path of J with ends t_2, t_3, and for each $e \in E(D)$ let R_e be an e-rung, with $v_i \in V(R_{e_i})$ for $i = 2, 3$. Then the hole

$$v_2\text{-}q\text{-}Q\text{-}y\text{-}a\text{-}v_3\text{-}R_D\text{-}v_2$$

contains exactly one neighbour of b in addition to those in v_2-q-Q-y, and so contains an even number, a contradiction. This proves (1).

For each $y \in Y$, let P_y be some minimal path of G between y and its other end (say p_y) such that a, b have no neighbours in $V(P_y \setminus y)$ and $X(p_y) \not\subseteq N[a, b]$, if there is such a path. If not, let P_y be the one-vertex path with vertex y, and let $p_y = y$. From the minimality of P_y, $X(P_y \setminus p_y) \subseteq N[a, b]$. (Note that there are two cases when $p_y = y$, the two extremes: when we don't need the path P_y, because y itself has a neighbour in $V(M) \setminus N[a, b]$; and when we can't find the path P_y, and therefore y is complete to one of $N[a] \cap V(M), N[b] \cap V(M)$ by (1).)

(2) *Let $t_1 \in \alpha$ and $t_2 \in \beta$, and let D be a path of J with ends t_1, t_2. Let $y \in Y$. If*

$$X(P_y) \setminus N[a, b] \not\subseteq \bigcup_{e \in E(D)} M_e,$$

there is a vertex d of D and a path Q of G with the following properties:

- *d is an internal vertex of D, incident with edges g_1, g_2 of D say;*

- *Q has ends y, d_3, where $d_3 \in M_d \setminus (M_{g_1} \cup M_{g_2})$;*
- *$V(Q) \subseteq V(P_y) \cup V(M)$; and*
- *Q^* is anticomplete to $\bigcup_{e \in E(D)} M_e$, and $V(Q \setminus y)$ is anticomplete to $\{a, b\}$.*

Since $X(P_y) \setminus N[a,b] \not\subseteq \bigcup_{e\in E(D)} M_e$, there exists $e_3 \in E(J) \setminus E(D)$ such that $X(P_y) \setminus N[a,b]$ meets M_{e_3}. Let C be a path of J, containing e_3 and edge-disjoint from D and with one end in $V(D)$; and choose e_3, C with C minimal. Let d be the end of C in $V(D)$. Choose an e-rung R_e for each $e \in E(C)$, choosing R_{e_3} to contain a vertex of $X(P_y) \setminus N[a,b]$. Then R_C is an induced path containing a vertex in $X(p_y) \setminus N[a,b]$, with ends c, d_3 say, and $d_3 \in M_d \setminus (M_{g_1} \cup M_{g_2})$, where g_1, g_2 are the two edges of D incident with d. Thus $R_C \setminus d_3$ is anticomplete to $\bigcup_{e\in E(D)} M_d$. Moreover no vertex of R_C belongs to $N[a,b]$ except possibly c, and in that case p_y has a neighbour in R_C different from c. Choose a minimal subpath S of R_C that has one end d_3 and the other adjacent to p_y. Then no vertex of S is adjacent to a or b, and so setting Q to be the path y-P_y-p_y-S-d_3 satisfies the claim. This proves (2).

(3) *Let $t_1 \in \alpha$ and $t_2 \in \beta$, and let D be a path of J with ends t_1, t_2. For each $e \in E(D)$ let R_e be an e-rung. For each $y \in Y$, either $X(P_y) \cap V(R_D) \neq \emptyset$, or*

$$\emptyset \neq X(P_y) \setminus N[a,b] \subseteq \bigcup_{d\in E(D)} M_d.$$

In either case, $X(P_y) \cap \bigcup_{d\in E(D)} M_d$ is nonempty.

If $X(P_y) \cap V(R_D) \neq \emptyset$ then the claim holds, so we may assume that $X(P_y) \cap V(R_D) = \emptyset$. Consequently y is not complete to either of $N[a] \cap V(M), N[b] \cap V(M)$, and so by (1), $X(P_y) \not\subseteq N[a,b]$. Suppose that

$$X(P_y) \setminus N[a,b] \not\subseteq \bigcup_{e\in E(D)} M_e.$$

Choose d, Q as in (2), and for $i = 1,2$ let D_i be the subpath of D between d and t_i. Let Q have ends y, d_3. Thus d_3 has two adjacent neighbours d_1, d_2 in R_D, where $d_i \in R_{D_i}$ for $i = 1,2$. But then there is a near-prism with bases $\{d_1, d_2, d_3\}$ and $\{a, b, y\}$, with constituent paths

$$a\text{-}R_{D_1}\text{-}d_1,$$

$$b\text{-}R_{D_2}\text{-}d_2,$$

$$y\text{-}Q\text{-}d_3,$$

a contradiction. This proves (3).

Choose distinct $a_1, a_2 \in \alpha$ and $b_1, b_2 \in \beta$ such that the paths D_1, D_2 are vertex-disjoint, where for $i = 1,2$, D_i is the path of J with ends a_i, b_i. (This is possible since J has at least two vertices that are not leaves, by hypothesis.) For $i = 1,2$, let $W_i = \bigcup_{e\in E(D_i)} M_e$. We observe that W_i is connected, because D_i has an internal vertex d, and $M_d \cap W_i$ is connected, and every other vertex of W_i can be joined to $M_d \cap W_i$ by a union of rungs.

Suppose that $y_1, y_2 \in Y$ are nonadjacent. For $i = 1,2$, let us say an induced path T of G between y_1, y_2 is *i-normal* if for every $v \in V(T^*) \setminus W_i$, there exists $j \in \{1,2\}$ such that $v \in V(P_{y_j} \setminus y_j)$ and

$X(P_{y_j} \setminus y_j) \cap W_i$ is nonempty.

(4) *For $i = 1, 2$, there is an i-normal path.*

Let $i \in \{1, 2\}$. For each $j \in \{1, 2\}$, (3) implies that $X(P_{y_j}) \cap W_i \neq \emptyset$; and so either y_j has a neighbour in W_i, or $X(P_{y_j} \setminus y_j) \cap W_i \neq \emptyset$. Hence there is a path S_j between y_j and a vertex of W_i, such that for every $v \in V(S_j)$, either

- $v \in \{y_j\} \cup W_i$; or
- $v \in X(P_{y_j} \setminus y_j)$, and $X(P_{y_j} \setminus y_j) \cap W_i \neq \emptyset$.

Since W_i is connected, it follows that there is an induced path joining y_1, y_2 with interior in $V(S_1 \cup S_2) \cup W_i$; and this is therefore i-normal. This proves (4).

(5) *For $i = 1, 2$, let T_i be an i-normal path. Then T_1^* is anticomplete to T_2^*.*

Suppose not. Since W_1 is anticomplete to W_2, we may assume (exchanging D_1, D_2 or y_1, y_2 if necessary) that there exist $v_1 \in V(P_{y_1} \setminus y_1) \cap T_1^*$, and $v_2 \in T_2^*$, such that v_1, v_2 are equal or adjacent. Hence $X(P_{y_1} \setminus y_1) \cap W_1 \neq \emptyset$. By 4.2, $X(P_{y_1} \setminus y_1)$ is local, and consequently $X(P_{y_1} \setminus y_1)$ is disjoint from W_2; and in particular $v_2 \notin W_2$, and so $v_2 \in V(P_{y_2} \setminus y_2) \cap T_2^*$. By the same argument with T_1, T_2 exchanged, $X(P_{y_2} \setminus y_2)$ meets W_2. But $Q = (P_{y_1} \setminus y_1) \cup (P_{y_2} \setminus y_2)$ is connected and $X(Q)$ meets both W_1 and W_2, and so is not local, contrary to 4.2. This proves (5).

From (4), for $i = 1, 2$ there is an i-normal path T_i. By (5), $T_1 \cup T_2$ is a hole, and so one of T_1, T_2 is odd and one is even; say T_1 is odd and T_2 is even. For every 1-normal path T_1', $T_1' \cup T_2$ is a hole, and so T_1' is odd, and similarly every 2-normal path is even.

(6) *Every 2-normal path meets both M_{a_2}, M_{b_2}. In particular, if $X(P_{y_2} \setminus y_2)$ meets W_2 then y_1 has no neighbour in $V(P_{y_2})$, and if $X(P_{y_1} \setminus y_1)$ meets W_2 then y_2 has no neighbour in $V(P_{y_1})$.*

Let T_2 be 2-normal. Since T_2 is even, y_1-T_2-y_2-a-y_1 is not a hole; and so a has a neighbour in T_2^*, and similarly so does b. But a has no neighbours in P_{y_1}, P_{y_2} different from y_1, y_2, and the set of neighbours of a in W_2 is M_{a_2}. Hence T_2^* meets M_{a_2}, and similarly it meets M_{b_2}. This proves the first claim. For the second, observe that if $X(P_{y_2} \setminus y_2)$ meets W_2 and y_1 has a neighbour in $V(P_{y_2})$ then there is a 2-normal path with interior in $V(P_{y_2})$ and therefore not meeting both (or indeed, either of) M_{a_2}, M_{b_2}, a contradiction. This proves (6).

For each $e \in E(D_1)$ choose an e-rung R_e.

(7) *One of y_1, y_2 has no neighbour in R_{D_1}.*

Suppose that y_1, y_2 both have a neighbour in $V(R_{D_1})$. By 5.1, since y_1, y_2 are nonadjacent, it follows that one of y_1, y_2 is adjacent to a_1, and the other to b_1, and neither has any more neighbours in $V(R_{D_1})$. Since R_{D_1} is even, adding y_1, y_2 to R_{D_1} gives a 1-normal path of even length, a contradiction. This proves (7).

Henceforth we assume that y_1 has no neighbour in R_{D_1}.

(8) *$X(P_{y_1} \setminus y_1) \cap W_2 = \emptyset$; $X(y_1) \cap W_2 \subseteq N[a,b]$; and $X(P_{y_1}) \setminus N[a,b] \not\subseteq W_2$.*

Since y_1 has no neighbour in R_{D_1}, it follows that y_1 is not complete to either of $N[a] \cap V(M), N[b] \cap V(M)$, and so by (1), $X(P_{y_1}) \not\subseteq N[a,b]$. Suppose that $X(P_{y_1} \setminus y_1) \cap W_2 \neq \emptyset$. Consequently P_{y_1} has length at least one, and so $X(y_1) \subseteq N[a,b]$. Moreover, $X(P_{y_1} \setminus y_1) \cap W_1 = \emptyset$, since $X(P_{y_1} \setminus y_1)$ is local. Since y_1 has no neighbour in R_{D_1}, it follows that $X(P_{y_1}) \cap V(R_{D_1}) = \emptyset$. By (3), $X(P_{y_1}) \setminus N[a,b] \subseteq W_1$. But $X(P_{y_1}) \setminus N[a,b] \subseteq X(P_{y_1} \setminus y_1)$, since $X(y_1) \subseteq N[a,b]$; and so

$$X(P_{y_1}) \setminus N[a,b] \subseteq X(P_{y_1} \setminus y_1) \cap W_1 = \emptyset,$$

a contradiction. This proves the first claim. For the second, suppose that y_1 has a neighbour in $W_2 \setminus N[a,b]$. From the minimality of P_{y_1}, $p_{y_1} = y_1$. Consequently $X(P_{y_1}) \cap V(R_{D_1}) = \emptyset$, and so by (3), $X(P_{y_1}) \setminus N[a,b] \subseteq W_1$, contradicting that y_1 has a neighbour in $W_2 \setminus N[a,b]$. This proves the second claim. The third claim follows, since we have shown that $X(P_{y_1}) \setminus N[a,b] \neq \emptyset$ and is disjoint from W_2. This proves (8).

For each $e \in E(D_2)$, choose an e-rung R_e, such that $X(P_{y_2})$ meets R_{D_2} (this is possible by (3)). Since $X(P_{y_1}) \setminus N[a,b] \not\subseteq W_2$ by (8), it follows from (3) that $X(P_{y_1})$ meets R_{D_2}; and since $X(P_{y_1} \setminus y_1) \cap W_2 = \emptyset$ by (8), it follows that y_1 has a neighbour in R_{D_2}. Thus there is a 2-normal path T_2 meeting W_2 in a subpath of R_{D_2}. By (6), both ends of R_{D_2} belong to T_2. Consequently a unique vertex of R_{D_2}, one of its ends, is adjacent to y_1, and a unique vertex of R_{D_2}, its other end, belongs to $X(P_{y_2})$. Let R_{D_2} have ends s,t where $s \in M_{a_2}$ and $t \in M_{b_2}$. Exchanging a,b if necessary, we may assume that y_1 is adjacent to s and to no other vertex of R_{D_2}, and $X(P_{y_2}) \cap V(R_{D_2}) = \{t\}$. Choose a minimal subpath P_2 of P_{y_2} with ends y_2, p_2 say, such that p_2 is adjacent to t. (Possibly $p_2 = y_2$.)

By (8), $X(P_{y_1}) \setminus N[a,b] \not\subseteq W_2$. By (2) there is a vertex d of D_2 and a path Q of G with the following properties:

- d is an internal vertex of D_2, incident with edges g_1, g_2 of D_2 say;
- Q has ends y_1, d_3, where $d_3 \in M_d \setminus (M_{g_1} \cup M_{g_2})$;
- $V(Q) \subseteq V(P_{y_1}) \cup V(M)$; and
- Q^* is anticomplete to W_2, and $V(Q \setminus y_1)$ is anticomplete to $\{a,b\}$.

In particular, d_3 has exactly two neighbours in $V(R_{D_2})$, say d_1, d_2 where s, d_1, d_2, t are in order in R_{D_2}, and d_1, d_2 are adjacent.

(9) *y_2 is nonadjacent to t, and $V(P_2 \setminus y_2)$ is not anticomplete to $V(Q)$, and y_1 has no neighbour in $V(P_{y_2})$.*

Suppose first that $V(P_2)$ is anticomplete to $V(Q)$. Then there is a near-prism with bases $\{d_1, d_2, d_3\}, \{a, s, y_1\}$ and constituent paths

$$y_1\text{-}Q\text{-}d_3,$$

$$a\text{-}y_2\text{-}P_2\text{-}p_2\text{-}t\text{-}R_{D_2}\text{-}d_2,$$

$$s\text{-}R_{D_2}\text{-}d_1,$$

contrary to 2.1.

Thus $V(P_2)$ is not anticomplete to $V(Q)$. Suppose next that $V(P_2 \setminus y_2)$ is anticomplete to $V(Q)$, and therefore y_2 has a neighbour in $V(Q)$. Let Q' be a path with ends y_2, d_3, where $Q' \setminus y_2$ is a subpath of Q. It follows that y_1 has no neighbour in $V(Q')$; for y_1 only has one neighbour in $V(Q)$, and that vertex is not adjacent to y_2 since otherwise there would be a 4-hole. If y_2 is adjacent to t, then there is a near-prism with bases $\{y_2, b, t\}, \{d_1, d_2, d_3\}$ and constituent paths

$$y_2\text{-}Q'\text{-}d_3,$$

$$b\text{-}y_1\text{-}s\text{-}R_{D_2}\text{-}d_1,$$

$$t\text{-}R_{D_2}\text{-}d_2.$$

If y_2 is not adjacent to t, there is a theta in G with ends y_2, t and constituent paths

$$y_2\text{-}P_2\text{-}p_2\text{-}t,$$

$$y_2\text{-}b\text{-}t,$$

$$y_2\text{-}Q'\text{-}d_3\text{-}d_2\text{-}R_{D_2}\text{-}t,$$

contrary to 2.1. This proves that $P_2 \setminus y_2$ is not anticomplete to $V(Q)$. In particular, P_2 has length at least one, and so y_2, t are nonadjacent. Hence $t \in X(P_2 \setminus y_2)$, and so by (6), y_1 has no neighbour in $V(P_{y_2})$. This proves (9).

By (9), we may choose $v_1 \in V(Q)$ and $v_2 \in V(P_2 \setminus y_2)$, such that v_1, v_2 are equal or adjacent. Since y_1 has no neighbour in $V(P_2)$ by (9), it follows that either $v_1 \in V(P_{y_1} \setminus y_1)$ or $v_1 \in V(M)$. Suppose that $v_1 \in V(P_{y_1} \setminus y_1)$. Then $F = G[V(P_{y_1} \setminus y_1) \cup V(P_{y_2} \setminus y_2)]$ is connected, and disjoint from $V(M) \cup Z$, and $X(F)$ includes both $X(P_{y_1} \setminus y_1)$ and $X(P_{y_2} \setminus y_2)$. But since P_{y_1} has length at least one, it follows that $X(P_{y_1} \setminus y_1) \setminus N[a, b]$ is nonempty, and is a subset of W_1 by (3). Hence $X(F)$ meets W_1, and contains t, and so $X(F)$ is not local, contrary to 4.2.

Thus $v_1 \in V(M)$. Since $V(Q)$ is anticomplete to $\{a, b\}$, it follows that $v_1 \in V(M) \setminus N[a, b]$. From the minimality of P_{y_2}, no vertex of P_{y_2} except y_2 has a neighbour in $V(M) \setminus N[a, b]$, and so $v_2 = p_{y_2}$, and in particular $P_2 = P_{y_2}$. But $X(P_{y_2} \setminus y_2)$ is local, and contains t and v_1. Since $t \in M_{t_2}$, and Q^* is anticomplete to W_2, it follows that $v_1, t \in M_{s_2}$, and hence $v_1 = d_3$ and $t = d_2$. Morover, $V(Q)$ is disjoint from $V(P_2 \setminus y_2)$, and the edge $p_{y_2}\text{-}d_3$ is the only edge joining them. (But y_2 might have neighbours in $V(Q)$.) Now y_2 is nonadjacent to d_3, since otherwise $y_2\text{-}d_3\text{-}d_2\text{-}b\text{-}y_2$ is a 4-hole. Then

$$b\text{-}y_1\text{-}s\text{-}R_{D_2}\text{-}d_1\text{-}d_3\text{-}p_{y_2}\text{-}P_{y_2}\text{-}y_2\text{-}b$$

is a hole, and $d_2 = t$ has exactly four neighbours in it, namley d_1, d_3, p_{y_2} and b, a contradiction. This proves 5.2. ∎

Finally, we have:

5.3 *Let ab be an edge of an even-hole-free graph G, and let (J, M) be optimal for ab. Let Z be the set of all common neighbours of a, b, and let $Y \subseteq Z$ be the set of all major vertices. If F is a component of $G \setminus (V(M) \cup Z)$, and some vertex in $Z \setminus Y$ has a neighbour in F, then there is a leaf t of $V(J)$, such that every vertex in $V(M)$ with a neighbour in $V(F)$ belongs to M_t.*

Proof. Let $z \in Z \setminus Y$ have a neighbour in $V(F)$. Let $X(F)$ be the set of vertices in $V(M)$ with a neighbour in $V(F)$. If one of a, b has a neighbour in $V(F)$, the claim follows from 4.2, so suppose not. If $X(F) \not\subseteq N[a, b]$ has a neighbour in $V(F)$, this contradicts that z is not major. So $X(F) \subseteq N[a, b]$, and then the claim follows since $X(F)$ is local by 4.2. This proves 5.3. ∎

Let us summarize the previous results. The vertices of G are partitioned into the following sets:

- The special vertices a, b.
- $V(M)$ (this is further partitioned into strips corresponding to the edges of J).
- The small components. Each small component F satisfies $X(F) \subseteq M_e$ for some $e \in E(J)$ or $X(F) \subseteq M(t)$ for some $t \in V(J)$. Moreover if $N(F)$ contains a or b, or a vertex in $Z \setminus Y$, then F must be peripheral, and if $N(F)$ contains only one of a, b, then $X(F) \subseteq N[a]$ or $N[b]$ correspondingly.
- The set Y of the major vertices. These form a clique, but we know nothing about their neighbours outside of Z.
- The vertices in $Z \setminus Y$. All their neighbours in $V(M)$ are in $N[a, b]$, and all their neighbours in small components belong to peripheral small components.

If we assume that a is splendid (which will be true in our application), we can simplify the theorem a little; let us see that next. We need:

5.4 *Let ab be an edge of an even-hole-free graph G, and let (J, M) be optimal for ab. If a is splendid, there is no small F such that a has a neighbour in $V(F)$.*

Proof. Let Z be the set of vertices of G adjacent to both a, b. Suppose that there is such an subgraph F, and we may assume that F is small component. If b has no neighbour in $V(F)$, then since by 4.2 every vertex in $V(M)$ with a neighbour in $V(F)$ belongs to $N[a]$, it follows that F is a component of $G \setminus N[a]$, contradicting that a is splendid. Thus b has a neighbour in $V(F)$. For the same reason, some vertex of $V(M)$ nonadjacent to a has a neighbour in $V(F)$; but by 4.2, every such vertex belongs to B.

Hence there is an induced path P of F such that a has a neighbour in $V(P)$, and some vertex in B has a neighbour in $V(P)$. Let P be minimal with this property. Let P have ends p_1, p_2, where a is adjacent to p_1 and to no other vertex of $V(P)$, and some vertex in B (v_2 say) is adjacent to p_2, and no vertex in B has a neighbour in $V(P \setminus p_2)$. Since p_1 is nonadjacent to b (because $p_1 \notin Z$) and there is no 4-hole, it follows that p_1 is anticomplete to B, and in particular $p_1 \neq p_2$. Let $v_2 \in M_{e_2}$, where $e_2 \in \beta$. From 4.2, there is at most one $e \in \alpha$ such that M_e is not anticomplete to $V(P \setminus p_2)$, and so there exists $d_1 \in \alpha$ such that M_{d_1} is anticomplete to $V(P \setminus p_2)$. Since M_{d_1} is anticomplete to p_2 by 4.2, it follows that M_{d_1} is anticomplete to $V(P)$.

There is a path D of J with end-edges d_1, e_2. Let R_e be an e-rung for each $e \in E(J)$, with $v_2 \in V(R_{e_2})$; then R_D is an induced path of G between a, v_2, with interior in $V(M)$ and anticomplete to $V(P)$. Hence $P \cup R_D$ is a hole, and b has two nonadjacent neighbour in $P \cup R_D$, namely v_2, a; and since G has no full star cutset, 3.2 applied to b and $P \cup R_D$ implies that b is adjacent to p_2 and has no other neighbour in $V(P)$. But then there is a short pyramid with apex a and base $\{b, v_2, p_2\}$, and constituent paths

$$a\text{-}b,$$

$$a\text{-}f_1\text{-}F\text{-}f_2,$$

$$a\text{-}R_D\text{-}v_2,$$

contradicting that a is splendid. This proves 5.4. ∎

We deduce an upgraded version of 4.2:

5.5 *Let G be even-hole-free, and ab be an edge of G, where a is splendid. Let (J, M) be optimal for ab. Let Z be the set of vertices of G adjacent to both a, b, and let Y be the set of major vertices. Then*

- *every vertex in $V(M)$ with a neighbour in $Z \setminus Y$ belongs to M_t for some $t \in \beta$; and*
- *for each $e = st \in \alpha$, $M_s \cap M_t = \emptyset$.*

Moreover, for every small subgraph F, let X be the set of vertices in $V(M)$ with a neighbour in $V(F)$; then

- *a has no neighbours in $V(F)$;*
- *if $V(F)$ is anticomplete to $\{b\} \cup (Z \setminus Y)$, then either $X \subseteq M_e$ for some $e \in E(J)$ or $X \subseteq M_t$ for some $t \in V(J) \setminus \alpha$;*
- *if either b or some vertex in $Z \setminus Y$ has a neighbour in $V(F)$, then $X \subseteq M_t$ for some $t \in \beta$.*

Proof. Since a is splendid, every vertex in Z is a-external, and therefore the vertices in $Z \setminus Y$ are not b-external. In particular, none of them has a neighbour in $V(M) \setminus N[b]$. That proves the first claim.

Suppose that there exists $e = st \in \alpha$ where t is a leaf of J, and $M_s \cap M_t \neq \emptyset$. Let $v \in M_s \cap M_t$. Let D be a path of J containing s, with one end in $\alpha \setminus \{t\}$ and the other in β. Choose an e-rung R_e for every $e \in E(D)$. Then the subgraph of G induced on $V(R_D) \cup \{a, b, v\}$ is a short pyramid with apex a, contradicting that a is splendid. This proves the second claim. The third claim, about small sets, follows from 4.2 and 5.4. This proves 5.5. ∎

6 Graphs with no extended near-prism

It would be nice if we had a decomposition theorem complementary to the results of the previous sections, describing a decomposition for even-hole-free graphs that do not contain a extended near-prism. We do not have that; we only have a decomposition theorem for such graphs that have a splendid vertex. (This is good enough for our purposes, since it is straightforward to show that every minimum counterexample to 1.2 has a splendid vertex.) Our next goal is to state and prove this decomposition theorem.

A *pyramid strip system* $\mathcal{S} = (a, S_1, \dots, S_k)$ in G consists of a set of proper strips $S_1, \dots, S_k$ with $k \geq 3$, pairwise vertex-disjoint (that is, the sets $V(S_1), \dots, V(S_k)$ are pairwise disjoint), and a vertex a of G called the *apex*, such that, setting $S_i = (A_i, B_i, C_i)$ for $1 \leq i \leq k$:

- for $1 \leq i < j \leq k$, B_i is complete to B_j, and there are no other edges between $V(S_i)$ and $V(S_j)$;
- a belongs to none of $V(S_1), \dots, V(S_k)$;
- for $1 \leq i \leq k$, a is complete to A_i, and anticomplete to $B_i \cup C_i$.

Let $V(\mathcal{S})$ denote $V(S_1) \cup \cdots \cup V(S_k) \cup \{a\}$. For an induced subgraph F of G with $V(F) \subseteq V(G) \setminus V(\mathcal{S})$, we say $v \in V(\mathcal{S})$ is an *attachment* of F if v has a neighbour in F, and we define $\mathcal{S}(F)$ to be the set of all attachments of F. A proper strip $S = (A, B, C)$ is *indecomposable* if $A \cup C$ is connected, and a pyramid strip system is *indecomposable* if all its strips are indecomposable.

If $a \in V(G)$ is the apex of a pyramid, then it is also the apex of an indecomposable pyramid strip system with $k = 3$ and with only one rung in each strip. That motivates the following:

6.1 *Let G be even-hole-free, and let $a \in V(G)$ be splendid. Suppose there is no extended near-prism contained in G such that a is an end of its cross-edge. Let $\mathcal{S} = (a, S_1, \dots, S_k)$ be an indecomposable strip system with apex a, with strips $S_i = (A_i, B_i, C_i)$ for $1 \leq i \leq k$, chosen with $V(\mathcal{S})$ maximal. Then for each component F of $G \setminus (V(\mathcal{S}) \cup N[a]$, either $\mathcal{S}(F)$ is a nonempty subset of $B_1 \cup \cdots \cup B_k$, or for some $i \in \{1, \dots, k\}$, $\mathcal{S}(F)$ is a subset of one of $V(S_i)$ and has nonempty intersection with $B_i \cup C_i$.*

Proof. First we observe:

(1) *For each component F of $G \setminus (V(\mathcal{S}) \cup N[a])$, $\mathcal{S}(F)$ has nonempty intersection with $B_i \cup C_i$ for some $i \in \{1, \dots, k\}$.*

If not, then F is a component of $G \setminus N[a]$, which is impossible since $G \setminus N[a]$ is connected (because a is splendid). This proves (1).

(2) *For each vertex f of $G \setminus (V(\mathcal{S}) \cup N[a])$, $\mathcal{S}(f)$ is either a subset of $B_1 \cup \cdots \cup B_k$ or a subset of $V(S_i)$ for some $i \in \{1, \dots, k\}$.*

Suppose not. We may assume f has a neighbour in $A_1 \cup C_1$, since $\mathcal{S}(F)$ is not a subset of $B_1 \cup \cdots \cup B_k$. Choose an S_1-rung R_1 in which f has a neighbour in $A_1 \cup C_1$, with ends $a_1 \in A_1$ and $b_1 \in B_1$. Suppose also that f has a neighbour in $A_2 \cup C_2$, and choose R_2, a_2, b_2 similarly. If f has a neighbour in $V(S_3)$, then there is a theta with ends f, a and constitutent paths

$$f\text{-}R_1\text{-}a_1\text{-}a,$$

$$f\text{-}R_2\text{-}a_2\text{-}a,$$

$$f_2\text{-}G[V(S_3)]\text{-}a,$$

contrary to 2.1. Thus f is anticomplete to $V(S_3),\dots,V(S_k)$. If f has two nonadjacent neighbours in R_1, there is a theta with ends a,f and constitutent paths

$$f\text{-}R_1\text{-}a_1\text{-}a,$$

$$f\text{-}R_2\text{-}a_2\text{-}a,$$

$$f\text{-}R_1\text{-}b_1\text{-}G[V(S_3)]\text{-}a,$$

contrary to 2.1. So f has either one or two adjacent neighbours in R_1, and similarly it has one or two adjacent in R_2. Since f is not adjacent to both a_1,a_2, we may assume by exchanging S_1,S_2 if necessary that f is not adjacent to a_1. If f has a unique neighbour u in R_1, there is a theta with ends u,a and constitutent paths

$$u\text{-}R_1\text{-}a_1\text{-}a,$$

$$u\text{-}f\text{-}R_2\text{-}a_2\text{-}a,$$

$$u\text{-}R_1\text{-}b_1\text{-}G[V(S_3)]\text{-}a,$$

contrary to 2.1. Thus f has exactly two adjacent neighbours in R_1, say p,q, where a_1,p,q,b_1 are in order in R_1. If f also has two adjacent neighbours in R_2, there is a 4-wheel with centre f and hole induced on $V(R_1\cup R_2)\cup\{a\}$, a contradiction. Thus f has a unique neighbour u in R_2. If $u\neq a_2$, we obtain a contradiction as before; and if $u=a_2$, the subgraph induced on $V(R_1\cup R_2)\cup\{a,f\}$ is an extended near-prism, and a is an end of its cross-edge, a contradiction.

This proves that f has no neighbour in $A_2\cup C_2$, and similarly none in $A_i\cup C_i$ for $2\le i\le k$. If f is complete to $B_2\cup\cdots\cup B_k$, we can add f to B_1, contrary to the maximality of $V(\mathcal{S})$. Thus f has a neighbour in $B_2\cup\cdots\cup B_k$, and a non-neighbour in this set. Since $k\ge 3$, we may assume that f has a neighbour $b_2\in B_2$ and a non-neighbour $b_3\in B_3$. But then there is a theta with ends b_2,a and constituent paths

$$b_2\text{-}f\text{-}G[A_1\cup C_1]\text{-}a,$$

$$b_2\text{-}G[A_2\cup C_2]\text{-}a,$$

$$b_2\text{-}b_3\text{-}G[A_3\cup C_3]\text{-}a,$$

contrary to 2.1. This proves (2).

Let us say a subset X of $V(\mathcal{S})$ is *local* if X is a subset of $A_1\cup\cdots\cup A_k$, or of $B_1\cup\cdots\cup B_k$ or of $V(S_i)$ for some $i\in\{1,\dots,k\}$. (Note that in (2) we did not include $A_1\cup\cdots\cup A_k$, but here we do.)

(3) *Every subset of $V(\mathcal{S})$ that is not local includes a 2-element subset that is not local.*

Suppose $X\subseteq V(\mathcal{S})$ is not local. If there exists $c\in X\cap C_1$, choose $d\in X\setminus V(S_1)$; then $\{c,d\}$ is not local. So we may assume that $X\cap C_i=\emptyset$ for $1\le i\le k$. There exists $c\in X\setminus(A_1\cup\cdots\cup A_k)$, say $c\in B_1$. If there exists $d\in X\cap A_i$ where $i\ge 2$ then $\{c,d\}$ is not local, so we may assume that $X\cap A_i=\emptyset$ for $2\le i\le k$. Since $X\not\subseteq B_1\cup\cdots\cup B_k$, there exists $c\in X\cap A_1$; and since $X\not\subseteq V(S_1)$, there exists $d\in X\cap B_i$ for some $i>1$, and then $\{c,d\}$ is not local. (The claim also follows from König's matching theorem.) This proves (3).

Suppose the theorem is false; then from (1) there is a minimal connected induced subgraph F of $G \setminus (V(\mathcal{S}) \cup N[a])$ such that $\mathcal{S}(F)$ is not local. By (3) there is a 2-element subset $\{v_1, v_2\}$ of $\mathcal{S}(F)$ that is not local. From the minimality of F, F is the interior of a path joining v_1, v_2. Let F have ends f_1, f_2, where v_i, f_i are adjacent for $i = 1, 2$.

(4) *No vertex in F^* has a neighbour in $A_1 \cup \cdots \cup A_k$.*

Suppose that some $f_3 \in V(F) \setminus \{f_1, f_2\}$ is adjacent to $a_1 \in A_1$ say. Let F_i be the subpath of F between f_i, f_3 for $i = 1, 2$. From the minimality of F, each of $\mathcal{S}(F_1), \mathcal{S}(F_2)$ is a subset of one of $V(S_1), A_1 \cup \cdots \cup A_k$; and since $\mathcal{S}(F)$ is not local, we may assume that $\mathcal{S}(F_1) \subseteq V(S_1)$ and $\mathcal{S}(F_2) \subseteq A_1 \cup \cdots \cup A_k$. Moreover, $v_1 \notin A_1 \cup \cdots \cup A_k$ and $v_2 \notin V(S_1)$. Thus $v_1 \in B_1 \cup C_1$, and we may assume that $v_2 \in A_2$. From the minimality of F, $\mathcal{S}(F \setminus f_1)$ is local and hence is a subset of $A_1 \cup \cdots \cup A_k$, and $\mathcal{S}(F \setminus f_2)$ is a subset of $V(S_1)$ (because they both contains a_1). Thus $\mathcal{S}(F \setminus \{f_1, f_2\}) \subseteq A_1$, and $\mathcal{S}(f_2) \subseteq A_2$ by (2). For $i = 1, 2$ let R_i be an S_i-rung with ends $a_i \in A_i$ and $b_i \in B_i$, containing v_i. Thus $v_2 = a_2$, and $v_1 \neq a_1$, and a_2 is the unique neighbour of f_2 in $V(R_2)$.

Let a_1 have t neighbours in $V(F \setminus f_1)$; thus $t > 0$. Choose a neighbour c of f_1 in $V(R_1)$, such that the subpath of R_1 between b_1, c is minimal. Thus $c \neq a_1$. If c, a_1 are nonadjacent we can add the interior of the path c_1-F-a_1 to C_1, contrary to the maximality of $V(\mathcal{S})$. So c, a_1 are adjacent, and hence a_1 has at least $t+1$ neighbours in the path b_1-R_1-c-F-a_2. (It would have $t+2$ if a_1, f_1 are adjacent, and $t+1$ otherwise.) This path can be completed to a hole via a_2-R_2-b_2-b_1 or via a_2-a-$G[V(S_3)]$-b_1, and the number of neighbours of a_1 in the second hole is one more than in the first. Since there is no even wheel, it follows that $t = 1$, and f_3 is the unique neighbour of a_1 in $V(F)$; but then there is a theta with ends f_3, c and constituent paths

$$f_3\text{-}F\text{-}c,$$

$$f_3\text{-}a_1\text{-}c_1,$$

$$f_3\text{-}F\text{-}a_2\text{-}R_2\text{-}b_2\text{-}b_1\text{-}R_1\text{-}c,$$

contrary to 2.1. This proves (4).

(5) *If f_1 has a neighbour in $A_1 \cup C_1$, then f_2 has a neighbour in $A_i \cup C_i$ for some $i \in \{2, \ldots, k\}$.*

Suppose not; then $\mathcal{S}(f_2)$ is a subset of $B_1 \cup \cdots \cup B_k$, and we may assume that f_2 has a neighbour in B_2. By (4), no vertex in A_2 has a neighbour in $V(F)$, and so from the minimality of F, $\mathcal{S}(F \setminus \{f_1, f_2\}) \subseteq B_1$. If f_2 has a nonneighbour $b_3 \in B_3$, there is a theta with ends b_2, a and constituent paths

$$b_2\text{-}G[A_2 \cup C_2]\text{-}a,$$

$$b_2\text{-}F\text{-}f_1\text{-}R_1\text{-}a,$$

$$b_2\text{-}b_3\text{-}G[A_3 \cup C_3]\text{-}a,$$

contrary to 2.1. So f_2 is complete to B_3 and similarly to B_i for $3 \leq i \leq k$; and since $k \geq 3$, it follows by exchanging S_2, S_3 that f_2 is complete to B_2. But then we can add f_2 to B_1 and $V(F \setminus f_2)$ to C_1, contrary to the maximality of $V(\mathcal{S})$. This proves (5).

(6) *For $1 \le i \le k$, f_1 has no neighbour in C_i, and does not have both a neighbour in A_i and one in B_i.*

Suppose that f_1 has either a neighbour in C_1, or a neighbour in A_1 and one in B_1. In the second case, if the neighbour of f_1 in A_1 is nonadjacent to the one in B_1, we could add f_1 to C_1, contrary to the maximality of $V(\mathcal{S})$. Thus in either case, there is an S_1-rung R_1, such that f_1 has either a neighbour in $V(R_1) \cap C_1$, or one in $V(R_1) \cap A_1$ and one in $V(R_1) \cap B_1$. Let R_1 have ends $a_1 \in A_1$ and $b_1 \in B_1$. If f_1 has two nonadjacent neighbours in R_1, we can add f_1 to C_1, again contrary to the maximality of $V(\mathcal{S})$. Thus f_1 has either a unique neighbour or exactly two adjacent neighbours in R_1. From the minimality of F, $\mathcal{S}(F \setminus f_2) \subseteq V(S_1)$.

By (5), we may assume that f_2 has a neighbour in $A_2 \cup C_2$, and so $\mathcal{S}(f_2)$ is a subset of $V(S_2)$ by (2). Hence $\mathcal{S}(F \setminus f_1) \subseteq V(S_2)$ by (4). Consequently $\mathcal{S}(F \setminus \{f_1, f_2\}) = \emptyset$. The only edges between $V(F)$ and $V(\mathcal{S})$ are the edges between f_1 and $V(S_1)$, and the edges between f_2 and $V(S_2)$. Choose an S_2-rung R_2 in which f_2 has a neighbour in $A_2 \cup C_2$, with ends $a_2 \in A_2$ and $b_2 \in B_2$. If f_2 has two nonadjacent neighbours in $V(R_2)$ we can add f_2 to C_2, a contradiction. Thus f_2 has one or exactly two adjacent neighbours in R_2. Let f_i have n_i neighbours in $V(R_i)$ for $i = 1, 2$; thus $n_i \in \{1, 2\}$.

If $n_1 = n_2 = 2$, there is a prism, so we may assume that either $n_1 = 1$ or $n_2 = 1$. If $n_1 = 1$, let c be the unique neighbour of f_1 in $V(R_1)$ (necessarily $c \in C_1$), and let R_3 be an S_3-rung with ends $a_3 \in A_3$ and $b_3 \in B_3$. Then there is a theta with ends c, a and constituent paths

$$c\text{-}R_1\text{-}a_1\text{-}a,$$

$$c\text{-}f_1\text{-}F\text{-}f_2\text{-}R_2\text{-}a_2\text{-}a,$$

$$c\text{-}R_1\text{-}b_1\text{-}b_3\text{-}R_3\text{-}a_3\text{-}a,$$

a contradiction. Thus $n_1 = 2$, and consequently $n_2 = 1$.

Let c be the unique neighbour of f_2 in $V(R_2)$. By the same argument with S_1, S_2 exchanged, it follows that $c \notin C_2$, and so $c = a_2$. Let R_3 be an S_3-rung; then the subgraph induced on $V(R_1 \cup R_2 \cup R_3 \cup F) \cup \{a\}$ is an extended near-prism, a contradiction. This proves (6).

From (6), no vertex of F has a neighbour in $C_1 \cup \cdots \cup C_k$, and since we may assume that f_1 has a neighbour in $A_1 \cup C_1$, it follows from (6) that $\mathcal{S}(f_1) \subseteq A_1$. Since $\{v_1, v_2\}$ is not local, it follows that $v_2 \in B_2 \cup \cdots \cup B_k$, and we may assume that f_2 has a neighbour in B_2. By (6), $\mathcal{S}(f_2) \subseteq B_1 \cup \cdots \cup B_k$, contrary to (5). This proves 6.1. ∎

The reader will observe that much of the generality of strip systems was not used in this proof; we never increased the number of strips, or changed the sets $A_1, \ldots, A_k$. That will come in the next proof, where we try to enlarge $V(\mathcal{S})$ by adding vertices from $N[a] \setminus V(\mathcal{S})$. The *parity* of a path or hole is the parity of its length.

For $1 \le i \le k$, let D_i be the union of the vertex sets of all components F of $G \setminus (V(\mathcal{S}) \cup N[a])$ such that $\mathcal{S}(F) \cap (A_i \cup C_i) \neq \emptyset$. For $v \in N[a] \setminus V(\mathcal{S})$, let us say v has:

- *type* α if for each $i \in \{1, \ldots, k\}$, either v has a neighbour in $B_i \cup C_i$ or v is complete to A_i;
- *type* α' if there exists $i \in \{1, \ldots, k\}$ such that v has a neighbour in D_i and none in $B_i \cup C_i$, and for all $j \in \{1, \ldots, k\} \setminus \{i\}$, v is complete to A_j and anticomplete to $B_j \cup C_j \cup D_j$ (we also call this *type* α'_i; it is "almost" a case of type α);

- *type β* if there exists $i \in \{1, \ldots, k\}$ such that v is anticomplete to $A_i \cup B_i \cup C_i$, and for all $j \in \{1, \ldots, k\} \setminus \{i\}$, v has a neighbour in $B_j \cup C_j$ (we also call this *type β_i*).

We also need one other type. In the usual notation, for $v \in N[a] \setminus V(\mathcal{S})$ and $1 \leq i \leq k$, let us say v has *type γ* or *type γ_i*, and Q is the *corresponding private path*, if

- Q is an induced path with one end v and the other q say, and $V(Q \setminus v)$ is disjoint from $V(\mathcal{S}) \cup N[a]$;
- q has a neighbour in B, and q is either complete or anticomplete to $B \setminus B_i$;
- v is complete to A_j for all $j \in \{1, \ldots, k\} \setminus \{i\}$, and v has no neighbours in $B_j \cup C_j \cup D_j$ for $1 \leq j \leq k$; and
- all edges between $V(\mathcal{S})$ and $V(Q \setminus v)$ are between q and B.

We will show:

6.2 *Let G be even-hole-free, and let $a \in V(G)$ be splendid. Suppose there is no extended near-prism contained in G such that a is an end of its cross-edge. Let $\mathcal{S} = (a, S_1, \ldots, S_k)$ be an indecomposable strip system with apex a, with strips $S_i = (A_i, B_i, C_i)$ for $1 \leq i \leq k$, chosen with $V(\mathcal{S})$ maximal. For each $v \in N[a] \setminus V(\mathcal{S})$, v has type α, α', β or γ.*

Proof. Let $D_1, \ldots, D_k$ be defined as before. We begin with:

(1) *The sets $D_1, \ldots, D_k$ are pairwise disjoint, and every component of $G[B_i \cup C_i \cup D_i]$ contains a vertex of B_i.*

This is immediate from 6.1.

Let $H \subseteq I$ be the set of $i \in \{1, \ldots, k\}$ such that v has a neighbour in $B_i \cup C_i$, and $J = \{1, \ldots, k\} \setminus H$. Let $I \subseteq \{1, \ldots, k\}$ be the set of $i \in \{1, \ldots, k\}$ such that v has a neighbour in $B_i \cup C_i \cup D_i$. (Thus $H \subseteq I$.)

(2) *If $I \neq \emptyset$, then either:*

- *v is complete to $\bigcup_{j \in J} A_j$ (and so v has type α), or*
- *$|I| = 1$ and v is complete to $\bigcup_{i \notin I} A_i$ (and so v has type α or α'),or*
- *$|J| = 1$, $J = \{j\}$ say, and v is anticomplete to A_j (and so v has type β_j).*

We may assume that $I \neq \emptyset$. Choose $h \in \{1, \ldots, k\}$ as follows:

- If $H \neq \emptyset$ choose $h \in H$;
- If $H = \emptyset$ and either $|I| = 1$ or v is complete to $A_1 \cup \cdots \cup A_k$, choose $h \in I$;
- If $H = \emptyset$ and $|I| > 1$ and v is not complete to $A_1 \cup \cdots \cup A_k$, choose $h \in I$ such that v is not complete to A_j for some $j \neq h$.

For notational simplicity let us assume $h = 1$. Suppose first that v is complete to A_j for all $j \in J\setminus\{1\}$. If $1 \in H$ then the claim holds, so we may assume that $1 \notin H$ and therefore $H = \emptyset$ from the choice of h. Also, from the choice of h, either $|I| = 1$ or v is complete to $A_1 \cup \cdots \cup A_k$, and in both cases the claim holds.

Hence we may assume that there exists $j \in J \setminus \{1\}$ such that v is not complete to A_j, say $j = 2$. Choose an induced path P between v and some $b_1 \in B_1$ with interior in $C_1 \cup D_1$ (this is possible by (1)). Choose $a_2 \in A_2$ nonadjacent to v, and let R_2 be an S_2-rung containing a_2, and let b_2 be its end in B_2. Now let $a_2' \in A_2$, and define R_2', b_2' similarly. The R_2, R_2' have the same parity, and so if v is adjacent to a_2' then the holes

$$v\text{-}P\text{-}b_1\text{-}b_2\text{-}R_2\text{-}a_2\text{-}a\text{-}v,$$

$$v\text{-}P\text{-}b_1\text{-}b_2'\text{-}R_2'\text{-}a_2'\text{-}v$$

have different parity, a contradiction. Thus v is nonadjacent to a_2' for each $a_2' \in A_2$, and therefore anticomplete to A_2. If $|J \setminus \{1\}| = 1$, then $|J| \leq 2$, and hence $H \neq \emptyset$, and so $1 \in H$ and $|J| = 1$. But then the claim holds. Thus we may assume that $|J \setminus \{1\}| \geq 2$; let $3 \in J$ say. Let R_3 be an S_3-rung with ends $a_3 \in A_3$ and $b_2 \in B_3$. If v is adjacent to a_3, then similarly the holes

$$v\text{-}P\text{-}b_1\text{-}b_2\text{-}R_2\text{-}a_2\text{-}a\text{-}v,$$

$$v\text{-}P\text{-}b_1\text{-}b_3\text{-}R_3\text{-}a_3\text{-}v$$

have different parity, a contradiction. So v is anticomplete to $\bigcup_{j\in J\setminus\{1\}} A_j$. For each $i \in I$, let P_i be an induced path between v and B_i with interior in $C_i \cup D_i$. Define

$$\begin{aligned} A_0 &= \{v\} \cup \bigcup_{i\in I} A_i; \\ B_0 &= \bigcup_{i\in I} B_i; \\ C_0 &= \bigcup_{i\in I} C_i \cup (V(P_i) \cap D_i); \end{aligned}$$

Then S_0 is a strip, and $(a, S_i\ (i \in J \cup \{0\}))$ is an indecomposable pyramid strip system contrary to the maximality of $V(\mathcal{S})$. This proves (2).

To complete the proof of the theorem, we therefore may assume that $I = \emptyset$; so now let $v \in N(a)$ with no neighbour in $B_i \cup C_i \cup D_i$ for $1 \leq i \leq k$. Since a is splendid, v has a neighbour $u \in V(G)\setminus N[a]$; and so $u \notin V(\mathcal{S}) \cup N[a]$. Let F be the component of $G \setminus (V(\mathcal{S}) \cup N[a])$ that contains u. Since F is contained in none of the sets D_i, it follows that $\mathcal{S}(F) \subseteq B_1 \cup \cdots \cup B_k$. Choose a minimal path Q of $G[V(F) \cup \{v\}]$ with one end v such that the other end, q say, has a neighbour in $B_1 \cup \cdots \cup B_k$. For $1 \leq i \leq k$ let $B_i' \subseteq B_i$ be the set of vertices in B_i adjacent to q, and let $B_i'' = B_i \setminus B_i'$. Let A_i' be the set of vertices in A_i adjacent to v, and $A_i'' = A_i \setminus A_i'$. The only edges between $V(\mathcal{S})$ and $V(Q)$ are the edges between v and $\{a\} \cup A_1 \cup \cdots \cup A_k$, and the edges between q and $B_1 \cup \cdots \cup B_k$, since $Q \setminus v$ is a subgraph of F and $\mathcal{S}(F) \subseteq B_1 \cup \cdots \cup B_k$.

By a *rung* we mean an S_i-rung for some $i \in \{1, \ldots, k\}$. For $1 \leq i \leq k$, let us say an S_i-rung R_i is *crooked* if it has one end in A_i and the other in B_i', or one end in A_i' and the other in B_i; and *straight* otherwise. Choose $x, y \in \{0, 1\}$ such that Q has length x modulo 2, and every rung has length y

modulo 2.

(3) *If $x \neq y$ then no rung is crooked, and either v is complete to $A_1 \cup \cdots \cup A_k$ (and v has type α), or for some i, v is complete to $\bigcup_{j \neq i} A_j$, and anticomplete to A_i, and q is complete to $\bigcup_{j \neq i} B_j$, and anticomplete to B_i (and so v has type γ_i, and Q is a private path).*

Suppose that R_1 is a crooked S_1-rung, with ends $a_1 \in A_1$ and $b_1 \in B_1$. If $a_1 \in A_1''$ and $b_1 \in B_1'$ then b_1-R_1-a_1-a-v-Q-q-b_1 is an even hole; so $a_1 \in A_1'$ and $b_1 \in B_1''$. If there exists $b_2 \in B_2'$, then b_1-R_1-a_1-v-Q-q-b_2-b_1 is an even hole, a contradiction; so $B_2', \ldots, B_k' = \emptyset$. Hence $B_1' \neq \emptyset$; and so for $2 \leq i \leq k$ there is no crooked S_i-rung, by the same argument with S_1, S_i exchanged, and so $A_2', \ldots, A_k' = \emptyset$. But then we can add v to A_1 and $V(Q) \setminus \{v\}$ to C_1 (note that the edge va_1 guarantees the indecomposability of the new strip), contrary to the maximality of $V(\mathcal{S})$.

Thus every rung is straight. Suppose that $A_1', A_1'' \neq \emptyset$. Let C_1' be the union of all interior of S_1-rungs between A_1', B_1', and let C_1'' be the union of all interiors of S_1-rungs between A_1'', B_1''. Since every S_1-rung is of one of these two types, $C_1' \cup C_1'' = C_1$. Since there is no S_1-rung with ends in A_1' and B_1'', it follows that $C_1' \cap C_1'' = \emptyset$ and C_1', C_1'' are anticomplete. For the same reason, the only edges between $A_1' \cup C_1'$ and $A_1'' \cup C_1''$ are between A_1' and A_2'. Since S_1 is indecomposable, there is an edge between some $a_1' \in A_1'$ and some $a_1'' \in A_1''$. Let R_1'' be an S_1-rung with ends a_1'' and some $b_1'' \in B_1''$. If there exists $a_2 \in A_2'$, let R_2' be an S_2-rung with ends a_2, b_2; then

$$b_1''\text{-}R_1''\text{-}a_1''\text{-}a_1'\text{-}v\text{-}a_2\text{-}R_2\text{-}b_2\text{-}b_1''$$

is an even hole, a contradiction. So $A_2', \ldots, A_k' = \emptyset$, and since every rung is straight, it follows that $B_2', \ldots, B_k' = \emptyset$. But then we can add v to A_1 and $V(Q \setminus v)$ to C_1, contrary to the maximality of $V(\mathcal{S})$.

This proves that for each $i \in \{1, \ldots, k\}$, either $A_i' = B_i' = \emptyset$, or $A_i'' = B_i'' = \emptyset$. Let I be the set of $i \in \{1, \ldots, k\}$ such that $A_i' \neq \emptyset$. Suppose that $|I| \leq k-2$, say $I = \{i+1, \ldots, k\}$ where $i \geq 3$. Define $S_0 = (A_0, B_0, C_0)$, where

$$\begin{aligned} A_0 &= \{v\} \cup \bigcup_{i \in I} A_i \\ B_0 &= \bigcup_{i \in I} B_i \\ C_0 &= V(Q \setminus v) \cup \bigcup_{i \in I} C_i. \end{aligned}$$

Then $(a, S_0, S_1, \ldots, S_i)$ is an indecomposable pyramid strip system, contrary to the maximality of $V(\mathcal{S})$. So $|I| \geq k-1$. This proves (3).

(4) *If $x = y$ then there exists i such that v is complete to $\bigcup_{j \neq i} A_j$, and q is anticomplete to $\bigcup_{j \neq i} B_j$ (and so v has type γ_i and Q is a private path).*

Suppose that R_1 is a straight S_1-rung, with ends $a_1 \in A_1$ and $b_1 \in B_1$. If $a_1 \in A_1'$ and $b_1 \in B_1'$ then $G[V(R_1 \cup Q)]$ is an even hole, which is impossible. Since R_1 is straight, it follows that $a_1 \in A_1''$ and $b_1 \in B_1''$. If there exists $b_2 \in B_2'$, then b_1-R_1-a_1-a-v-Q-q-b_2-b_1 is an even hole, a contradiction;

so $B'_2, \dots, B'_k = \emptyset$. Hence $B'_1 \neq \emptyset$; and so for $2 \leq i \leq k$ there is no straight S_i-rung, by the same argument with S_1, S_i exchanged. Hence $A''_2, \dots, A''_k = \emptyset$, and the claim holds.

Thus we may assume that every rung is crooked. Suppose that $A'_1, A''_1 \neq \emptyset$. Let C'_1 be the union of all interior of S_1-rungs between A'_1, B''_1, and let C''_1 be the union of all interiors of S_1-rungs between A''_1, B'_1. Since every S_1-rung is of one of these two types, $C'_1 \cup C''_1 = C_1$. Since there is no S_1-rung with ends in A'_1 and B'_1, it follows that $C'_1 \cap C''_1 = \emptyset$ and C'_1, C''_1 are anticomplete. For the same reason, the only edges between $A'_1 \cup C'_1$ and $A''_1 \cup C''_1$ are between A'_1 and A'_2. Since S_1 is indecomposable, there is an edge between some $a'_1 \in A'_1$ and some $a''_1 \in A''_1$. Let R_1 be an S_1-rung with ends a''_1 and some $b'_1 \in B'_1$. If there exists $a_2 \in A'_2$, let R_2 be an S_2-rung with ends a_2, b_2; then

$$b'_1\text{-}R_1\text{-}a''_1\text{-}a'_1\text{-}v\text{-}a_2\text{-}R_2\text{-}b_2\text{-}b'_1$$

is an even hole, a contradiction. So $A'_2, \dots, A'_k = \emptyset$, and since every rung is crooked, it follows that $B''_2, \dots, B''_k = \emptyset$. But then we can add v to A_1, q to B_1, and Q^* to C_1, contrary to the maximality of $V(\mathcal{S})$.

This proves that for each $i \in \{1, \dots, k\}$, either $A'_i = B''_i = \emptyset$, or $A''_i = B'_i = \emptyset$. Let I be the set of $i \in \{1, \dots, k\}$ such that $A'_i \neq \emptyset$. If $I = \emptyset$, define $S_0 = (\{v\}, \{q\}, Q^*)$, then $(a, S_0, S_1, \dots, S_k)$ is an indecomposable pyramid strip system, contrary to the maximality of $V(\mathcal{S})$. So $I \neq \emptyset$. Suppose that $|I| \leq k - 2$, say $I = \{i+1, \dots, k\}$ where $3 \leq i \leq k$. Define $S_0 = (A_0, B_0, C_0)$, where

$$\begin{aligned} A_0 &= \{v\} \cup \bigcup_{i \in I} A_i \\ B_0 &= \{q\} \cup \bigcup_{i \in I} B_i \\ C_0 &= Q^* \cup \bigcup_{i \in I} C_i. \end{aligned}$$

Then $(a, S_0, S_1, \dots, S_i)$ is an indecomposable pyramid strip system, contrary to the maximality of $V(\mathcal{S})$. So $|I| \geq k - 1$ and again the claim holds. This proves (4).

From (3) and (4) it follows that v has type γ_i, and Q is the corresponding private path. In view of (2), this proves 6.2. ∎

We say A *meets* B if $A \cap B \neq \emptyset$.

6.3 *Let G be even-hole-free, and let $a \in V(G)$ be splendid. Suppose there is no extended near-prism contained in G such that a is an end of its cross-edge. Let $\mathcal{S} = (a, S_1, \dots, S_k)$ be an indecomposable strip system with apex a, with strips $S_i = (A_i, B_i, C_i)$ for $1 \leq i \leq k$, chosen with $V(\mathcal{S})$ maximal. Then $N[a] \setminus V(\mathcal{S})$ is a clique.*

Proof. For $X, Y \subseteq V(G)$, an $X - Y$ *path* means (in this proof) an induced path P of G with ends x, y say, where $X \cap V(P) = \{x\}$ and $Y \cap V(P) = \{y\}$ (possibly $x = y$ and $V(P) = \{x\}$, if $x \in X \cap Y$). If $X \subseteq V(G)$, a path of G is said to be *within* X if $V(P) \subseteq X$. Let $B = B_1 \cup \dots \cup B_k$.

(1) *For each $v \in N[a] \setminus V(\mathcal{S})$ there exists $x(v) \in \{0, 1\}$ such that for $1 \leq i \leq k$, every $N(v) - B$ path within $V(S_i)$ has parity $x(v)$.*

Since $v \in N(a) \setminus V(\mathcal{S})$, 6.2 implies that there are at least two values of $i \in \{1,\dots,k\}$ such that $N(v) \cap V(S_i) \neq \emptyset$; and for each such i there is an $N(v)-B$ path within $V(S_i)$. Let $N(v) \cap V(S_i) \neq \emptyset$ for $i = 1, 2$ say, and for $i = 1, 2$ let P_i be an $N(v) - B$ path within $V(S_i)$. Then $G[V(P_1 \cup P_2) \cup \{v\}]$ is a hole, and so P_1, P_2 have the same parity, say $x(v) \in \{0, 1\}$. We claim that for $1 \leq j \leq k$, every $N(v) - B$ path P in $V(S_j)$ has parity $x(v)$. To see this, choose $i \in \{1, 2\}$ different from j; then $G[V(P_i \cup P) \cup \{v\}]$ is a hole, and the claim follows. This proves (1).

In particular, $x(a)$ exists, and so for $1 \leq i \leq k$, all S_i-rungs have parity $x(a)$. Suppose that $u, v \in N(a) \setminus V(\mathcal{S})$ are nonadjacent.

(2) *If X_1, X_2 are connected subsets of $V(G)$, disjoint and anticomplete, and u, v both have neighbours in X_i for $i = 1, 2$, then all $N(u) - N(v)$ paths within X_1 have the same parity, and all $N(u) - N(v)$ paths within X_2 have the opposite parity.*

For $i = 1, 2$, let P_i be an $N(u) - N(v)$ path P_i within X_i; then $G[V(P_1 \cup P_2) \cup \{u, v\}]$ is a hole, and so P_1, P_2 have opposite parity. This proves (2).

(3) *There do not exist three connected subsets X_1, X_2, X_3 of $V(G)$, pairwise disjoint and pairwise anticomplete, such that for $i = 1, 2, 3$, u, v both have neighbours in X_i.*

This is immediate from (2).

(4) *There is at most one $i \in \{1, \dots, k\}$ such that $N(u) \cap (A_i \cap C_i) = \emptyset$, and the same for $N(v)$.*

Suppose that $N(u)$ is disjoint from $A_i \cup C_i$ for $i = 1, 2$. By 6.2, $N(a)$ meets at least $k - 1$ of $V(S_1), \dots, V(S_k)$, so we may assume there exists $b_1 \in B_1 \cap N(u)$. Choose $b_2 \in B_2$, and for $i = 1, 2$ let R_i be an S_i-rung containing b_i. If b_2, u are adjacent, there is a short pyramid with apex a, with base $\{b_1, b_2, u\}$ and constituent paths R_1, R_2 and the edge u-a, which is impossible since a is splendid. If b_2, u are nonadjacent, there is a theta with ends b_1, a and constituent paths b_1-$R_1 i$-a, b_1-u-a, and b_1-b_2-R_2-a, contrary to 2.1. This proves (4).

(5) *$k = 3$, and there exists $i \in \{1, 2, 3\}$ such that u, v both have neighbours in $A_i \cup C_i$.*

Since each S_i is indecomposable, there are only at most two values of i such that $N(u), N(v)$ both meet $A_i \cup C_i$, by (3). Then both claims follow from (4). This proves (5).

As before, for $i = 1, 2, 3$, let D_i be the union of all components F of $G \setminus (V(\mathcal{S} \cup N[a])$ such that $\mathcal{S}(F) \cap (A_i \cup C_i) \neq \emptyset$. From (3), there exists $i \in \{1, 2, 3\}$ such that not both u, v have neighbours in $A_i \cup C_i \cup D_i$.

(6) *If v is anticomplete to $V(S_3) \cup D_3$ then v has type β_3.*

Suppose not. Certainly v does not have type α or α', since it has no neighbour in $V(S_3) \cup D_3$. It does not have type β_1 or β_2 since it has no neighbour in $B_3 \cup C_3$; and not type γ_1, γ_2 since it is not complete to A_3. So v has type γ_3; let Q be the corresponding private path, between v and q say, and let p be the neighbour of v in this path. Also, since v is complete to A_1 and anticomplete to $B_1 \cup C_1$, it follows that $x(v) = x(a)$.

For $i = 1, 2$, if $N(u)$ meets $A_i \cup C_i \cup D_i$, then there is an $N(u) - \{a\}$ path R within $\{a\} \cup A_i \cup C_i \cup D_i$; and since its ends are adjacent to u, it has odd length. Hence $R \setminus a$ is an $N(u) - N(v)$ path (since v is complete to A_i and anticomplete to $B_i \cup C_i$), and has even length. By (2), $N(u)$ is disjoint from one of $A_1 \cup C_1 \cup D_1, A_2 \cup C_2 \cup D_2$, say $A_2 \cup C_2 \cup D_2$; and by (4), $N(u)$ meets $A_1 \cup C_1$. Let P_1 be an even $N(u) - N(v)$ path within $A_1 \cup C_1$.

Now u has no neighbour in $A_2 \cup C_2 \cup D_2$. Suppose that u has a neighbour in the connected set $C_3 \cup B_3 \cup B_2 \cup V(Q \setminus v)$, and let T be an $N(u) - \{a\}$ path within

$$C_3 \cup B_3 \cup V(Q \setminus v) \cup B_2 \cup C_2 \cup A_2 \cup \{a\}.$$

This path has odd length (because its ends are neighbours of u), and it contains no neighbour of v except the one in A_2 (because p is nonadjacent to u). Consequently the path $T \setminus a$ is an $N(u) - N(v)$-path of even length anticomplete to P_1, a contradiction. So u has no neighbour in $C_3 \cup B_3 \cup V(Q) \cup B_2$. Since u is anticomplete to $V(S_2) \cup D_2$, 6.2 implies that u has type γ_2, and in particular, u is complete to A_3 and has no neighbour in C_3. Let T be an $N(v) - \{a\}$ path within $V(Q) \cup V(S_3) \cup \{a\}$; again it has odd length (since its ends are adjacent to v), and $T \setminus a$ is an even $N(u) - N(v)$-path anticomplete to P_1, a contradiction. This proves (6).

(7) *There is only one $i \in \{1, 2, 3\}$ such that both $N(u), N(v)$ meet $A_i \cup C_i \cup D_i$.*

Suppose that $N(u), N(v)$ both meet $A_i \cup C_i \cup D_i$ for $i = 1, 2$. Then by (3), one of u, v has no neighbours in $V(S_3) \cup D_3$, say v. By (6), v has type β_3, and so has a neighbour in $B_1 \cup C_1$ and one in $B_2 \cup C_2$. For $i = 1, 2$, let P_i be an $N(u) - N(v)$ path within $A_i \cup C_i$. By exchanging S_1, S_2 if necessary, we may assume that P_1 has odd length, and so P_2 is even. Hence there is no $N(u) - N(v)$ path within the connected set $B_2 \cup B_3 \cup C_2 \cup C_3 \cup D_2 \cup D_3$, because we could combine it with one of u-a-v and u-P_1-v to make an even hole. Since v has a neighbour in this set, u does not. So u does not have type β. By (6), u has a neighbour $a_3 \in A_3$. Let R_3 be an S_3-rung containing a_3, and for $i = 1, 2$, let R_i be an $N(v) - B_i$ path within $B_i \cup C_i$. For $i = 1, 2, 3$, let b_i be the end of R_i in B_i. Thus R_1, R_2 both have parity $x(v)$. For $i = 1, 2$, let Q_i be the induced path R_i-b_i-b_3-R_3. Thus Q_2 is an $N(u) - N(v)$ path, but Q_1 might not be. Now Q_1, Q_2 have the same parity. Since Q_2 is anticomplete to P_1 it follows that Q_2 is even, and hence Q_1 is even; and since Q_1 is anticomplete to P_2, it follows that Q_1 is not an $N(u) - N(v)$ path. But it has one end in $N(v)$ and no other vertex in $N(v)$; and its other end is in $N(u)$. Consequently some internal vertex is in $N(u)$, and so u has a neighbour in $V(R_1)$.

If u has a unique neighbour $t \in V(R_1)$, there is a theta with ends t, v and constituent paths

$$t\text{-}R_1\text{-}v,$$

$$t\text{-}u\text{-}a\text{-}v,$$

$$t\text{-}R_1\text{-}b_1\text{-}b_2\text{-}R_2\text{-}v,$$

contrary to 2.1. (Note that t, v are nonadjacent since u, v have no common neighbour nonadjacent to a.) If u has two nonadjacent neighbours in $V(R_1)$, there is a theta with ends u, v and constituent paths

$$u\text{-}R_1\text{-}v,$$

$$u\text{-}a\text{-}v,$$

$$u\text{-}R_1\text{-}b_1\text{-}b_2\text{-}R_2\text{-}v,$$

contrary to 2.1. If u has exactly two adjacent neighbours $p, q\ inV(R_1)$, where v, p, q, b_1 are in order in R_1, there is a near-prism with bases $\{v, p, q\}$ and $\{b_1, b_2, b_3\}$ and constituent paths

$$p\text{-}R_1\text{-}v\text{-}R_2\text{-}b_2,$$

$$u\text{-}R_3\text{-}b_3,$$

$$q\text{-}R_1\text{-}b_1,$$

contrary to 2.1. This proves (7).

In view of (4), (5) and (7), we may assume that u, v both have neighbours in $A_1 \cup C_1$; v has a neighbour in $A_2 \cup C_2$ and none in $A_3 \cup C_3 \cup D_3$, and u has a neighbour in $A_3 \cup C_3$ and none in $A_2 \cup C_2 \cup D_2$.

(8) *u has no neighbour in B_2, and v has no neighbour in B_3.*

Suppose that v has a neighbour in B_3, say b_3, and so $x(v) = 0$. Let R_3 be an S_3-rung with ends a_3, b_3. The path $a\text{-}a_3\text{-}R_3\text{-}b_3$ is odd, since its ends are neighbours of v, and so $x(a) = 0$.

Suppose first that $x(u) = 0$. There is an $N(u) - N(v)$ path with one end b_3 and otherwise contained in $A_3 \cup C_3$. Its length has parity $x(u)$, and it is anticomplete to P_1, where P_1 is an $N(u) - N(v)$ path within $A_1 \cup C_1$; so P_1 has odd length by (2). Hence there is no $N(u) - N(v)$ path within the connected set $B_2 \cup C_2 \cup D_2 \cup B_3 \cup C_3 \cup D_3$, and so u is anticomplete to this set. By (6) u has type β_2, a contradiction since u has no neighbour in $B_3 \cup C_3$.

This shows that $x(u) = 1$, and hence u has no neighbour in B. Let R_1 be an S_1-rung with ends $a_1 \in A_1$ and $b_1 \in B_1$, that contains a neighbour of u, and let T be an $N(u) - B_1$ subpath of R_1. Thus T has parity $x(u)$ and hence is odd, and so $a_1 \notin V(T)$ since $x(a) = 0$. Consequently u has a neighbour in R_1^*. Since the connected sets $\{a\}$, R_1^* and $V(S_3)$ are pairwise anticomplete, (3) implies that v has no neighbour in R_1^*. But the path $T\text{-}b_1\text{-}b_3$ is even, and anticomplete to $\{a\}$; and so this path is not an $N(u) - N(v)$ path, and so v has a neighbour in T, and therefore v, b_1 are adjacent. Since R_1 is even, and v has no neighbour in R_1^*, it follows that v, a_1 are not adjacent. But then there is a short pyramid with apex a, base $\{v, b_1, b_3\}$, and constituent paths

$$a\text{-}a_1\text{-}R_1\text{-}b_1,$$

$$a\text{-}v,$$

$$a\text{-}a_3\text{-}R_3\text{-}b_3,$$

contradicting that a is splendid. This proves (8).

Thus u has no neighbour in $V(S_2) \cup D_2$, and v has no neighbour in $V(S_3) \cup D_3$. By (6), u has type β_2 and v has type β_3. Since v has a neighbour in $B_2 \cup C_2$, there is an S_2-rung R_2 with ends $a_2 \in A_2$ and $b_2 \in B_2$, such that v has a neighbour in R_2 different from a_2. Choose an S_3-rung R_3 with ends a_3, b_3 similarly for u. Now v has two nonadjacent neighbours in the hole

$$a\text{-}a_2\text{-}R_2\text{-}b_2\text{-}b_3\text{-}R_3\text{-}a_3\text{-}a,$$

and hence it has at least three, and an odd number; and they all belong to R_2 except a. Similarly R_3 contains a positive even number of neighbours of u. Also, the hole

$$v\text{-}R_2\text{-}b_2\text{-}b_3\text{-}R_3\text{-}a_3\text{-}a\text{-}v$$

is odd, and so $x(v) \neq x(a)$, and similarly $x(u) \neq x(a)$.

(9) *Every S_1-rung contains an even number of neighbours of v, and an even number of neighbours of u.*

Let R_1 be an S_1-rung with ends $a_1 \in A_1$ and $b_1 \in B_1$. Since

$$a\text{-}a_1\text{-}R_1\text{-}b_1\text{-}b_2\text{-}R_2\text{-}a_2\text{-}a$$

is a hole, and the path R_2-a_2-a contains an odd number at least three of neighbours of v, and the total cannot be even and at least three, it follows that there is an even number of neighbours of v in R_1. Similarly R_1 contains an even number of neighbours of u. This proves (9).

(10) *For every $N(u) - N(v)$ path P_1 within $A_1 \cup C_1$, P_1 has even length, and either $V(P_1) \subseteq A_1$, or one end of P_1 belongs to A_1 and its other vertices belong to C_1. In particular, $A_1 \cap V(P) \neq \emptyset$.*

There is an $N(u) - N(v)$ path Q within $B_2 \cup C_2 \cup B_3 \cup C_3$, and it is anticomplete to $\{a\}$ and so odd; and it is also anticomplete to P_1, and so P_1 is even. Now u-P_1-v-Q-u is a hole H say, and the neighbours of a in it are u, v, and all vertices of $V(P_1) \cap A_1$. Since a is splendid and therefore $V(G) \setminus N[a]$ is connected, 3.2 implies that either

- a is complete to H; or
- the subgraph induced on the set of vertices of H adjacent to a is a path; or
- a has exactly three neighbours in H, and two of them are adjacent.

The first is impossible since a is not complete to $V(Q)$. The second implies that $V(P_1)$ is complete to a, that is, $V(P_1) \subseteq A_1$; and the third implies that one end of P_1 belongs to A_1 and the others belong to C_1. This proves (10).

(11) *No S_1-rung meets both $N(v_1)$ and $N(v_2)$.*

Let R_1 be an S_1-rung with ends $a_1 \in A_1$ and $b_1 \in B_1$. By (10), not both $N(u), N(v)$ meet R_1^*, so we may assume that $N(v) \cap V(R_1) = \{a_1, b_1\}$ (since it has even cardinality by (9)). Thus $b_1 \notin N(u)$, and so $N(u)$ meets R_1^* by (9). Since u has an even number of neighbours in $V(R_1)$, and v-a_1-R_1-b_1-v is a hole, and there is no even wheel and no theta, it follows that u has exactly two neighbours in R_1 and they are adjacent. But then the subgraph induced on $V(R_1) \cup \{u, v, a\}$ is a near-prism, contrary to 2.1. This proves (11).

(12) *There is no $N(u) - N(v)$ path within $A_1 \cup C_1$ with one end in A_1 and all other vertices in C_1.*

Suppose there is such a path, P say. Let P have ends $p \in A_1 \cap N(u)$ and $q \in N(v)$ (possibly $p = q$), with $V(P) \setminus \{p\} \subseteq C_1$. If $p = q$, an S_1-rung with one end p contradicts (11); so $p \neq q$. Let R_1 be an S_1-rung with ends $a_1 \in A_1$ and $b_1 \in B_1$, containing q. The path p-P-q-R_1-b_1 includes an S_1-rung with one end in $N(u)$, and therefore contains another neighbour of u by (9). This does not belong to $V(P)$, so it belongs to $V(R_1)$; and so $V(R_1)$ meets both $N(u)$ and $N(v)$, contrary to (11). This proves (12).

From (10) and (12), every $N(u) - N(v)$ path within $A_1 \cup C_1$ is within A_1. Choose P_1 as in (10) to have as few vertices in A_1 as possible. It follows that $V(P_1) \subseteq A_1$. Let P_1 have ends p, q, where p is adjacent to u and q to v. From (11) $p \neq q$. Let R_1 be an S_1-rung with one end p, and let b_1 be the end of R_1 in B_1. By (11), v has no neighbour in $V(R_1)$. Now $V(P_1 \setminus p)$ is disjoint from $V(R_1 \setminus p)$; suppose these two sets are anticomplete. Then q-P_1-p-R_1-b_1 is an $N(v) - B_1$ path, and so it has parity $x(v)$. But its parity is the same as that of R_1, since P_1 is even; and so $x(v) = x(a)$, a contradiction. Hence $V(P_1 \setminus p)$ is not anticomplete to $V(R_1 \setminus p)$.

Suppose that $V(P_1 \setminus p)$ is not anticomplete to R_1^*. Since every $N(u) - N(v)$ path within $A_1 \cup C_1$ is within A_1, it follows that no vertex of R_1^* is adjacent to u. But from (11), at least two vertices of R_1 are adjacent to u, and so b_1 is adjacent to u. Since $V(P_1 \setminus p)$ is not anticomplete to $V(R_1 \setminus p)$, there is an S_1-rung with one end b_1 and the other in $V(P_1 \setminus p)$, and this S_1-rung therefore contains a unique neighbour of u, contrary to (9).

Thus $V(P_1 \setminus p)$ is anticomplete to R_1^*, and so b_1 has a neighbour $r \in V(P_1 \setminus p)$. By (9), u has a neighbour in $V(R_1 \setminus p)$, and so there is an induced path Q between u, b_1 with interior in R_1^*. Hence Q has parity $x(u) + 1$, and since the path r-b_1 is an S_1-rung and so has parity $x(a) \neq x(u)$, it follows that a-u-Q-b_1-r-a is an even hole, a contradiction. This proves 6.3. ∎

7 Using the decomposition theorems

Let $S = (A, B, C)$ be a strip in a graph G, and let $a \in V(G) \setminus V(S)$ be complete to A and anticomplete to $B \cup C$. Let D be the union of all the vertex sets of all components F of $G \setminus (V(S) \cup N[a])$ such that F is not anticomplete to $A \cup C$, and let Z be the set of all vertices in $V(G) \setminus V(S)$ that are adjacent or equal to a and have a neighbour in $A \cup C \cup D$. For $v \in Z$, a *backdoor* for v is an induced path R of G with ends v, b say, such that R^* is anticomplete to $V(S) \cup D$, and b is complete to B and has no neighbours in $A \cup C \cup D$. We say (S, a, D, Z) is a *completed strip* if

- S is proper;
- Z is a clique; and
- every vertex in Z has a backdoor.

We will see that both our decomposition theorems yield completed strips; and completed strips are good for finding bisimplicial vertices by induction, because of the following.

7.1 *Let G be even-hole-free, such that 1.2 holds for all graphs with fewer vertices than G. Let (S, a, D, Z) be a completed strip in G, where $S = (A, B, C)$. Let there be at least three vertices in G that are not in $A \cup C \cup D$ and have no neighbour in this set. Then some vertex in $A \cup C \cup V(F)$ is bisimplicial in G.*

Proof. For each $z \in Z$, let R_z be a backdoor for z. Let Z_1 be the set of all $z \in Z$ such that R_z has odd length, and Z_2 the set for which R_z has even length.

(1) *If $v \in A \cup C \cup D$, then every neighbour of v in G belongs to $V(S) \cup D \cup \{a\} \cup Z$.*

Suppose $u \in V(G)$ is adjacent to v, and $u \notin V(S) \cup D \cup \{a\} \cup Z$. Thus u is not adjacent to a, since $u \notin Z$ and $v \in A \cup C \cup D$. If $v \in A \cup C$ then $u \in D$ from the definition of D; and if $v \in D$, let $v \in V(F)$ where F is a component of $G \setminus V(S)$ such that F is anticomplete to a and not anticomplete to $A \cup C$; then u also belongs to $V(F)$ and hence to D, a contradiction. This proves (1),

(2) *If $z \in Z_2$, every induced path between z, B with interior in $A \cup C \cup D$ is even.*

Let P be an induced path between z and some $b' \in B$ with interior in $A \cup C \cup D$; then $V(P \cup R_z)$ induces an odd hole, and since R_z is even it follows that P is even. This proves (2).

(3) *If $z \in Z_1$, every induced path between z and B with interior in $Z_2 \cup A \cup C \cup D$ is odd.*

Let P be an induced path between z and some $b' \in B$ with interior in $Z_2 \cup A \cup C \cup D$. If $Z_2 \cap V(P) = \emptyset$, then $V(P \cup R_z)$ induces an odd hole, and since R_z is odd it follows that P is odd. So we may assume that there exists $z_2 \in V(P) \cap Z_2$. Since $Z_1 \cup Z_2$ is a clique, z_2 is unique, and is the neighbour of z_1 in P. Thus $P \setminus z_1$ is an induced path between z_2 and b with interior in $A \cup C \cup D$, and so is even by (2); and so P is odd. This proves (3).

Let G' be the graph obtained from $G[V(S) \cup D \cup Z]$ by adding two new vertices b, c, where b is complete to $B \cup Z_1$ and c is complete to $Z \cup \{b\}$. We claim that G' is even-hole-free. To see this, suppose that H is an even hole in G'. Since G is even-hole-free, H contains at least one of b, c; and if H contains c then it also contains b since the other G'-neighbours of c are a clique. Thus $b \in V(H)$. If both H-neighbours of b belong to B, then there is an induced subgraph of the even-hole-free graph $G[V(S) \cup D \cup Z \cup V(R_a)]$ isomorphic to H, which is impossible. Thus b is H-adjacent to some vertex $z_1 \in Z_1 \cup \{c\}$. Since b is G'-complete to $Z_1 \cup \{c\}$, only one vertex of H belongs to this set. Consequently the other H-neighbour of b belongs to B, and $|V(H) \cap B| = 1$. If $z_1 \in Z_1$ then $c \notin V(H)$ and $H \setminus b$ is an even induced path of G between z_1 and B_1 with interior in $Z_2 \cup A \cup C \cup D$, contrary to (3). Thus $z_1 = c$, and hence $V(H) \cap Z_1 = \emptyset$, and the other H-neighbour of c is some $z_2 \in Z_2$. But then $H \setminus \{b, c\}$ is an odd induced path of G between z_2, B with interior in $A \cup C \cup D$, contrary to (2). This proves that G' is even-hole-free.

Now $A \neq \emptyset$, and so bc is a non-dominating clique of G', since S is proper. But $|V(G')| < |V(G)|$, since every vertex of G' except b, c belongs to $V(G)$ and is not anticomplete to $A \cup C \cup D$. From the inductive hypothesis, there is a vertex $v \in V(G') \setminus N_{G'}[b, c]$ that is bisimplicial in G. Consequently $v \in A \cup C \cup D$. Since v is nonadjacent to b, all edges of G' with both ends in $N_{G'}(v)$ are edges of G. But all neighbours of v in G are neighbours of v in G', by (1); and so v is bisimplicial in G. This proves 7.1. ∎

In order to prove 1.2, we will show:

7.2 *Let G be an even-hole-free graph, such that 1.2 holds for all graphs with fewer vertices than G. Let K be a non-dominating clique in G with $|K| \leq 2$. Then some vertex in $V(G) \setminus N[K]$ is* *bisimplicial in G.*

We divide the proof into four parts. First we need:

7.3 *Let G be an even-hole-free graph, such that 1.2 holds for all graphs with fewer vertices than G. Let K be a non-dominating clique in G with $|K| \le 2$, and let $a \in V(G) \setminus N[K]$ be splendid, and such that there is an extended near-prism in G with cross-edge ab for some b. Then some vertex in $V(G) \setminus N[K]$ is bisimplicial in G.*

Proof. Choose a tree J and a J-strip system M in G with the same cross-edge ab, with (J, M) optimal for ab. Let Z be the set of all vertices adjacent to both a, b, and Y the set of major vertices. Let (α, β) be the corresponding partition. For each $e = st \in E(J)$ with $t \in \alpha$, let D_e be the union of the vertex sets of all components of $G \setminus (V(M) \cup Z)$ that are not anticomplete to $M_e \setminus M_s$. By 5.5, if F' is such a component then a, b have no neighbour in F', and every vertex in $V(M)$ with a neighbour in F' belongs to M_e.

(1) *For each edge $e = st$ of J with $t \in \alpha$, there is a bisimplicial vertex of G in $(M_e \setminus M_s) \cup D_e$, where D_e is the union of the vertex sets of all components of $G \setminus (V(M) \cup Z)$ that are anticomplete to a and not anticomplete to $M_e \setminus M_s$.*

Let $A = M_t \cap M_e$, $B = M_s \cap M_e$, $C = M_e \setminus (M_s \cup M_t)$ and $D = D_e$; then $S = (A, B, C)$ is a strip, and it is proper, by 5.5. Let Z' be the set of all vertices in $V(G) \setminus V(S)$ that are adjacent or equal to a and have a neighbour in $A \cup C \cup D$. We claim that all vertices in Z' are major. Let $z \in Z'$. Then $\{z\}$ is not small, since a has a neighbour in $\{z\}$, and so b, z are adjacent; and hence $z \in Z$. Since z has a neighbour in $V(S)$, and b has no neighbour in $V(S)$, it follows that z is b-external; and since a is splendid, every vertex in $N(a)$ is a-external. This proves that $z \in Y$. Consequently Z' is a clique, by 5.2.

Choose $t' \in \beta$, and let P be a path of J with ends s, t'. Choose an f-rung R_f for each $f \in E(P)$. Let u, v be the ends of R_P, where $u \in M_{t'}$ and $v \in M_s$. For each $z \in Z'$, since z is adjacent to b, there is a path from z to v with interior in $V(R_P) \cup \{b\}$; and this is a backdoor for z since v is complete to B and anticomplete to $A \cup C$.

Now D is the union of the vertex sets of all components F of $G \setminus (V(M) \cup Z)$ that are not anticomplete to $M_e \setminus M_s$. By 5.5, for each such F, a has no neighbour in $V(F)$; and so D is the union of the vertex sets of all components F of $G \setminus (V(S) \cup N[a])$ such that F is not anticomplete to $A \cup C$. Hence (S, a, D, Z') is a completed strip, and there are at least three vertices of G that are anticomplete to $A \cup C \cup D$, namely b and at two vertices of $M_{e'}$ (the latter has at least two vertices, since the corresponding strip is proper by 5.5). From 7.1, there is a bisimplicial vertex of G in $A \cup C \cup D$. This proves (1).

Choose edges $e = st$ and $e' = s't'$ of J where $t, t' \in \alpha$ are distinct; then by (1), there are bisimplicial vertices $v \in (M_e \setminus M_s) \cup D_e$, and $v' \in (M_{e'} \setminus M_{s'}) \cup D_{e'}$, defining $D_e, D_{e'}$ as in (1). Suppose they both belong to $N[K]$. Now for $k \in K$, k is not adjacent to a since $a \in V(G) \setminus N[K]$ by hypothesis; and so $k \notin Z$. We may choose $k \in K$ adjacent or equal to v, and so k is not anticomplete to $(M_e \setminus M_s) \cup D_e$. Consequently $k \in M_e \cup D_e$. Similarly there exists $k' \in K \cap (M_{e'} \cup D_{e'})$. But $M_e \cup D_e$ is anticomplete to $M_{e'} \cup D_{e'}$ by 4.2, a contradiction. This proves that one of v, v' is anticomplete to K, and so satisfies the theorem. This proves 7.3. ∎

Second, we need:

7.4 *Let G be an even-hole-free graph, such that 1.2 holds for all graphs with fewer vertices than G. Let K be a non-dominating clique in G with $|K| \le 2$, and let $a \in V(G) \setminus N[K]$ be splendid. Suppose that there is no extended near-prism in G such that a is an end of its cross-edge, and there is a pyramid in G with apex a. Then some vertex in $V(G) \setminus N[K]$ is bisimplicial in G.*

Proof. From 6.2 since there is a pyramid with apex a, and all its constituent paths have length at least two (because a is splendid), there is an indecomposable strip system with apex a. Let $\mathcal{S} = (a, S_1, \ldots, S_k)$ be an indecomposable strip system with apex a, with strips $S_i = (A_i, B_i, C_i)$ for $1 \le i \le k$, chosen with $V(\mathcal{S})$ maximal. In the notation of 6.2, for $1 \le i \le k$, let D_i be the union of the vertex sets of all components F of $G \setminus (V(\mathcal{S} \cup N[a])$ such that $\mathcal{S}(F) \cap (A_i \cup C_i) \neq \emptyset$.

(1) *For $1 \le i \le k$, there is a bisimplicial vertex of G in $A_i \cup C_i \cup D_i$.*

Let $1 \le i \le k$, $i = 1$ say; and let Z be the set of all $z \in N(a) \setminus V(\mathcal{S})$ such that z has a neighbour in $A_1 \cup C_1 \cup D_1$. Thus Z is a clique by 6.3. We need to show that each $z \in Z$ has a backdoor. By 6.2, z has type α, α', β or γ, and hence for some $2 \le j \le k$, z has a neighbour in $V(S_j)$. Choose an S_j-rung R in which z has a neighbour, with an end $b \in B_j$ say; then a path between z, b with interior in $V(R)$ provides a backdoor. Thus each $z \in Z$ has a backdoor; and there are at least three vertices in G that are anticomplete to $A_1 \cup C_1 \cup D_1$, for instance all vertices of $A_2, \ldots, A_k$ and $B_2, \ldots, B_k$. By 7.1, this proves (1).

Since $|K| \le 2$ and $k \ge 3$, we may assume that K is disjoint from $S_1 \cup D_1$. Let $v \in A_1 \cup C_1 \cup D_1$ be bisimplicial. Since K is anticomplete to a, it follows from 6.2 that K is anticomplete to v, and so v satisfies the theorem. This proves 7.4. ∎

Third, we need:

7.5 *Let G be an even-hole-free graph, such that 1.2 holds for all graphs with fewer vertices than G. Let K be a non-dominating clique in G with $|K| \le 2$, and let $a \in V(G) \setminus N[K]$ be splendid. Suppose that there is no pyramid in G with apex a. Then some vertex in $V(G) \setminus N[K]$ is bisimplicial in G.*

Proof. By 3.1, we may assume that G does not admit a full star cutset. We begin with:

(1) *There do not exist distinct $y_1, y_2, y_3 \in N(a)$, pairwise nonadjacent.*

Suppose such y_1, y_2, y_3 exist. Now $G \setminus N[a]$ is connected, and y_1, y_2, y_3 all have neighbours in it, since a is splendid. Let S be a minimal connected induced subgraph of $G \setminus N[a]$ such that y_1, y_2, y_3 all have neighbours in S. No two of y_1, y_2, y_3 have a common neighbour in $V(S)$, since such a vertex would make a 4-hole with a and two of y_1, y_2, y_3. Consequently $|V(S)| \ge 2$, and so there are at least two vertices $x \in V(S)$ such that $S \setminus x$ is connected. Choose two such vertices x_1, x_2 say. From the minimality of S, for $i = 1, 2$ one of y_1, y_2, y_3 has no neighbour in $V(S) \setminus \{x_i\}$, and so we may assume that for $i = 1, 2$, x_i is the unique neighbour of y_i in $V(S)$. Let $P = p_1\text{-}\cdots\text{-}p_k$ be an induced path of S with $p_1 = x_1$ and $p_k = x_2$. Now y_3 might or might not have neighbours in $V(P)$. Let $Q = q_0\text{-}q_1\text{-}\cdots\text{-}q_\ell$ be a minimal path in $G[S \cup \{y_3\}]$ where $q_0 = y_3$ and q_ℓ has a neighbour in $V(P)$.

(Thus if y_3 has a neighbour in $V(P)$ then $\ell = 0$.) If q_ℓ has a unique neighbour $p_i \in V(P)$, there is a theta in G with ends a, p_i and constituent paths

$$a\text{-}y_1\text{-}P\text{-}p_i,$$

$$a\text{-}y_2\text{-}P\text{-}p_i,$$

$$a\text{-}y_3\text{-}Q\text{-}p_i,$$

contrary to 2.1.

Suppose that q_ℓ has two nonadjacent neighbours in $V(P)$. Then $\ell = 0$ by the minimality of S (because if $\ell > 0$, we could delete from S a vertex of P between the first and last neighbour of q_ℓ in P). Let H be the hole induced on $V(P) \cup \{a, y_1, y_2\}$. Then y_3 is adjacent to a and not to its neighbours in H; and y_3 has two other neighbours in $V(H)$, nonadjacent to each other. Since G admits no full star cutset, this is contrary to 3.2.

Thus q_ℓ has exactly two neighbours in $V(P)$ and they are adjacent, say p_i, p_{i+1}. But then there is a pyramid with apex a, base $\{q_\ell, p_i, p_{i+1}\}$ and constituent paths

$$a\text{-}Q\text{-}q_\ell,$$

$$a\text{-}y_1\text{-}P\text{-}p_i,$$

$$a\text{-}y_2\text{-}P\text{-}y_{i+1},$$

a contradiction. This proves (1).

We suppose that a is not bisimplicial, and so the graph complement of $G[N(a)]$ is not bipartite, and hence has an induced odd cycle. It has no induced cycle of length at least six, since $G[N(a)]$ has no 4-hole; and none of length three by (1). Thus it has an induced cycle of length five, and hence so does $G[N(a)]$. Let $v_1\text{-}\cdots\text{-}v_5\text{-}v_1$ be a 5-hole of G where $v_1, \ldots, v_5$ are adjacent to a. Choose a connected subgraph S with $V(S) \cap N(a) = \emptyset$, minimal such that at least four of $v_1, \ldots, v_5$ have a neighbour in $V(S)$.

(2) *If $u, v \in \{v_1, \ldots, v_5\}$ are nonadjacent then they have no common neighbour in $V(S)$.*

Because if $s \in V(S)$ is adjacent to both u, v then $s\text{-}u\text{-}a\text{-}v\text{-}s$ is a 4-hole. This proves (2).

(3) *If $P = p_1\text{-}\cdots\text{-}p_k$ is a path of S such that p_1v_2 and p_kv_4 are edges, then one of v_1, v_5 has a neighbour in $\{p_1, \ldots, p_k\}$.*

Suppose not, and choose k minimum. Thus $v_2\text{-}p_1\text{-}p_k\text{-}v_4$ is an induced path. If v_3 is nonadjacent to $p_1, \ldots, p_k$ then there is a theta with ends v_2, v_4 and constituent paths

$$v_2\text{-}v_3\text{-}v_4,$$

$$v_2\text{-}v_1\text{-}v_5\text{-}v_4,$$

$$v_2\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v_4,$$

contrary to 2.1. So v_3 is adjacent to at least one of $p_1, \ldots, p_k$. Let v_3 be adjacent to $n \geq 1$ of $p_1, \ldots, p_k$. If n is odd then there is an even wheel with centre v_3 and hole a-v_2-p_1-$\cdots$-p_k-v_4-a; and if n is even there is an even wheel with centre v_3 and hole v_1-v_2-p_1-$\cdots$-p_k-v_4-v_5-v_1, in both cases contrary to 2.1. This proves (3).

From (2) it follows that $|V(S)| \geq 2$. Let X be the set of vertices $x \in V(S)$ such that $S \setminus x$ is connected. For each $x \in X$, let $T(x)$ be the set of $v \in \{v_1, \ldots, v_5\}$ such that x is the unique neighbour of v in $V(S)$. The minimality of S implies that $T(x) \neq \emptyset$ for each $x \in X$, and (2) implies that $T(x)$ is a clique.

(4) *Exactly four of $v_1, \ldots, v_5$ have a neighbour in $V(S)$.*

Suppose $v_1, \ldots, v_5$ all have a neighbour in $V(S)$. From the minimality of S, it follows that $|T(x)| \geq 2$ for each $x \in X$, and since the sets $T(x)$ $(x \in X)$ are pairwise disjoint, it follows that $|X| \leq 2$. Since $|V(S)| \geq 2$ and S is connected, it follows that S is a path of length at least one, and X consists of the ends of S. Let S have vertices s_1-$\cdots$-s_k in order. Now $T(s_1)$ is a clique, so we may assume that $T(s_1) = \{v_1, v_2\}$. Since $T(s_1), T(s_k)$ are disjoint, similarly we may assume that $T(s_k) = \{v_4, v_5\}$. Thus each of v_1, v_2, v_4, v_5 has a unique neighbour in $V(S)$, and v_3 has at least one such neighbour. But then there is a 4-hole with centre v_3 and hole s_1-S-v_3-a-v_1-s_1, contrary to 2.1. This proves (4).

We may therefore assume that v_3 has no neighbour in $V(S)$. If $x \in X$, then $T(x) \neq \{v_2\}$, since otherwise $S \setminus x$ would contain a path in which v_1, v_4 have neighbours and v_2, v_3 do not, contrary to (3). Similarly $T(x) \neq \{v_4\}$, and $T(x) \neq \{v_2, v_4\}$ since $T(x)$ is a clique. Thus $T(x)$ contains one of v_1, v_5. Hence $|X| = 2$, and so S is a path s_1-$\cdots$-s_k say, where $v_1 \in T(s_1)$ and $v_5 \in T(S_k)$. If both v_2, v_4 have a neighbour in S^*, there is a theta with ends v_2, v_4 and constituent paths

$$v_2\text{-}v_3\text{-}v_4,$$

$$v_1\text{-}v_1\text{-}v_5\text{-}v_4,$$

$$v_2\text{-}G[S^*]\text{-}v_4,$$

contrary to 2.1. From the symmetry we may therefore assume that v_2 has no neighbour in S^*. Also by (2), v_2 is nonadjacent to s_k, so $v_2 \in T(s_1)$. Let v_4 have n neighbours in $V(S)$. If n is even then there is an even wheel with centre v_4 and hole a-v_2-s_1-$\cdots$-s_k-v_5-a, and if n is odd and $n > 1$ then there is an even wheel with centre v_4 and hole v_1-s_1-$\cdots$-s_k-v_5. Thus $n = 1$. Let s_i be the unique neighbour of v_4 in $V(S)$. If $i = k$, there is a prism with bases $\{v_1, v_2, s_1\}$, $\{v_4, v_5, s_k\}$ and constituent paths

$$v_1\text{-}v_5,$$

$$v_2\text{-}v_3\text{-}v_4,$$

$$s_1\text{-}\cdots\text{-}s_k,$$

contrary to 2.1. If $i < k$ there is a theta with ends s_i, v_5 and constituent paths

$$s_i\text{-}\cdots\text{-}s_k\text{-}v_5,$$

$$s_i\text{-}\cdots\text{-}s_1\text{-}v_1\text{-}v_5,$$

$$s_i\text{-}v_4\text{-}v_5,$$

contrary to 2.1. This proves 7.5. ∎

Finally, the fourth part of the proof of 7.2; we will show:

7.6 *Let G be even-hole-free, and let K be a non-dominating clique in G with $|K| \leq 2$. Suppose that 1.2 holds for all graphs with fewer vertices than G, but there is no bisimplicial vertex of G in $V(G) \setminus N[K]$. Then there is a splendid vertex in $V(G) \setminus N[K]$.*

Proof. If $K \neq \emptyset$ let Z be the set of all vertices in $V(G) \setminus K$ that are complete to K, and if $K = \emptyset$ let $Z = \emptyset$. Choose $a \in V(G) \setminus N[K]$ with as few neighbours in Z as possible; and subject to that, with degree as small as possible. We claim that a is splendid. By 3.1 we may assume that G admits no full star cutset, and so for every vertex v, the subgraph induced on $V(G) \setminus N[v]$ is connected. In particular, this holds when $v = a$, which is the first requirement to be splendid.

(1) *Every vertex in $N(a)$ has a neighbour in $V(G) \setminus N[a]$.*

Suppose that $v \in N(a)$ has no neighbour in $V(G) \setminus N[a]$. Then every neighbour of v belongs to $N[a]$, and in particular, $v \notin N[K]$, and every vertex in Z adjacent to v is also adjacent to a, and the degree of v is at most that of a. From the choice of a, equality holds, and so a, v have the same neighbours (except for a, v themselves). Let $G' = G \setminus v$. Since K is non-dominating in G', the inductive hypothesis implies that there exists $u \in V(G') \setminus N_{G'}[K]$ that is bisimplicial in G'. If $u = a$, then since v is adjacent to every neighbour of a, it follows that a is bisimplicial in G; so we may assume that u, v, a are all distinct. If u, v are nonadjacent, then u is bisimplicial in G. If u, v are adjacent, then u, a are adjacent, and since v, a have the same neighbours in $N[u]$, it follows that u is bisimplicial in G. In each case this is impossible. This proves (1).

Suppose there is a short pyramid in G with apex a; with base $\{b_1, b_2, b_3\}$ say, and constituent paths R_1, R_2, R_3 where R_i has ends a, b_i for $i = 1, 2, 3$, and R_3 has length one. Thus R_1, R_2 have length at least three. For $i = 1, 2$ let y_i be the neighbour of a in R_i. Let S be the set of vertices of G nonadjacent to both a, b_3.

(2) *If $P = p_1\text{-}\cdots\text{-}p_k$ is a path with $p_1, \ldots, p_k \in S$, of minimum length such that p_1 has a neighbour in $R_1^* \setminus \{y_1\}$ and p_k has a neighbour in $V(R_2)$, then p_1 has exactly two adjacent neighbours in $V(R_1)$ and y_2 is the unique neighbour of p_k in $V(R_2)$, and these three edges are the only edges between $\{p_1, \ldots, p_k\}$ and $V(R_1 \cup R_2 \cup R_3)$.*

From the minimality of k, none of $p_2, \ldots, p_k$ has a neighbour in $R_1^* \setminus \{y_1\}$, but they might be adjacent to b_1 or y_1. Also none of $p_1, \ldots, p_{k-1}$ has a neighbour in $V(R_2)$. (Note that possibly $k = 1$.) Suppose that p_k has two nonadjacent neighbours in $V(R_2)$. Then there is a theta with ends p_k, a and constituent paths

$$p_k\text{-}R_2\text{-}a,$$

$$p_k\text{-}R_2\text{-}b_2\text{-}b_3\text{-}a$$

$$p_k\text{-}(P \cup R_1 \setminus b_1)\text{-}a,$$

contrary to 2.1. If p_k has exactly two neighbours x, y in R_2 and they are adjacent (and a, x, y, b_2 are in this order in R_2, say), there is a near-prism with bases $\{b_1, b_2, b_3\}$ and $\{p_k, x, y\}$, with constituent paths

$$x\text{-}R_2\text{-}a\text{-}b_3,$$

$$p_k\text{-}(P \cup R_1 \setminus a)\text{-}b_1,$$

$$y\text{-}R_2\text{-}b_2,$$

contrary to 2.1. Thus p_k has a unique neighbour u say in $V(R_2)$. If $u \neq y_2$, there is a theta with ends u, a and constituent paths

$$u\text{-}R_2\text{-}a,$$

$$u\text{-}R_2\text{-}b_2\text{-}b_3\text{-}a,$$

$$u\text{-}(P \cup R_1 \setminus b_1)\text{-}a,$$

contrary to 2.1. So $u = y_2$. Hence p_k is not adjacent to y_1, because otherwise there would be a 4-hole $p_k\text{-}y_2\text{-}a\text{-}y_1\text{-}p_k$. If b_1 is the unique neighbour of p_k in $V(R_1)$, there is a theta with ends y_1, b_1 and constituent paths

$$y_1\text{-}a\text{-}R_1\text{-}b_1,$$

$$y_1\text{-}R_2\text{-}b_2\text{-}b_1,$$

$$y_1\text{-}p_k\text{-}b_1,$$

contrary to 2.1. So if p_k has a neighbour in $V(R_1)$ then $k = 1$. Thus the only edges between $\{p_1, \ldots, p_k\}$ and $V(R_1 \cup R_2 \cup R_3)$ are the edges between p_1 and $V(R_1)$, and the edge $p_k y_2$. If p_1 has two nonadjacent neighbours in R_1, say x, y where a, x, y, b_1 are in order in R_1, then there is a theta with ends p_1, a and constituent paths

$$p_1\text{-}x\text{-}R_1\text{-}a,$$

$$p_1\text{-}y\text{-}R_1\text{-}b_1\text{-}b_3\text{-}a,$$

$$p_1\text{-}P\text{-}p_k\text{-}y\text{-}a,$$

contrary to 2.1. If p_1 has a unique neighbour say v in $V(R_1)$, then since $v \neq y_1$ (because by hypothesis, p_1 has a neighbour in $R_1^* \setminus \{y_1\}$), there is a theta with ends v, a and constituent paths

$$v\text{-}R_1\text{-}a,$$

$$v\text{-}R_1\text{-}b_1\text{-}b_3\text{-}a,$$

$$v\text{-}P\text{-}y_2\text{-}a,$$

contrary to 2.1. So p_1 has exactly two neighbours in $V(R_1)$ and they are adjacent. Ths proves (2).

(3) *There is no path $p_1, \ldots, p_k$ with $p_1, \ldots, p_k \in S$, such that p_1 has a neighbour in $R_1^* \setminus \{y_1\}$ and p_2 has a neighbour in $R_2^* \setminus \{y_2\}$.*

Suppose $P = p_1, \ldots, p_k$ is such a path, chosen with k minimum. Note that y_1, y_2, b_1, b_2 may have neighbours in the interior of P, but from the minimality of k, $p_1, \ldots, p_{k-1}$ have no neighbours in $R_2^* \setminus \{y_2\}$, and $p_2, \ldots, p_k$ have no neighbours in $R_1^* \setminus \{y_1\}$. Choose $i \in \{1, \ldots, k\}$ minimum such that p_i has a neighbour in $V(R_2)$. From (2) applied to the path $p_1\text{-}\cdots\text{-}p_i$, it follows that p_1 has exactly two neighbours in $V(R_1)$, say x_1, y_1, and they are adjacent, and y_2 is the unique neighbour of p_i in $V(R_2)$, and these three edges are the only edges between $\{p_1, \ldots, p_i\}$ and $V(R_1 \cup R_2 \cup R_3)$. In particular $i < k$. Choose $j \in \{1, \ldots, k\}$ maximum such that p_j has a neighbour in $V(R_1)$; then similarly p_k has exactly two neighbours in $V(R_2)$, say x_2, y_2, and they are adjacent, and y_1 is the unique neighbour of p_j in $V(R_1)$, and these three edges are the only edges between $\{p_j, \dots, p_k\}$ and $V(R_1 \cup R_2 \cup R_3)$. Thus $j > i$, and since $1 \le i < j \le k$ it follows that $k \ge 2$. Let Q be the path p_i-p_{i+1}-$\cdots$-p_j. Thus the only edges between $\{p_1, \dots, p_k\}$ and $V(R_1 \cup R_2 \cup R_3)$ are edges between p_1 and $V(R_1)$, edges between p_k and $V(R_2)$, the edges p_iy_2, p_jy_1, and edges between Q^* and $\{y_1, y_2, b_1, b_2\}$. If b_1 has a neighbour in Q^*, there is a theta with ends b_1, y_1 and constituent paths

$$b_1\text{-}R_1\text{-}y_1,$$

$$b_1\text{-}b_3\text{-}a\text{-}y_1,$$

$$b_1\text{-}Q\text{-}y_1,$$

contrary to 2.1. So b_1 has no neighbour in $\{p_2, \dots, p_k\}$, and similarly b_2 has no neighbour in $\{p_1, \dots, p_{k-1}\}$. If y_1, y_2 both have neighbours in P^*, there is a theta with ends y_1, y_2 and constituent paths

$$y_1\text{-}G[P^*]\text{-}y_2,$$

$$y_1\text{-}a\text{-}y_2,$$

$$y_1\text{-}R_1\text{-}b_1\text{-}b_2\text{-}R_2\text{-}y_2,$$

contrary to 2.1. Thus we may assume that y_2 has no neighbour in P^*, and in particular $i = 1$. Consequently p_1, y_1 are nonadjacent, since p_1-y_1-a-y_2-p_1 is not a 4-hole. Then there is a theta with ends p_1, y_1 and constituent paths

$$p_1\text{-}R_1\text{-}y_1,$$

$$p_1\text{-}R_1\text{-}b_1\text{-}b_3\text{-}a\text{-}y_1,$$

$$p_1\text{-}P\text{-}y_1,$$

contrary to 2.1. This proves (3).

For $i = 1, 2$, let S_i be the component of $G[S]$ that contains $R_i \setminus \{a, y_i, b_i\}$. So S_1, S_2 are nonempty since R_1, R_2 have length at least three; and S_1, S_2 are distinct by (3). For $i = 1, 2$, let B_i be the set of vertices adjacent to b_3 and not to a, with a neighbour in S_i. So $b_i \in B_i$ for $i = 1, 2$. If there exists $v \in B_1 \cap B_2$, there is a theta with ends v, a and constituent paths

$$v\text{-}b_3\text{-}a,$$

$$v\text{-}S_1\text{-}y_1\text{-}a,$$

$$v\text{-}S_2\text{-}y_2\text{-}a,$$

contrary to 2.1. So $B_1 \cap B_2 = \emptyset$.

The only vertices of G not in $V(S_1)$ but with a neighbour in $V(S_1)$ belong to $B_1 \cup N[a]$. From the inductive hypothesis, applied to the graph $G' = G[S_1 \cup B_1 \cup N[a] \cup \{b_3\}]$, since the edge ab_3 is non-dominating in G', it follows that some vertex in S_1 is bisimplicial in G' and hence in G. Since there is no bisimplicial vertex of G in $V(G) \setminus N[K]$, it follows that $N[K] \cap S_1 \neq \emptyset$, and similarly $N[K] \cap S_2 \neq \emptyset$. But $K \cap N[a] = \emptyset$ from the choice of a; and so $K \cap (V(S_i) \cup B_i) \neq \emptyset$ for $i = 1, 2$. Since the sets $V(S_1), B_1, B_2, V(S_2)$ are pairwise disjoint, and there are no edges between $B_2 \cup V(S_2)$

and $V(S_1)$, it follows that $K \cap S_1 = \emptyset$, and so $K \cap B_1 \neq \emptyset$; and similarly $K \cap B_2 \neq \emptyset$. In particular $|K| = 2$. Let $K \cap B_i = b_i'$ for $i = 1, 2$.

We recall that Z is the set of all vertices adjacent to both b_1', b_2', and so $b_3 \in Z$. Now a, b_3 are adjacent. But $y_1 \notin N[K]$ (because if y_1, b_i' are adjacent then there is a 4-hole y_1-b_i'-b_3-a-y_1), and y_1, b_3 are nonadjacent. From the choice of a, y_1 has at least as many neighbours in Z as does a; and since b_3 is adjacent to a and not to y_1, there exists $z \in Z$ adjacent to y_1 and not to b_3. Since z-y_1-a-b_3-z is not a 4-hole, z, b_3 are nonadjacent. Since $b_2' \in B_2$ and hence $b_2' \notin B_1$, and b_2' is adjacent to z, it follows that $z \notin V(S_1)$. But then there is a theta with ends b_1', y_1 and constituent paths

$$b_1'\text{-}z\text{-}y_1,$$

$$b_1'\text{-}b_3\text{-}a\text{-}y_1,$$

$$b_1'\text{-}S_1\text{-}y_1,$$

contrary to 2.1. This proves 7.6. ∎

From 7.3, 7.4, 7.5 and 7.6, this completes the proof of 7.2, and hence of 1.2.

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