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Graph Minors.
IV. Tree-Width and Well-Quasi-Ordering

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K. Wagner conjectured that if $G_1, G_2, ...$ is any countable sequence of finite graphs, then there exist i, j with $j > i \geqslant 1$ such that G_i is isomorphic to a minor of G_j. Kruskal proved this when $G_1, G_2, ...$ are all trees. We prove a strengthening of Kruskal's result—Wagner's conjecture is true for all sequences in which G_1 is planar. We hope to show in a future paper that Wagner's conjecture is true in general, and the results of this paper will be needed for that proof.

1. INTRODUCTION

By a "graph" we shall mean (except when we say otherwise) in this paper a finite, undirected graph which may have loops and multiple edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by edge-contraction.

There is an ever-growing collection of excluded minor theorems in graph theory. (By an "excluded minor theorem," we mean a result asserting that a graph has a specified property if and only if it has no minor isomorphic to a member of a constructively characterized set of graphs.) Perhaps the most imposing of these to date is the result of Archdeacon [1] and Glover, Huneke, and Wang [2], that a graph may be embedded in the projective plane if and only if it has no minor isomorphic to one of 35 specified graphs. An interesting aspect of all excluded minor theorems known is that not only is the set of excluded minors constructively characterized, but also

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it is *finite*. One possible explanation for this is that no one tries to find excluded minor theorems for ugly properties, and that nice properties might be expected to produce nice sets of excluded minors. But there is another possible explanation, contained in the following conjecture of K. Wagner (unpublished), which we hope to prove in a future paper.

(1.1) CONJECTURE. *If $G_1, G_2, \ldots$ is any countable sequence of graphs then there exist $j > i \geqslant 1$ such that G_i is isomorphic to a minor of G_j.*

This is connected with excluded minor theorems as follows. Let P be any property of graphs which can in principle be characterized by excluded minors; that is, any graph isomorphic to a minor of a graph with property P also has property P. Let L' be the class of all minor-minimal graphs not possessing property P, and let $L \subseteq L'$ contain exactly one representative of each isomorphism class of L'. Then we can assert "a graph has property P if and only if it has no minor isomorphic to a graph in L." But no member of L is isomorphic to a minor of another, and so Wagner's conjecture implies that L is finite.

In this paper we prove a special case of Wagner's conjecture, that the conjecture holds for a sequence $G_1, G_2, \ldots$ when G_1 is planar. We also prove some variations and extensions, which will be important in future papers of this series. (At the time of writing, we believe that we have a proof of (1.1) itself, and indeed of a similar conjecture, due to Nash–Williams, the "immersion" conjecture, although the latter has not yet been carefully written out. These proofs in particular rely heavily on the results of the present paper.)

A *quasi-order* Ω is a pair $(E(\Omega), \leqslant)$ where $E(\Omega)$ is a class and $\leqslant$ is a binary relation on $E(\Omega)$ which is reflexive and transitive. (This would become a partial order if we made the third requirement of antisymmetry.) We shall study the quasi-orders on classes of graphs defined by the relation "is isomorphic to a minor of." We treat this as a quasi-order rather than as a partial order, because it will be necessary to distinguish between isomorphism and equality. A quasi-order Ω is a *well-quasi-order* if for every countable sequence $x_1, x_2, \ldots$ of elements of $E(\Omega)$, there exist $i' > i \geqslant 1$ such that $x_i \leqslant x_{i'}$.

The reader may perhaps consider it more natural to study quasi-orders with no infinite antichain to approach the finiteness phenomenon for excluded minor theorems, rather than to study well-quasi-orders. However, it comes to the same thing; for if Ω has no infinite descending chain (which is obvious in our context) then it is a well-quasi-order if and only if it has no infinite antichain, as may easily be verified. We choose the sequence formulation because we find it easier to manipulate.

Incidentally, there are some other natural quasi-orders of the class of all graphs. The most obvious is the subgraph order; but that is not a well-quasi-order. (We mention in passing that it *is* a well-quasi-order when restricted to the class of all graphs with no path of length $\geqslant k$, for any fixed k. We omit the proof, which is easy.) A more plausible candidate for a well-quasi-order is "topological containment." We say that G_1 *topologically contains* G_2 if some subgraph of G_1 is isomorphic to a subdivision of G_2. (A *subdivision* of a graph is a graph which may be constructed from the first by repeatedly replacing edges by pairs of edges in series.) Kruskal [5] proved the following.

(1.2) *The class of all trees is well-quasi-ordered by topological containment.*

Using this, Mader [6] showed that

(1.3) *For any integer $k \geqslant 0$, the class of all graphs without k vertex-disjoint circuits is well-quasi-ordered by topological containment.*

However, the class of *all* graphs is not well-quasi-ordered by topological containment, as the following counterexample (basically due to Jenkyns and Nash–Williams [4]) shows. For $j \geqslant 3$, let G_j be the graph with j vertices $v_0, v_1, ..., v_j = v_0$, and with two edges joining v_i and v_{i+1} $(0 \leqslant i \leqslant j-1)$. Then no G_i is topologically contained in any G_j for any $i, j \geqslant 3$ with $j > i$. For that reason Wagner's conjecture uses minors.

When $\mathscr{C}$ is a class of graphs, we say that $\mathscr{C}$ is *well-quasi-ordered by minors* if the quasi-order $(\mathscr{C}, \leqslant)$ is a well-quasi-order, where $G \leqslant G'$ means that G is isomorphic to a minor of G'. One of the main results of this paper is

(1.4) *For any planar graph H, the class of all graphs with no minor isomorphic to H is well-quasi-ordered by minors.*

It is easy to see that (1.4) is equivalent to the result stated earlier in this section and in the abstract. We derive it as a corollary of some theorems about "tree-width," which is a graph invariant defined in Section 5. We prove that

(1.5) *For any integer k the class of all graphs with tree-width $\leqslant k$ is well-quasi-ordered by minors.*

This implies (1.4) because it is proved in [10] that the class of graphs involved in (1.4) has bounded tree-width.

Thus, one object of this paper is to prove (1.5). The basic idea of the proof is that we can regard graphs of bounded tree-width as "tree-shaped," and we can adapt Nash–Williams' proof [7] of Kruskal's theorem (1.2) about trees to apply to "tree-shaped graphs" instead.

The paper falls into two parts. In the first part (Sections 2–5) we prove a generalization of Kruskal's theorem (1.2); and apply it to prove that, for every containment relation defined on a class of rooted hypergraphs satisfying certain axioms, if it forms a well-quasi-order (in a sense) on the hypergraphs in the class of bounded size, then it forms a well-quasi-order on the hypergraphs in the class of bounded tree-width. (1.5) is a consequence of this where our hypergraphs are graphs. In the second part of the paper (Sections 6–9) we discuss a concrete (but rather complex) containment relation on a class of hypergraphs with additional structure, so-called *patchworks*. We verify that this relation does indeed satisfy our axioms for a containment relation, and so the results of the first part apply to it. These results, applied to patchworks, will be important in future papers.

2. A Lemma about Trees

In this section we prove a lemma about rooted trees generalizing (1.2). Most of the trees we will need in this paper are "rooted" trees, and so for convenience we define an *undirected tree* to be a non-null connected finite undirected graph without circuits, and a *tree* T to be a directed graph, the undirected graph underlying which is an undirected tree, such that every vertex of T is the head of at most one edge. It follows that there is a unique vertex of each tree T (called the *root* and denoted by $o(T)$) which is the head of no edge of T, and every edge of T is directed away from the root.

We begin with a preliminary form of our lemma. If M is a (possibly infinite) graph, its vertex- and edge-sets are denoted by $V(M)$ and $E(M)$, respectively. If $v_1, v_2 \in V(M)$ are adjacent in M we say they are *M-adjacent.* A subset $X \subseteq V(M)$ is *M-stable* if no two elements of X are M-adjacent, and $X \subseteq V(M)$ is *M-rich* if no infinite subset of X is M-stable.

(2.1) *Let $T_1, T_2, \ldots$ be a countable sequence of disjoint trees. Let M be an infinite graph with $V(M) = V(T_1 \cup T_2 \cup \cdots)$, such that for $i' > i \geqslant 1$, if $u \in V(T_i)$ is M-adjacent to $w \in V(T_{i'})$ and $v \in V(T_{i'}) - \{o(T_{i'})\}$ lies on the path of $T_{i'}$ from $o(T_{i'})$ to w then u is M-adjacent to v. Let $\{o(T_1), o(T_2), \ldots\}$ be M-stable. Then there is an infinite M-stable set $X \subseteq V(M)$ such that $|X \cap V(T_i)| \leqslant 1$ for each $i \geqslant 1$ and such that the set of heads of all edges of $T_1 \cup T_2 \cup \cdots$ with tails in X is M-rich.*

Proof. We proceed by a variation on Nash–Williams' [7] "minimal bad sequence" argument. Let us say a sequence $z_1, z_2, \ldots$ of elements of $V(M)$ is *increasing* if $i' > i$ for all $j' > j \geqslant 1$ where $z_j \in V(T_i)$ and $z_{j'} \in V(T_{i'})$. Let $V_0 = V(M) - \{o(T_1), o(T_2), \ldots\}$. A *section* is a countable increasing sequence $z_1, z_2, \ldots$ of elements of V_0 such that $\{z_1, z_2, \ldots\}$ is M-stable. We may assume that there is a section, for otherwise V_0 is M-rich and the sequence $o(T_1)$, $o(T_2)$, ... satisfies the theorem.

For each $v \in V_0$, let T^v be the maximal subtree of T_i (where $v \in V(T_i)$) with root v. Inductively we shall define a countable sequence $x_1, x_2, \ldots$ of elements of V_0, such that

(1) *For each $i \geqslant 1$ there is a section with first i terms $x_1, x_2, \ldots, x_i$, and*

(2) *For each $i \geqslant 1$, there is no section with first i terms $x_1, \ldots, x_{i-1}, x$ where $x \in V(T^{x_i}) - \{x_i\}$.*

For suppose that $k \geqslant 1$ and we have chosen $x_1, \ldots, x_{k-1}$ to satisfy (1) and (2) for $1 \leqslant i \leqslant k-1$. Since there is a section with first $k-1$ terms $x_1, \ldots, x_{k-1}$ we may choose x_k such that there is a section with first k terms $x_1, \ldots, x_k$ and with T^{x_k} minimal, in the sense that there is no section with first terms $x_1, \ldots, x_{k-1}, x$ for any $x \in V(T^{x_k}) - \{x_k\}$. But then (1) and (2) remain satisfied for $1 \leqslant i \leqslant k$. This completes our inductive definition of $x_1, x_2, \ldots$. Let $X = \{x_1, x_2, \ldots\}$. We shall show that X satisfies the theorem. We observe

(3) *$x_1, x_2, \ldots$ is a section.*

This is immediate from (1).

(4) *For $j' > j \geqslant 1$, if z is the head of an edge of $T_1 \cup T_2 \cup \cdots$ with tail $x_{j'}$, then x_j is not M-adjacent to z.*

For let $x_{j'} \in V(T_{i'})$. Then $x_{j'} \neq o(T_i)$ by (3), and $x_{j'}$ lies on the path of T_i from $o(T_i)$ to z. But x_j is not M-adjacent to $x_{j'}$ by (3), and hence x_j is not M-adjacent to z, from a hypothesis of the theorem.

Let Y be the set of heads of all edges of $T_1 \cup T_2 \cup \cdots$ with tails in X.

(5) *No sequence of elements of Y is a section.*

For suppose that $y_1, y_2, \ldots$ is a section of elements of Y. Since $y_1 \in Y$ there is an edge of $T_1 \cup T_2 \cup \cdots$ with head y_1 and tail x_k for some $k \geqslant 1$. Then $x_1, x_2, \ldots, x_{k-1}, y_1, y_2, \ldots$ is a section; for it is increasing, and $\{x_1, x_2, \ldots, x_{k-1}\}$ and $\{y_1, y_2, \ldots\}$ are both M-stable, and for $1 \leqslant j \leqslant k-1$ and $j' \geqslant 1$, x_j is not M-adjacent to $y_{j'}$ by (4). But $y_1 \in V(T^{x_k}) - \{x_k\}$, contrary to (2).

From (5) it follows that Y is M-rich, and so X satisfies the theorem, as required. ∎

If G is a graph or directed graph, $G \backslash F$ denotes the result of deleting F from G, where F may be an edge or a set of edges. If T is a tree and

$F \subseteq E(T)$, we define the *contraction of* T *onto* F to be the tree S with $E(S) = F$ and $V(S)$ the set of roots of the components of $T\backslash F$, in which $v \in V(S)$ is the head (or tail, respectively) of $f \in F$ if the head (or tail, respectively) of f in T belongs to the component of $T\backslash F$ with root v.

The main result of this section is the following generalization of (2.1) ((2.1) is the special case when $n=0$). Let T be a tree, $n \geqslant 0$ be an integer, and $\phi: E(T) \to \{0, ..., n\}$ be a function. For $v, w \in V(T)$ we say that v *precedes* w (*with respect to* ϕ) if $v \neq o(T)$, there is a directed path P of T from v to w, and $\phi(e) = \phi(f)$ where e, f are the edges of T with heads v, w, respectively, and $\phi(g) \geqslant \phi(f)$ for every edge g of P.

(2.2) *Let $T_1, T_2, ...$ be a countable sequence of disjoint trees, let $n \geqslant 0$ be an integer, and for each $i \geqslant 1$ let ϕ_i: $E(T_i) \to \{0, 1, ..., n\}$ be some function. Let M be an infinite graph with $V(M) = V(T_1 \cup T_2 \cup \cdots)$ such that for $i' > i \geqslant 1$, if $u \in V(T_i)$ is M-adjacent to $w \in V(T_{i'})$, and $v \in V(T_{i'})$ precedes w (with respect to $\phi_{i'}$) then u is M-adjacent to v. Let $\{o(T_1), o(T_2), ...\}$ be M-stable. Then there is an infinite M-stable set $X \subseteq V(M)$ such that $|X \cap V(T_i)| \leqslant 1$ for each $i \geqslant 1$ and such that the set of heads of all edges of $T_1 \cup T_2 \cup \cdots$ with tails in X is M-rich.*

Proof. We proceed by induction on n. If $n=0$ the result follows from (2.1), and so we may assume that $n>0$ and the result holds for $n-1$. For each $i \geqslant 1$, let $F_i = \{e \in E(T_i) : \phi_i(e) = 0\}$ and let S_i be the contraction of T_i onto F_i. Let N be the restriction of M to $V(S_1) \cup V(S_2) \cup \cdots$.

(1) *For all $i' > i \geqslant 1$, if $u \in V(S_i)$ is N-adjacent to $w \in V(S_{i'})$ and $v \neq o(S_{i'})$ lies on the path of $S_{i'}$ from $o(S_{i'})$ to w then u is N-adjacent to v.*

For let e, f be the edges of $T_{i'}$ with heads v, w respectively. Since $v, w \in V(S_{i'})$ it follows that $e, f \in F_{i'}$ and so $\phi_{i'}(e) = \phi_{i'}(f) = 0$. Since $\phi_{i'}(g) \geqslant 0$ for every $g \in E(T_{i'})$ it follows that v precedes w (with respect to $\phi_{i'}$), and so u is M-adjacent (and hence N-adjacent) to v, as required.

Let $T = T_1 \cup T_2 \cup \cdots$. From (1) we may apply (2.1) to $S_1, S_2, ...$ and N, and we deduce (defining *increasing* as in (2.1)) that

(2) *There is a countable increasing sequence $z_1, z_2, ...$ of vertices of M such that $\{z_1, z_2, ...\}$ is M-stable, and each z_j is the root of some component R_j of some $T_i\backslash F_i$, and the set A of all heads (in T) of all edges in $F_1 \cup F_2 \cup \cdots$ with tails (in T) in $R_1 \cup R_2 \cup \cdots$ is M-rich.*

For each $j \geqslant 1$ let ϕ'_j: $E(R_j) \to \{0, 1, ..., n-1\}$ be defined by $\phi'_j(e) = \phi_i(e) - 1$ $(e \in E(R_j))$, where $z_j \in V(T_i)$. From our inductive hypothesis applied to $R_1, R_2, ...$ and to the restriction of M to $V(R_1 \cup R_2 \cup \cdots)$, we deduce that

(3) *There is an infinite M-stable set* $X \subseteq V(R_1 \cup R_2 \cup \cdots)$ *such that* $|X \cap V(T_i)| \leqslant 1$ *for each* $i \geqslant 1$, *and the set B of heads of all edges of* $R_1 \cup R_2 \cup \cdots$ *with tails in X is M-rich.*

We claim that X satisfies the theorem. For let C be the set of heads of all edges of T with tails in X, and let $z \in C$. Then z is the head of some edge f of some T_i with tail in X. If $\phi_i(f) = 0$ then $f \in F_i$ and so $z \in A$. If $\phi_i(f) > 0$ then f is an edge of some R_j, since its tail is in some $V(R_j)$, and so $z \in B$. Thus $C \subseteq A \cup B$. But A is M-rich from (2), and B is M-rich from (3), and so C is M-rich. Thus X satisfies the theorem. ∎

To illustrate the use of (2.2) we show that it (indeed, (2.1)) implies Kruskal's theorem (1.2). Let $T_1, T_2, \ldots$ be a countable sequence of disjoint trees, and let M be the infinite graph with $V(M) = V(T_1 \cup T_2 \cup \cdots)$ in which for $i' > i \geqslant 1$, $u \in V(T_i)$ is M-adjacent to $v \in V(T_{i'})$ if (with notation as in (2.1)) T^v topologically contains T^u. (We define topological containment for directed graphs in the natural way; however, we do not demand that root be mapped to root.) Now the conclusion of (2.1) cannot be satisfied, by Higman's "finite sequence" theorem ((8.3), later); and so its hypothesis that $\{o(T_1), o(T_2), \ldots\}$ is M-stable is false. This proves Kruskal's theorem.

One may derive similarly a theorem of Friedman [12] using (2.2) in place of (2.1). By adjusting the definition of M accordingly, one may also derive the versions of these theorems when the vertices are labelled from a well-quasi-order.

3. Hypergraphs and Tree-Decompositions

A *hypergraph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges* with $E(G) \cap V(G) = \emptyset$, and an *incidence* relation, a subset of $V(G) \times E(G)$. The vertices incident with an edge e are its *ends*, and the set of ends of e in G is denoted by $V(e)$ (or $V_G(e)$ in cases of ambiguity). If G, G' are hypergraphs and $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, and $V_G(e) = V_{G'}(e)$ for every $e \in E(G')$, then we call G' a *subhypergraph* of G and write $G' \subseteq G$. If $G_1, G_2 \subseteq G$ we define $G_1 \cap G_2$, $G_1 \cup G_2$ in the natural way. A *separation* of G is a pair (G_1, G_2) of subhypergraphs of G with $G_1 \cup G_2 = G$ and $E(G_1 \cap G_2) = \emptyset$; its *order* is $|V(G_1 \cap G_2)|$.

A *march* in a set V is a finite sequence of distinct elements of V. If π is the march $v_1, \ldots, v_k$ we denote $\{v_1, \ldots, v_k\}$ by $\bar{\pi}$. A *rooted hypergraph* $G = (G^-, \pi(G))$ consists of a hypergraph G^- and a march $\pi(G)$ in $V(G^-)$. We define $V(G) = V(G^-)$, $E(G) = E(G^-)$. A *separation* of a rooted hypergraph G is a pair (G_1, G_2) of rooted hypergraphs such that (G_1^-, G_2^-) is a separation of G^-, and $\pi(G_2) = \pi(G)$, and $\bar{\pi}(G_1) = V(G_1^- \cap G_2^-)$. Its *order* is

$|V(G_1^- \cap G_2^-)|$. A *tree-decomposition* (T, τ) of a rooted hypergraph G consists of a tree T and a function τ which assigns to each $t \in V(T)$ a rooted hypergraph $\tau(t)$, such that

(i) for each $t \in V(T)$, $\tau(t)^- \subseteq G^-$

(ii) $\bigcup_{t \in V(T)} \tau(t)^- = G^-$

(iii) for distinct $t_1, t_2 \in V(T)$, $E(\tau(t_1)^- \cap \tau(t_2)^-) = \varnothing$

(iv) for $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1 and t_3 then $\tau(t_1)^- \cap \tau(t_3)^- \subseteq \tau(t_2)^-$

(v) $\pi(\tau(o(T))) = \pi(G)$

(vi) for each edge $f \in E(T)$ with head t_1 and tail t_2, $\bar{\pi}(\tau(t_1)) = V(\tau(t_1)^- \cap \tau(t_2)^-)$.

If T is a tree and $f \in E(T)$ we denote by T^f, T_f the two components of $T \backslash f$, where T^f has root the head of f and T_f has root $o(T)$. If $v \in V(T)$, we denote by T^v the maximal subtree of T with root v; thus if $f \in E(T)$ has head v then $T^v = T^f$. Let (T, τ) be a tree-decomposition of some rooted hypergraph. If T' is a subtree of T we denote by $\tau \times T'$ the rooted hypergraph G' defined by

$$G'^- = \bigcup_{t \in V(T')} \tau(t)^-$$

$$\pi(G') = \pi(\tau(o(T'))).$$

(3.1) *Let (T, τ) be a tree-decomposition of some rooted hypergraph G. Let $f \in E(T)$ with head t_1 and tail t_2. Then $(\tau \times T^f, \tau \times T_f)$ is a separation of G.*

This is almost identical with [9, Theorem (3.4)] or [8, Theorem (2.4)] and we omit the proof. The order of the separation in (3.1) is called the *order of f in (T, τ)*.

(3.2) *Let (T, τ) be a tree-decomposition of a rooted hypergraph G, and let $F \subseteq E(T)$. Let S be the contraction of T onto F, and for each $s \in V(S)$, let $\sigma(s)$ be $\tau \times T_s$ where T_s is the component of $T \backslash F$ with root s. Then (S, σ) is a tree-decomposition of G.*

The proof is clear, and we omit it.

A *star* S is a tree such that $o(S)$ is the tail of every edge of S, and we define $U(S) = V(S) - \{o(S)\}$. A *star-decomposition* is a tree-decomposition (S, σ) where S is a star. If (S, σ) is a star-decomposition and $s \in U(S)$ we call $\sigma(s)$ a *tip* of (S, σ). Let (T, τ) be a tree-decomposition of G, and let $s \in V(T)$. Let S be the maximal star with root s which is a subtree of T.

Define $\sigma(s) = \tau(s)$, and for each $t \in U(S)$ let $\sigma(t)$ be $\tau \times T^t$. Then (S, σ) is a star-decomposition of $\tau \times T^s$, and we call it the *branching of* (T, τ) *at* s.

We wish eventually to apply (2.2) to deduce results about a certain containment relation defined on a class of rooted hypergraphs. This relation is very unwieldy, however, and it is convenient to postpone its introduction as long as possible. We shall therefore proceed with a general containment relation ("simulation") satisfying certain "axioms;" and we shall verify that our concrete relation does indeed satisfy these axioms later.

Axiom 1. *$\mathscr{R}$ is a class of rooted hypergraphs, and if $G \in \mathscr{R}$ and H is a rooted hypergraph with $H^- \subseteq G^-$ then $H \in \mathscr{R}$.*

Axiom 2. *The relation "G is simulated in H" defines a quasi-order on $\mathscr{R}$.*

We assume henceforth that Axioms 1 and 2 are satisfied.

A subclass of $\mathscr{R}$ is *well-simulated* if it is well-quasi-ordered by simulation. The *index* of a star-decomposition (S, σ) is $\max(|\bar{\pi}(\sigma(s))| : s \in V(S))$, and the *index* of a class $\mathscr{S}$ of star-decompositions is the maximum index of its members (or ∞ if there is no such maximum). A class $\mathscr{S}$ of star-decompositions is *good* if

(a) $\sigma \times S \in \mathscr{R}$ for each $(S, \sigma) \in \mathscr{S}$

(b) $\mathscr{S}$ has finite index, and

(c) for every countable sequence (S_i, σ_i) $(i = 1, 2, ...)$ of members of $\mathscr{S}$ such that the set of all tips of all the (S_i, σ_i)'s is well-simulated, there exist $i' > i \geq 1$ such that $\sigma_i \times S_i$ is simulated in $\sigma_{i'} \times S_{i'}$.

Let (T, τ) be a tree-decomposition with $\tau \times T \in \mathscr{R}$. A subset $F \subseteq E(T)$ is *linked* in (T, τ) if for every directed path P of T with first edge $f_2 \in F$ and last edge $f_1 \in F$ such that f_1, f_2 have the same order and no element of $E(P) \cap F$ has smaller order, $\tau \times T^{f_1}$ is simulated in $\tau \times T^{f_2}$. We say that (T, τ) is *linked* if $E(T)$ is linked in (T, τ).

(3.3) *Let $\mathscr{S}$ be good, and let (T_i, τ_i) $(i = 1, 2, ...)$ be a countable sequence of linked tree-decompositions, each branching of which is in $\mathscr{S}$. Then there exist $i' > i \geq 1$ such that $\tau_i \times T_i$ is simulated in $\tau_{i'} \times T_{i'}$.*

Proof. Let n be the index of $\mathscr{S}$. We may assume that $T_1, T_2, ...$ are mutually disjoint. Let M be the infinite graph with $V(M) = V(T_1 \cup T_2 \cup \cdots)$ such that for $i' > i \geq 1$, $u \in V(T_i)$ is M-adjacent to $v \in V(T_{i'})$ if $\tau_i \times T_i^u$ is simulated in $\tau_{i'} \times T_{i'}^v$. For each $i \geq 1$, and each $f \in E(T_i)$, let $\phi_i(f)$ be the order of f in (T_i, τ_i). Then $0 \leq \phi_i(f) \leq n$ since each branching belongs to $\mathscr{S}$. Define "precedes" as before (2.2).

(1) *If $i' > i \geqslant 1$ and $u \in V(T_i)$ is M-adjacent to $w \in V(T_{i'})$, and $v \in V(T_{i'})$ precedes w, then u is M-adjacent to v.*

For $\tau_i \times T_i^u$ is simulated in $\tau_{i'} \times T_{i'}^w$ since $i' > i$ and u, w are M-adjacent. Since v precedes w it follows that $v \neq o(T_{i'})$; let e, f be the edges of $T_{i'}$ with heads v, w, respectively. Since v precedes w it follows that e and f have the same order and no edge of the path of $T_{i'}$ between them has smaller order. Since $(T_{i'}, \tau_{i'})$ is linked, we deduce that $\tau_{i'} \times T_{i'}^w$ is simulated in $\tau_{i'} \times T_{i'}^v$. By Axiom 2 it follows that $\tau_i \times T_i^u$ is simulated in $\tau_{i'} \times T_{i'}^v$, and so u is M-adjacent to v, as required.

(2) *There is no infinite subset $X \subseteq V(M)$ such that $|X \cap V(T_i)| \leqslant 1$ for each $i \geqslant 1$, and X is M-stable, and the set of heads of all edges of $T_1 \cup T_2 \cup \cdots$ with tails in X is M-rich.*

For suppose that such a set X exists; let $X = \{x_1, x_2, ...\}$ where for all $j' > j \geqslant 1$, if $x_j \in V(T_i)$ and $x_{j'} \in V(T_{i'})$, then $i' > i$. For each $j \geqslant 1$, let (S_j, σ_j) be the branching of (T_i, τ_i) at x_j, where $x_j \in V(T_i)$; then $\sigma_j \times S_j = \tau_i \times T_i^{x_j}$. Since X is M-stable, $\sigma_j \times S_j$ is not simulated in $\sigma_{j'} \times S_{j'}$ for $j' > j \geqslant 1$. Let $\mathscr{L}$ be the set of all tips of all the (S_j, σ_j)'s. Since $\mathscr{S}$ is good, $\mathscr{L}$ is not well-simulated, and so there is a countable sequence $L_1, L_2, ...$ of members of $\mathscr{L}$ such that for $j' > j \geqslant 1$, L_j is not simulated in $L_{j'}$. For each $j \geqslant 1$ there exists $i \geqslant 1$ and a vertex $v_j \in V(T_i)$ such that $L_j = \tau_i \times T_i^{v_j}$; and v_j is the head of an edge of T_i with tail in X. Now $v_1, v_2, ...$ are all distinct, since if $v_j = v_{j'}$ for some $j' > j$ then $L_j = L_{j'}$ and hence L_j is simulated in $L_{j'}$ by Axiom 2. Hence $\{v_1, v_2, ...\}$ is infinite and M-stable, contrary to our assumption that the set of heads of all edges of $T_1 \cup T_2 \cup \cdots$ with tails in X is M-rich. This proves (2).

From (1), (2) and (2.2), $\{o(T_1), o(T_2), ...\}$ is not M-stable, and the theorem follows. ∎

4. Rotundity

Now we introduce a third axiom.

Axiom 3. Let $G \in \mathscr{R}$ and let (G_1, G_2) be a separation of G of order $|\bar{\pi}(G)|$. Suppose that there is no separation (H_1, H_2) of G with $G_1^- \subseteq H_1^-$ of order less than $|\bar{\pi}(G)|$. Then there is a march π_1 with $\bar{\pi}_1 = \bar{\pi}(G_1)$ such that (G_1^-, π_1) is simulated in G.

Let (T, τ) be a tree-decomposition. A subset $F \subseteq E(T)$ is *rotund* in (T, τ) if for every directed path P of T with first edge $f_2 \in F$ and last edge $f_1 \in F$ such that f_1, f_2 have the same order, k say, and no element of $F \cap E(P)$ has smaller order, there is no separation (H_1, H_2) of $(\tau \times T)^-$ with

$(\tau \times T^{f_1})^- \subseteq H_1$ and $(\tau \times T_{f_2})^- \subseteq H_2$ of order smaller than k. We say that (T, τ) is *rotund* if $E(T)$ is rotund in (T, τ).

(4.1) *Let (T, τ) be a tree-decomposition with $\tau \times T \in \mathscr{R}$, and let $F \subseteq E(T)$ be rotund in (T, τ). Then there is a tree-decomposition (T, τ') of $\tau \times T$, such that $\tau(t)^- = \tau'(t)^-$ for each $t \in V(T)$, and such that F is linked in (T, τ').*

Proof. For each $f \in F$ we shall choose a march π_f with $\bar{\pi}_f = \bar{\pi}(\tau \times T^f)$ in such a way that

(1) *For every directed path P in T with first edge $e \in F$ and last edge $f \in F$ such that e and f have the same order and no edge of P has smaller order, $((\tau \times T^f)^-, \tau_f)$ is simulated in $((\tau \times T^e)^-, \pi_e)$.*

We do so inductively, working out from the root. At a step of the induction, there is a subtree S of T with $o(S) = o(T)$ such that π_f is defined for each $f \in F \cap E(S)$, and not defined for $f \in F - E(S)$; and (1) is satisfied for paths in S. If $F \subseteq E(S)$ the inductive definition is complete, and so we may assume that $F \not\subseteq E(S)$; and hence we may choose $f \in F - E(S)$ such that every other edge in F of the path of T from $o(T)$ to f lies in $E(S)$. Let Q be the maximal directed path of T with last edge f such that every edge of Q has order $\geqslant k$, where f has order k. If no edge of Q in F has order k except f we set $\pi_f = \pi(\tau \times T^f)$ and the inductive step is complete. We assume then that $p \geqslant 2$, where $f_1, f_2, ..., f_p$ are the edges of $E(Q) \cap F$ of order k, listed in the order in which they appear in Q (whence $f_p = f$).

(2) *There is a march π_f with $\bar{\pi}_f = \bar{\pi}(\tau \times T^f)$ such that $((\tau \times T^f)^-, \pi_f)$ is simulated in $((\tau \times T^{f_{p-1}})^-, \pi_{f_{p-1}})$.*

For suppose that (H_1, H_2) is a separation of $\tau \times T^{f_{p-1}}$ of order $<k$ with $(\tau \times T^f)^- \subseteq H_1^-$. Then $(K_1, K_2) = (H_1^-, H_2^- \cup (\tau \times T_{f_{p-1}})^-)$ is a separation of $(\tau \times T)^-$ of order $<k$, contrary to the rotundity of F in (T, τ), for f_{p-1} and $f_p = f$ both have order k and no edge of the path between them has order $<k$, and yet $(\tau \times T^f)^- \subseteq K_1$ and $(\tau \times T_{f_{p-1}})^- \subseteq K_2$. Hence there is no such separation (H_1, H_2), and (2) follows from Axiom 3 applied to the separation $(\tau \times T^f, ((\tau \times (T^{f_{p-1}})_f)^-, \pi_{f_{p-1}}))$ of $((\tau \times T^{f_{p-1}})^-, \pi_{f_{p-1}})$.

Let $S' = S \cup Q$; we must verify that (1) remains true for every directed path P of S'; that is, that for $1 \leqslant i \leqslant p-1$, $((\tau \times T^f)^-, \pi_f)$ is simulated in $((\tau \times T^{f_i})^-, \pi_{f_i})$. But from (1), $((\tau \times T^f)^-, \pi_f)$ is simulated in $((\tau \times T^{f_{p-1}})^-, \pi_{f_{p-1}})$, and that in turn is simulated in $((\tau \times T^{f_i})^-, \pi_{f_i})$ because (1) is satisfied for all directed paths P of S. By Axiom 2, the desired result follows. This completes our inductive definition. We define $\tau'(t) = \tau(t)$ if t is not the head of an edge in F, and $\tau'(t) = (\tau(t)^-, \pi_f)$ if t is the head of $f \in F$; then τ' satisfies the theorem. ∎

Let us say that a class $\mathscr{S}$ of star-decompositions is *symmetric* if for every $(S, \sigma) \in \mathscr{S}$, and every star-decomposition (S, σ') with $\sigma'(s)^- = \sigma(s)^-$ and $\bar{\pi}(\sigma'(s)) = \bar{\pi}(\sigma(s))$ for each $s \in V(S)$, we have $(S, \sigma') \in \mathscr{S}$.

(4.2) *Let $\mathscr{S}$ be good and symmetric, and let (T_i, τ_i) $(i = 1, 2, ...)$ be a countable sequence of rotund tree-decompositions, each branching of which is in $\mathscr{S}$. Then there exist $i' > i \geqslant 1$ such that $\tau_i \times T_i$ is simulated in $\tau_{i'} \times T_{i'}$.*

Proof. Since $\mathscr{S}$ is good, and contains the branching of (T_i, τ_i) at $o(T_i)$, it follows that each $\tau_i \times T_i \in \mathscr{R}$. From (4.1) for each $i \geqslant 1$ we may choose a linked tree-decomposition (T_i, τ_i') of $\tau_i \times T_i$ such that $\tau_i'(t)^- = \tau_i(t)^-$ for each $t \in V(T_i)$. Then $\bar{\pi}(\tau_i'(t)) = \bar{\pi}(\tau_i(t))$ for all $t \in V(T_i)$, by conditions (v) and (vi) in the definition of a tree-decomposition. From the symmetry of $\mathscr{S}$, every branching of every (T_i, τ_i') is in $\mathscr{S}$. Since $\mathscr{S}$ is good, by (3.3) there exist $i' > i \geqslant 1$ such that $\tau_i' \times T_i = \tau_i \times T_i$ is simulated in $\tau_{i'}' \times T_{i'} = \tau_{i'} \times T_{i'}$, as required. ∎

5. A Lemma of Thomas

It is convenient to make use of a result of R. Thomas [13]. (We should perhaps comment on who did what in this area, because of a certain amount of circularity of reference. Theorem (1.5) was first proved in the original draft of this paper in 1982. Thomas, having heard of our result but not having seen the proof, worked out his own proof, and indeed extended (1.5) to infinite graphs and to better-quasi-ordering. He proved a lemma which we had not been able to prove in our early work and for which we had been forced to construct a clumsy substitute. Thomas' lemma is clearly better than our substitute, and we see no reason to stick to our original method.)

The *width* of a tree-decomposition (T, τ) of a rooted hypergraph G is

$$\max(|V(\tau(t))| - 1 : t \in V(T))$$

and the *tree-width* of G is the minimum $w \geqslant 0$ such that G has a tree-decomposition of width $\leqslant w$. Thomas [13] proved (where 0 denotes the null sequence) that

(5.1) *Let G be a graph. If $(G, 0)$ has a tree-decomposition of width $\leqslant w$, then it has a tree-decomposition (T, τ) of width $\leqslant w$ such that for all distinct $t_1, t_2 \in V(T)$ and all $k \geqslant 0$ either there are k mutually vertex-disjoint paths of G from $V(\tau(t_1))$ to $V(\tau(t_2))$, or some edge of the path of T between t_1 and t_2 has order $< k$.*

Indeed, Thomas found a decomposition with even stronger properties, which will not concern us. We should perhaps stress that there is no assumption in (5.1) that t_1, t_2 and $o(T)$ lie on a directed path (in contrast with our definition of rotundity) and so which vertex of the underlying undirected tree of T is chosen as the root is irrelevant to (5.1), as are the arbitrary choices of the $\pi(\tau(t))$'s. Indeed, (5.1) is most naturally stated in terms of a different kind of tree-decomposition, where T is undirected and the $\tau(t)$'s are (unrooted) hypergraphs.

We apply (5.1) to deduce

(5.2) *Let G be a rooted hypergraph of tree-width $\leqslant w$. Then there is a rotund tree-decomposition of G of width $\leqslant w$.*

Proof. Let K be the simple graph with $V(K) = V(G)$, in which distinct v_1, v_2 are adjacent if either some edge of G is incident with them both, or v_1, v_2 are both terms of $\pi(G)$.

(1) $(K, 0)$ *has tree-width* $\leqslant w$.

For let (T, τ) be a tree-decomposition of G of width $\leqslant w$. Since for each edge of K there exists $t \in V(T)$ such that both ends of the edge are in $V(\tau(t))$, we may choose edge-disjoint rooted subgraphs $\tau'(t)$ $(t \in V(T))$ of K such that $V(\tau'(t)) = V(\tau(t))$ for each $t \in V(T)$, and such that (T, τ') is a tree-decomposition of $(K, 0)$. Then (1) follows.

From (1) and (5.1), we deduce

(2) *There is a tree-decomposition (T, τ) of $(K, 0)$ of width $\leqslant w$, such that for all distinct $t_1, t_2 \in V(T)$ and all $k \geqslant 0$, either there are k mutually vertex-disjoint paths of K between $V(\tau(t_1))$ and $V(\tau(t_2))$, or some edge of the path of T between t_1 and t_2 has order $< k$.*

(3) *For every complete subgraph X of K there exists $t \in V(T)$ such that $V(X) \subseteq V(\tau(t))$.*

For if $x \in V(X)$ the set $\{t \in V(T): x \in V(\tau(t))\}$ is the vertex set of a subtree T_x of T, and any two of these subtrees T_x, $T_{x'}$ have a common vertex (since x, x' are adjacent in G). By an elementary property of subtrees of a tree, it follows that all the T_x's have a common vertex, as required.

From (3) and the fact that (2) does not depend on which vertex of T is the root, we may choose (T, τ) (redefining the $\pi(\tau(t))$'s suitably) so that

(4) $\bar{\pi}(G) \subseteq V(\tau(o(T)))$.

Again, from (3) and (4) we deduce that

(5) *There is a tree-decomposition (T, τ') of G such that for each $t \in V(T)$, $V(\tau'(t)) = V(\tau(t))$.*

We claim that (T, τ') is rotund. For let P be a directed path of T with first edge f_2 and last edge f_1, such that f_1, f_2 have the same order k and every other edge of P has order $\geqslant k$ in (T, τ'). Let the first and last vertices of P be t_2, t_1. By (2), there are k mutually vertex-disjoint paths $Q_1, ..., Q_k$ of K between $V(\tau(t_1))$ and $V(\tau(t_2))$. Let (H_1, H_2) be a separation of G^- with $(\tau' \times T^{f_1})^- \subseteq H_1$ and $(\tau' \times T_{f_2})^- \subseteq H_2$. Each Q_i has one end in $V(H_1)$ and the other end in $V(H_2)$, and for each $e \in E(Q_i)$ either both ends of e in K are ends in G of some $e' \in E(G)$ or both ends of e in K are in $\bar{\pi}(G)$; and in either case both ends of e in K lie in some $V(\tau'(t))$ and hence either both lie in $V(H_1)$ or both lie in $V(H_2)$. Hence each $V(Q_i)$ meets $V(H_1 \cap H_2)$, and so $|V(H_1 \cap H_2)| \geqslant k$. Thus (H_1, H_2) has order $\geqslant k$; and hence (T, τ') is rotund, as required. ▮

We introduce a fourth axiom. For each $n \geqslant 0$, let $\mathscr{S}_n$ denote the class of all star-decompositions (S, σ) with $\sigma \times S \in \mathscr{R}$ and with $|V(\sigma(o(S)))| \leqslant n$.

Axiom 4. *For each* $n \geqslant 0$, $\mathscr{S}_n$ *is good.*

From (5.2) we deduce

(5.3) *Let* G_i $(i = 1, 2, ...)$ *be a countable sequence of elements of* $\mathscr{R}$, *each of tree-width* $\leqslant n$. *Then there exist* $i' > i \geqslant 1$ *such that* G_i *is simulated in* $G_{i'}$.

Proof. From (5.2), for each $i \geqslant 1$ there is a rotund tree-decomposition of G_i of width $\leqslant n$, and hence with all its branchings in $\mathscr{S}_{n+1}$. Since $\mathscr{S}_{n+1}$ is good and symmetric, the result follows from (4.2). ▮

Proof of (1.5) (*Sketch*). Let G_i $(i = 1, 2, ...)$ be a countable sequence of graphs, each of tree-width $\leqslant n$. Let $\mathscr{R}$ be the class of all rooted graphs G such that G^- is a subgraph of some G_i. For $H_1, H_2 \in \mathscr{R}$ we say that H_1 is simulated in H_2 if $|\bar{\pi}(H_1)| = |\bar{\pi}(H_2)|$ and there exists $H \in \mathscr{R}$ with $H^- \subseteq H_2^-$ and $\pi(H) = \pi(H_2)$, such that a rooted graph isomorphic to H_1 can be obtained from H by edge-contraction. We verify Axioms 1–4, and (1.5) follows from (5.3). ▮

We have omitted verifying Axioms 1–4, because it is easy and we shall later carry out the verification for a more general "concrete" definition of simulation.

6. Patchworks

Now we come to the second part of the paper. We introduce a concrete containment relation on a class of rooted hypergraphs, and verify that it satisfies the axioms.

If V is a finite set we denote by K_V the complete graph on V, that is, the simple graph with vertex set V and edge set the set of all subsets of V of cardinality 2, with the natural incidence relation. A *grouping* in V is a subgraph of K_V every component of which is complete. A *pairing* in V is a grouping in V every component of which has at most two vertices. If K is a pairing in V, we say that K *pairs* X, Y if $X, Y \subseteq V$ are disjoint and

(i) every 2-vertex component of K has one vertex in X and the other in Y, and

(ii) every vertex of $X \cup Y$ belongs to some 2-vertex component of K.

A *patch* in V is a collection Δ of groupings in V, each with the same vertex set $V(\Delta) \subseteq V$. A patch Δ is *free* if it contains every grouping in V with vertex set $V(\Delta)$; and it is *robust* if for every choice of $X, Y \subseteq V(\Delta)$ with $|X| = |Y|$ and $X \cap Y = \emptyset$, there is a pairing in Δ which pairs X, Y.

Let Ω be a quasi-order. An *Ω-patchwork* is a quadruple (G, μ, Δ, ϕ), where

(i) G is a rooted hypergraph

(ii) μ is a function with domain $\mathrm{dom}(\mu) \subseteq E(G)$; and for each $e \in \mathrm{dom}(\mu)$, $\mu(e)$ is a march with $\bar{\mu}(e) = V(e)$.

(iii) Δ is a function with domain $E(G)$, and for each $e \in E(G)$, $\Delta(e)$ is a patch with $V(\Delta(e)) = V(e)$; and for each $e \in E(G) - \mathrm{dom}(\mu)$, $\Delta(e)$ is free

(iv) ϕ is a function from $E(G)$ into $E(\Omega)$.

The Ω-patchwork is *robust* if each $\Delta(e)$ $(e \in E(G))$ is robust. (This is automatic if $e \notin \mathrm{dom}(\mu)$, since free patches are robust.)

If V is a finite set, N_V denotes the graph with vertex set V and no edges. A *realization* of an Ω-patchwork (G, μ, Δ, ϕ) is a subgraph of $K_{V(G)}$ expressible in the form

$$N_{V(G)} \cup \bigcup (\delta_e : e \in E(G))$$

where $\delta_e \in \Delta(e)$ for each $e \in E(G)$. The significance of robustness is that for robust Ω-patchworks we can prove a form of Menger's theorem, as follows.

(6.1) *Let $P = (G, \mu, \Delta, \phi)$ be a robust Ω-patchwork, let $X_1, X_2 \subseteq V(G)$, and let $k \geq 0$ be an integer. The following are equivalent*:

(i) *every separation (G_1, G_2) of G^- with $X_i \subseteq V(G_i)$ $(i = 1, 2)$ has order $\geq k$*

(ii) *there is a realization of P such that k of its components have nonempty intersection with both X_1 and X_2*

(iii) *there is a realization of P such that k of its components are paths from X_1 to X_2 with no vertex except their first in X_1, and no vertex except their last in X_2, and the remainder of the components of this realization are isolated vertices.*

Proof. Let K be the simple graph with $V(K)=V(G)$ in which distinct $a, b \in V(G)$ are adjacent if and only if some edge of G is incident with both a and b. By Menger's theorem, (i) is equivalent to statement (i)′, that there are k paths of K from X_1 to X_2, mutually vertex-disjoint. We must prove the equivalence of (i)′, (ii), and (iii).

Obviously (iii) implies (ii), and (ii) implies (i)′; it remains to show that (i)′ implies (iii). Suppose then that $P_1, \ldots, P_k$ are mutually vertex-disjoint paths of K from X_1 to X_2, and let us choose them with $\sum |E(P_i)|$ minimum. Then for $1 \leqslant i \leqslant k$, no vertex of P_i except its first is in X_1, and no vertex except its last is in X_2. Moreover, for each $e \in E(G)$, at most two vertices of P_i are incident with e in G, since any two such vertices are adjacent in K; and if there are two such vertices then they are consecutive in P_i.

For $1 \leqslant i \leqslant k$ and each $f \in E(P_i)$, let the ends of f be f^+, f^- where f^- is before f^+ in P_i. Choose $e(f) \in E(G)$ such that f^+, f^- are both incident with $e(f)$ in G. For each $e \in E(G)$, let $A_e = \{f^- : e = e(f)\}$ and $B_e = \{f^+ : e = e(f)\}$. Then $A_e \cap B_e = \emptyset$ and $|A_e| = |B_e|$. Since $\Delta(e)$ is robust there is a pairing $\delta_e \in \Delta(e)$ which pairs A_e, B_e. The graph

$$N_{V(G)} \cup \bigcup (\delta_e : e \in E(G))$$

is a realization of P satisfying (iii), as is easily seen (for example by rerouting the paths patch by patch). This completes the proof. ∎

7. Patchwork Containment

We wish now to introduce our containment relation on patchworks. Before we do so we attempt to motivate it by giving in the same spirit a definition of when a graph is isomorphic to a minor of another; our patchwork relation is not much different. If a graph G is isomorphic to a minor of a graph G', then each edge e of G is represented by an edge $\eta(e)$ of G', and each vertex v of G is "formed" by identifying under contraction a nonempty subset $\eta(v)$ of the vertex set of G'. Moreover, there is a connected subgraph of G' with vertex set $\eta(v)$ all edges of which are to be contracted in producing this minor. Thus, in summary:

(i) η is a function with domain $V(G) \cup E(G)$; for each $v \in V(G)$, $\eta(v)$ is a non-empty subset of $V(G')$ and for each $e \in E(G)$, $\eta(e) \in E(G')$

(ii) $\eta(v_1) \cap \eta(v_2) = \emptyset$ for distinct $v_1, v_2 \in V(G)$, and $\eta(e_1) \neq \eta(e_2)$ for distinct $e_1, e_2 \in E(G)$

(iii) for each $v \in V(G)$ and $e \in E(G)$, e is incident with v in G if and only if $\eta(v)$ contains an end of $\eta(e)$ in G', and e is a loop of G with end v if and only if $\eta(v)$ contains every end of $\eta(e)$ in G'

(iv) for each $v \in V(G)$ there is a connected subgraph of $G' \backslash \eta(E(G))$ with vertex set $\eta(v)$.

(If $\eta: A \to B$ is a function and $X \subseteq A$ we denote $\{\eta(x) : x \in X\}$ by $\eta(X)$.) If our graphs were directed, and we wanted our minor relation to preserve edge-directions, we would replace (iii) by

(iii)′ for each $v \in V(G)$ and $e \in E(G)$, v is the head (respectively, tail) of e in G if and only if $\eta(v)$ contains the head (respectively, tail) of $\eta(e)$ in G'.

If in addition we wanted no non-loop edge of G' to correspond to a loop of G we would add

(v) for each $e \in E(G)$, e and $\eta(e)$ have the same number of ends.

If G, G' were rooted graphs and we wanted our relation to take roots to roots, we would demand

(vi) $\pi(G)$ and $\pi(G')$ have the same length k say, and for $1 \leqslant i \leqslant k$, $\eta(v)$ contains the ith term of $\pi(G')$ where v is the ith term of $\pi(G)$.

Thus, we can regard (i), (ii), (iii)′, (iv), (v), (vi) as natural. As we said, our Ω-patchwork relation is not much different. There are three principal differences :

(i) Our edges are labelled from Ω, and we demand that the relation respect this ordering.

(ii) Edges in Ω-patchworks may have more than two ends. For graphs, if an edge is to be removed (when producing a minor) it is either deleted or contracted. For patchworks, an edge e may be removed in a greater variety of ways; in effect, we choose a member δ of $\Delta(e)$ and contract each component of δ to a single vertex. (Thus, a graph may be "mimicked" by a patchwork by defining $\Delta(e) = \{K_{V(e)}, N_{V(e)}\}$ for each edge e of the graph.)

(iii) An edge may also "shrink"—become incident with only some of the vertices with which it was previously incident. However, this is *only* permitted for edges not in dom(μ). (We remark that the "shrinking" feature is not needed for Wagner's conjecture, but seems to be required for Nash–Williams' conjecture. Our approach to the latter is to show that the class of all Ω-patchworks with a bounded number of roots and in which all patches are free is well-quasi-ordered by our containment relation, provided that Ω is a well-quasi-order; and in particular, it is important that we do not require the size of edges to be bounded. It is clear, therefore, that if our relation is to yield a well-quasi-order, it must permit edges to change size.)

The definition is as follows. If π, π' are marches of the same length, we denote by $\pi \to \pi'$ the bijection from $\bar{\pi}$ onto $\bar{\pi}'$ mapping π to π'. Let $P = (G, \mu, \Delta, \phi)$, $P' = (G', \mu', \Delta', \phi')$ be Ω-patchworks. An *expansion* of P in P' is a function η with domain $V(G) \cup E(G)$ such that

(i) for each $v \in V(G)$, $\eta(v)$ is a non-empty subset of $V(G')$, and for each $e \in E(G)$, $\eta(e) \in E(G')$

(ii) for distinct $v_1, v_2 \in V(G)$, $\eta(v_1) \cap \eta(v_2) = \varnothing$

(iii) for distinct $e_1, e_2 \in E(G)$, $\eta(e_1) \neq \eta(e_2)$

(iv) for each $e \in E(G)$, $e \in \mathrm{dom}(\mu)$ if and only if $\eta(e) \in \mathrm{dom}(\mu')$

(v) for each $e \in E(G) - \mathrm{dom}(\mu)$, if v is an end of e in G then $\eta(v)$ contains an end of $\eta(e)$ in G'

(vi) for each $e \in \mathrm{dom}(\mu)$, $\mu(e)$ and $\mu'(\eta(e))$ have the same length, k, say, and for $1 \leqslant i \leqslant k$, $\eta(v)$ contains the ith term of $\mu'(\eta(e))$ where v is the ith term of $\mu(e)$

(vii) $\pi(G)$ and $\pi(G')$ have the same length, k, say, and for $1 \leqslant i \leqslant k$, $\eta(v)$ contains the ith term of $\pi(G')$ where v is the ith term of $\pi(G)$

(viii) for each $e \in \mathrm{dom}(\mu)$, $\mu(e) \to \mu'(\eta(e))$ maps $\Delta(e)$ to $\Delta'(\eta(e))$

(ix) for each $e \in E(G)$, $\phi(e) \leqslant \phi'(\eta(e))$.

If G is a hypergraph and $F \subseteq E(G)$, $G \backslash F$ denotes the subhypergraph with the same vertex set and edge set $E(G) - F$. If G is a rooted hypergraph, $G \backslash F$ denotes $(G^- \backslash F, \pi(G))$. If $P = (G, \mu, \Delta, \phi)$ is an Ω-patchwork and $F \subseteq E(G)$, $P \backslash F$ denotes the Ω-patchwork $(G \backslash F, \mu', \Delta', \phi')$ where μ', Δ', ϕ' are the restrictions of μ, Δ, ϕ to $\mathrm{dom}(\mu) \cap E(G \backslash F)$, $E(G \backslash F)$, $E(G \backslash F)$, respectively. Let η be an expansion of $P = (G, \mu, \Delta, \phi)$ in $P' = (G', \mu', \Delta', \phi')$. A realization H of $P' \backslash \eta(E(G))$ is said to *realize* η if for every $v \in V(G)$, $\eta(v)$ is the vertex set of some component of H; and if there is such a realization, η is said to be *realizable*. Let us say that P is *simulated* in P' if there is a realizable expansion of P in P'. This is our containment relation.

(7.1) *Let η be a expansion of $P = (G, \mu, \Delta, \phi)$ in $P' = (G', \mu', \Delta', \phi')$, and let η' be an expansion of P' in $P'' = (G'', \mu'', \Delta'', \phi'')$. Let η'' be defined by*

$$\eta''(v) = \bigcup (\eta'(v') : v' \in \eta(v)) \qquad (v \in V(G))$$

$$\eta''(e) = \eta'(\eta(e)) \qquad (e \in E(G)).$$

Then η'' is an expansion of P in P'', and if η and η' are realizable then so is η''. In particular, the simulation relation provides a quasi-order of the class of all Ω-patchworks.

Proof. To verify that η'' is an expansion of P in P'' we check conditions (i), ..., (ix). Conditions (i), (iii), (iv), and (ix) are clear.

(1) *For distinct $v_1, v_2 \in V(G)$, $\eta''(v_1) \cap \eta''(v_2) = \emptyset$.*

For

$$\begin{aligned}\eta''(v_1) \cap \eta''(v_2) &= \bigcup (\eta'(v_1'): v_1' \in \eta(v_1)) \cap \bigcup (\eta'(v_2'): v_2' \in \eta(v_2)) \\ &= \bigcup (\eta'(v_1') \cap \eta'(v_2'): v_1' \in \eta(v_1) \text{ and } v_2' \in \eta(v_2)) \\ &= \emptyset\end{aligned}$$

since $v_1' \neq v_2'$ for $v_1' \in \eta(v_1)$ and $v_2' \in \eta(v_2)$, and hence $\eta'(v_1') \cap \eta'(v_2') = \emptyset$.

(2) *For each $e \in E(G) - \mathrm{dom}(\mu)$, if v is an end of e in G then $\eta''(v)$ contains an end of $\eta''(e)$ in G''.*

For $\eta(v)$ contains an end v' of $\eta(e)$, and $\eta(e) \in E(G') - \mathrm{dom}(\mu')$, and so $\eta'(v') \subseteq \eta''(v)$ contains an end of $\eta'(\eta(e)) = \eta''(e)$.

(3) *For each $e \in \mathrm{dom}(\mu)$, $\mu(e)$ and $\mu''(\eta''(e))$ have the same length, k, say, and for $1 \leqslant i \leqslant k$, $\eta''(v)$ contains the ith term of $\mu''(\eta''(e))$ where v is the ith term of $\mu(e)$.*

For $\mu(e)$ and $\mu'(\eta(e))$ have the same length, k, say, and $\eta(e) \in \mathrm{dom}(\mu')$, and so $\mu''(\eta'(\eta(e)))$ also has length k. For $1 \leqslant i \leqslant k$, let v, v', v'' be the ith terms of $\mu(e)$, $\mu'(\eta(e))$, and $\mu''(\eta''(e))$; then $v' \in \eta(v)$ and $v'' \in \eta'(v')$, and so $v'' \in \eta''(v)$.

(4) *$\pi(G)$ and $\pi(G'')$ have the same length, k, say, and for $1 \leqslant i \leqslant k$ $\eta''(v)$ contains the ith term of $\pi(G'')$ where v is the ith term of $\pi(G)$.*

The proof is similar to (3).

(5) *For each $e \in \mathrm{dom}(\mu)$, $\mu(e) \to \mu''(\eta''(e))$ maps $\Delta(e)$ to $\Delta''(\eta''(e))$.*

For $\mu(e) \to \mu''(\eta''(e))$ is the composition of $\mu(e) \to \mu'(\eta(e))$ and $\mu'(\eta(e)) \to \mu''(\eta''(e))$.

From (1), ..., (5) we deduce that η'' is an expansion of P in P''. Now suppose that η, η' are realized by H, H', respectively. For each $e \in E(G') - \eta(E(G))$ choose $\delta_e \in \Delta'(e)$ such that

$$H = N_{V(G')} \cup \bigcup (\delta_e: e \in E(G') - \eta(E(G)))$$

and for each $e \in E(G'') - \eta'(E(G'))$ choose $\delta_e' \in \Delta''(e)$ such that

$$H' = N_{V(G'')} \cup \bigcup (\delta_e': e \in E(G'') - \eta'(E(G'))).$$

For each $e \in E(G'') - \eta''(E(G))$ we define δ''_e as follows. If $e \notin \eta'(E(G'))$ we define $\delta''_e = \delta'_e$. If $e = n'(f)$ where $f \in E(G')$ we define δ''_e to be the subgraph of $K_{V(G'')}$ with vertex set $V_{G''}(e)$, in which distinct ends a, b of e are adjacent if and only if $a \in \eta'(p)$ and $b \in \eta'(q)$ for some p, q in the same component of δ_f.

(6) *For each* $e \in E(G'') - \eta''(E(G))$, $\delta''_e \in \Delta''(e)$.

For if $e \notin \eta'(E(G'))$, this is clear since $\delta''_e = \delta'_e$. Let $e = \eta'(f)$. We claim that δ''_e is a grouping. For suppose that $a, b, c \in V_{G''}(e)$ are distinct and a, b are adjacent in δ''_e and so are b, c. Choose p, q in the same component of δ_f with $a \in \eta'(p)$ and $b \in \eta'(q)$; and choose r, s in the same component of δ_f with $a \in \eta'(r)$ and $b \in \eta'(s)$. Then $p = r$ since $\eta'(p) \cap \eta'(r) \neq \emptyset$, and so q and s are in the same component of δ_f. Hence a is adjacent to c in δ''_e. This proves that δ''_e is a grouping. If $e \notin \mathrm{dom}(\mu'')$ then $\delta''_e \in \Delta''(e)$ since $\Delta''(e)$ is free, and we may therefore assume that $e \in \mathrm{dom}(\mu'')$. But then δ''_e is the image under $\mu'(f) \to \mu''(\eta'(f))$ of δ_f, and therefore belongs to $\Delta''(e)$ since $\delta_f \in \Delta'(f)$ and $\mu'(f) \to \mu''(\eta'(f))$ maps $\Delta'(f)$ to $\Delta''(e)$. This verifies (6).

Let H'' be

$$N_{V(G'')} \cup \bigcup (\delta''_e : e \in E(G'') - \eta''(E(G))).$$

Then H'' is a realization of $G'' \backslash \eta''(E(G))$, and we shall show that it realizes η''. Let $v \in V(G)$. We must show that $\eta''(v)$ is the vertex of some component of H''.

(7) *No edge of* H'' *joins a vertex of* $\eta''(v)$ *to a vertex of* $V(G'') - \eta''(v)$.

For suppose that some edge of H'' has ends $a \in \eta''(v)$ and $b \in V(G'')$. We shall show that $b \in \eta''(v)$. Choose $v' \in \eta(v)$ with $a \in \eta'(v')$, and choose $e \in E(G'') - \eta''(E(G))$ with a, b adjacent in δ''_e. If $e \notin \eta'(E(G'))$ then a, b are adjacent in $\delta'_e \subseteq H'$ and since H' realizes η' and $a \in \eta'(v')$ we deduce that $b \in \eta'(v') \subseteq \eta''(v)$ as required. If $e \in \eta'(E(G'))$ and $e = \eta'(f)$ where $f \in E(G')$, then there exist p, q in the same component of δ_f such that $a \in \eta'(p)$ and $b \in \eta'(q)$. Since $a \in \eta'(v') \cap \eta'(p)$ we deduce that $p = v'$, and so v', q are in the same component of δ_f and hence of H. Since H realizes η and $v' \in \eta(v)$ we deduce that $q \in \eta(v)$ and hence $b \in \eta'(q) \subseteq \eta''(v)$ as required.

(8) *If* $p, q \in V(H)$ *are adjacent in* H *then there exist* $a \in \eta'(p)$ *and* $b \in \eta'(q)$ *adjacent in* H''.

For choose $f \in E(G')$ with p, q adjacent in δ_f. Let $e = \eta'(f)$ and choose ends a, b of e such that $a \in \eta'(p)$, $b \in \eta'(q)$. Then a, b are adjacent in δ''_e and hence in H'' as required.

(9) *If* $a, b \in \eta''(v)$ *then* a, b *belong to the same component of* H''.

For choose $p, q \in \eta(v)$ with $a \in \eta'(p)$, $b \in \eta'(q)$. Since H realizes η there is a path of H joining p and q with t edges, say. By applying (8) to each edge of this path, we deduce that there is a sequence $a = a_0, b_1, a_1, b_2, ..., a_{t-1}, b_t, a_t, b_{t+1} = b$ of vertices of H'' such that for $1 \leqslant i \leqslant t$, a_i and b_i are adjacent in H'' and for $0 \leqslant i \leqslant t$, a_i and b_{i+1} both belong to $\eta'(r)$ for some vertex r of our path. Since H' realizes η' we deduce that for $0 \leqslant i \leqslant t$, a_i and b_{i+1} belong to the same component of H', and hence of H'' since $H' \subseteq H''$. We deduce that a and b belong to the same component of H'' as required.

From (7) and (9) we deduce that H'' realizes η'', and hence η'' is realizable. This completes the proof of the second statement of the theorem; and the third follows, for the simulation relation has been shown to be transitive, and it is reflexive, because the "identity" expansion of an Ω-patchwork $P = (G, \mu, \Delta, \phi)$ in itself is realized by the realization $N_{V(G)}$ of $P \backslash E(G)$. ∎

8. Stars with Small Hearts

The results of the previous two sections will be used to show that our containment relation satisfies Axioms 1, 2, and 3. In this section we prove a result will be used for Axiom 4. We begin with the following lemma.

Let $P = (G, \mu, \Delta, \phi)$ be an Ω-patchwork. If G' is a rooted hypergraph with $G'^{-} \subseteq G^{-}$, and μ', Δ', ϕ' are the restrictions of μ, Δ, ϕ to $\mathrm{dom}(\mu) \cap E(G')$, $E(G')$, $E(G')$, respectively, then $(G', \mu', \Delta', \phi')$ is an Ω-patchwork which we denote by $P \mid G'$.

(8.1) *Let $P = (G, \mu, \Delta, \phi)$, $P' = (G', \mu', \Delta', \phi')$ be Ω-patchworks, and let (G_1, G_2), (G'_1, G'_2) be separations of G, G' respectively. Let η_1 be a realizable expansion of $P \mid G_1$ in $P' \mid G'_1$ (whence $|\bar{\pi}(G_1)| = |\bar{\pi}(G'_1)| = k$, say) and let η_2 be a realizable expansion of $P \mid G_2$ in $P' \mid G'_2$ such that for $1 \leqslant i \leqslant k$, $\eta_2(v)$ contains the* ith *term of $\pi(G'_1)$, where v is the* ith *term of $\pi(G_1)$. Define η by*

$$\begin{aligned}
\eta(v) &= \eta_1(v) && (v \in V(G_1) - V(G_2)) \\
&= \eta_2(v) && (v \in V(G_2) - V(G_1)) \\
&= \eta_1(v) \cup \eta_2(v) && (v \in V(G_1) \cap V(G_2)) \\
\eta(e) &= \eta_1(e) && (e \in E(G_1)) \\
&= \eta_2(e) && (e \in E(G_2)).
\end{aligned}$$

Then η is a realizable expansion of P in P'.

Proof. For $1 \leqslant i \leqslant k$, let v_i, v'_i be the ith terms of $\pi(G_1)$, $\pi(G'_1)$, respectively.

(1) *For* $1 \leqslant i \leqslant k$ *and all* $v \in V(G)$, *if* $v'_i \in \eta(v)$ *then* $v = v_i$.

For either $v \in V(G_1)$ and $v'_i \in \eta_1(v)$, or $v \in V(G_2)$ and $v'_i \in \eta_2(v)$. But $v'_i \in \eta_1(v_i)$ and $v'_i \in \eta_2(v_i)$ (the first since η_1 is an expansion of P_1 in P'_1, and the second by hypothesis) and so either $v \in V(G_1)$ and $\eta_1(v_i) \cap \eta_1(v) \neq \emptyset$, or $v \in V(G_2)$ and $\eta_2(v_i) \cap \eta_2(v) \neq \emptyset$. In either case $v = v_i$ as required.

(2) *For distinct* $u_1, u_2 \in V(G)$, $\eta(u_1) \cap \eta(u_2) = \emptyset$.

For suppose that $v' \in \eta(u_1) \cap \eta(u_2)$. If $v' \in V(G'_1 \cap G'_2)$ then $v' = v'_i$ for some i, and by (1) $u_1 = v_i$ and $u_2 = v_i$, a contradiction. If $v' \in V(G'_1) - V(G'_2)$ then $u_1, u_2 \in V(G_1)$ and $v' \in \eta_1(u_1) \cap \eta_1(u_2)$, whence $u_1 = u_2$, a contradiction, and similarly we obtain a contradiction if $v' \in V(G'_2) - V(G'_1)$.

From (2) it follows that η is an expansion of P in P'; for conditions (i), ..., (ix) are all clearly satisfied except (ii), and the truth of (ii) follows from (2).

Now let H_i be a realization of $(P' \mid G'_i) \backslash \eta_i(E(G_i))$ which realizes η_i ($i = 1, 2$). Let $H = H_1 \cup H_2$; then H is a realization of $P' \backslash \eta(E(G))$ and we shall show that it realizes η. Let $v \in V(G)$; we must show that $\eta(v)$ is the vertex set of a component of H. If $v \in V(G_1) - V(G_2)$, then $\eta(v) = \eta_1(v)$, and hence is the vertex set of a component of H_1; and since $\eta(v) \cap V(G'_2) = \emptyset$ (by (1)) it follows that this component is also a component of H, as required. The claim follows similarly if $v \in V(G_2) - V(G_1)$. If $v \in V(G_1 \cap G_2)$, let $v = v_i$ where $1 \leqslant i \leqslant k$. Then $\eta_1(v)$ is the vertex set of a component of H_1, and contains no vertex of H_2 except v'_i by (1); and similarly, $\eta_2(v)$ is the vertex set of a component of H_2 containing no vertex of H_1 except v'_i. Hence $\eta(v) = \eta_1(v) \cup \eta_2(v)$ is the vertex set of a component of H as required. ∎

Let $P = (G, \mu, \Delta, \phi)$ be an Ω-patchwork. We say that $e \in E(G)$ is *removable* if $N_{V(e)} \in \Delta(e)$; and P is *removable* if each $e \in E(G)$ is removable. Evidently robust patchworks are removable.

(8.2) *Let* $P = (G, \mu, \Delta, \phi)$, $P' = (G', \mu', \Delta', \phi')$ *be* Ω*-patchworks where* P' *is removable. Let* (S, σ), (S', σ') *be star-decompositions of* G, G', *respectively. Let* γ *be a bijection from* $V(\sigma(o(S)))$ *to* $V(\sigma'(o(S')))$ *mapping* $\pi(G)$ *to* $\pi(G')$. *Let* α *be an injection from* $E(\sigma(o(S)))$ *into* $E(\sigma'(o(S')))$, *such that for each* $e \in E(\sigma(o(S)))$

(i) $\alpha(e) \in \text{dom}(\mu')$ *if and only if* $e \in \text{dom}(\mu)$

(ii) *if* $e \in \text{dom}(\mu)$ *then* γ *maps* $\mu(e)$ *to* $\mu'(\alpha(e))$

(iii) *if* $e \notin \text{dom}(\mu)$ *then* γ *maps* $V_G(e)$ *to* $V_{G'}(\alpha(e))$

(iv) $\phi(e) \leqslant \phi'(\alpha(e))$.

Let β be a bijection from $U(S)$ to $U(S')$, such that for each $s \in U(S)$,

(v) *γ maps $\pi(\sigma(s))$ to $\pi(\sigma'(\beta(s)))$*

(vi) *$P \mid \sigma(s)$ is simulated in $P' \mid \sigma'(\beta(s))$.*

Then P is simulated in P'.

Proof. Let $V(S) = \{s_0, s_1, ..., s_n\}$ where $s_0 = o(S)$, and let $V(S') = (s'_0, s'_1, ..., s'_n)$ where $\beta(s_i) = s'_i$ $(1 \leqslant i \leqslant n)$. For each $i \geqslant 1$, let η_i be a realizable expansion of $P \mid \sigma(s_i)$ in $P' \mid \sigma'(s'_i)$. Let η_0 be the expansion of $P \mid \sigma(s_0)$ in $P' \mid \sigma'(s'_0)$ defined by

$$\eta_0(v) = \{\gamma(v)\} \qquad (v \in V(\sigma(s_0)))$$

$$\eta_0(e) = \alpha(e) \qquad (e \in E(\sigma(s_0))).$$

Now since P' is removable, N_X is a realization of $(P' \mid \sigma'(s'_0)) \backslash \eta_0(E(\sigma(s_0)))$, where $X = V(\sigma'(s'_0))$; and it realizes η_0. Thus η_0 is realizable.

For $i \geqslant 0$, let S_i be the subtree of S with vertex set $\{s_0, s_1, ..., s_i\}$ and let $G_i = \sigma \times S_i$; and define S'_i, G'_i similarly. For each $j \geqslant 0$, let ζ_j be defined by

$$\zeta_j(e) = \eta_i(e) \qquad \text{if} \quad 0 \leqslant i \leqslant j \text{ and } e \in E(\sigma(s_i))$$

$$\zeta_j(v) = \bigcup (\eta_i(v) : 0 \leqslant i \leqslant j, v \in V(\sigma(s_i))).$$

We claim that ζ_j is a realizable expansion of $P \mid G_j$ in $P' \mid G'_j$, for $0 \leqslant j \leqslant n$. This is true if $j = 0$; we assume that it holds for some j with $0 \leqslant j < n$, and we shall prove that it holds for $j+1$. Now for $1 \leqslant i \leqslant |\bar{\pi}(\sigma(s_{j+1}))|$, let v, v' be the ith terms of $\pi(\sigma(s_{j+1}))$, $\pi(\sigma'(s'_{j+1}))$, respectively. By condition (v) above, $\gamma(v) = v'$, and so $\eta_0(v) = \{v'\}$. Since $\eta_0(v) \subseteq \zeta_j(v)$, it follows that $v' \in \zeta_j(v)$. Hence we may apply (8.1) to the separations $(\sigma(s_{j+1}), G_j)$ and $(\sigma(s'_{j+1}), G'_j)$ of G_{j+1}, G'_{j+1} (replacing P, P' by $P \mid G_{j+1}$, $P' \mid G'_{j+1}$, and replacing η_1, η_2 by η_{j+1}, ζ_j); and we deduce that ζ_{j+1} is a realizable expansion of $P \mid G_{j+1}$ in $P' \mid G'_{j+1}$. This proves our claim, by induction on j. In particular, $P \mid G_n = P$ is simulated in $P' \mid G'_n = P'$, as required. ∎

We shall need the following theorem of Higman [3].

(8.3) *Let Ω be a well-quasi -order, and let X_i $(i = 1, 2, ...)$ be a countable sequence of finite subsets of $E(\Omega)$. Then there exists an infinite subset $I \subseteq \{1, 2, ...\}$ with the property that for all i, $i' \in I$ with $i < i'$ there is a injection $\alpha: X_i \to X_{i'}$ such that $x \leqslant \alpha(x)$ for all $x \in X_i$.*

(8.4) *Let $n \geqslant 0$, let Ω be a well-quasi-order, and let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, ...)$ be a countable sequence of removable Ω-patchworks. For each $i \geqslant 1$ let (S_i, σ_i) be a star-decomposition of G_i with $|V(\sigma_i(o(S_i)))| \leqslant n$.*

Suppose that $\{P_i|\sigma_i(s): i \geqslant 1, s \in U(S_i)\}$ *is well-quasi-ordered by simulation. Then there exist* $i' > i \geqslant 1$ *such that* P_i *is simulated in* $P_{i'}$.

Proof. Since there are only $n+1$ possibilities for each $|V(\sigma_i(o(S_i)))|$ we may choose an infinite subset $I_1 \subseteq \{1, 2, ...\}$ and $m \geqslant 0$ such that

(1) *For all* $i \in I_1$, $|V(\sigma_i(o(S_i)))| = m$.

Similarly, we may choose an infinite $I_2 \subseteq I_1$ and $k \geqslant 0$ such that

(2) *For all* $i \in I_2$, $|\bar{\pi}(G_i)| = k$.

By replacing each P_i by an "isomorphic" Ω-patchwork, we may therefore assume, to simplify notation, that for some set V_0 and march π_0 in V_0,

(3) $V(\sigma_i(o(S_i))) = V_0$ *and* $\pi(G_i) = \pi_0$ *for all* $i \in I_2$.

Let ρ be a march in V_0. Since Ω is well-quasi-ordered and there are only finitely many different patches with a given vertex set, we may choose by (8.3) an infinite $I_3 \subseteq I_2$ such that

(4) *For all* i, $i' \in I_3$ *with* $i' > i$ *there is an injection* α *from* $\{e \in \mathrm{dom}(\mu_i) \cap E(\sigma_i(o(S_i))): \mu_i(e) = \rho\}$ *into* $\{e' \in \mathrm{dom}(\mu_{i'}) \cap E(\sigma_{i'}(o(S_{i'}))): \mu_{i'}(e') = \rho\}$, *such that for each* e, $\phi_i(e) \leqslant \phi_{i'}(\alpha(e))$ *and* $\Delta_i(e) = \Delta_{i'}(\alpha(e))$.

Since V_0 is finite, we may assume (by repeating this procedure for all ρ) that (4) holds for all ρ. We may similarly choose an infinite subset $I_4 \subseteq I_3$ such that for all subsets Y of V_0,

(5) *For all* $i, i' \in I_4$ *with* $i' > i$ *there is an injection* α *from* $\{e \in E(\sigma_i(o(S_i))) - \mathrm{dom}(\mu_i): V_{G_i}(e) = Y\}$ *into* $\{e' \in E(\sigma_{i'}(o(S_{i'}))) - \mathrm{dom}(\mu_{i'}): V_{G_{i'}}(e') = Y\}$, *such that* $\phi_i(e) \leqslant \phi_{i'}(\alpha(e))$ *for each* e.

By piecing together the injections of (4) and (5), we deduce

(6) *For all* i, $i' \in I_4$ *with* $i' > i$ *there is an injection* $\alpha: E(\sigma_i(o(S_i))) \to E(\sigma_{i'}(o(S_{i'})))$ *such that for each* $e \in E(\sigma_i(o(S_i)))$

- (i) $\alpha(e) \in \mathrm{dom}(\mu_{i'})$ *if and only if* $e \in \mathrm{dom}(\mu_i)$
- (ii) *if* $e \in \mathrm{dom}(\mu_i)$ *then* $\mu_i(e) = \mu_{i'}(\alpha(e))$ *and* $\Delta_i(e) = \Delta_{i'}(\alpha(e))$
- (iii) *if* $e \notin \mathrm{dom}(\mu_i)$ *then* $V_{G_i}(e) = V_{G_{i'}}(\alpha(e))$
- (iv) $\phi_i(e) \leqslant \phi_{i'}(\alpha(e))$.

Let ρ be a march in V_0. Since $\{P_i|\sigma_i(s): i \geqslant 1, s \in U(S_i)\}$ is well-quasi-ordered by simulation we may choose by (8.3) an infinite $I_5 \subseteq I_4$ such that

(7) *For all* i, $i' \in I_5$ *with* $i' > i$ *there is an injection* β *from* $\{s \in U(S_i): \pi(\sigma_i(s)) = \rho\}$ *into* $\{s' \in U(S_{i'}): \pi(\sigma_{i'}(s')) = \rho\}$, *such that* $P_i|\sigma_i(s)$ *is simulated in* $P_{i'}|\sigma_{i'}(\beta(s))$ *for each* s.

By repeating this for each ρ and piecing together the resulting injections β, we deduce that there is an infinite $I_6 \subseteq I_4$ such that

(8) *For all i, $i' \in I_6$ with $i' > i$ there is an injection $\beta: U(S_i) \to U(S_{i'})$ such that for each $s \in U(S_i)$,*

(i) $\pi(\sigma_i(s)) = \pi(\sigma_{i'}(\beta(s)))$

(ii) *there is a realizable expansion of $P_i | \sigma_i(s)$ in $P_{i'} | \sigma_{i'}(\beta(s))$.*

Choose $i, i' \in I_6$ with $i' > i$. We claim that P_i is simulated in $P_{i'}$. For let α, β be as in (6), (8). Let η_0 be defined by

$$\eta_0(v) = \{v\} \qquad (v \in V_0),$$

$$\eta_0(e) = \alpha(e) \qquad (e \in E(\sigma_i(o(S_i)))).$$

Then η_0 is an expansion of $P_i | \sigma_i(o(S_i))$ in $P_{i'} | \sigma_{i'}(o(S_{i'}))$. Moreover, N_{V_0} is a realization of

$$(P_{i'} | \sigma_{i'}(o(S_{i'}))) \backslash \eta_0(E(\sigma_i(o(S_i))))$$

since $P_{i'}$ is removable, and N_{V_0} realizes η_0.

By applying (8.2) to P_i, $P_{i'} | (\sigma_{i'} \times T)$ we deduce that P_i is simulated in $P_{i'} | (\sigma_{i'} \times T)$, where T is the subtree of $S_{i'}$ with

$$V(T) = \{o(S_{i'})\} \cup \beta(U(S_i)).$$

But $P_{i'} | (\sigma_{i'} \times T)$ is simulated in $P_{i'}$ since $P_{i'}$ is removable, and hence P_i is simulated in $P_{i'}$ by (7.1). This completes the proof. ∎

9. Verifying the Axioms

Let Ω be a well-quasi-order, and let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, ...)$ be a countable sequence of robust Ω-patchworks, where for $i' > i \geqslant 1$, G_i and $G_{i'}$ are disjoint. Let $\mathscr{R}$ be the set of all rooted hypergraphs G such that $G^- \subseteq G_i^-$ for some i. For each $G \in \mathscr{R}$ we define $P(G)$ to be $P_i | G$ where $G^- \subseteq G_i^-$. (This is well-defined since such a value of i is unique unless G is null.) Let us say that $G \in \mathscr{R}$ is simulated in $G' \in \mathscr{R}$ if $P(G)$ is simulated in $P(G')$. We verify Axioms 1–4.

Axiom 1. This is clear from the definition of $\mathscr{R}$.

Axiom 2. This is immediate from (7.1).

Axiom 3. Let $G \in \mathscr{R}$, and let (G_1, G_2) be a separation of G of order $k = |\bar{\pi}(G)|$, such that there is no separation (H_1, H_2) of G of order $< k$ with $G_1^- \subseteq H_1^-$. Then every separation (J_1, J_2) of G_2^- with $V(G_1^- \cap G_2^-) \subseteq V(J_1)$

and $\bar{\pi}(G_2) \subseteq V(J_2)$ has order $\geqslant k$. From (6.1) and the robustness of $P(G_2)$ there is a realization H of $P(G_2)$, k components of which contain a vertex of $\bar{\pi}(G_2)$ and a vertex of $V(G_1^- \cap G_2^-)$. Since

$$|V(G_1^- \cap G_2^-)| = |\pi(G_2)| = k$$

there is a march π_1 with $\bar{\pi}_1 = V(G_1^- \cap G_2^-)$ such that for $1 \leqslant i \leqslant k$ some component D_i say of H contains the ith vertex of $\pi(G_2)$ and the ith vertex of π_1. Let π_1 be $v_1, v_2, ..., v_k$. Define η by

$$\eta(v) = \{v\} \qquad (v \in V(G_1) - V(G_2))$$

$$\eta(v_i) = V(D_i) \qquad (1 \leqslant i \leqslant k)$$

$$\eta(e) = e \qquad (e \in E(G_1)).$$

Then η is an expansion of (G_1^-, π_1) in G, and it is realized by $H \cup N_{V(G_1)}$ which is a realization of $G \backslash \eta(E(G_1))$. Thus η is realizable, and so (G_1^-, π_1) is simulated in G as required.

Axiom 4. Let $n \geqslant 0$; we must show that $\mathscr{S}_n$ is good. Certainly $\sigma \times S \in \mathscr{R}$ for each $(S, \sigma) \in \mathscr{S}_n$, and $\mathscr{S}_n$ has index $\leqslant n$, by definition of $\mathscr{S}_n$. Let (S_i, σ_i) $(i = 1, 2, ...)$ be a countable sequence of members of $\mathscr{S}_n$ such that the set of all tips of all the (S_i, σ_i)'s is well-simulated. Let $G_i = \sigma_i \times S_i$ $(i \geqslant 1)$. It remains to show that there exist $i' > i \geqslant 1$ such that G_i is simulated in $G_{i'}$, that is, $P(G_i)$ is simulated in $P(G_{i'})$. Since $P_1, P_2, ...$ are all robust, it follows that $P(G)$ is robust for all $G \in \mathscr{R}$, and in particular each $P(G_i)$ is removable. Now for each $i \geqslant 1$, (S_i, σ_i) is a star-decomposition of G_i with $|V(\sigma_i(o(S_i)))| \leqslant n$; and $\{P(G_i) | \sigma_i(s) : i \geqslant 1, s \in U(S_i)\}$ is well-quasi-ordered by simulation, since the set of all tips of all the (S_i, σ_i)'s is well-simulated. From (8.4), it follows that there exist $i' > i \geqslant 1$ such that $P(G_i)$ is simulated in $P(G_{i'})$. This verifies Axiom 4.

Consequently, we have

(9.1) *Let $n \geqslant 0$, let Ω be a well-quasi-order, and let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, ...)$ be a countable sequence of robust Ω-patchworks, where each G_i has tree-width $\leqslant n$. Then there exist $i' > i \geqslant 1$ such that there is a realizable expansion of P_i in $P_{i'}$.*

Proof. We may assume that $G_1, G_2, ...$ are mutually disjoint, and hence we may define $\mathscr{R}$, etc. as at the start of this section; and the claim follows from (5.3). ∎

As we mentioned earlier, (1.5) is a consequence of (9.1). Let Ω be the well-quasi-order with $E(\Omega) = \{1\}$. If G is a graph, let $P(G)$ be the Ω-patchwork $((G, 0), \mu, \Delta, \phi)$, where

(i) 0 is the null sequence

(ii) $\mathrm{dom}(\mu) = E(G)$

(iii) for each $e \in E(G)$, $\mu(e)$ is a march with $\bar{\mu}(e) = V(e)$, and $\Delta(e) = \{N_{V(e)}, K_{V(e)}\}$. (If $|V(e)| = 1$ then $N_{V(e)} = K_{V(e)}$.)

(iv) $\phi(e) = 1$ for each $e \in E(G)$.

If $P(G)$ is simulated in $P(G')$ then G is isomorphic to a minor of G', and hence (9.1) implies (1.5).

A second application of (9.1) will be important in a future paper [11], and it is convenient to deal with that application here rather than to redefine all the apparatus of patchworks in that paper. A *surface* is a compact 2-manifold without boundary. A *painting* in a surface Σ is a pair (U, N), where $U \subseteq \Sigma$ is closed and $N \subseteq U$ is finite, such that

(i) $U - N$ has only finitely many arc-wise connected components, called *cells*

(ii) for each cell e, its closure $\bar{e}$ is homeomorphic to a closed disc, and $\bar{e} \cap N = \bar{e} - e$ is a finite subset of $bd(\bar{e})$.

From (9.1) we deduce

(9.2) *Let Σ be a surface, let $n \geqslant 0$, and let Ω be a well-quasi-order. For $i \geqslant 1$ let $(U_i, N_i, \pi_i, \mu_i, \phi_i)$ be such that*

(i) *(U_i, N_i) is a painting in Σ with set of cells E_i, say, and π_i is a march in N_i*

(ii) *for each $e \in E_i$, $\mu_i(e)$ is a march with $\bar{\mu}_i(e) = \bar{e} - e$, and $\phi_i(e) \in E(\Omega)$*

(iii) *the rooted hypergraph G_i with $V(G_i) = N_i$, $E(G_i) = E_i$, with the natural incidence relation and $\pi(G_i) = \pi_i$, has tree-width $\leqslant n$.*

Then there exist $i' > i \geqslant 1$ and a drawing Ξ of a forest in Σ with vertex set $N_{i'}$, and a function σ with domain $N_i \cup E_i$, such that

(a) *every edge of Ξ is a subset of a cell of $(U_{i'}, N_{i'})$*

(b) *for each $v \in N_i$, $\sigma(v)$ is a component of Ξ, and for each $e \in E_i$, $\sigma(e) \in E_{i'}$*

(c) *for distinct $e_1, e_2 \in E_i$, $\sigma(e_1) \neq \sigma(e_2)$, and for $v_1, v_2 \in N_i$, $\sigma(v_1)$ and $\sigma(v_2)$ are disjoint*

(d) *for each $v \in N_i$ and $e \in E_i$, no edge of $\sigma(v)$ intersects $\sigma(e)$*

(e) *for each $e \in E_i$, $|\bar{e} - e| = |\overline{\sigma(e)} - \sigma(e)| = k$, say, and for $1 \leqslant j \leqslant k$ the jth term of $\mu_{i'}(\sigma(e))$ is a vertex of $\sigma(v)$ where v is the jth term of $\mu_i(e)$*

(f) *π_i and $\pi_{i'}$ have the same length k, say, and for $1 \leqslant j \leqslant k$ the jth term of $\pi_{i'}$ is a vertex of $\sigma(v)$ where v is the jth term of π_i*

(g) *for each $e \in E_i$, $\phi_i(e) \leqslant \phi_{i'}(\sigma(e))$.*

Proof. For each $i \geqslant 1$, let G_i be as in (iii) above, and for each $e \in E_i$ let $\Delta_i(e)$ be the set of all pairings in $\bar{\mu}_i(e)$ no two edges of which cross (where $\{a, c\}$ *crosses* $\{b, d\}$ if $a, b, c, d \in \bar{e} - e$ are all distinct and occur in that order, or the reverse, in the natural cyclic order of $\bar{e} - e$). Then $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ is an Ω-patchwork, and it is robust. By (9.1) there exist $i' > i \geqslant 1$ and a realizable expansion η of P_i in $P_{i'}$. Let H be a realization of $P_{i'} \backslash \eta(E(G_i))$ realizing η, and let F be a spanning forest of H. From our choice of the $\Delta_i(e)$'s and the fact that H is a realization of $P_{i'} \backslash \eta(E(G_i))$, there is a drawing Ξ of F in Σ such that each vertex of F is represented by itself (we recall that $V(F) = V(H) = N_{i'} \subseteq \Sigma$) and each edge is drawn within some cell $e' \in E_i' - \eta(E_i)$. Define $\sigma(e) - \eta(e)$ $(e \in E_i)$ and for each $v \in N_i$, let $\sigma(v)$ be the component of Ξ with vertex set $\eta(v)$. Then $(a), ..., (g)$ are satisfied, as required. ▮

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