Graph minors
XIX. Well-quasi-ordering on a surface

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Abstract

In a previous paper [4] we showed that for any infinite set of (finite) graphs drawn in a fixed surface, one of the graphs is isomorphic to a minor of another. In this paper we extend that result in two ways:

- we generalize from graphs to hypergraphs drawn in a fixed surface, in which each edge has two or three ends, and
- the edges of our hypergraphs are labelled from a well-quasi-order, and the minor relation is required to respect this order.

This result is another step in the proof of Wagner's conjecture, that for any infinite set of graphs, one is isomorphic to a minor of another.

1 Introduction

This paper is the penultimate step in the proof of Wagner's conjecture, that for any infinite set of finite graphs, one is isomorphic to a minor of another. Roughly speaking (we shall define our terms later), we wish to show that for any fixed surface Σ and fixed well-quasi-order Ω, if $H_1, H_2, \ldots$ is an infinite sequence of hypergraphs drawn in Σ where each edge has two or three ends, and for each $i \geq 1$, $\phi_i : E(H_i) \to E(\Omega)$ is some function, then there exist $j > i \geq 1$ such that H_i is isomorphic to a "minor" of H_j, and for each edge e of H_i the corresponding edge f of H_j satisfies $\phi_i(e) \leq \phi_j(f)$. There are also special rules concerning the boundary of Σ and edges drawn touching the boundary (roughly, we ask that when we do contractions in producing a minor, any such edge remain in contact with the boundary during the contraction process).

Our approach is by a grand induction on the complexity of the surface and the well-quasi-order Ω. Thus, we first assume the result for all simpler surfaces, that is, with fewer handles and crosscaps, even if the surface boundary has more components ("cuffs") and the well-quasi-order is bigger. Second, for a fixed number of handles and crosscaps, we assume the result for sets of labelled hypergraphs in which the labels of the "internal" edges (that is, not on the boundary of Σ) all come from some proper subideal of our well-quasi-order, even if the labels on the boundary come from some larger well-quasi-order. Third, we proceed by induction on the number of cuffs, and there is a fourth of the same kind. To make this induction work, we find it necessary to divide the boundary of our surface into segments, and then have different restrictions on the labels of edges bordering each segment; and also, some edges drawn on the boundary have to be regarded as fixed.

The method of proof is to apply two theorems of earlier papers concerned with "tangles" in hypergraphs. Let $H_1, H_2, \ldots$ be a "bad" sequence of hypergraphs all drawn in Σ, with labelling functions $\phi_1, \phi_2, \ldots$, such that each pair (H_i, ϕ_i) satisfies the restrictions on labels described above. A theorem of [7] implies that if $\mathcal{T}$ is a tangle in H_i, and (H_i, ϕ_i) is "sufficiently general" relative to $\mathcal{T}$, then (H_i, ϕ_i) contains (H_1, ϕ_1) in the required way, a contradiction. It follows that, for every tangle $\mathcal{T}$ in every H_i, there is one of a bounded number of "structural deficiencies". But a theorem of [10] says that if relative to every tangle in every H_i we have a suitable kind of decomposition of H_i, then again some (H_j, ϕ_j) contains some (H_i, ϕ_i). Thus, it remains to show that our structural deficiencies can be converted to the right kind of decompositions, and that is the main part of the proof. This is where the grand induction is used—since the pieces into which we propose to decompose our hypergraphs are simpler than the originals (that is, are covered by the inductive hypothesis), we can infer that these pieces satisfy the theorem; and inferring that is a large part of showing that our decompositions are indeed of the "right kind".

The paper is organized as follows. We begin in sections 2 and 3 with definitions, and in section 4 explain the overall induction. In sections 5 and 6 we reconcile our containment relation with that of the theorem of [10] (unfortunately, they are not quite the same). Sections 7 and 8 are more definitions, introducing "tangles" and "tie-breakers", and in section 9 we state the theorem of [10] that we wish to apply, and begin the application. This is continued in sections 10-14. Finally, in section 15 we use a theorem of [7] to complete the proof.

2 Surfaces

In this paper, we mean by a *surface* a compact, connected 2-manifold Σ, with (possibly null) boundary, denoted by $bd(\Sigma)$. An O-arc in Σ is a subset of Σ homeomorphic to a circle, and a *line* is a subset of Σ homeomorphic to $[0, 1]$. A *closed disc* (or just "disc") in Σ is a subset of Σ homeomorphic to the unit disc in the real plane $\{(x, y) : x^2 + y^2 \leq 1\}$, and an *open disc* is defined similarly. Each component of $bd(\Sigma)$ is an O-arc, and we call them the *cuffs* of Σ. The surface obtained from Σ by "pasting" a disc onto each cuff we denote by $\hat{\Sigma}$. (It will not be necessary to distinguish between the different surfaces obtained by pasting different discs onto the cuffs.) If $X \subseteq \Sigma$, we denote its closure by $\bar{X}$, and define $\tilde{X} = \bar{X} \setminus X$.

A *drawing* G in Σ is a pair (U, V), where $U \subseteq \Sigma$ is closed and $V \subseteq U$ is finite, such that

- $U \setminus V$ has only finitely many arc-wise connected components, called *edges*
- for each edge e, $|\tilde{e}| = 2$, and $\bar{e}$ is a line with ends the two members of $\tilde{e}$
- for each edge e, either $e \subseteq bd(\Sigma)$ or $e \cap bd(\Sigma) = \emptyset$.

Thus, our drawings may have multiple edges, but not loops. We write $U(G) = U$ and $V(G) = V$. The set of edges of G is denoted by $E(G)$, and the elements of $V(G)$ are the *vertices* of G. The components of $\Sigma \setminus U(G)$ are called the *regions* of G in Σ (we shall occasionally also need to discuss the regions of G in $\hat{\Sigma}$). *Paths* and *circuits* have no "repeated" vertices. The remainder of our graph-theory terminology is standard.

A *march* in a set V is a sequence of distinct elements of V. If μ is the march $v_1, \ldots, v_k$ we denote the set $\{v_1, \ldots, v_k\}$ by $\bar{\mu}$. A *painting* Γ in Σ is a triple (U, V, γ), where $U \subseteq \Sigma$ is closed and $V \subseteq U$ is finite, such that

- $bd(\Sigma) \subseteq U$, and $U \setminus V$ has only finitely many arc-wise connected components, called *edges*
- for each edge e, either $|\tilde{e}| = 2$ and $\bar{e}$ is a line with ends the two members of $\tilde{e}$, or $|\tilde{e}| = 3$ and $\bar{e}$ is a disc with $\tilde{e} \subseteq bd(\bar{e})$
- γ is a function assigning to each edge e a march $\gamma(e)$ with $\bar{\gamma}(e) = \tilde{e}$; we call the ith term of $\gamma(e)$ the *ith end* of e, and the first and last terms of $\gamma(e)$ are the *tail* and *head* of e
- every edge e with $e \cap bd(\Sigma) \neq \emptyset$ satisfies $|\tilde{e}| = 2$ and $e \subseteq bd(\Sigma)$.

We write $U(\Gamma) = U, V(\Gamma) = V, \gamma_\Gamma = \gamma$, and we denote the set of edges of Γ by $E(\Gamma)$. Again, the elements of V are the *vertices* of Γ, and the connected components of $\Sigma \setminus U$ are the *regions* of Γ. We call $|\tilde{e}|$ the *size* of an edge e. Let

$$U' = V \cup \bigcup(e : e \in E(\Gamma), |\tilde{e}| = 2) \cup \bigcup(bd(\bar{e}) : e \in E(\Gamma), |\tilde{e}| = 3).$$

Then (U', V) is a drawing in Σ which we denote by $sk(\Gamma)$ (it is the "1-skeleton" of Γ).

(This definition is a little different from the definition of a painting in [9], but since we do not use here any results from [9] about paintings, we do not have to pay for the discrepancy yet.) For a painting or drawing in Σ, an edge e is a *border* edge if $e \cap bd(\Sigma) \neq \emptyset$, and otherwise is *internal*. (It is possible that $\tilde{e} \cap bd(\Sigma) \neq \emptyset$ for internal edges e.) An edge e *borders* a cuff Θ if $e \cap \Theta \neq \emptyset$. Note that all border edges in paintings have size 2.

Let Γ, Γ' be paintings in Σ. An *inflation* of Γ in Γ' is a function σ with domain $V(\Gamma) \cup E(\Gamma)$, satisfying

- $\sigma(e) \in E(\Gamma')$ and has the same size as e, for each $e \in E(\Gamma)$, and if $e_1, e_2 \in E(\Gamma)$ are distinct then $\sigma(e_1) \neq \sigma(e_2)$
- $\sigma(v)$ is a non-null connected induced subdrawing of $sk(\Gamma')$, for each $v \in V(\Gamma)$; and if $v_1, v_2 \in V(\Gamma)$ are distinct then $\sigma(v_1), \sigma(v_2)$ are disjoint
- for each $e \in E(\Gamma)$ and $1 \leq i \leq |\tilde{e}|$, if v is the ith end of e then $\sigma(v)$ contains the ith end of $\sigma(e)$.

We remark that it follows that for $e \in E(\Gamma)$ and $1 \leq i \leq |\tilde{e}|$, if v is not the ith end of e then $\sigma(v)$ does not contain the ith end of $\sigma(e)$, because $\sigma(v), \sigma(v')$ are disjoint where v' is the ith end of e. Consequently $\sigma(v)$ contains at most one end of $\sigma(e)$, for all $v \in V(\Gamma)$ and $e \in E(\Gamma)$, and in particular every edge of each $\sigma(v)$ is a subset of some $e \in E(\Gamma') \setminus \sigma(E(\Gamma))$. (We denote $\{\sigma(e) : e \in E(\Gamma)\}$ by $\sigma(E(\Gamma))$.)

An inflation σ of Γ in Γ' is *linear* if

- for each $e \in E(\Gamma)$ and for each cuff Θ, e borders Θ if and only if $\sigma(e)$ borders Θ (and so e is internal if and only if $\sigma(e)$ is internal)
- for each border edge e' of Γ', if $e' \notin \sigma(E(\Gamma))$ then e' is an edge of $\sigma(v)$ for some $v \in V(\Gamma)$
- for each $e \in E(\Gamma)$ bordering a cuff Θ, if we orient Θ such that the tail of e immediately precedes e under this orientation, then the tail of $\sigma(e)$ immediately precedes $\sigma(e)$ under the same orientation of Θ.

It follows that if $e_1, \ldots, e_k$ are the edges of Γ bordering a cuff Θ, in cyclic order, then $\sigma(e_1), \ldots, \sigma(e_k)$ occur in the same cyclic order around Θ.

Let Γ be a drawing or painting in Σ. A subset $X \subseteq \Sigma$ is Γ-*normal* if $X \cap U(\Gamma) \subseteq V(\Gamma)$. We say that Γ is *internally 3-connected* if $E(\Gamma) \neq \emptyset$ and every Γ-normal O-arc $F \subseteq \Sigma$ with

$$|F \cap V(\Gamma)| + |F \cap bd(\Sigma)| \leq 2$$

bounds a disc $\Delta \subseteq \Sigma$ with $\Delta \cap V(\Gamma) \subseteq F$.

A *quasi-order* Ω consists of a set $E(\Omega)$ and a reflexive, transitive relation $\leq$. It is a *well-quasi-order* if for every countable sequence ω_i $(i = 1, 2 \ldots)$ of elements of $E(\Omega)$ there exist $j > i \geq 1$ such that $\omega_i \leq \omega_j$. If Γ, Γ' are paintings in Σ, and σ is an inflation of Γ in Γ', and $\phi : E(\Gamma) \to E(\Omega)$, $\phi' : E(\Gamma') \to E(\Omega)$ are functions, we write $\phi \leq \phi' \circ \sigma$ if $\phi(e) \leq \phi'(\sigma(e))$ for every $e \in E(\Gamma)$.

The following is a version of the main theorem of this paper, although it is not yet in the most convenient form for us to prove.

2.1 *Let Σ be a surface, and let Ω be a well-quasi-order. Let Γ_i $(i = 1, 2 \ldots)$ be a countable sequence of internally 3-connected paintings in Σ, and for each $i \geq 1$ let $\phi_i : E(\Gamma_i) \to E(\Omega)$ be some function. Then there exist $j > i \geq 1$ and a linear inflation σ of Γ_i in Γ_j satisfying $\phi_i \leq \phi_j \circ \sigma$.*

3 Frames and colour schemes

In this section we cast 2.1 into a more convenient form. Let Σ be a surface. A *directed drawing* in Σ means a drawing in Σ with a direction assigned to each edge. A *frame* Φ in Σ consists of a directed drawing in Σ (which we also denote by Φ) with $U(\Phi) = bd(\Sigma)$, together with a designation of each edge of Φ as *long* or *short*, such that no two long edges have a common end. We call the edges of Φ the *sides* of the frame. A painting Γ in Σ is said to *fit* a frame Φ if

- $V(\Phi) \subseteq V(\Gamma)$
- each short side of Φ is an edge of Γ
- for every border edge e of Γ, if S is the side of Φ with $e \subseteq S$ then the tail of e precedes its head as S is traversed from its tail in Φ to its head (that is, briefly, the direction of e defined by $\gamma_\Gamma(e)$ *agrees* with the direction of S in Φ)
- Γ is internally 3-connected
- if $e \in E(\Gamma)$ and $|\tilde{e}| = 3$ and r is a region of Γ in Σ with $|\bar{r} \cap V(\Gamma)| \geq 3$, then $\bar{f} \cap bd(\Sigma) \neq \emptyset$ for every component f of $e \cap \bar{r}$.

The fourth and fifth conditions have nothing to do with the frame, but this is a convenient place to introduce them. The fifth condition is a very strong requirement; in particular, it says that for any edge e of Γ with size 3, if no vertex of $\tilde{e}$ is in $bd(\Sigma)$, then for any two $u, v \in \tilde{e}$, there is a second edge f with $u, v \in \tilde{f}$. This might seem to be very restrictive, but really it is not. For given a general painting, we can augment it to construct another satisying this restriction (for every edge e of size 3, just add three edges of size 2 joining the pairs of vertices of $\tilde{e}$, drawn close to e); and it turns out that one of these augmented paintings contains another in the required way, if and only if one of the unaugmented paintings contains another. So if we only want to prove well-quasi-ordering, it suffices to prove it for augmented paintings. And it turns out to be a convenient technical device for the proof, later.

If e is a border edge of Γ and S is a side of Φ with $e \subseteq S$, we say that e *borders* S. If Γ, Γ' are paintings in Σ both fitting a frame Φ, and σ is an inflation of Γ in Γ', we say that σ *respects* Φ if for every border edge e of Γ, $\sigma(e)$ is a border edge and e and $\sigma(e)$ border the same side of Φ. If σ respects Φ, it follows that $\sigma(e) = e$ for every short side e of Φ, and $v \in V(\sigma(v))$ for every $v \in V(\Phi)$. If σ respects Φ, and satisfies the second condition in the definition of "linear", then σ is linear, as the reader should verify.

A *colour scheme* χ consists of a surface Σ_χ, a frame Φ_χ in Σ_χ, two well-quasi-orders $\Omega_\chi(2)$ and $\Omega_\chi(3)$, and a well-quasi-order $\Omega_\chi(S)$ for each side S of Φ_χ, such that $E(\Omega_\chi(S)) = \{S\}$ for every short side S. (Thus, to specify a colour scheme, we need not define $\Omega_\chi(S)$ for short sides S.) If χ is a colour scheme, a χ-*coloured painting* is a pair (Γ, ϕ), where

- Γ is a painting in Σ_χ fitting the frame Φ_χ
- ϕ is a function with domain $E(\Gamma)$, such that if $e \in E(\Gamma)$ is internal then $\phi(e) \in \Omega_\chi(|\tilde{e}|)$, and if e borders a side S then $\phi(e) \in \Omega_\chi(S)$.

If $(\Gamma, \phi), (\Gamma', \phi')$ are χ-coloured paintings, an *inflation* of (Γ, ϕ) in (Γ', ϕ') is an inflation σ of Γ in Γ' respecting Φ_χ, such that $\phi \leq \phi' \circ \sigma$ (with the natural meaning); and *linear inflations* are defined similarly. The following is the main theorem of the paper.

3.1 *For every colour scheme χ and every countable sequence (Γ_i, ϕ_i) $(i = 1, 2, \ldots)$ of χ-coloured paintings, there exist $j > i \geq 1$ and a linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j).*

We begin the proof of 3.1 in the next section. But let us first verify that it implies 2.1. If Ω_1, Ω_2 are quasi-orders, $\Omega_1 \times \Omega_2$ denotes their product, with element set $\{(x_1, x_2) : x_1 \in E(\Omega_1), x_2 \in E(\Omega_2)\}$, in which $(x_1, x_2) \leq (x_1', x_2')$ if $x_1 \leq x_1'$ in Ω_1 and $x_2 \leq x_2'$ in Ω_2.

Proof of 2.1, assuming 3.1.

Let $\Sigma, \Omega, \Gamma_i, \phi_i (i \geq 1)$ be as in 2.1. Let Φ be a frame for Σ, such that each component of Φ is a 2-edge circuit, one edge of which is long and one short. Now $bd(\Sigma) \subseteq U(\Gamma_i)$ for each i. By replacing each Γ_i by its image under a suitable homeomorphism of Σ to itself, we may therefore assume that for each cuff Θ, the short side of Φ included in Θ is an edge of each Γ_i.

(1) *We may assume that for each short side s of Φ,*

- *$\phi_1(s) \leq \phi_2(s) \leq \ldots$, and*
- *for each $i \geq 1$ the direction of s under $\gamma_{\Gamma_i}(s)$ agrees with its direction in Φ.*

Subproof. There is an infinite subset $I \subseteq \{1, 2, \ldots\}$ such that for all $i, j \in I$ with $i \leq j$ and every short side s of Φ, $\gamma_{\Gamma_i}(s) = \gamma_{\Gamma_j}(s)$ and $\phi_i(s) \leq \phi_j(s)$. We may assume that $I = \{1, 2, \ldots\}$ by replacing our original sequence by this subsequence, and we may reverse the direction in Φ of each short side of Φ if necessary. This proves (1).

For each $i \geq 1$, let $\Gamma_i = (U_i, V_i, \gamma_i)$. For $k = 2, 3$, let $E_i^k = \{e \in E(\Gamma_i) : |\tilde{e}| = k\}$. For each $i \geq 1$ and each $e \in E_i^3$, choose a disc $\Delta \subseteq \bar{e}$ with $\Delta \cap bd(\bar{e}) = \tilde{e}$, and define $s_i(e) = \Delta \setminus \tilde{e}$. Let

$$U_i' = U(sk(\Gamma_i)) \cup \bigcup (s_i(e) : e \in E_i^3)$$
$$V_i' = V_i$$

and define γ_i' as follows. For $e \in E_i^3$, let $\gamma_i'(s_i(e)) = \gamma_i(e)$. For each border edge e of Γ_i, let $\gamma_i'(e)$ agree with the direction of the side of Φ containing e. For each internal edge e of Γ_i with $|\tilde{e}| = 2$, let $\gamma_i'(e) = \gamma_i(e)$. For each internal edge e of $sk(\Gamma_i)$ with $e \notin E(\Gamma_i)$, let $\gamma_i'(e)$ be an arbitrary march with $\bar{\gamma}_i'(e) = \tilde{e}$. Let $\Gamma_i' = (U_i', V_i', \gamma_i')$. We see that Γ_i' is a painting in Σ fitting the frame Φ.

Let R be the well-quasi-order with $E(R) = \{-1, 0, 1, 2, 3\}$, ordered by equality. Let χ be the colour scheme with $\Sigma_\chi = \Sigma, \Phi_\chi = \Phi, \Omega_\chi(3) = \Omega_\chi(2) = \Omega \times R$, and $\Omega_\chi(S) = \Omega \times R$ for every long side S of Φ. For each edge e of Γ_i', define $\phi_i'(e)$ as follows. If $|\tilde{e}| = 3$, choose $c \in E(\Gamma_i)$ with $e = s_i(c)$, and let $\phi_i'(e) = (\phi_i(c), 3)$. If $e \in E_i^2$ and e is internal, let $\phi_i'(e) = (\phi_i(e), 2)$. If e is a border edge of Γ_i' (and therefore $e \in E_i^2$), but not a short side, let $\phi_i'(e) = (\phi_i(e), \pm 1)$, where -1 is chosen if $\gamma_i'(e) \neq \gamma_i(e)$. If $|\tilde{e}| = 2$ and $e \notin E_i^2$, let $\phi_i'(e) = (x, 0)$ where $x \in E(\Omega)$ is arbitrary. For each short side e of Φ, let $\phi_i'(e) = e$. It follows easily that for each $i \geq 1, (\Gamma_i', \phi_i')$ is a χ-coloured painting.

From 3.1, there exists $j > i \geq 1$ and a linear inflation σ' of (Γ_i', ϕ_i') in (Γ_j', ϕ_j'). We claim that there is a linear inflation σ of Γ_i in Γ_j such that $\phi_i \leq \phi_j \circ \sigma$. We construct σ as follows. For each

$v \in V(\Gamma_i)$ let $\sigma(v) = \sigma'(v) \cap sk(\Gamma_j)$. Since each $\sigma'(v)$ is induced and every edge of $sk(\Gamma'_j)$ either is an edge of $sk(\Gamma_j)$ or is parallel to an edge of $sk(\Gamma_j)$, it follows that each $\sigma(v)$ is a non-null connected induced subgraph of $sk(\Gamma_j)$. For $e \in E_i^3$, let $\sigma(e) = f$, where $\sigma'(s_i(e)) = s_j(f)$ (such an f exists since $\sigma'(s_i(e))$ has the same size as $s_i(e)$). For $e \in E_i^2$, let $\sigma(e) = \sigma'(e)$. Let us verify that σ is a linear inflation σ of Γ_i in Γ_j such that $\phi_i \leq \phi_j \circ \sigma$. First, by examining the second terms of ϕ', it follows that $\sigma(e) = e$ for every short side e. Moreover, if $e \in E_i^2$ is not a short side then since the second term of $\phi'_i(e)$ is non-zero, so is the second term of $\phi'_j(s'(e))$, since $\phi'_i(e) \leq \phi'_j(\sigma'(e))$, and hence $\sigma'(e) \in E_j^2$. By examining the first term of $\phi'_i(e)$ and using (1), we see that $\phi_i \leq \phi_j \circ \sigma$. To complete the proof, we must check that

(2) *For each $e \in E(\Gamma_i)$ and $1 \leq k \leq |\tilde{e}|$, if v is the kth end of e then $\sigma(v)$ contains the kth end of $\sigma(e)$.*

Subproof. Now v is the kth term of $\gamma_i(e)$. If $|\tilde{e}| = 3$, then $\gamma'_i(s_i(e)) = \gamma_i(e)$, and therefore $V(\sigma'(v)) = V(\sigma(v))$ contains the kth term of $\gamma'_j(\sigma'(s_i(e))) = \gamma_j(\sigma(e))$ as required. Similarly, if $|\tilde{e}| = 2$ and e is internal or a short side of Φ, then $\gamma'_i(e) = \gamma_i(e)$, and therefore $V(\sigma'(v)) = V(\sigma(v))$ contains the kth term of $\gamma'_j(\sigma'(e)) = \gamma_j(\sigma(e))$ as required. Finally, suppose that e is a border edge of Γ_i and not a short side of Φ. Let S be the long side of Φ with $e, \sigma(e) \subseteq S$. Since the second terms of $\phi'_i(e)$ and $\phi'_j(\sigma(e))$ are equal, it follows that $\gamma'_i(e) = \gamma_i(e)$ if and only if $\gamma'_j(\sigma(e)) = \gamma_j(\sigma(e))$, and so there exists $l (l = 1$ or $2)$ such that the kth term of $\gamma_i(e)$ is the lth term of $\gamma'_i(e)$, and the kth term of $\gamma_j(\sigma(e))$ is the lth term of $\gamma'_j(\sigma(e))$. Therefore v is the lth term of $\gamma'_i(e)$, and so $\sigma'(v)$ contains the lth term of $\gamma'_j(\sigma(e))$. Consequently $\sigma(v)$ contains the kth term of $\gamma_j(\sigma(e))$. This proves (2).

From (2), it follows that σ is an inflation of Γ_i in Γ_j. Moreover, it is clearly linear, and satisfies $\phi_i \leq \phi_j \circ \sigma$, as required. ■

4 The induction

The remainder of the paper is devoted to proving 3.1. We proceed by an induction on χ which we shall explain in this section.

A *bad sequence* for a colour scheme χ is a countable sequence $(\Gamma_i, \phi_i)(i = 1, 2, \ldots)$ of χ-coloured paintings, such that for all $j > i \geq 1$ there is no linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j). We say that a colour scheme is *bad* if there is a bad sequence for it. Thus, we wish to show that no colour scheme is bad. We wish to consider the following four statements $\mathbf{S_1}$–$\mathbf{S_4}$ about a colour scheme χ.

A surface without boundary is *simpler* than another if the second can be obtained from the first by adding at least one handle or crosscap. Our first statement is

$\mathbf{S_1}$ *There is no bad colour scheme χ' with $\hat{\Sigma}_{\chi'}$ simpler than $\hat{\Sigma}_{\chi}$.*

A set $\{(\Gamma_i, \phi_i) : i \in I\}$ of χ-coloured paintings is said to be *similarly oriented* if either Σ_χ is not orientable, or there is an orientation ω of Σ_χ such that for all $i \in I$ and every $e \in E(\Gamma_i)$ with $|\tilde{e}| = 3$, the orientations of e given by ω and given by the cyclic order of the terms of $\gamma_{\Gamma_i}(e)$ coincide. A colour scheme χ is *orientedly bad* if there is a similarly oriented bad sequence for it.

If Ω, Ω' are quasi-orders we write $\Omega' \preceq \Omega$ if

- $E(\Omega') \subseteq E(\Omega)$

- for each $x' \in E(\Omega')$ and $x \in E(\Omega)$, if $x \leq x'$ in Ω then $x \in E(\Omega')$, and
- for $x_1, x_2 \in E(\Omega'), x_1 \leq x_2$ in Ω' if and only if $x_1 \leq x_2$ in Ω.

We write $\Omega' \prec \Omega$ if $\Omega' \preceq \Omega$ and $\Omega' \neq \Omega$. We use $\cong$ to denote homeomorphism. Our second statement is

S$_2$ *There is no orientedly bad colour scheme χ' with $\hat{\Sigma}_{\chi'} \cong \hat{\Sigma}_\chi$ such that $\Omega_{\chi'}(3) \preceq \Omega_\chi(3)$ and $\Omega_{\chi'}(2) \preceq \Omega_\chi(2)$ where at least one of the inclusions is strict.*

We denote the number of cuffs of a surface Σ by $c(\Sigma)$.

S$_3$ *There is no orientedly bad colour scheme χ' with $\hat{\Sigma}_{\chi'} \cong \hat{\Sigma}_\chi, \Omega_{\chi'}(k) = \Omega_\chi(k)(k = 2, 3)$, and $c(\Sigma_{\chi'}) < c(\Sigma_\chi)$.*

Now let χ, χ' be colour schemes. We say that χ' is a *refinement* of χ if $\Sigma_{\chi'} \cong \Sigma_\chi, \Omega_{\chi'}(k) = \Omega_\chi(k)(k = 2, 3)$, and there is a function f from the set of long sides of $\Phi_{\chi'}$ to the set of long sides of Φ_χ, such that

- $\Omega_{\chi'}(R) \preceq \Omega_\chi(f(R))$ for each long side R of $\Phi_{\chi'}$
- if R_1, R_2 are distinct long sides of $\Phi_{\chi'}$ with $f(R_1) = f(R_2)$ then $\Omega_{\chi'}(R_1) \prec \Omega_\chi(f(R_1))$
- if for each long side S of Φ_χ there is a long side R of $\Phi_{\chi'}$ with $f(R) = S$ and $\Omega_{\chi'}(R) = \Omega_\chi(S)$, then $\Phi_{\chi'}$ has fewer short sides then Φ_χ.

In this case, we call f an *embedding* of χ' in χ.

S$_4$ *There is no orientedly bad colour scheme which is a refinement of χ.*

The purpose of **S$_1$**–**S$_4$** is that to prove 3.1, it suffices to prove the following.

4.1 *If χ satisfies* **S$_1$**–**S$_4$** *then it is not orientedly bad.*

We prove 4.1 in section 15. The remainder of this section is devoted to proving that 4.1 implies 3.1. To show that, we need several lemmas, which follow.

4.2 *If χ is a bad colour scheme, there is an orientedly bad colour scheme χ' with $\Sigma_{\chi'} = \Sigma_\chi$.*

Proof. If Σ_χ is not orientable then χ itself is orientedly bad. Thus we may assume that ω is an orientation of Σ_χ. Let $(\Gamma_i, \phi_i)(i = 1, 2 \ldots)$ be a bad sequence for χ, and let $\Gamma_i = (U_i, V_i, \gamma_i)(i \geq 1)$. Now for each $e \in E(\Gamma_i)$ with $|\tilde{e}| = 3$, let $\gamma'_i(e)$ be a march with $\bar{\gamma}'_i(e) = \tilde{e}$, such that the orientations of e given by ω and given by the cyclic order of the terms of $\gamma'_i(e)$ coincide, and $\gamma_i(e), \gamma'_i(e)$ have the same first term. For each $e \in E(\Gamma_i)$ with $|\tilde{e}| = 2$ let $\gamma'_i(e) = \gamma_i(e)$. Let $\Gamma'_i = (U_i, V_i, \gamma'_i)$. Let $\phi'_i(e)$ be defined by

$$\phi'_i(e) = \begin{cases} \phi_i(e) & \text{if } |\tilde{e}| = 2 \\ (\phi_i(e), \pm 1) & \text{if } |\tilde{e}| = 3 \end{cases}$$

where we choose -1 if $\gamma_i(e) \neq \gamma'_i(e)$. Let R be the well-quasi-order with $E(R) = \{-1, 1\}$, ordered by equality. Let χ' be the colour scheme with $\Sigma_{\chi'} = \Sigma_\chi, \Phi_{\chi'} = \Phi_\chi, \Omega_{\chi'}(3) = \Omega_\chi(3) \times R, \Omega_{\chi'}(2) = \Omega_\chi(2)$, and $\Omega_{\chi'}(S) = \Omega_{\chi}(S)$ for each long side S. Then for each $i \geq 1$, (Γ'_i, ϕ'_i) is a χ'-coloured painting, and $\{(\Gamma'_i, \phi'_i) : i \geq 1\}$ is similarly oriented. Suppose that for some i, j with $j > i \geq 1$, there is a linear inflation σ of (Γ'_i, ϕ'_i) in (Γ'_j, ϕ'_j). We claim that σ is also a linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j). This is mostly clear, but let us show that the third condition in the definition of "inflation" is satisfied, that is, that for each $e \in E(\Gamma_i)$ and $1 \leq k \leq |\tilde{e}|$, if v is the kth term of $\gamma_i(e)$, then $\sigma(v)$ contains the kth term of $\gamma_j(\sigma(e))$. If $\gamma_i(e) = \gamma'_i(e)$ and $\gamma_j(\sigma(e)) = \gamma'_j(\sigma(e))$ then the claim holds since σ is an inflation of Γ'_i in Γ'_j. We may assume then that $|\tilde{e}| = 3$. Since the second terms of $\phi'_i(e)$ and of $\phi'_j(\sigma(e))$ are equal, and we may assume they are not both equal to 1, it follows that they are both -1, and consequently $\gamma'_i(e)$ is obtained from $\gamma_i(e)$ by exchanging the second and third terms, and the same holds between $\gamma'_j(\sigma(e))$ and $\gamma_j(\sigma(e))$. If $k = 1$ let $k' = 1$, and if $k = 2$ or 3 let k' be 3 or 2 respectively. Consequently the kth term of $\gamma_i(e)$ (that is, v) is the k'th term of $\gamma'_i(e)$, and the kth term of $\gamma_j(\sigma(e))$ is the k'th term of $\gamma'_j(\sigma(e))$. Since σ is an inflation of (Γ'_i, ϕ'_i) in (Γ'_j, ϕ'_j) and v is the k'th term of $\gamma'_i(e)$, it follows that $\sigma(v)$ contains the k'th term of $\gamma'_j(\sigma(e))$, and therefore contains the kth term of $\gamma_j(\sigma(e))$, as required. This proves our claim that σ is a linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j), which is impossible since $(\Gamma_i, \phi_i)(i = 1, 2 \ldots)$ is a bad sequence for χ. Thus there is no such choice of i, j, σ, and so χ' is orientedly bad, as required. ■

4.3 *There is no countable sequence Ω_i $(i = 1, 2, \ldots)$ of well-quasi-orders such that $\Omega_{i+1} \prec \Omega_i$ for all $i \geq 1$.*

The result is well-known and the proof is easy, and so we omit it.

4.4 *There is no countable sequence χ_i $(i = 1, 2, \ldots)$ of colour schemes such that χ_{i+1} is a refinement of χ_i for all $i \geq 1$.*

Proof. Suppose that χ_i $(i = 1, 2, \ldots)$ is such a sequence. For each $i \geq 1$, let $\mathcal{S}_i$ be the set of long sides of Φ_{χ_i}, and let $f_i : \mathcal{S}_{i+1} \to \mathcal{S}_i$ be an embedding of χ_{i+1} in χ_i. Let t_0 be a new vertex, and let T be the infinite tree with

$$V(T) = \{t_0\} \cup \{(i, S) : i \geq 1, S \in \mathcal{S}_i\}$$
$$E(T) = \{(t_0, (1, S)) : S \in \mathcal{S}_1\} \cup \{((i, f_i(S)), (i+1, S)) : i \geq 1, S \in \mathcal{S}_{i+1}\}.$$

(1) *T has infinitely many vertices with degree $\neq 2$.*

Subproof. Otherwise f_i is a bijection for all sufficiently large i, say for all $i > n$. For all $i \geq n$, let $\mathcal{S}_i = \{S_i^1, \ldots, S_i^k\}$, numbered such that $f_i(S_{i+1}^j) = S_i^j$ for all $i > n$ and $1 \leq j \leq k$. For all j with $1 \leq j \leq k$ it follows from the first condition in the definition of "refinement" that

$$\Omega_{\chi_{i+1}}(S_{i+1}^j) \preceq \Omega_{\chi_i}(S_i^j)$$

for all $i > n$, and so by 4.3, equality holds here for all sufficiently large i; and we may therefore assume, by increasing n, that equality holds for all $i > n$. From the third condition in the definition of "refinement", it follows that for all $i > n$, $\Phi_{\chi_{i+1}}$ has strictly fewer short sides than Φ_{χ_i}, which is impossible. This proves (1).

Since every vertex of T has finite degree, we can apply applying König's lemma to the infinite tree obtained from T by suppressing all vertices of degree 2. We deduce that T has a path containing infinitely many vertices of degree $\neq 2$. Thus for all i there exists $S_i \in \mathcal{S}_i$ such that $f_i(S_{i+1}) = S_i$, and such that (i, S_i) has degree ≥ 3 in T for infinitely many values of i. We deduce that

$$\Omega_{\chi_{n+1}}(S_{n+1}) \succeq \Omega_{\chi_{n+2}}(S_{n+2}) \succeq \dots$$

with strict inclusion infinitely many times, contrary to 4.3. Thus there is no such sequence χ_i ($i = 1, 2, \dots$). This proves 4.4. ■

Proof of 3.1, assuming 4.1.

Suppose that 3.1 is false; then there is a bad colour scheme χ, and we may choose it to satisfy $\mathbf{S_1}$. By 4.2, we may choose χ orientedly bad. By 4.3, we may choose χ so that in addition it satisfies $\mathbf{S_2}$. By choosing such a χ with $c(\Sigma_\chi)$ minimum, we find it also satisfies $\mathbf{S_3}$. By 4.4, we may choose χ to satisfy $\mathbf{S_4}$ as well. But then 4.1 is contradicted. Thus 3.1 holds. ■

5 Surface homeomorphisms

Let Φ be a frame in Σ. A homeomorphism $\alpha : \Sigma \to \Sigma$ is Φ-*preserving* if

- $\alpha(v) = v$ for all $v \in V(\Phi)$, and $\alpha(S) = S$ for all $S \in E(\Phi)$ (and hence α preserves the direction of each side, and maps each cuff onto itself in an orientation-preserving way)
- if Σ is orientable then α preserves the orientation of Σ.

Let (Γ, ϕ) be a χ-coloured painting, and let $\alpha : \Sigma_\chi \to \Sigma_\chi$ be a Φ_χ-preserving homeomorphism. We define *the image of* (Γ, ϕ) *under* α in the natural way; that is, it is the χ-coloured painting (Γ', ϕ') where $U(\Gamma') = \alpha(U(\Gamma)), V(\Gamma') = \alpha(V(\Gamma))$, and for each $e \in E(\Gamma), \gamma_{\Gamma'}(\alpha(e)) = \alpha(\gamma_\Gamma(e))$ and $\phi'(\alpha(e)) = \phi(e)$. The proof of the following lemma is clear.

5.1 *Let* $(\Gamma_1, \phi_1), (\Gamma_2, \phi_2)$ *be* χ*-coloured paintings, and let* α_1, α_2 *be* Φ_χ*-preserving homeomorphisms of* Σ. *For* $i = 1, 2$ *let* (Γ'_i, ϕ'_i) *be the image of* (Γ_i, ϕ_i) *under* α_i. *Then there is a (linear) inflation of* (Γ'_1, ϕ'_1) *in* (Γ'_2, ϕ'_2) *if and only if there is one of* (Γ_1, ϕ_1) *in* (Γ_2, ϕ_2).

Thus, for example, if we suppose that $(\Gamma_i, \phi_i) (i = 1, 2, \dots)$ is a bad sequence for χ, then we may replace each (Γ_i, ϕ_i) by its image under some Φ_χ-preserving homeomorphism, and obtain another bad sequence; and if the terms of the first sequence are similarly oriented, then so are the terms of the second. This method will be used in combination with the following lemma. Let Φ be a frame in Σ. A drawing K in Σ is a *feature* (in Φ) if K has no border edges and no isolated vertices, and $U(K) \cap S = \emptyset$ for every short side S of Φ. ($U(K)$ may intersect the long sides of Φ.) Two features K_1, K_2 are *equivalent* if there is a Φ-preserving homeomorphism α of Σ such that α maps K_1 to K_2. Hence if K_1, K_2 are equivalent they have the same size. (The *size* of a drawing G is $|V(G)| + |E(G)|$.) This defines an equivalence relation, and it follows easily (by choosing a line to represent an edge of K, cutting the surface along it, and using induction on $|E(K)|$—see section 3 of [3] for similar proofs) that

5.2 *For every surface* Σ, *every frame* Φ *in* Σ *and every integer* $n \geq 0$, *there are only finitely many equivalence classes of features in* Φ *with size* $\leq n$.

Now let Φ be a frame in Σ, and let K be a feature in Φ. If we cut Σ along $U(K)$ we obtain one or several components $\Sigma_1, \ldots, \Sigma_k$ say, which we call *fragments* of Σ after cutting along $U(K)$. There is a natural surjection of $\psi : \Sigma_1 \cup \ldots \cup \Sigma_k \to \Sigma$, which we shall call the *associated surjection.* Let Σ_1 be a fragment. We denote by $\psi^{-1}(\Phi) \cap \Sigma_1$ the frame for Σ_1 consisting of the drawing

$$(bd(\Sigma_1), \psi^{-1}(V(\Phi) \cup V(K)) \cap \Sigma_1)$$

where for each edge R of this drawing

- if $\psi(R) \subseteq S$ for some side S of Φ, then R is directed such that ψ maps its direction onto that of S, and R is designated as long if and only if S is long
- if there is no such S, then R is directed arbitrarily and is designated as short.

Now let Γ be a painting fitting Φ, such that $U(\Gamma) \cap U(K) = V(\Gamma) \cap V(K)$. We denote by $\psi^{-1}(\Gamma) \cap \Sigma_1$ the painting $\Gamma_1 = (U_1, V_1, \gamma_1)$ in Σ_1, where

$$\begin{aligned} U_1 &= (\psi^{-1}(U(\Gamma)) \cap \Sigma_1) \cup bd(\Sigma_1) \\ V_1 &= \psi^{-1}(V(\Gamma) \cup V(K)) \cap \Sigma_1 \end{aligned}$$

and for each edge e of Γ_1, if $e \subseteq bd(\Sigma_1)$ then $\gamma_1(e)$ is defined such that the direction of e agrees with the direction of the side of $\psi^{-1}(\Phi) \cap \Sigma_1$ including it, while if $e \not\subseteq bd(\Sigma_1)$ then $\gamma_1(e)$ is defined such that ψ maps it to $\gamma_\Gamma(\psi(e))$.

5.3 *With Γ, K, etc. as above, if each component K' of K satisfies*

$$|V(K') \cap V(\Gamma)| + |V(K') \cap bd(\Sigma)| \geq 2$$

then $\psi^{-1}(\Gamma) \cap \Sigma_1$ fits the frame $\psi^{-1}(\Phi) \cap \Sigma_1$.

Proof. We observe first that the hypothesis implies that for every component K' of K, $V(K')$ contains a vertex of Γ, since $V(K) \cap bd(\Sigma) \subseteq V(K) \cap U(\Gamma) \subseteq V(\Gamma)$. Verifying the first three conditions in the definition of "fit" is easy and is omitted. Let us verify the fourth and fifth conditions. Let $\Gamma_1 = \psi^{-1}(\Gamma) \cap \Sigma_1$, and let F_1 be a Γ_1-normal O-arc in Σ_1 with $|F_1 \cap V(\Gamma_1)| + |F_1 \cap bd(\Sigma_1)| \leq 2$. Then $|F_1 \cap bd(\Sigma_1)| \leq 1$ since $F_1 \cap bd(\Sigma_1) \subseteq F_1 \cap V(\Gamma_1)$, and so $F = \psi(F_1)$ is a Γ-normal O-arc in Σ with $|F \cap V(\Gamma)| + |F \cap bd(\Sigma)| \leq 2$. Since Γ is internally 3-connected, there is a disc $\Delta \subseteq \Sigma$ bounded by F with $\Delta \cap V(\Gamma) \subseteq F$. If $U(K) \cap (\Delta \setminus F) = \emptyset$, then $\psi^{-1}(\Delta)$ includes a disc in Σ_1 bounded by F_1 as required. We assume then (for a contradiction) that $U(K) \cap (\Delta \setminus F) \neq \emptyset$. Let K' be a component of K with $U(K') \cap (\Delta \setminus F) \neq \emptyset$. Since $V(K')$ contains a vertex of Γ, and $\Delta \setminus F$ is disjoint from $V(\Gamma)$, it follows that $U(K') \not\subseteq \Delta \setminus F$, and since K' is connected, there is an edge e of K' with $\bar{e} \cap F \neq \emptyset$ and with $e \cap (\Delta \setminus F) \neq \emptyset$. Choose $u \in \bar{e} \cap F$, and let $u_1 \in F_1$ with $\psi(u_1) = u$. Since $u \in U(K')$, it follows that $u_1 \in bd(\Sigma_1) \subseteq U(\Gamma_1)$. Since F_1 is Γ_1-normal, we deduce that $u_1 \in V(\Gamma_1)$. Consequently $u \in V(\Gamma) \cup V(K)$, and since also $u \in U(K)$ and $V(\Gamma) \cap U(K) \subseteq V(K)$, it follows that $u \in V(K)$. Since u_1 belongs to both $F_1 \cap V(\Gamma_1)$ and $F_1 \cap bd(\Sigma_1)$, and $|F_1 \cap V(\Gamma_1)| + |F_1 \cap bd(\Sigma_1)| \leq 2$, we deduce that $F_1 \cap V(\Gamma_1) = F_1 \cap bd(\Sigma_1) = \{u_1\}$; and so $F \cap (U(K') \cup bd(\Sigma)) = \{u\}$.

Let $L_1 \subseteq F_1$ be a closed line segment with u_1 in its interior. Since F_1 is an O-arc, there is a second line segment $L_1' \subseteq \Sigma_1$ with the same ends as L_1, such that $L_1 \cup L_1'$ is an O-arc in Σ_1, bounding a disc $D_1 \subseteq \Sigma_1$ with $D_1 \cap F_1 = L$. Let $D = \psi(D_1)$, $L = \psi(L_1)$, and $L' = \psi(L'_1)$. Then D is a disc in Σ, with boundary $L \cup L'$, where L is a line segment in F with u in its interior, and $D \cap F = L$. Moreover, the interior of D is disjoint from $U(K)$, and in particular, disjoint from e. It follows that $D \cap \Delta = L$. Since the interiors of both D and Δ are disjoint from $bd(\Sigma)$, we deduce that $u \notin bd(\Sigma)$; and since the interior of D is disjoint from $U(K)$, it follows that there is no edge f of K with $f \subseteq \Sigma \setminus \Delta$ incident with u. We deduce that $U(K') \subseteq \Delta$. Since $u \notin bd(\Sigma)$, it follows that $F \cap bd(\Sigma) = \emptyset$, and so $V(K') \cap bd(\Sigma) = \emptyset$. Moreover, since $V(\Gamma) \cap \Delta \subseteq F$, it follows that

$$V(K') \cap V(\Gamma) = V(K') \cap V(\Gamma) \cap \Delta \subseteq V(K') \cap F \subseteq \{u\}.$$

Consequently,

$$|V(K') \cap V(\Gamma)| + |V(K') \cap bd(\Sigma)| \leq 1$$

contrary to the hypothesis. This verifies the fourth condition.

For the fifth, let $e_1 \in E(\Gamma_1)$ with $|\tilde{e}_1| = 3$, let r_1 be a region of Γ_1 in Σ_1, let f_1 be a component of $\bar{r}_1 \cap e_1$, and let $\bar{f}_1 \cap \tilde{e}_1 = \{v_1, v'_1\}$; we suppose that $v_1, v'_1 \notin bd(\Sigma_1)$, and will show that $|\bar{r}_1 \cap V(\Gamma_1)| = 2$. Let $e = \psi(e_1), f = \psi(f_1), v = \psi(v_1), v' = \psi(v'_1)$, and let r be the region of Γ in Σ including $\psi(r_1)$. Then f is a component of $\bar{r} \cap e$, and $|\bar{f} \cap \tilde{e}| = \{v, v'\}$. Since $v_1, v'_1 \notin bd(\Sigma_1)$, it follows that $v, v' \notin bd(\Sigma)$. Since Γ fits Φ, it follows from the fifth condition in the definition of "fit", applied to Γ, Φ, that $|\bar{r} \cap V(\Gamma)| = 2$ and so $\bar{r} \cap V(\Gamma) = \{v, v'\}$. Hence $U(K) \cap bd(\bar{r}) = \emptyset$, and since each component K' of K contains a vertex of Γ it follows that $U(K) \cap \bar{r} = \emptyset$. Consequently $\psi(r) = r_1$, and so $|\bar{r}_1 \cap V(\Gamma_1)| = 2$ as required. ■

A painting Γ in a surface Σ is 2-*cell* if every region of Γ in Σ is homeomorphic to an open disc. Every internally 3-connected painting is 2-cell. Let Γ be a 2-cell painting with $E(\Gamma) \neq \emptyset$. A drawing Γ^* in $\hat{\Sigma}$ is a *radial drawing* of Γ if it satisfies the four conditions following, where R^* denotes the set of vertices of Γ^* that are not in $V(\Gamma)$:

- $U(\Gamma) \cap U(\Gamma^*) = V(\Gamma) \subseteq V(\Gamma^*)$
- every region of Γ in $\hat{\Sigma}$ contains a unique vertex of Γ^*
- Γ^* is bipartite, and $(V(\Gamma), R^*)$ is a bipartition of it
- for every $v \in V(\Gamma)$, the edges of Γ and of Γ^* incident with v alternate in their cyclic order around v.

It is easy to see that such a drawing Γ^* exists, and is unique up to homeomorphisms of $\hat{\Sigma}$ to itself that fix $U(\Gamma)$ pointwise. If r is a region of Γ, the unique vertex of Γ^* contained in r is denoted by r^*; and if $r \not\subseteq \Sigma$ (and hence r is a component of $\hat{\Sigma} \setminus \Sigma$) we call r^* a *pole* . We shall use the Γ^*, R^*, r^* notation without further explanation.

If Γ is an internally 3-connected painting in Σ, we define $dist(\Gamma)$ to be the minimum of $\frac{1}{2}|E(F)|$, taken over all paths F of Γ^* with ends distinct poles. (If $c(\Sigma) \leq 1$ we set $dist(\Gamma) = \infty$.) As a first application of 5.1 and 5.2 we prove the following.

5.4 *Let χ satisfy $\mathbf{S_3}$, and let (Γ_i, ϕ_i) $(i = 1, 2, \ldots)$ be a similarly oriented bad sequence for χ. Then for all $n \geq 0$, there exists $h \geq 0$ such that $dist(\Gamma_i) > n$ for all $i \geq h$.*

Proof. Suppose that for some n there is no such h. Then $c(\Sigma_\chi) \geq 2$ and $dist(\Gamma_i) \leq n$ for infinitely many values of i, and we may assume that $dist(\Gamma_i) \leq n$ for all $i \geq 1$, by replacing our original sequence by the appropriate subsequence. For each $i \geq 1$, let F_i be a path of Γ_i^* with $\frac{1}{2}|E(F_i)| \leq n$ joining distinct poles, such that every vertex of F_i is in $\Sigma_\chi \setminus bd(\Sigma_\chi)$ except the first two and the last two. Then $K_i = F_i \cap \Sigma_\chi$ is a feature in Φ_χ of size $\leq 4n$, and by 5.2 (by replacing our sequence by a subsequence) we may assume that all the K_i's are equivalent. By 5.1 we may assume that all the K_i's are equal, to some K say. Let Σ_1 be the (unique) fragment obtained by cutting Σ_χ along K, with associated surjection ψ. Let χ' be the colour scheme defined by $\Sigma_{\chi'} = \Sigma_1$, $\Phi_{\chi'} = \psi^{-1}(\Phi_\chi) \cap \Sigma_1$, $\Omega_{\chi'}(k) = \Omega_\chi(k)$ $(k = 2, 3)$, and for each long side S' of $\Phi_{\chi'}, \Omega_{\chi'}(S') = \Omega_\chi(S)$ where S is the long side of Φ_χ with $\psi(S') \subseteq S$. Then χ' is not orientedly bad, since χ satisfies $\mathbf{S_3}$. For each $i \geq 1$, let $\Gamma_i' = \psi^{-1}(\Gamma_i) \cap \Sigma_1$, and for each $e \in E(\Gamma_i')$, let $\phi_i'(e) = \phi_i(e)$ if $e \in E(\Gamma_i)$, and $\phi_i'(e) = e$ if $e \notin E(\Gamma_i)$ (so that e is a short side of $\psi^{-1}(\Phi_\chi) \cap \Sigma_1$). Then (Γ_i', ϕ_i') is a χ'-coloured painting, by 5.3, and the sequence (Γ_i', ϕ_i') $(i = 1, 2, \ldots)$ is similarly oriented (for if $\Sigma_{\chi'}$ is orientable then so is Σ_χ). Thus, since χ' is not orientedly bad, there exist $j > i \geq 1$ and a linear inflation σ' of (Γ_i', ϕ_i') in (Γ_j', ϕ_j'). Define σ by

$$\sigma(e) = \psi(\sigma'(\psi^{-1}(e))) \ (e \in E(\Gamma_i))$$
$$V(\sigma(v)) = \bigcup_{v' \in \psi^{-1}(v)} \psi V((\sigma'(v')))i \ (v \in V(\Gamma_i)),$$

where each $\sigma(v)$ is an induced subdrawing of $sk(\Gamma_j)$ with given vertex set as given. If $v \in V(\Gamma_i)$ satisfies $|\psi^{-1}(v)| > 1$, then every $v' \in \psi^{-1}(v)$ belongs to $V(\Phi_\chi')$, and consequently satisfies $v' \in V(\sigma'(v'))$. So v belongs to $V(\psi(\sigma'(v')))$ for each such v', and it follows that $\sigma(v)$ is connected. It is easy to deduce that σ is a linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j) (for similar arguments, see for example section 8 of [1]). This is a contradiction. ■

Let Γ be a 2-cell painting in Σ with $E(\Gamma) \neq \emptyset$. If $\hat{\Sigma}$ is not a sphere, we define $rep(\Gamma)$ to be the minimum of $\frac{1}{2}|E(F)|$ over all non-null-homotopic circuits F of Γ^*. (This exists, by theorem 11.10 of [3].). If $\hat{\Sigma}$ is a sphere we set $rep(\Gamma) = \infty$.

5.5 *Let χ satisfy $\mathbf{S_1}$, $\mathbf{S_3}$, and let (Γ_i, ϕ_i) $(i = 1, 2, \ldots)$ be a similarly oriented bad sequence for χ. Then for all $n \geq 0$ there exists $h \geq 1$ such that $rep(\Gamma_i) > n$ for all $i \geq h$.*

Proof. Suppose that for some n there is no such h. Then $\hat{\Sigma_\chi}$ is not a sphere, and for infinitely many $i \geq 1$ there is a circuit F_i of Γ_i^* with $\frac{1}{2}|E(F_i)| \leq n$ such that $U(F_i)$ bounds no disc in $\hat{\Sigma_\chi}$. If we choose F_i with $|V(F_i) \cap \Sigma_\chi|$ minimum, it is easy to see that for each pole $r^*, V(F)$ contains at most two neighbours of r^*, and at most one if $r^* \notin V(F_i)$. As in 5.4 we may assume that F_i exists for all $i \geq 1$, and all the F_i are equal to some F say. Let $F \cap \Sigma_\chi = K$. Then K is a feature.

If for some $i \geq 1$ there is a component K' of K with

$$|V(K') \cap V(\Gamma_i)| + |V(K') \cap bd(\Sigma_\chi)| \leq 1$$

then $V(K') \cap bd(\Sigma_\chi) = \emptyset$, and so $K' = K = F$, and $|F \cap V(\Gamma_i)| \leq 1$, which is impossible since Γ_i is internally 3-connected. Thus 5.3 can be applied. Now by 5.4, $U(K)$ meets at most one cuff (for K is a subgraph of each Γ_i^*, and yet $dist(\Gamma_i) > \frac{1}{2}|E(F)|$ for all sufficiently large i). Moreover,

$|U(K) \cap bd(\Sigma_\chi)| \leq 2$, with equality only if $K \neq F$ (from our choice of the F_i's.) In particular, when we cut along $U(K)$ (with surjection ψ) we obtain either one or two fragments. For each fragment Σ_1 say, $\hat{\Sigma}_1$ is simpler than $\hat{\Sigma_\chi}$ because $U(F)$ is non-null-homotopic in $\hat{\Sigma}$, and so if there is only one fragment the proof is completed as for 5.4, using $\mathbf{S_1}$ in place of $\mathbf{S_3}$. (Since we are applying $\mathbf{S_1}$ it does not matter whether the fragment is orientable.) Suppose then that there are two fragments Σ_1, Σ_2. For $t = 1, 2$, let χ_t be the colour scheme defined by $\Sigma_{\chi_t} = \Sigma_t, \Phi_{\chi_t} = \psi^{-1}(\Phi) \cap \Sigma_t$ etc., as in 5.4. For each $i \geq 1$, let $\Gamma'_i = \psi^{-1}(\Gamma_i) \cap \Sigma_1, \Gamma''_i = \psi^{-1}(\Gamma_i) \cap \Sigma_2$. For $e \in E(\Gamma'_i)$, define $\phi'_i(e) = \phi_i(\psi(e))$ if $\psi(e) \in E(\Gamma_i)$, and $\phi'_i(e) = e$ otherwise. Define ϕ''_i similarly. Then each (Γ'_i, ϕ'_i) is a χ_1-coloured painting, and each (Γ''_i, ϕ''_i) is a χ_2-coloured painting, and χ_1, χ_2 are not bad since χ satisfies $\mathbf{S_1}$. It follows that there exist $j > i \geq 1$ such that there is a linear inflation σ' of (Γ'_i, ϕ'_i) in (Γ'_j, ϕ'_j), and a linear inflation σ'' of (Γ''_i, ϕ''_i) in (Γ''_j, ϕ''_j). Define σ by

$$\sigma(e) = \begin{cases} \psi(\sigma'(\psi^{-1}(e))) & \text{if } \psi^{-1}(e) \subseteq \Sigma_1 \\ \psi(\sigma''(\psi^{-1}(e))) & \text{if } \psi^{-1}(e) \subseteq \Sigma_2 \end{cases}$$

$$\sigma(v) = \bigcup_{v' \in \psi^{-1}(v) \cap \Sigma_1} \psi(\sigma'(v')) \cup \bigcup_{v'' \in \psi^{-1}(v) \cap \Sigma_2} \psi(\sigma''(v'')).$$

Then σ is a linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j), a contradiction, as required. ■

6 Inflations and linear inflations

Our strategy to prove 4.1 is to apply a theorem of [10], which we describe later. That, however, applies to inflations rather than linear inflations, and the objective of this section is to smooth over the discrepancy, by means of the following.

6.1 *Let χ satisfy $\mathbf{S_1}$, $\mathbf{S_3}$, and let (Γ_i, ϕ_i) $(i = 1, 2 \ldots)$ be a similarly oriented bad sequence for χ. Then there exists $h \geq 1$ such that for all $j > i \geq h$ there is no inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j).*

6.1 is a consequence of 5.5 and the following.

6.2 *For any surface Σ there is a number n with the following property. Let Φ be a frame in Σ, and let Γ_1, Γ_2 be paintings in Σ fitting Φ, with $rep(\Gamma_1) \geq n$. Let σ be an inflation of Γ_1 in Γ_2 respecting Φ. Then there is a linear inflation σ' of Γ_1 in Γ_2 respecting Φ, such that $\sigma(e) = \sigma'(e)$ for all $e \in E(\Gamma_1)$, and $\sigma(v) \subseteq \sigma'(v)$ for all $v \in V(\Gamma_1)$.*

Proof of 6.1, assuming 6.2.
Choose n to satisfy 6.2, taking $\Sigma = \Sigma_\chi$. Choose $h \geq 1$ such that $rep(\Gamma_i) \geq n$ for all $i \geq h$ (this is possible by 5.5). Suppose that $j > i \geq h$ and σ is an inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j). Then σ is an inflation of Γ_i in Γ_j respecting Φ_χ, and $\phi_i \leq \phi_j \circ \sigma$. From 6.2, there is a linear inflation σ' of Γ_i in Γ_j respecting Φ_χ with $\sigma(e) = \sigma'(e)$ for all $e \in E(\Gamma_1)$, and hence with $\phi_i \leq \phi_j \circ \sigma'$. But then σ' is a linear inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j), a contradiction. Thus there are no such i, j, σ, as required. ■

The proof of 6.2 will require several lemmas, however. If G is a drawing in a surface Σ, not a sphere, we define $rep(G)$ to be the minimum of $|F \cap V(G)|$, taken over all non-null-homotopic G-normal O-arcs F in Σ. If Σ is a sphere, we set $rep(G) = \infty$. If G, G' are drawings in Σ, Σ' respectively we say that G, G' are *isomorphic* if there is a bijection $\beta : V(G) \cup E(G) \to V(G') \cup E(G')$, mapping vertices to vertices and edges to edges and preserving incidence. (This is the usual definition of isomorphism for non-embedded graphs, and it takes no note of the way the graphs are drawn in the surface.) An *enlargement* of G in G' is a function σ with domain $V(G) \cup E(G)$, such that

- for each $e \in E(G), \sigma(e) \in E(G')$, and if $e_1, e_2 \in E(G)$ are distinct then $\sigma(e_1) \neq \sigma(e_2)$
- for each $v \in V(G), \sigma(v)$ is a non-null connected subdrawing of G', and if $v_1, v_2 \in V(G)$ are distinct then $\sigma(v_1), \sigma(v_2)$ are disjoint
- for each $e \in E(G)$ and $v \in V(G), \sigma(v)$ contains an end of $\sigma(e)$ if and only if v is an end of e.

(Thus, there is an enlargement of G in G' if and only if G is isomorphic to a minor of G', with the usual definition of "minor" for graphs.)

6.3 *Let G, G' be drawings in a surface Σ with $bd(\Sigma) = \emptyset$, such that there is an enlargement of G in G'. Then G is isomorphic to a drawing H in Σ with $rep(H) \leq rep(G')$.*

Proof. Let σ be an enlargement of G in G'. We may assume that $\sigma(v)$ is a tree, for each $v \in V(G)$. Let H' be the subdrawing of G' with

$$V(H') = \bigcup (V(\sigma(v)) : v \in V(G))$$
$$E(H') = \bigcup (E(\sigma(v)) : v \in V(G)) \cup \{\sigma(e) : e \in E(G)\}.$$

Then $rep(H') \leq rep(G')$. Let Σ' be obtained from Σ by identifying all the elements of $U(\sigma(v))$, for each $v \in V(G)$, and let H be the image of H' under this identification. Since each $\sigma(v)$ is a tree it follows that $\Sigma' \cong \Sigma$; and since σ is an enlargement of G it follows that H is isomorphic to G. But $rep(H) \leq rep(H')$, and so $rep(H) \leq rep(G')$, as required. ■

Theorem 9.2 of [3] asserts the following.

6.4 *For every surface Σ with $bd(\Sigma) = \emptyset$, not a sphere, and every drawing H in Σ, there is a number k such that for every drawing G in Σ with $rep(G) \geq k$ there is an enlargement of H in G.*

We deduce

6.5 *For every surface Σ with $bd(\Sigma) = \emptyset$, not a sphere, there is a number $k \geq 0$ such that if G is a drawing in Σ with $rep(G) \geq k$ and G' is a drawing in Σ such that there is an enlargement of G in G', then $rep(G') \geq 1$.*

Proof. Let H be a connected drawing in Σ which is not isomorphic to any drawing in any simpler surface. (Such a drawing H exists; for instance, it follows by considering the Euler characteristic of the surfaces that any triangulation of Σ without parallel edges has the desired property.)

Choose k as in 6.4. Let G, G' be as in the theorem. By 6.4, there is an enlargement of H in G, and hence an enlargement of H in G'. By 6.3, H is isomorphic to a drawing H' in Σ with $rep(H') \leq rep(G')$. If $rep(H') = 0$, then we can cut Σ along a non-null-homotopic O-arc F with $F \cap U(H') = \emptyset$, to obtain a drawing isomorphic to H in a surface simpler than Σ, a contradiction. Hence $rep(H') \geq 1$, and so $rep(G') \geq 1$. ■

A drawing G in Σ is a *block* if G is non-null and connected, and $G \setminus v$ is connected for every vertex $v \in V(G)$.

6.6 *For every surface Σ with $bd(\Sigma) = \emptyset$ there is a number $k \geq 0$ with the following property. Let G, G' be blocks in Σ with $rep(G) > k$, and let σ be an enlargement of G in G'. Then there is an enlargement σ' of G in G' such that*

- *$\sigma(e) = \sigma'(e)$ for every $e \in E(G)$, and $\sigma(v) \subseteq \sigma'(v)$ for every $v \in V(G)$*
- *$\bigcup(V(\sigma'(v)) : v \in V(G)) = V(G')$, and each $\sigma'(v)$ is an induced subgraph of G'*
- *for every region r of G', and every $v \in V(G), \sigma'(v) \cap \bar{r}$ is null or connected.*

Proof. If Σ is a sphere, let $k = 0$. If not, choose k to satisfy 6.5. Now let G, G', σ be as in the theorem. Choose an enlargement σ' of G in G' satisfying the first statement of the theorem, in such a way that

$$\bigcup(\sigma'(v) : v \in V(G))$$

is maximal. Since G' is connected and G is non-null because they are blocks, it follows that the second statement of the theorem holds. It remains to verify the third. Let r be a region of G', let $v \in V(G)$, and let $a, b \in \bar{r} \cap V(\sigma'(v))$. We shall show that there is a path P of $\sigma'(v)$ with ends a, b and with $U(P) \subseteq \bar{r}$. For certainly there is a path Q of $\sigma'(v)$ with ends a, b since $\sigma'(v)$ is connected. Let $H = G \setminus v$. Now $rep(H) \geq k$, since $rep(G) > k$; and because there is an enlargement of H in $G' \setminus V(Q)$, it follows that $rep(G' \setminus V(Q)) \geq 1$, since k satisfies 6.5 unless Σ is a sphere. Let F be an O-arc in Σ with $U(Q) \subseteq F$ and $F \setminus U(Q) \subseteq r$. Since $F \cap U(G' \setminus V(Q)) = \emptyset$ and $rep(G' \setminus V(Q)) \geq 1$, it follows that there is a disc $\Delta \subseteq \Sigma$ bounded by F.

(1) *If $G' \cap \Delta \subseteq \sigma'(v)$ then there is a path P of $\sigma'(v)$ with ends a, b and with $U(P) \subseteq \bar{r}$.*

Subproof. $r \cap F$ is connected, and so there is a path of $G' \cap (\bar{r} \cap \Delta)$ between a and b; and so if $G' \cap \Delta \subseteq \sigma'(v)$ we may take P to be this path. This proves (1).

(2) *If $U(\sigma'(u)) \subseteq \Delta$ for every $u \in V(G) \setminus \{v\}$ then there is a path P of $\sigma'(v)$ with ends a, b and with $U(P) \subseteq \bar{r}$.*

Subproof. Let G'' be $G' \cap \Delta$, regarded as a drawing in Σ. If Σ is not a sphere then there is a non-null-homotopic O-arc disjoint from Δ, and hence $rep(G'') = 0$, which is impossible under the hypothesis of (2), because there is an enlargement of H in G'', and $rep(H) \geq k$, and k satisfies 6.5. Thus Σ is a sphere. Let $\Delta' \subseteq \Sigma$ be the disc bounded by F with $\Delta \neq \Delta'$. Then $G' \cap \Delta' \subseteq \sigma'(v)$, since $U(\sigma'(u)) \subseteq \Delta$ for every $u \in V(G) \setminus \{v\}$ and the second statement of the theorem holds. But then the claim follows by (1) applied to Δ'. This proves (2).

From (1) and (2) we may suppose, for a contradiction, that there exist $u_1, u_2 \in V(G) \setminus \{v\}$ with $U(\sigma'(u_1)) \cap \Delta \neq \emptyset$ and $U(\sigma'(u_2)) \not\subseteq \Delta$. Since $U(\sigma'(u_i)) \cap F = \emptyset$ and $\sigma'(u_i)$ is connected $(i = 1, 2)$, it follows that $U(\sigma'(u_1)) \subseteq \Delta \setminus F$ and $U(\sigma'(u_2)) \cap \Delta = \emptyset$. In particular, $u_1 \neq u_2$. Now every path of G' from $V(\sigma'(u_1))$ to $V(\sigma'(u_2))$ passes through $V(\sigma'(v))$, and so every path of G from u_1 to u_2 passes through v. But G is a block, a contradiction, as required. ■

6.7 *If* Γ *is an internally 3-connected painting in a surface* Σ*, then* $sk(\Gamma)$ *is a block and* $rep(sk(\Gamma)) = rep(\Gamma)$*.*

Proof. We may assume that $|V(\Gamma)| \geq 3$. Now since for every region r of Γ in Σ, r is an open disc and $r \cup \{v\}$ is simply-connected for every vertex v of Γ with $v \in \bar{r}$, it follows that the closed curve tracing the perimeter of r has no repeated vertices, and since $|V(\Gamma)| \geq 3$ this curve is an O-arc. Thus for each region r of Γ in Σ there is a circuit C_r of $sk(\Gamma)$ with $U(C_r) = bd(\bar{r})$. Hence the same is true for each region r of $sk(\Gamma)$ in $\hat{\Sigma}$. Now if $v \in V(sk(\Gamma))$ and $sk(\Gamma) \setminus v$ is disconnected, then there are two vertices $u_1, u_2 \neq v$ belonging to the same circuit C_r, and belonging to different components of $sk(\Gamma) \setminus v$, which is impossible. Thus $sk(\Gamma)$ is a block. The second claim is clear. ∎

Proof of 6.2.

Choose n such that 6.6 holds with Σ and k replaced by $\hat{\Sigma}$ and $n - 1$. Let $\Phi, \Gamma_1, \Gamma_2, \sigma$ be as in 6.2. For each $e \in E(sk(\Gamma_1))$ define $\tau(e)$ as follows. If $e \in E(\Gamma_1)$ let $\tau(e) = \sigma(e)$. If e is a component of $bd(\bar{c}_1) \setminus \tilde{c}_1$ for some $c_1 \in E(\Gamma_1)$ with $|\tilde{c}_1| = 3$, let $c_2 = \sigma(c_1)$, let the ends of e be u, v and let $\tau(e)$ be the edge of $sk(\Gamma_2)$ with $\tau(e) \subseteq bd(\bar{c}_2)$ and with ends in $\sigma(u), \sigma(v)$. For each $v \in V(sk(\Gamma_1))$ let $\tau(v) = \sigma(v)$. Then τ is an enlargement of $sk(\Gamma_1)$ in $sk(\Gamma_2)$.

Now by 6.7 $sk(\Gamma_1), sk(\Gamma_2)$ are blocks and $rep(sk(\Gamma_1)) = rep(\Gamma_1) \geq n > k$. By 6.6, there is an enlargement τ' of $sk(\Gamma_1)$ in $sk(\Gamma_2)$ such that

- $\tau'(e) = \tau(e)$ for every $e \in E(sk(\Gamma_1))$, and $\tau(v) \subseteq \tau'(v)$ for every $v \in V(sk(\Gamma_1))$
- $\bigcup(V(\tau'(v)) : v \in V(sk(\Gamma_1))) = V(sk(\Gamma_2))$, and each $\tau'(v)$ is an induced subgraph of $sk(\Gamma_2)$
- for every region r of $sk(\Gamma_2)$ in $\hat{\Sigma}$, and every $v \in V(sk(\Gamma_1))$, $\tau'(v) \cap \bar{r}$ is null or connected.

For each $e \in E(\Gamma_1)$, let $\sigma'(e) = \sigma(e)$, and for each $v \in V(\Gamma_1)$, let $\sigma'(v) = \tau'(v)$. Since all the $\sigma'(v)$'s are disjoint, and for $e \in E(\Gamma_1)$ if v is the ith end of e then $\sigma'(v)$ contains the ith end of $\sigma'(e)$ (because $\sigma(v)$ does, and $\sigma(v) = \tau(v) \subseteq \tau'(v) = \sigma'(v)$), it follows that σ' is an inflation of Γ_1 in Γ_2 respecting Φ. We claim that σ' is linear. For let Θ be a cuff of Σ and let C_i be the circuit of $sk(\Gamma_i)$ with $U(C_i) = \Theta$ $(i = 1, 2)$. Let

$$E(C_1) = \{e_1, \ldots, e_t\}$$
$$V(C_1) = \{v_1, \ldots, v_t\}$$

where e_i has ends v_i and v_{i+1} $(1 \leq i \leq t)$, where v_{t+1} means v_1. Now $\sigma'(e_i) = \sigma(e_i) \subseteq \Theta$ for $1 \leq i \leq t$, and for $1 \leq i \leq t$ $\sigma(v_i)$, and hence $\tau'(v_i)$, contains an end of both $\sigma'(e_{i-1})$ and $\sigma'(e_i)$, where e_0 means e_t. Thus, by the third statement above, there is a path P_i of $\tau'(v_i)$ joining an end of $\sigma'(e_{i-1})$ and an end of $\sigma'(e_i)$, with $U(P_i) \subseteq \Theta$. Since $\tau'(v_1), \ldots, \tau'(v_t)$ are disjoint, it follows that $P_1, \ldots, P_t$ are disjoint, and so

$$V(P_1) \cup \ldots \cup V(P_t) = V(C_2)$$
$$E(P_1) \cup \ldots \cup E(P_t) \cup \{\sigma'(e_1), \ldots, \sigma'(e_t)\} = E(C_2).$$

Hence every edge $e \in E(\Gamma_2) \setminus \sigma'(E(\Gamma_1))$ with $e \subseteq \Theta$ is an edge of some P_i and hence of some $\sigma'(v_i)$. Thus σ' is linear, as required. ∎

7 Tangles

A *hypergraph* H consists of a set $V(H)$ of *vertices*, a set $E(H)$ of *edges*, and an incidence relation between them; the vertices incident with an edge are its *ends.* A hypergraph G is a *subhypergraph* of H (written $G \subseteq H$) if $V(G) \subseteq V(H), E(G) \subseteq E(H)$, and each edge of G has the same ends in G and in H. If $A, B \subseteq H$, we define $A \cap B, A \cup B$ in the natural way. A *separation* of H is an ordered pair (A, B) of subhypergraphs with $A \cup B = H$ and $E(A \cap B) = \emptyset$; its *order* is $|V(A \cap B)|$. A *tangle* of *order* $\theta \geq 1$ in H is a set $\mathcal{T}$ of separations of H, all of order $< \theta$, such that

- for every separation (A, B) of H of order $< \theta, \mathcal{T}$ contains either (A, B) or (B, A)
- if $(A_i, B_i) \in \mathcal{T}$ $(i = 1, 2, 3)$ then $A_1 \cup A_2 \cup A_3 \neq H$
- if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(H)$.

We refer to these as the "tangle axioms". We define $ord(\mathcal{T})$ to be the order of $\mathcal{T}$.

If Γ is a painting then $(V(\Gamma), E(\Gamma))$ (with the natural incidence relation) is a hypergraph, and to avoid proliferating notation we shall also call this hypergraph Γ. Thus, we may speak of subhypergraphs, tangles etc. of a painting Γ.

Let Γ be a 2-cell painting in Σ with $E(\Gamma) \neq \emptyset$, and let $\mathcal{T}$ be a tangle in Γ. We define $rep(\mathcal{T})$ to be the maximum $k \leq ord(\mathcal{T})$ such that for every circuit F of Γ^* with $\frac{1}{2}|E(F)| < k$ there is a disc $\Delta \subseteq \hat{\Sigma}$ bounded by $U(F)$, such that $(\Gamma \cap \Delta, \Gamma \cap \Delta') \in \mathcal{T}$, where Δ' is the closure of $\hat{\Sigma} \setminus \Delta$. If $\frac{1}{2}|E(F)| < rep(\mathcal{T})$ and Δ is as described, we write $\Delta = ins(F), \Delta' = out(F)$. We make the convention that when we are dealing with more than one tangle in the same painting, $ins(F)$ and $out(F)$ will always be defined with reference to the tangle currently called $\mathcal{T}$ (the others will be called $\mathcal{T}', \mathcal{T}_1$ etc.). When there is only one tangle specified in the painting, $ins(F)$ and $out(F)$ are defined with reference to that.

Let Γ be a 2-cell painting in Σ. The *atoms* of Γ are the sets $\{v\} (v \in V(\Gamma))$, the edges of Γ, and the regions of Γ in $\hat{\Sigma}$; and the set of atoms is denoted by $A(\Gamma)$. To every atom of Γ there corresponds an atom of Γ^* in the natural way. Now assume in addition that $E(\Gamma) \neq \emptyset$, and let $\mathcal{T}$ be a tangle in Γ. If W is a closed walk in Γ^*, we define $\Gamma^*|W$ to be the subdrawing of Γ^* consisting of all vertices and edges in W. If W has length $< 2\,rep(\mathcal{T})$, we define $ins(W)$ to be the union of $U(\Gamma^*|W)$ and all the closed discs $ins(F)$ where F is a circuit of $\Gamma^*|W$. It is easy to see that if $x \in A(\Gamma)$ and x^* is the corresponding atom of Γ^*, then $x \cap ins(W) \neq \emptyset$ if and only if $x^* \subseteq ins(W)$, and we frequently use this fact without further explanation.

In theorem 9.1 of [6], we defined a metric on the set of atoms of Γ^*, using so-called "restraints". However, from theorems 8.5 and 9.2 of that paper, the same metric can be defined using the sets $ins(W)$ instead of general restraints; and since every atom of Γ corresponds to an atom of Γ^*, this induces a metric on $A(\Gamma)$. In summary, for $a, b \in A(\Gamma)$, we define $d(a, b)$ as follows:

- if $a = b$ then $d(a, b) = 0$
- if $a \neq b$ and there is a closed walk W of Γ^* with length $< 2rep(\mathcal{T})$ and with $a \cap ins(W), b \cap ins(W) \neq \emptyset$, we define $d(a, b)$ to be half the minimum length of such a walk
- if neither of the above applies then $d(a, b) = rep(\mathcal{T})$.

We call d the *metric* of $\mathcal{T}$ (it is indeed a metric on $A(\Gamma)$, as explained earlier.) When $v \in V(\Gamma)$ and $z \in A(\Gamma)$, we often write $d(v, z)$ for $d(\{v\}, z)$

In theorem 9.2 of [8], we proved a useful result about this metric, but only for drawings, not paintings. Now we need to generalize that to paintings, as follows.

7.1 *Let Γ be a 2-cell painting with $E(\Gamma) \neq \emptyset$ in a surface Σ, and let $\mathcal{T}$ be a tangle in Γ with metric d. Let $z \in A(\Gamma)$, and let κ be an integer with $4 \leq \kappa \leq rep(\mathcal{T}) - 6$. Then there is a circuit C of $sk(\Gamma)$, bounding an open disc $\Lambda \subseteq \hat{\Sigma}$ with $z \subseteq \Lambda$, such that*

- $d(z, x) \leq \kappa + 5$ *for every $x \in A(\Gamma)$ with $x \cap \bar{\Lambda} \neq \emptyset$*
- $d(z, x) \geq \kappa$ *for every $x \in A(\Gamma)$ with $x \cap \Lambda = \emptyset$*
- $x \subseteq \Lambda$ *for for every $x \in A(\Gamma)$ with $d(z, x) \leq \kappa - 2$*
- $ins(C^*) \subseteq \bar{\Lambda}$ *for every circuit C^* of Γ^* with $U(C^*) \subseteq \bar{\Lambda}$ and $|E(C^*)| < 2(rep(\mathcal{T}) - \kappa - 5)$.*

Proof. For $1 \leq i \leq rep(\mathcal{T})$, let $Z(i)$ be the union of $ins(W)$, taken over all closed walks W of Γ^* with length $< 2i$ and with $z^* \subseteq ins(W)$, where z^* is the atom of Γ^* corresponding to z. By theorems 8.5, 8.10, 8.12 and 9.2 of [6], $Z(rep(\mathcal{T}))$ is simply-connected and $\neq \hat{\Sigma}$, and so by theorems 4.2 and 11.9 of [3], there is a closed disc $\Delta \subseteq \hat{\Sigma}$ with $Z(rep(\mathcal{T})) \subseteq \Delta \setminus bd(\Delta)$. Since $\kappa \geq 4$, it follows that $z \subseteq Z(\kappa) \subseteq Z(rep(\mathcal{T})) \subseteq \Delta$.

(1) *Let r be a region of $sk(\Gamma)$ in $\hat{\Sigma}$ with $\bar{r} \cap Z(\kappa) \neq \emptyset$, and let $s \in A(\Gamma)$ with $r \subseteq s$; then $d(z, s) \leq \kappa + 2$ and $s \subseteq \Delta \setminus bd(\Delta)$.*

Subproof. Since $\bar{r} \cap Z(\kappa) \neq \emptyset$ it follows that $\bar{s} \cap Z(\kappa) \neq \emptyset$, and so there is a vertex v of Γ with $v \in \bar{s} \cap Z(\kappa)$. Hence $d(s, v) \leq 3$ and $d(z, v) < \kappa$, and so $d(z, s) \leq \kappa + 2$. Suppose that $s \not\subseteq Z(rep(\mathcal{T}))$. Since either $r = s$ or $s \in E(\Gamma)$ with $|\tilde{s}| = 3$, s is a subset of the closure of the union of the regions of Γ^* that meet s; and so there is a region e^* of Γ^* with $s \cap e^* \neq \emptyset$ and $e^* \not\subseteq Z(rep(\mathcal{T}))$. Let e be the corresponding edge of Γ; then $d(z, e) = rep(\mathcal{T})$ since $e^* \not\subseteq Z(rep(\mathcal{T}))$. But $d(s, e) \leq 3$ since $s \cap e^* \neq \emptyset$, and so

$$d(z, s) \geq rep(\mathcal{T}) - 3 \geq \kappa + 3,$$

a contradiction. Thus $s \subseteq Z(rep(\mathcal{T})) \subseteq \Delta \setminus bd(\Delta)$. This proves (1).

From (1) and theorem 5.2 of [8] it follows that there is a circuit C of $sk(\Gamma)$ with $U(C) \subseteq \Delta$, bounding an open disc $\Lambda \subseteq \Delta$ including $Z(\kappa)$, such that every edge of C is incident with a region r of $sk(\Gamma)$ in $\hat{\Sigma}$ with $\bar{r} \cap Z(\kappa) \neq \emptyset$. We claim that C satisfies the theorem. Certainly $z \subseteq \Lambda$, since $z \subseteq Z(\kappa)$. To verify the first assertion of the theorem, we shall show that $\bar{\Lambda} \subseteq Z(\kappa + 6)$. Let $e \in E(C)$, and let r be a region of $sk(\Gamma)$ in $\hat{\Sigma}$ with $e \subseteq \bar{r}$ and with $\bar{r} \cap Z(\kappa) \neq \emptyset$. Let $s \in A(\Gamma)$ with $r \subseteq s$. Then by (1), $d(z, s) \leq \kappa + 2$. Let $f \in E(\Gamma)$ with $e \subseteq f$; then $e \subseteq \bar{s} \cap f$ and so $d(s, f) \leq 3$. Hence $d(z, f) \leq \kappa + 5$, and so $f \subseteq f^* \subseteq Z(\kappa + 6)$, where $f^* \in A(\Gamma^*)$ corresponds to f. Consequently $e \subseteq Z(\kappa + 6)$. Since this holds for every edge e of C it follows that $U(C) \subseteq Z(\kappa + 6)$. By theorem 8.10 of [6], $Z(\kappa + 6)$ is simply-connected, and so there is a closed disc $D \subseteq Z(\kappa + 6) \subseteq \Delta$ bounded by $U(C)$. Thus D and $\bar{\Lambda}$ are both closed discs in Δ bounded by $U(C)$, and hence $D = \bar{\Lambda}$; and consequently $\bar{\Lambda} \subseteq Z(\kappa + 6)$. Now let $x \in A(\Gamma)$ with $x \cap \bar{\Lambda} \neq \emptyset$. Then $x \cap Z(\kappa + 6) \neq \emptyset$, and so $d(z, x) \leq \kappa + 5$. Hence the first assertion of the theorem holds.

To verify the second, let $x \in A(\Gamma)$ with $x \cap \Lambda = \emptyset$, and let x^* be the corresponding atom of Γ^*. Since $x \cap x^* \neq \emptyset$ and $x \cap \Lambda = \emptyset$ it follows that $x^* \not\subseteq \Lambda$, and so $x^* \not\subseteq Z(\kappa)$. Hence $d(z, x) \geq \kappa$. This verifies the second assertion.

For the third, let $x \in A(\Gamma)$ with $x \not\subseteq \Lambda$. If $x \cap \Lambda = \emptyset$ then because the second assertion of the theorem holds, $d(z, x) \geq \kappa$ as required. We may assume then that $x \cap \Lambda \neq \emptyset$, and since $x \not\subseteq \Lambda$ it follows that $x \cap U(C) \neq \emptyset$. Since $x \in A(\Gamma)$ and $x \cap \Lambda \neq \emptyset$ and $x \cap U(C) \neq \emptyset$, it follows that x is an edge of Γ with $|\tilde{x}| = 3$, and there is a vertex $v \in \tilde{x} \cap V(C)$. The atoms of Γ^* corresponding to $\{v\}$ and to x are adjacent in Γ^*, and so $d(x, \{v\}) \leq 1$. But since $\{v\} \cap \Lambda = \emptyset$, it follows from the second assertion that $d(z, \{v\}) \geq \kappa$, and so

$$d(z, x) \geq d(z, \{v\}) - d(x, \{v\}) \geq \kappa - 1.$$

This proves the third assertion.

To verify the fourth, let $f \in E(\Gamma)$ with $d(z, f) = rep(\mathcal{T})$. (This exists by theorem 8.12 of [6].) Let C^* be a circuit of Γ^* with $U(C^*) \subseteq \bar{\Lambda}$ and with $|E(C^*)| < 2(rep(\mathcal{T}) - \kappa - 5)$. Let D be the closed disc in $\bar{\Lambda}$ bounded by $U(C^*)$, and let $v \in V(C^*) \cap V(\Gamma)$. Then $d(z, v) \leq \kappa + 5$, and so $d(v, f) \geq rep(\mathcal{T}) - \kappa - 5$. Hence $f \not\subseteq ins(C^*)$, because $|E(C^*)| < 2(rep(\mathcal{T}) - \kappa - 5)$, and so $ins(C^*)$ and D are both closed discs in $\Sigma \setminus f$ bounded by $U(C^*)$. Consequently $D = ins(C^*)$, and so $ins(C^*) \subseteq \bar{\Lambda}$. This verifies the fourth assertion, and so completes the proof of 7.1. ■

8 Tie-breakers

Throughout this section Γ is an internally 3-connected painting in Σ. Let $e \in E(\Gamma^*)$ with ends $v \in V(\Gamma)$ and $r^* \in R^*(\Gamma)$, and let C be the circuit of $sk(\Gamma)$ with $U(C)$ bounding r. (We assume for the moment that C exists.) Let f_1, f_2 be the two edges of C incident with v, and let c_1, c_2 be the two edges of Γ with $f_1 \subseteq c_1, f_2 \subseteq c_2$. Now if (A, B) is a separation of Γ, we say that (A, B) *splits* e if $c_1 \in E(A)$ and $c_2 \in E(B)$ or vice versa. (If the circuit C does not exist, then, since the perimeter of r has no "repeated" vertices because Γ is internally 3-connected, it follows that $|V(\Gamma)| = 2$ and $|E(\Gamma)| = 1$. In this case we say that (A, B) does not split e.) We shall need the following lemma.

8.1 *Let (A, B) be a separation of Γ of order $< rep(\Gamma)$, with $E(A), E(B) \neq \emptyset$. Then there is a circuit F of Γ^* such that (A, B) splits every edge of F and there is a disc $\Delta \subseteq \hat{\Sigma}$ bounded by $U(F)$ such that either $\Gamma \cap \Delta \subseteq A$ or $\Gamma \cap \Delta \subseteq B$.*

Proof. Let G be the subdrawing of Γ^* with $V(G) = V(\Gamma^*)$ and edges those edges of Γ^* with are split by (A, B). Now $E(G) \neq \emptyset$ since $E(A), E(B) \neq \emptyset$; and every vertex of G has even degree from the definition of G. Moreover, for any circuit F of G, half its vertices belong to $V(A \cap B)$, and so $\frac{1}{2}|E(F)| < rep(\Gamma)$; and therefore there is a disc in $\hat{\Sigma}$ bounded by $U(F)$. Let us choose a minimal disc $\Delta \subseteq \hat{\Sigma}$ with $bd(\Delta) \subseteq U(G)$; and let F be the circuit of Γ^* with $U(F) = bd(\Delta)$. Since every edge of G is contained in a circuit of G, and $G \cap \Delta$ has no circuit except F, it follows that $G \cap \Delta = F$. But then $\Gamma \cap \Delta \subseteq A$ or $\Gamma \cap \Delta \subseteq B$, as required. ■

In [5] we discussed "tie-breakers" in general hypergraphs. In this paper we only need a particular kind of tie-breaker, chosen to work nicely in paintings, but we use the same name. Thus, for each $e \in E(\Gamma^*)$ let $\lambda(e) > 0$ be some real number, and for each $v \in V(\Gamma)$ let $\lambda(v) > 0$ be some real number, such that all the $\lambda(e)$'s and $\lambda(v)$'s are rationally independent; that is,

$$\sum(\alpha(e)\lambda(e) : e \in E(\Gamma^*)) = \sum(\beta(v)\lambda(v) : v \in V(\Gamma))$$

for rationals $\alpha(e), \beta(v)$ only if each $\alpha(e) = 0$ and each $\beta(v) = 0$. Moreover, let $\lambda(e) < \lambda(f)$ for all $e, f \in E(\Gamma^*)$ such that $e \subseteq \hat{\Sigma} \setminus \Sigma$ and $f \subseteq \Sigma$. We call λ a *tie-breaker* in Γ. Throughout this section, λ is a fixed tie-breaker in Γ. We define the λ-*order* of a separation (A, B) of Γ to be the triple (N_1, N_2, N_3), where

$$\begin{aligned} N_1 &= |V(A \cap B)|, \\ N_2 &= \sum(\lambda(e) : e \text{ is split by } (A, B)), \\ N_3 &= \sum(\lambda(v) : v \in V(A \cap B)). \end{aligned}$$

We order λ-orders lexicographically; thus, if $(A, B), (A', B')$ have λ-orders (N_1, N_2, N_3) and (N'_1, N'_2, N'_3) respectively, we say that (A, B) has smaller λ-order than (A', B') if either $N_1 < N'_1$, or $N_1 = N'_1$ and $N_2 < N'_2$, or $N_1 = N'_1$ and $N_2 = N'_2$ and $N_3 < N'_3$. The next two results prove that the tie-breakers in this paper are indeed tie-breakers in the sense of [5]. It is easy to prove that

8.2 *If $(A, B), (A', B')$ are separations with the same λ-order, then $(A, B) = (A', B')$ or (B', A').*

Moreover,

8.3 *If $(A, B), (A', B')$ are separations then so are $(A \cup A', B \cap B'), (A \cap A', B \cup B')$, and either $(A \cup A', B \cap B')$ has smaller λ-order than (A, B), or $(A \cap A', B \cup B')$ has λ-order at most that of (A', B').*

Proof. Let these four separations have λ-orders (N_1, N_2, N_3), (N'_1, N'_2, N'_3), (N''_1, N''_2, N''_3), and (N'''_1, N'''_2, N'''_3) respectively. Then $N_1 + N'_1 = N''_1 + N'''_1$, $N_2 + N'_2 \geq N''_2 + N'''_2$, and $N_3 + N'_3 = N''_3 + N'''_3$, and the result follows. ■

Let $\mathcal{T}, \mathcal{T}'$ be tangles in Γ, with $\mathcal{T} \not\subseteq \mathcal{T}'$ and $\mathcal{T}' \not\subseteq \mathcal{T}$. Then there exists $(A, B) \in \mathcal{T}$ with $(B, A) \in \mathcal{T}'$, and there is a unique such (A, B) with minimum λ-order, called the $(\mathcal{T}, \mathcal{T}')$-*distinction.*

8.4 *Let $\mathcal{T}, \mathcal{T}'$ be tangles in Γ and let (A, B) be the $(\mathcal{T}, \mathcal{T}')$-distinction. Suppose that $|V(A \cap B)| <$ rep$(\mathcal{T})$. Then there is a circuit F in Γ^* such that $A = \Gamma \cap ins(F), B = \Gamma \cap out(F)$.*

Proof. By the second tangle axiom, $E(A), E(B) \neq \emptyset$ since $(A, B) \in \mathcal{T}$ and $(B, A) \in \mathcal{T}'$. By 8.1 there is a circuit F of Γ^* such that (A, B) splits every edge of F and there is a disc $\Delta \subseteq \hat{\Sigma}$ bounded by $U(F)$ with either $\Gamma \cap \Delta \subseteq A$ or $\Gamma \cap \Delta \subseteq B$. Let Δ' be the closure of $\hat{\Sigma} \setminus \Delta$, and let $H = \Gamma \cap \Delta$ and $H' = \Gamma \cap \Delta'$. Then $H \cap H' \subseteq A \cap B$.

The three separations (H, H'), $(A \cap H', B \cup H)$ and $(A \cup H, B \cap H')$ all have λ-order at most that of (A, B), because $H \cap H' \subseteq A \cap B$ and every edge of Γ^* split by one of these separations is split by (A, B).

If $H \subseteq A$ then $A \cup H' = \Gamma$, and so $(H', H) \notin \mathcal{T}$ since $(A, B) \in \mathcal{T}$; while if $H \subseteq B$ then $B \cup H = \Gamma$, and so $(H', H) \notin \mathcal{T}'$ since $(B, A) \in \mathcal{T}'$. In either case it follows that (H', H) does not belong to both $\mathcal{T}, \mathcal{T}'$, and so (H, H') belongs to at least one of them.

Suppose that (H, H') belongs to both of $\mathcal{T}, \mathcal{T}'$. Now $(A \cap H', B \cup H) \notin \mathcal{T}'$, from the second tangle axiom, since $(H, H'), (B, A) \in \mathcal{T}'$ and $H \cup B \cup (A \cap H') = \Gamma$. Consequently $(B \cup H, A \cap H') \in \mathcal{T}'$. But $(A \cap H', B \cup H) \in \mathcal{T}$, since $(A, B) \in \mathcal{T}$; and since $(A \cap H', B \cup H)$ has λ-order at most that of (A, B), and (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction, it follows that equality holds, and so $(A \cap H', B \cup H) = (A, B)$, that is, $A \subseteq H'$ and $H \subseteq B$. Similarly, $(B \cap H', A \cup H) \notin \mathcal{T}$, since $(H, H'), (A, B), \in \mathcal{T}$, and $H \cup A \cup (B \cap H') = \Gamma$. Consequently $(A \cup H, B \cap H') \in \mathcal{T}$. But $(B \cap H', A \cup H) \in \mathcal{T}'$, since $(B, A) \in \mathcal{T}'$, and so this separation has the same λ-order as (A, B), and therefore $(A \cup H, B \cap H') = (A, B)$, that is, $H \subseteq A$ and $B \subseteq H'$. But we already showed that $A \subseteq H'$ and $H \subseteq B$, and so $E(H) = \emptyset$. Since Δ includes a region of Γ^* and hence an edge of Γ, this is impossible.

It follows that (H, H') belongs to exactly one of $\mathcal{T}, \mathcal{T}'$. Since its λ-order is at most that of (A, B), and (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction, it follows that equality holds, and so $(H, H') = (A, B)$ or (B, A). The first is the desired result. If the second holds, then since (H, H') has order less than $rep(\mathcal{T})$ and $(H, H') \notin \mathcal{T}$, it follows that there is a disc bounded by F different from Δ; and so Δ' is a disc. Since in this case $\Delta' = ins(F)$, we deduce that again the desired result holds. ■

Given a tangle $\mathcal{T}$ in Γ, a circuit F of Γ^* is a $\mathcal{T}$-*enclave* if $\frac{1}{2}|E(F)| < rep(\mathcal{T})$, and there is a tangle $\mathcal{T}'$ in Γ for which $(\Gamma \cap ins(F), \Gamma \cap out(F))$ is the $(\mathcal{T}, \mathcal{T}')$-distinction. We also call F a $\mathcal{T}$-*enclave around* $\mathcal{T}'$. If K is a subgraph of Γ^*, $\lambda(K)$ denotes $\sum_{e \in E(K)} \lambda(e)$.

8.5 *Let $\mathcal{T}$ be a tangle in Γ, and let F be a $\mathcal{T}$-enclave around $\mathcal{T}'$, and let $u, v \in V(F)$ be distinct. Let F_1, F_2 be the two paths of F between u and v, and let P be a path of Γ^* between u and v with no other vertex or edge in common with F. Suppose that $|E(P)| \leq |E(F_1)|$, and if equality holds then $\lambda(P) \leq \lambda(F_1)$. Then*

- $(\Gamma \cap ins(P \cup F_2), \Gamma \cap out(P \cup F_2)) \in \mathcal{T}'$
- *if $U(P) \not\subseteq ins(F)$ then $ins(P \cup F_2) \cap ins(F) = U(F_2)$*
- *$|E(P)| \geq |E(F_2)|$, and if equality holds then $\lambda(P) > \lambda(F_2)$.*

Proof. Let C_i be the circuit $P \cup F_i$ $(i = 1, 2)$. Certainly $|E(C_2)| \leq |E(F)|$, and if equality holds then $\lambda(C_2) < \lambda(F)$, because the $\lambda(e)$'s are rationally independent. Since F is a $\mathcal{T}$-enclave around $\mathcal{T}'$, it follows that $(\Gamma \cap ins(C_2), \Gamma \cap out(C_2)) \notin \mathcal{T} \setminus \mathcal{T}'$. Hence the first assertion of the theorem holds. It follows that $ins(F) \not\subseteq ins(C_2)$, because $(\Gamma \cap out(F), \Gamma \cap ins(F)) \in \mathcal{T}'$, and so the second assertion holds. Suppose that the third is false. Then by the same argument $(\Gamma \cap ins(C_1), \Gamma \cap out(C_1)) \in \mathcal{T}'$, and so $ins(F) \not\subseteq ins(C_1) \cup ins(C_2)$ from the second tangle axiom applied to $\mathcal{T}'$. Hence $U(P) \not\subseteq ins(F)$, and so $ins(C_i) \cap ins(F) = U(F_i) (i = 1, 2)$. But then $ins(C_1) \cup ins(C_2) \cup ins(F) = \hat{\Sigma}$, contrary to the second tangle axiom applied to $\mathcal{T}$. Thus the third assertion holds. This proves 8.5. ■

8.6 *Let $\mathcal{T}$ be a tangle in Γ, and let F_1, F_2 be $\mathcal{T}$-enclaves. Then either $ins(F_1) \subseteq ins(F_2)$, or $ins(F_2) \subseteq ins(F_1)$, or $ins(F_1) \cap ins(F_2) = U(F_1) \cap U(F_2)$.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2$ be tangles such that F_i is a $\mathcal{T}$-enclave around $\mathcal{T}_i$ $(i = 1, 2)$. Let $A_i = \Gamma \cap ins(F_i), B_i = \Gamma \cap out(F_i) (i = 1, 2)$. Since (A_i, B_i) is the $(\mathcal{T}, \mathcal{T}_i)$- distinction $(i = 1, 2)$ it follows from theorems 9.4 and 10.2 of [5] that one of $E(A_1 \cap A_2), E(A_1 \cap B_2), E(B_1 \cap A_2), E(B_1 \cap B_2)$ is empty. Since $(A_i, B_i) \in \mathcal{T}(i = 1, 2)$ it follows that $E(B_1 \cap B_2) \neq \emptyset$, and so the fourth alternative is false. From the symmetry between the second and third alternatives, we may assume without loss of generality that if either holds then the second does. Thus, one of the first two alternatives holds.

If $E(A_1 \cap B_2) = \emptyset$, then every edge of A_1 is included in $ins(F_2) \setminus U(F_2)$ and so every edge of Γ^* split by (A_1, B_1) is included in $ins(F_2)$; therefore $U(F_1) \subseteq ins(F_2)$, and hence $ins(F_1) \subseteq ins(F_2)$ as required. On the other hand, if $E(A_1 \cap A_2) = \emptyset$, then no edge of Γ^* split by (A_1, B_1) is included in $ins(F_2) \setminus U(F_2)$, and so $U(F_1) \cap ins(F_2) = U(F_1) \cap U(F_2)$. Similarly, $U(F_2) \cap ins(F_1) = U(F_1) \cap U(F_2)$, and so $ins(F_1) \cap ins(F_2) = U(F_1) \cap U(F_2)$, as required. ■

If $\mathcal{T}$ is a tangle in Γ, a separation $(A, B) \in \mathcal{T}$ is said to be λ-*linked* to $\mathcal{T}$ if there is no $(A', B') \in \mathcal{T}$ of smaller λ-order with $A \subseteq A'$ and $B' \subseteq B$. If F is a circuit of Γ^* with $\frac{1}{2}|E(F)| < \min(ord(\mathcal{T}), rep(\Gamma))$, let Σ_1, Σ_2 be the closures of the two components of $\hat{\Sigma} \setminus U(F)$; then $\mathcal{T}$ contains one of $(\Gamma \cap \Sigma_1, \Gamma \cap \Sigma_2), (\Gamma \cap \Sigma_2, \Gamma \cap \Sigma_1)$, and if that separation is λ-linked to $\mathcal{T}$ we say that F is λ-*linked* (to $\mathcal{T}$).

8.7 *Let $\mathcal{T}$ be a tangle in Γ, let F be a circuit of Γ^* with $\frac{1}{2}|E(F)| < \min(ord(\mathcal{T}), rep(\Gamma))$, and let Σ_1, Σ_2 be the closures of the two components of $\hat{\Sigma} \setminus U(F)$, where $(\Gamma \cap \Sigma_1, \Gamma \cap \Sigma_2) \in \mathcal{T}$. Then there is a λ-linked circuit F' in Γ^* such that*

- $|E(F')| \leq |E(F)|$ *and if equality holds then* $\lambda(F') \leq \lambda(F)$
- $\Sigma_1 \subseteq \Sigma'_1$ *and* $\Sigma'_2 \subseteq \Sigma_2$, *where* Σ'_1, Σ'_2 *are the closures of the two components of* $\hat{\Sigma} \setminus U(F')$ *and* $(\Gamma \cap \Sigma'_1, \Gamma \cap \Sigma'_2) \in \mathcal{T}$.

Proof. Choose $(A, B) \in \mathcal{T}$ with $\Gamma \cap \Sigma_1 \subseteq A$ and $B \subseteq \Gamma \cap \Sigma_2$ of minimum λ-order. Then $E(B) \neq \emptyset$ since $(A, B) \in \mathcal{T}$, and $E(A) \neq \emptyset$ since $E(\Gamma \cap \Sigma_1) \neq \emptyset$. From 8.1 there is a circuit F' of Γ^* such that (A, B) splits every edge of F', and such that $U(F')$ bounds a disc $\Delta \subseteq \hat{\Sigma}$ with either $\Gamma \cap \Delta \subseteq A$ or $\Gamma \cap \Delta \subseteq B$. Let Σ'_1, Σ'_2 be the closures of the two components of $\hat{\Sigma} \setminus U(F')$, where $(\Gamma \cap \Sigma'_1, \Gamma \cap \Sigma'_2) \in \mathcal{T}$. It follows that $(\Gamma \cap \Sigma'_1, \Gamma \cap \Sigma'_2)$ has λ-order at most that of (A, B).

Suppose that $\Sigma_1 \subseteq \Sigma'_2$. Then Σ'_2 includes an edge of A (since $\Gamma \cap \Sigma_1 \subseteq A$). Now $(A, B) \in \mathcal{T}$, and $(\Gamma \cap \Sigma'_2, \Gamma \cap \Sigma'_1) \notin \mathcal{T}$, and yet the second separation has order $< ord(\mathcal{T})$, and $\mathcal{T}$ has order $> \frac{1}{2}|E(F)| \geq 1$, and so by the third assertion of theorem 2.9 of [5], not every edge of $\Gamma \cap \Sigma'_2$ belongs to A. Consequently, Σ'_2 includes an edge of B. We deduce that $\Gamma \cap \Sigma'_2 \not\subseteq A$ and $\Gamma \cap \Sigma'_2 \not\subseteq B$. Hence $\Sigma'_2 \neq \Delta$, and so $\Sigma'_1 = \Delta$. Furthermore,

$$(A \cap \Sigma'_2, B \cup (\Gamma \cap \Sigma'_1)), (A \cup (\Gamma \cap \Sigma'_1), B \cap \Sigma'_2) \in \mathcal{T}$$

because they both have order at most that of (A, B), and $(\Gamma \cap \Sigma'_1, \Gamma \cap \Sigma'_2) \in \mathcal{T}$. Yet if $\Gamma \cap \Sigma'_1 \subseteq A$, the first separation has smaller λ-order than (A, B), while if $\Gamma \cap \Sigma'_1 \subseteq B$ the second does (for in both cases the first term of the λ-order does not increase, and the second term strictly decreases). In either case this is contrary to our choice of (A, B); and one of these occurs since $\Sigma'_1 = \Delta$, a contradiction. Thus, $\Sigma_1 \not\subseteq \Sigma'_2$.

Now no edge of F' lies in $\Sigma_1 \setminus U(F)$, because every edge of F' is split; and so $\Sigma_1 \subseteq \Sigma'_1$ and $\Sigma'_2 \subseteq \Sigma_2$. Hence $(\Gamma \cap \Sigma'_1, \Gamma \cap \Sigma'_2)$ has λ-order at least that of (A, B), from the choice of (A, B). Since we already shown the reverse inequality, it follows that the λ-orders are equal and so from 8.2, $(A, B) = (\Gamma \cap \Sigma'_1, \Gamma \cap \Sigma'_2)$. Hence F' is λ-linked to $\mathcal{T}$, and (from the choice of (A, B)) $|E(F')| \leq |E(F)|$, and if equality holds then $\lambda(F') \leq \lambda(F)$. This proves 8.7. ■

An *arm* A of a painting Γ is a pair $(A^-, \pi(A))$, where A^- is a subhypergraph of Γ and $\pi(A)$ is a march in $V(A^-)$, with the property that there exists $B \subseteq \Gamma$ such that (A^-, B) is a separation of Γ and $V(A^- \cap B) = \bar{\pi}(A)$. (In other words, for each edge e of Γ not in $E(A^-), \bar{\pi}(A)$ contains every end of e in $V(A^-)$.) In this case, we call B the *complement* of A (it is unique) and write $B = A^c$. We define $V(A) = V(A^-), E(A) = E(A^-)$. We say that A is λ-*linked* to $\mathcal{T}$ if (A^-, A^c) is λ-linked to $\mathcal{T}$. The *order* of A is $|\bar{\pi}(A)|$, that is, the order of (A^-, A^c).

A *rooted location* in Γ is a set $\mathcal{L}$ of arms, such that if $A_1, A_2 \in \mathcal{L}$ with $A_1^- \neq A_2^-$ then $A_1^- \subseteq A_2^c$. Its *order* $ord(\mathcal{L})$ is the maximum order of its members (or zero, if $\mathcal{L} = \emptyset$). We define $\mathcal{L}^- = \{(A^-, A^c) : A \in \mathcal{L}\}$. A rooted location $\mathcal{L}$ with $\mathcal{L}^- \subseteq \mathcal{T}$ is λ-*linked* to $\mathcal{T}$ if each of its members is λ-linked to $\mathcal{T}$. A rooted location $\mathcal{L}$ θ-*isolates* a tangle $\mathcal{T}$ if $\mathcal{L}^- \subseteq \mathcal{T}, ord(\mathcal{L}) < \theta \leq ord(\mathcal{T})$, and for every $A \in \mathcal{L}$ and for every tangle $\mathcal{T}'$ of order $\geq \theta$ with $(A^c, A^-) \in \mathcal{T}'$, the $(\mathcal{T}, \mathcal{T}')$-distinction (C, D) satisfies $C \subseteq A^-$ and $A^c \subseteq D$. It follows by applying theorem theorem 7.1 of [10] to $\mathcal{L}^-$ that

8.8 *If $\mathcal{T}$ is a tangle in Γ, and $\mathcal{L}$ is a rooted location with $\mathcal{L}^- \subseteq \mathcal{T}$, and $ord(\mathcal{L}) < \theta \leq ord(\mathcal{T})$, and $\mathcal{L}$ is λ-linked to $\mathcal{T}$, then $\mathcal{L}$ θ-isolates $\mathcal{T}$.*

9 An application of patchworks

The reader familiar with [1, 10] will see that we can regard a painting Γ as a patchwork, by assigning to $e \in E(\Gamma)$ the free patch on $\tilde{e}$; and by doing so, we can apply theorem 6.7 of [10]. Our object now is to state that theorem in the terminology of paintings.

If $\Omega_1, \ldots, \Omega_k$ are well-quasi-orders with $E(\Omega_1), \ldots, E(\Omega_k)$ mutually disjoint, we define their *union* to be the well-quasi-order Ω with $E(\Omega) = E(\Omega_1) \cup \ldots \cup E(\Omega_k)$, in which $x \leq y$ if and only if $x, y \in E(\Omega_i)$ for some i and $x \leq y$ in Ω_i. A colour scheme χ is *disjoint* if $\Omega_\chi(3)$, $\Omega_\chi(2)$ and all the $\Omega_\chi(S)$'s (over all sides S of Φ_χ) have mutually disjoint element sets, and if χ is disjoint we denote the union of these well-quasi-orders by Ω_χ. If χ is disjoint and (Γ, ϕ) is a χ-coloured painting, we may regard ϕ as a function from $E(\Gamma)$ into $E(\Omega_\chi)$.

9.1 *Let $(\Gamma, \phi), (\Gamma', \phi')$ be χ-coloured paintings, where χ is disjoint. Let σ be an inflation of Γ in Γ', such that $\phi(e) \leq \phi'(\sigma(e))$ (in Ω_χ) for every $e \in E(\Gamma)$. Then σ is an inflation of (Γ, ϕ) in (Γ', ϕ').*

Proof. If $e \in E(\Gamma)$, then since $\phi(e) \leq \phi'(\sigma(e))$ in Ω_χ and Ω_χ is disjoint it follows that $\sigma(e)$ is internal if and only if e is internal, and $\sigma(e)$ borders a side S if and only if e does. Thus σ respects Φ_χ, as required. ■

In section 2 we defined an inflation of one painting in another. Let us broaden that definition a little. If Γ, Γ' are paintings in Σ, and $H \subseteq \Gamma$ is a subhypergraph, an *inflation* of H in Γ' is a function σ with domain $V(H) \cup E(H)$ satisfying the three conditions of the definition of inflation in section 2 with Γ replaced by H.

If $\mathcal{L}$ is a rooted location in Γ we define $M(\Gamma, \mathcal{L})$ to be

$$\Gamma \cap \bigcap (A^c : A \in \mathcal{L}).$$

Thus $\bar{\pi}(A) \subseteq V(M(\Gamma, \mathcal{L}))$ for each $A \in \mathcal{L}$. Let $(\Gamma, \phi), (\Gamma', \phi')$ be χ-coloured paintings, where χ is disjoint, and let $\mathcal{L}, \mathcal{L}'$ be rooted locations in Γ, Γ' respectively. A function $\tau : \mathcal{L} \to \mathcal{L}'$ is an *outline* of $(\Gamma, \phi, \mathcal{L})$ in $(\Gamma', \phi', \mathcal{L}')$ if there is an inflation σ of $M(\Gamma, \mathcal{L})$ in Γ', such that

- $\sigma(e) \in E(M(\Gamma', \mathcal{L}'))$ for each $e \in E(M(\Gamma, \mathcal{L}))$
- for each $A \in \mathcal{L}, |\bar{\pi}(\tau(A))| = |\bar{\pi}(A)|$, and if $A_1, A_2 \in \mathcal{L}$ are distinct then $\tau(A_1) \neq \tau(A_2)$
- for each $A \in \mathcal{L}$ and $1 \leq i \leq |\bar{\pi}(A)|$, if v is the ith term of $\pi(A)$ then $\sigma(v)$ contains the ith term of $\pi(\tau(A))$
- for each $A \in \mathcal{L}$ and $v \in V(M(\Gamma, \mathcal{L}))$, $|V(\sigma(v)) \cap V(\tau(A))| \leq 1$
- for each $e \in E(M(\Gamma, \mathcal{L})), \phi(e) \leq \phi'(\sigma(e))$ in Ω_χ.

We stress that the $\sigma(v)$'s are subgraphs of $sk(\Gamma')$, but not necessarily of $sk(M(\Gamma', \mathcal{L}'))$.

If χ is disjoint, (Γ, ϕ) is a χ-coloured painting, and $\mathcal{L}$ is a rooted location in Γ, we call $(\Gamma, \phi, \mathcal{L})$ a *χ-place*. A set $\mathcal{P}$ of χ-places is *well-behaved* if for every well-quasi-order Ω and every countable sequence $(\Gamma_i, \phi_i, \mathcal{L}_i) (i = 1, 2, \ldots)$ of members of $\mathcal{P}$, and for all functions $\xi_i : \mathcal{L}_i \to E(\Omega)$, there exist $j > i \geq 1$ and an outline τ of $(\Gamma_i, \phi_i, \mathcal{L}_i)$ in $(\Gamma_j, \phi_j, \mathcal{L}_j)$ such that $\xi_i(A) \leq \xi_j(\tau(A))$ for all $A \in \mathcal{L}_i$. Theorem 6.7 of [10], together with 9.1, imply the following.

9.2 *Let χ be a disjoint colour scheme, let (Γ_i, ϕ_i) $(i = 1, 2, \ldots)$ be a countable sequence of χ-coloured paintings and for each $i \geq 1$ let λ_i be a tie-breaker in Γ_i. Let $\theta \geq 1$, and for each $i \geq 1$ and every tangle $\mathcal{T}$ in Γ_i of order $\geq \theta$, let $\mathcal{L}(\mathcal{T})$ be a rooted location in Γ_i which θ-isolates $\mathcal{T}$. Let the set of all these $(\Gamma_i, \phi_i, \mathcal{L}(\mathcal{T}))$ (over all i and $\mathcal{T}$) be well-behaved. Then there exist $j > i \geq 1$ such that there is an inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j).*

Now we can give the reader a little better intuition as to how the proof of 4.1 will work. Let χ be a colour scheme satisfying $\mathbf{S_1}, \ldots, \mathbf{S_4}$, and let (Γ_i, ϕ_i) $(i = 1, 2, \ldots)$ be a countable sequence of χ-coloured paintings. By 6.1, it will suffice to show that there exist $j > i \geq 1$ such that there is an inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j), and to show this we will apply 9.2. We therefore need to produce an infinite subsequence of this sequence, and a well-behaved set of rooted locations, so that the subsequence satisfies the hypotheses of 9.2. To get the subsequence, discard from the given sequence the first term, and all terms (Γ_i, ϕ_i) with $dist(\Gamma_i)$ or $rep(\Gamma_i)$ at most some appropriately-chosen number that depends only on (Γ_1, ϕ_1). By 5.4 and 5.5, an infinite sequence still remains, and this is the one we need. Now we need to produce the well-behaved set of rooted locations. Let (Γ_i, ϕ_i) be some term of the sequence that still remains. We know that there is no inflation of (Γ_1, ϕ_1) in (Γ_i, ϕ_i); and a theorem of [7] therefore can be applied. That theorem implies that for every large-order tangle $\mathcal{T}$ in Γ_i (the meaning of "large" depending only on (Γ_1, ϕ_1)), the triple $(\Gamma_i, \phi_i, \mathcal{T})$ is "insufficiently general", in one of only a few possible ways. We deduce that there is a "flaw" in $(\Gamma_i, \phi_i, \mathcal{T})$, of one of only a few possible kinds (and in particular, of only finitely many different kinds). For each kind of flaw, we shall show that there corresponds a well-behaved set of rooted locations, such that if $(\Gamma_i, \phi_i, \mathcal{T})$ admits the flaw then some rooted location in this set θ-isolates $\mathcal{T}$ (for appropriate θ); and then the union of these finitely many well-behaved sets is another well-behaved set, now satisfying the hypotheses of 9.2, as required.

In sections 10-14 we look at the various kinds of flaw, and in each case construct the desired well-behaved set, and in section 15 we complete the proof by applying the theorem of [7].

We shall need the following lemmas.

9.3 *Let χ be a disjoint colour scheme, and let $\mathcal{P}$ be a set of χ-places. For each $(\Gamma, \phi, \mathcal{L}) \in \mathcal{P}$ let $\pi(\mathcal{L})$ be a march with $\bar{\pi}(\mathcal{L}) = \bigcup(\bar{\pi}(A) : A \in \mathcal{L})$. Suppose that*

- *there exist m, n such that $|\mathcal{L}| \leq m$ and $|\bar{\pi}(\mathcal{L})| \leq n$ for all $(\Gamma, \phi, \mathcal{L}) \in \mathcal{P}$*
- *for every countable sequence $(\Gamma_i, \phi_i, \mathcal{L}_i)(i = 1, 2, \ldots)$ of members of $\mathcal{P}$ there exist $j > i \geq 1$ and an inflation σ of $M(\Gamma_i, \mathcal{L}_i)$ in Γ_j, such that*
  - *for each $v \in V(M(\Gamma_i, \mathcal{L}_i)), V(\sigma(v)) \subseteq V(M(\Gamma_j, \mathcal{L}_j))$*
  - *for each $e \in E(M(\Gamma_i, \mathcal{L}_i)), \sigma(e) \in E(M(\Gamma_j, \mathcal{L}_j))$ and $\phi_i(e) \leq \phi_j(\sigma(e))$*
  - *$|\bar{\pi}(\mathcal{L}_i)| = |\bar{\pi}(\mathcal{L}_j)|$, and for $1 \leq h \leq |\bar{\pi}(\mathcal{L}_i)|$ if v is the hth term of $\pi(\mathcal{L}_i)$ then $\sigma(v)$ contains the hth term of $\pi(\mathcal{L}_j)$.*

Then $\mathcal{P}$ is well-behaved.

The proof is easy and we omit it (see [10] for the proofs of several similar results).

We define the image of a χ-place under a Φ_χ-preserving homeomorphism in the natural way.

9.4 *Let χ be a disjoint colour scheme, and let $\mathcal{C}$ be a well-behaved set of χ-places. Let $\mathcal{C}'$ be the set of all images of members of $\mathcal{C}$ under Φ_χ-preserving homeomorphisms of Σ_χ. Then $\mathcal{C}'$ is well-behaved.*

Again, the proof is clear.

10 Tangle flaws

Our next objective is to produce some well-behaved sets of χ-places. The proofs of several of these theorems are similar, and so we give only the first in detail. Throughout this section and the next, χ is a disjoint colour scheme, and $\mathcal{S}$ is a similarly oriented set of χ-coloured paintings, such that if $(\Gamma, \phi) \in \mathcal{S}$ and α is a Φ_χ-preserving homeomorphism of Σ_χ, then $\mathcal{S}$ contains the image of (Γ, ϕ) under α (briefly, $\mathcal{S}$ is *closed under Φ_χ-preserving homeomorphisms*). Let $\mathcal{D}$ be the set of all quadruples $(\Gamma, \phi, \lambda, \mathcal{T})$ such that $(\Gamma, \phi) \in \mathcal{S}, \lambda$ is a tie-breaker in Γ, and $\mathcal{T}$ is a tangle in Γ.

Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$, such that $rep(\Gamma), ord(\mathcal{T}) > rep(\mathcal{T})$. Then $\hat{\Sigma_\chi}$ is not a sphere, and there is a circuit F of Γ^* with $\frac{1}{2}|E(F)| \leq rep(\mathcal{T})$ such that Σ_1 is not a disc, where Σ_1, Σ_2 are the closures of the two components of $\hat{\Sigma}_\chi \setminus U(F)$ and $(\Gamma \cap \Sigma_1, \Gamma \cap \Sigma_2) \in \mathcal{T}$. (It follows that $\frac{1}{2}|E(F)| = rep(\mathcal{T})$ for every such F.) Let us choose such a circuit F with $\lambda(F)$ minimal. We call F a *representativeness flaw* for $(\Gamma, \phi, \lambda, \mathcal{T})$.

10.1 *With $\Gamma, \phi, \lambda, \mathcal{T}, F, \Sigma_1, \Sigma_2$ as above,*

- *Σ_2 is a disc*
- *either $|U(F) \cap bd(\Sigma_\chi)| \leq 1$ and $U(F) \subseteq \Sigma_\chi$, or $|U(F) \cap bd(\Sigma_\chi)| = 2$ and $U(F) \not\subseteq \Sigma_\chi$, or $dist(\Gamma) \leq \frac{1}{4}|E(F)| + 1$*
- *F is λ-linked to $\mathcal{T}$.*

Proof. Since $rep(\Gamma) > rep(\mathcal{T}) = \frac{1}{2}|E(F)|$, it follows that one of Σ_1, Σ_2 (and hence Σ_2) is a disc, and so the first assertion of the theorem holds. If $U(F)$ meets two distinct cuffs then $dist(\Gamma) \leq \frac{1}{4}|E(F)|+1$, while if $|U(F) \cap bd(\Sigma_\chi)| \leq 1$ then $U(F) \subseteq \Sigma_\chi$, and in either case the second assertion holds. Thus, to prove the second assertion, we may assume that $U(F)$ intersects a unique cuff $\bar{r} \setminus r$ say, where r^* is a pole, and $|U(F) \cap (\bar{r} \setminus r)| \geq 2$. Let $v_1, v_2 \in U(F) \cap (\bar{r} \setminus r)$ be distinct, and let $e_i \in E(\Gamma^*)$ have ends r^*, v_i $(i = 1, 2)$. We shall prove that $e_1, e_2 \in E(F)$, from which the second assertion follows. For suppose that $e_1 \notin E(F)$. Let $u = r^*$ if $r^* \in V(F)$, and $u = v_2$ if $r^* \notin V(F)$. Let Q be a path of Γ^* between v_1 and u with $E(Q) \subseteq \{e_1, e_2\}$. Let F_1, F_2 be the two paths of F with ends v_1, u. Now $|E(Q)| \leq |E(F_1)|, |E(F_2)|$, and $\lambda(Q) < \lambda(F_i)(i = 1, 2)$ by the second condition in the definition of a tie-breaker. Thus if $e_1 \subseteq \Sigma_2$ then one of $F_1 \cup Q, F_2 \cup Q$ contradicts the choice of F. If $e_1 \not\subseteq \Sigma_2$ then one of $U(F_1 \cup Q), U(F_2 \cup Q)$ bounds a disc in $\hat{\Sigma}_\chi$ including Σ_2, because $\hat{\Sigma}$ is not a sphere and $rep(\Gamma) > \frac{1}{2}|E(F)|$, and again the choice of F is contradicted. This proves the second assertion of the theorem.

For the third assertion, choose F', Σ_1', Σ_2' as in 8.7; then $(\Gamma \cap \Sigma_1', \Gamma \cap \Sigma_2') \in \mathcal{T}$, and Σ_1' is not a disc. From the choice of F, $\lambda(F') = \lambda(F)$ and therefore $F' = F$; and so F is λ-linked to $\mathcal{T}$. This proves 10.1. ■

10.2 *Let χ satisfy $\mathbf{S_1}$, let F be a circuit in $\hat{\Sigma}_\chi$, with vertex set $\{v_1, \ldots, v_{2n}\}$, numbered in order in F, and let π be the march $v_2, v_4, \ldots, v_{2n}$. Then there is a well-behaved set $\mathcal{C}(F, \pi)$ of χ-places with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ satisfy*

- *$F \subseteq \Gamma^*$, and $U(F) \cap V(\Gamma) = \bar{\pi}$*
- *F is a representativeness flaw for $(\Gamma, \phi, \lambda, \mathcal{T})$, and*
- *$dist(\Gamma) > \frac{1}{4}|(E(F)| + 1$.*

Then there is a rooted location $\mathcal{L}$ with $ord(\mathcal{L}^-) = \frac{1}{2}|E(F)|$, which $(\frac{1}{2}|E(F)| + 1)$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}(F, \pi)$.

Proof. Let $\mathcal{D}_1$ be the set of members $(\Gamma, \phi, \lambda, \mathcal{T})$ of $\mathcal{D}$ satisfying the three displayed statements of the theorem. We may assume that $\mathcal{D}_1 \neq \emptyset$, and so $\hat{\Sigma}_\chi$ is not a sphere, and, defining Σ_1 and Σ_2 as before, Σ_2 is a disc by 10.1. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}_1$, let A be the rooted hypergraph with $A^- = \Gamma \cap \Sigma_1$ and $\pi(A) = \pi$, and let $\mathcal{L} = \{A\}$. Then $\mathcal{L}$ $(\frac{1}{2}|E(F)| + 1)$-isolates $\mathcal{T}$, by 8.8, since F is λ-linked to $\mathcal{T}$ by 10.1. Let $\mathcal{C}(F, \pi)$ be the set of all such $(\Gamma, \phi, \{A\})$; we must show that $\mathcal{C}(F, \pi)$ is well-behaved. By 9.3, it suffices to show that for any countable sequence $(\Gamma_i, \phi_i, \mathcal{L}_i)(i = 1, 2, \ldots)$ of members of $\mathcal{C}(F, \pi)$, there exist $j > i \geq 1$ and an inflation σ of $\Gamma_i \cap \Sigma_2$ in Γ_j such that

(a) for each $v \in V(\Gamma_i \cap \Sigma_2)$, $V(\sigma(v)) \subseteq V(\Gamma_j \cap \Sigma_2)$

(b) for each $e \in E(\Gamma_i \cap \Sigma_2)$, $\sigma(e) \in E(\Gamma_j \cap \Sigma_2)$ and $\phi_i(e) \leq \phi_j(\sigma(e))$

(c) for each $v \in \bar{\pi}, \sigma(v)$ contains v.

Now by 10.1 $U(F)$ meets at most one cuff of Σ_χ, and either $|U(F) \cap bd(\Sigma_\chi)| \leq 1$ and $U(F) \subseteq \Sigma_\chi$, or $|U(F) \cap bd(\Sigma_\chi)| = 2$ and $U(F) \not\subseteq \Sigma_\chi$. Thus there are two fragments obtained by cutting Σ_χ along $U(F) \cap \Sigma_\chi$; let Σ' be the fragment with $\psi(\Sigma') \subseteq \Sigma_2$, where ψ is the associated surjection.

Let χ' be the colour scheme with $\Sigma_{\chi'} = \Sigma'$, $\Phi_{\chi'} = \psi^{-1}(\Phi_\chi) \cap \Sigma'$, $\Omega_{\chi'}(k) = \Omega_\chi(k)$ $(k = 2, 3)$, and $\Omega_{\chi'}(S') = \Omega_\chi(S)$ for each long side S' of $\Phi_{\chi'}$, where S is the long side of Φ_χ with $\psi(S') \subseteq S$. Since $\hat{\Sigma}_\chi$ is not a sphere, and $\hat{\Sigma}_{\chi'}$ is a sphere, and χ satisfies $\mathbf{S_1}$, it follows that χ' is not bad. Now for each $i \geq 1$, let $\Gamma'_i = \psi^{-1}(\Gamma_i) \cap \Sigma'$, and for each $e \in E(\Gamma'_i)$, let $\phi'_i(e) = \phi_i(\psi(e))$ if e is not a short side of $\Phi_{\chi'}$, and $\phi'_i(e) = e$ if e is a short side. Then (Γ'_i, ϕ'_i) is a χ'-coloured painting. Since χ' is not bad, there exist $j > i \geq 1$ and a linear inflation σ' of (Γ'_i, ϕ'_i) in (Γ'_j, ϕ'_j). For each $e \in E(\Gamma_j \cap \Sigma_2)$, let $\sigma(e) = \psi(\sigma'(\psi^{-1}(e)))$, and for each $v \in V(\Gamma_j \cap \Sigma_2)$, let $\sigma(v) = \psi(\sigma'(v'))$, where $\psi^{-1}(v) = \{v'\}$. Then σ is an inflation satisfying (a), (b), (c) above, as required. This proves 10.2. ■

10.3 *Let χ satisfy $\mathbf{S_1}$, and let $n \geq 1$ be an integer. Then there is a well-behaved set $\mathcal{C}_1(n)$ of χ-places with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ satisfy $rep(\Gamma), ord(\mathcal{T}) \geq n > rep(\mathcal{T})$, and $dist(\Gamma) \geq \frac{1}{2}n+1$. Then there is a rooted location $\mathcal{L}$ which n-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}_1(n)$.*

Proof. By 5.2, there are finitely many pairs $(F_i, \pi_i)(i \in I)$, such that each F_i is an even length circuit in $\hat{\Sigma}_\chi$ with $\frac{1}{2}|E(F_i)| < n$, and π_i is a march with $\bar{\pi}_i \subseteq V(F_i)$ consisting of every second vertex of F_i, with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T})$ be as in the theorem, and let F be a representativeness flaw. Then there exist $i \in I$ and a Φ_χ-preserving homeomorphism α of Σ_χ which maps F to F_i and $U(F) \cap V(\Gamma)$ to $\bar{\pi}_i$. Let $\mathcal{C}_1(n)$ be the union, over all $i \in I$, of the set of all images of members of $\mathcal{C}(F_i, \pi_i)$ (defined in 10.2) under Φ_χ-preserving homeomorphisms. By 9.4, $\mathcal{C}_1(n)$ is well-behaved (since I is finite). Let $(\Gamma, \phi, \lambda, \mathcal{T}), F, i, \alpha$ be as before, and let $(\Gamma', \phi', \lambda', \mathcal{T}')$ be the image of $(\Gamma, \phi, \lambda, \mathcal{T})$ under α (in the natural sense). Since $\mathcal{S}$ is closed under Φ_χ-preserving homeomorphisms, it follows that $(\Gamma', \phi', \lambda', \mathcal{T}') \in \mathcal{D}$, and F_i is a representativeness flaw for it with $U(F_i) \cap V(\Gamma') = \bar{\pi}_i$. By 10.2, there is a rooted location $\mathcal{L}'$ with $ord(\mathcal{L}') = \frac{1}{2}|E(F)| < n$ which n-isolates $\mathcal{T}'$ and with $(\Gamma', \phi', \mathcal{L}') \in \mathcal{C}(F_i, \pi_i)$. Let $\mathcal{L}$ be the image of $\mathcal{L}'$ under α^{-1}. Then it satisfies the theorem. ■

The sets $\mathcal{C}_1(n)$ will handle failure of representativeness. Now we turn to another possible failure when $rep(\mathcal{T})$ is large but $d(r_1^*, r_2^*)$ is small for two poles r_1^*, r_2^*. More precisely, for $n \geq 1$ an integer, let us say that $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is *n-flawed in distance* if $rep(\mathcal{T}) > n$, and $dist(\Gamma) > \frac{1}{2}n+1$, and there are distinct poles r_1^*, r_2^* with $d(r_1^*, r_2^*) \leq n$. Suppose that $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is n-flawed in distance. Then since $dist(\Gamma) \geq \frac{1}{2}n + 1$, it follows from theorem 9.2 of [6] that there exists $F \subseteq \Gamma^*$ such that one of the following holds:

- F is a circuit with $|E(F)| \leq 2n$, satisfying
  - either $|U(F) \cap bd(\Sigma_\chi)| \leq 1$ and $U(F) \subseteq \Sigma_\chi$, or $|U(F) \cap bd(\Sigma_\chi)| = 2$ and $U(F) \not\subseteq \Sigma_\chi$, and
  - $ins(F)$ contains at least two poles;
- $F = F_0 \cup F_1$ with $2|E(F_0)| + |E(F_1)| \leq 2n$, where F_0 is a path with distinct ends and F_1 is a circuit, satisfying
  - one end of F_1 is a pole r^* say, and the other end is the unique element of $F_0 \cap F_1$
  - every vertex of F_0 is in $\Sigma_\chi \setminus bd(\Sigma_\chi)$ except r^* and its neighbour,
  - every vertex of $F_1 \setminus F_0$ is in $\Sigma_\chi \setminus bd(\Sigma_\chi)$, and
  - $ins(F_1)$ contains exactly one pole and does not contain r^*;

- $F = F_0 \cup F_1 \cup F_2$ with $2|E(F_0)| + |E(F_1)| + |E(F_2)| \le 2n$, where F_0 is a path and F_1, F_2 are circuits, satisfying
  - $F_0 \cap F_1 = \{v_1\}$ and $F_0 \cap F_2 = \{v_2\}$ where v_1, v_2 are the ends of F_0,
  - $U(F_0 \cup F_1) \subseteq \Sigma_\chi \setminus bd(\Sigma_\chi)$,
  - $U(F_2) \subseteq \Sigma_\chi$ and $|U(F_2) \cap bd(\Sigma_\chi)| \le 1$,
  - either $E(F_0) = \emptyset$ and $F_0 = F_1 \cap F_2$ or $E(F_0) \neq \emptyset$ and $F_1 \cap F_2$ is null, and
  - for $i = 1, 2$, $ins(F_i)$ contains exactly one pole r_i^*, say, and $r_1^* \neq r_2^*$.

By 8.5 we may choose F such that every circuit of F is λ-linked to $\mathcal{T}$. (To see this, choose F with $ins(F)$ maximal.) In this case we call $(F, ins(F))$ a *distance flaw* for $(\Gamma, \phi, \lambda, \mathcal{T})$.

10.4 *Let χ satisfy $\mathbf{S_3}$, and let $n \ge 1$ be an integer. Then there is a well-behaved set $\mathcal{C}_2(n)$ of χ-places with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ be n-flawed in distance. Then there is a rooted location $\mathcal{L}$ which $(n+1)$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}_2(n)$.*

Proof. By the argument of 10.3, it suffices to prove that if $\mathcal{D}' \subseteq \mathcal{D}$, and all members of $\mathcal{D}'$ have the same distance flaw, (F, X) say, and $U(F) \cap V(\Gamma)$ is the same for all $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, then there is a well-behaved set $\mathcal{C}$ of χ-places, such that for all $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ there is a rooted location $\mathcal{L}$ which $(n+1)$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}$.

For each circuit C of F, let π_C be a march with $\bar{\pi}_C = U(C) \cap V(\Gamma)$ for every $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, and let $\Sigma_C \subseteq X$ be the disc in $\hat{\Sigma_\chi}$ bounded by $U(C)$ such that $\Sigma_C = ins(C)$ for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$. Let Σ_0 be obtained from $\hat{\Sigma_\chi}$ by deleting the interior of each Σ_C. For each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, let $\mathcal{L}$ be the set of all rooted hypergraphs $(\Gamma \cap \Sigma_C, \pi_C)$, as C ranges over the (one or two) circuits of F. Let $\mathcal{C}$ be the set of all such $(\Gamma, \phi, \mathcal{L})$. As in the proof of 10.2, it suffices to show that for any countable sequence $(\Gamma_i, \phi_i, \mathcal{L}_i)$ $(i = 1, 2, \ldots)$ of members of $\mathcal{C}$ there exist $j > i \ge 1$ and an inflation σ of $\Gamma_i \cap \Sigma_0$ in Γ_j such that

- for each $v \in V(\Gamma_i \cap \Sigma_0), V(\sigma(v)) \subseteq V(\Gamma_j \cap \Sigma_0)$
- for each $e \in E(\Gamma_i \cap \Sigma_0), \sigma(e) \in E(\Gamma_j \cap \Sigma_0)$ and $\phi_i(e) \le \phi_j(\sigma(e))$
- for each circuit C of F and each $v \in \bar{\pi}_C, \sigma(v)$ contains v.

If we cut Σ_χ along $U(F)$ we obtain two or three fragments; let Σ' be the fragment with $\psi(\Sigma') \subseteq \Sigma_0$, where ψ is the associated surjection. Let χ' be defined as in the proof of 10.2. Since χ satisfies $\mathbf{S_3}$ and since $\hat{\Sigma}_{\chi'} \cong \hat{\Sigma}_\chi$ and $\Omega_{\chi'}(k) = \Omega_\chi(k)(k = 2, 3)$, and $c(\Sigma_{\chi'}) < c(\Sigma_\chi)$, it follows that χ' is not orientedly bad. For each $i \ge 1$ define Γ_i', ϕ_i' as in the proof of 10.2. The sequence $(\Gamma_i', \phi_i')(i = 1, 2, \ldots)$ is similarly oriented, because $\mathcal{S}$ is similarly oriented and if $\Sigma_{\chi'}$ is orientable then so is Σ_χ. Thus there exist $j > i \ge 1$ and a linear inflation of (Γ_i', ϕ_i') in (Γ_j', ϕ_j'). The result follows as in the proof of 10.2. ■

We say that $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is *n-flawed in freedom* if $rep(\mathcal{T}) > n$, and $d(r_1^*, r_2^*) > n$ for every two distinct poles r_1^*, r_2^*, and there is a circuit F of Γ^* with $\frac{1}{2}|E(F)| \le n$ such that $ins(F)$ includes a long side of Φ_χ or more than $\frac{1}{2}|E(F)|$ vertices of Φ_χ. Then F may be chosen to be λ-linked to $\mathcal{T}$ (by choosing F with $ins(F)$ maximal), and in this case we call it a *freedom flaw* for $(\Gamma, \phi, \lambda, \mathcal{T})$. By adapting the proofs of 10.2, 10.3, 10.4 in the natural way, using $\mathbf{S_4}$ in place of $\mathbf{S_1}$ and $\mathbf{S_3}$, we obtain

10.5 *Let χ satisfy $\mathbf{S_4}$, and let $n \geq 0$ be an integer. Then there is a well-behaved set $\mathcal{C}_3(n)$ of χ-places with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ be n-flawed in freedom. Then there is a rooted location $\mathcal{L}$ which $(n+1)$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}_3(n)$.*

11 Border label flaws

So far we have examined flaws in our paintings (Γ, ϕ) due to some kind of lack of generality in Γ — representativeness, distance and freedom flaws. There are two other flaws we must investigate, both concerned with a lack of generality in ϕ, and it is to deal with these that $\mathbf{S_2}$ and the full strength of $\mathbf{S_4}$ are needed. In this section we examine the situation when for some long side S, the values of $\phi(e)$ over edges e bordering S fail to be sufficiently general.

Let $\chi, \mathcal{S}, \mathcal{D}$ be as in section 10. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$, and let r^* be a pole. By a *bite* at r^* we mean a circuit F of Γ^* with $\frac{1}{2}|E(F)| < rep(\mathcal{T})$, such that $r^* \in V(F)$. If e_1, e_2 are edges of Γ bordering $\bar{r} \setminus r$, we define $l(e_1, e_2)$ to be the minimum of $\frac{1}{2}|E(F)|$ over all bites F at r^* with $e_1, e_2 \subseteq ins(F)$, if there is such a bite, and otherwise $l(e_1, e_2) = rep(\mathcal{T})$.

Let S be a long side of Φ_χ, let $m \geq 1$, let $\omega_1, \ldots, \omega_m$ be a sequence of elements of $\Omega_\chi(S)$, and let $n \geq 4$. We say that $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is *$(n, (\omega_1, \ldots, \omega_m))$-flawed on S* if

- $rep(\mathcal{T}) \geq 2(m+1)n + 8$, and $dist(\Gamma) \geq \frac{1}{2}(m+1)n + 1$,
- $(\Gamma, \phi, \lambda, \mathcal{T})$ is not $(m+1)n$-flawed in distance,
- $(\Gamma, \phi, \lambda, \mathcal{T})$ is not $(m+1)n$-flawed in freedom, and
- there do not exist distinct edges $e_1, \ldots, e_m \in E(\Gamma)$ bordering S in order, such that $\phi(e_i) \geq \omega_i$ $(1 \leq i \leq m)$, and $l(e_i, e_j) > n$ $(1 \leq i < j \leq m)$ and $l(e_i, s) > n$ $(1 \leq i \leq m)$ for every short side s bordering the same cuff as S.

The main result of this section is

11.1 *Let χ satisfy $\mathbf{S_4}$, let S be a long side of Φ_χ, let $\omega_1, \ldots, \omega_m \in E(\Omega_\chi(S))$ with $m \geq 1$, and let $n \geq 4$. Then there is a well-behaved set $\mathcal{C}_4(S, (\omega_1, \ldots, \omega_m), n)$ of χ-places with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ be $(n, (\omega_1, \ldots, \omega_m))$-flawed on S. Then there is a rooted location $\mathcal{L}$ which $(m+1)n$-isolates $\mathcal{T}$, and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}_4(S, (\omega_1, \ldots, \omega_m), n)$.*

The proof of 11.1 will require some lemmas, however, which follow. Throughout the section, $\chi, S, \omega_1, \ldots, \omega_m$ and n are as in 11.1.

Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$, and let r^* be the pole with $S \subseteq \bar{r}$. If F is a bite at r^*, we define $I(F)$ to be the I-arc in $ins(F) \cap (\bar{r} \setminus r)$ joining the two neighbours in F of r^*. If $\mathcal{F}$ is a set of bites at r^*, we define $I(\mathcal{F}) = \bigcup(I(F) : F \in \mathcal{F})$. The *order* of a set $\mathcal{F}$ of bites at r^* is $\Sigma(\frac{1}{2}|E(F)| : F \in \mathcal{F})$. A set $\mathcal{F}$ of bites at r^* is a *feast* if

- $|\mathcal{F}| \leq m+1$
- $I(\mathcal{F})$ includes the short sides of Φ_χ with an end in common with S, and
- for each component X of $S \setminus I(\mathcal{F})$ there exists h with $1 \leq h \leq m$ such that $\phi(e) \not\geq \omega_h$ for all $e \in E(\Gamma)$ with $e \subseteq X$.

11.2 *If $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is $(n, (\omega_1, \ldots, \omega_m))$-flawed on S, then there is a feast of order $\leq (m+1)n$.*

Proof. Let the edges of Γ bordering S be $e_1, \ldots, e_k$ in order, and define e_0 and e_{k+1} to be the short sides of Φ_χ with an end in common with S, numbered such that for $0 \leq i \leq k$, e_i and e_{i+1} have a common end. (It is possible that $e_0 = e_{k+1}$.) For $0 \leq i < j \leq k+1$, we write $i \ll j$ if there is no bite F at r^* with $\frac{1}{2}|E(F)| \leq n$ and $e_i, e_{i+1}, \ldots, e_j \subseteq ins(F)$. Since $(\Gamma, \phi, \lambda, \mathcal{T})$ is not $(m+1)n$-flawed in freedom, it follows that $1 \ll k$. (To see this, observe that since $(\Gamma, \phi, \lambda, \mathcal{T})$ is $(n, (\omega_1, \ldots, \omega_m))$-flawed on S, we have $rep(\mathcal{T}) \geq 2(m+1)n + 8$, and $dist(\Gamma) \geq \frac{1}{2}(m+1)n + 1$. In particular, $rep(\mathcal{T}) > n$ and $d(r_1^*, r_2^*) > n$ for every two poles r_1^*, r_2^*.) Let us choose p maximum with $0 \leq p \leq m$ such that there is a sequence of integers

$$0 = b_0 < a_1 < b_1 < a_2 < \ldots < b_p < k+1$$

satisfying

- $b_0 \ll b_1 \ll \ldots \ll b_p \ll k+1$
- $b_{i-1} \not\ll a_i$ for $1 \leq i \leq p$
- $\phi(e_{b_i}) \geq \omega_i$ for $1 \leq i \leq p$
- for $1 \leq i \leq p$ there is no j with $a_i < j < b_i$ such that $\phi(e_j) \geq \omega_i$.

(This is possible, for there is such a sequence with $p = 0$.)

(1) $p \leq m - 1$.

Subproof. If $p = m$, then by the fourth condition in the definition of "$(n, (\omega_1, \ldots, \omega_m))$-flawed on S" there exist $1 \leq i \leq m$ and a bite F at r^* with $\frac{1}{2}|E(F)| \leq n$ such that $e, e_{b_i} \subseteq ins(F)$, where either $e = e_{b_j}$ for some $j \neq i$ or e is a short side. But then $ins(F)$ includes one of

$$e_{b_{i-1}} \cup e_{b_{i-1}+1} \cup \cdots \cup e_{b_i}$$
$$e_{b_i} \cup e_{b_i+1} \cup \cdots \cup e_{b_{i+1}}$$

(where b_{p+1} means $k+1$), contrary to the first statement above. Thus (1) holds.

(2) *There is no j with $b_p < j < k+1$ such that $b_p \ll j \ll k+1$ and $\phi(e_j) \geq \omega_{p+1}$.*

Subproof. Suppose that some such j exists. Choose i with $b_p < i < j$, maximal such that $b_p \not\ll i$. (This is possible because $b_p \not\ll b_p + 1$, since $n \geq 3$, and so $j \neq b_p + 1$.) Set $a_{p+1} = i$. Choose j' with $i < j' \leq j$, minimal such that $\phi(e_{j'}) \geq \omega_{p+1}$. (This is possible because $\phi(e_j) \geq \omega_{p+1}$.) Set $b_{p+1} = j'$. Then $j' > i$, and so $b_p \ll b_{p+1}$ by the maximality of i; and $j' \leq j$, and so $b_{p+1} \ll k+1$ since $j \ll k+1$. Moreover $b_p \not\ll a_{p+1}$ from the choice of i; and $\phi(e_{b_{p+1}}) \geq \omega_{p+1}$ from the choice of j'; and there is no j'' with $a_{p+1} < j'' < b_{p+1}$ such that $\phi(e_{j''}) \geq \omega_{p+1}$, from the choice of j'. But then the sequence

$$0 = b_0 < a_1 < b_1 < \ldots < b_p < a_{p+1} < b_{p+1} < k+1$$

disproves the maximality of p, a contradiction. Thus (2) holds.

(3) *There exist* a_{p+1}, b_{p+1} *with* $b_p < a_{p+1} < b_{p+1} < k+1$, *such that* $b_p \not\ll a_{p+1}$ *and* $b_{p+1} \not\ll k+1$, *and such that there is no* j *with* $a_{p+1} < j < b_{p+1}$ *and with* $\phi(e_j) \geq \omega_{p+1}$.

Subproof. Choose i with $b_p < i < k+1$ maximum such that $b_p \not\ll i$. (This is possible since $b_p + 1 < k+1$ and $b_p \not\ll b_p + 1$, since $n \geq 3$.) Suppose first that $i < k$. Since $k \not\ll k+1$, we may choose i' with $i < i' < k+1$ minimum such that $i' \not\ll k+1$. Set $a_{p+1} = i, b_{p+1} = i'$. From (2), it follows that there is no j with $i < j < i'$ such that $\phi(e_j) \geq \omega_p + 1$, and so we may satisfy (3) by setting $a_{p+1} = i$ and $b_{p+1} = i'$. Now assume that $i = k$. Since $b_p \ll k+1$ and $n \geq 4$, it follows that $k - 1 > b_p$; and so we may set $a_{p+1} = k-1$ and $b_{p+1} = k$ to satisfy (3). This proves (3).

For $0 \leq i \leq p$ let F_i be a bite at r^* with $\frac{1}{2}|E(F_i)| \leq n$ such that $e_{b_i}, e_{b_i+1}, \ldots, e_{a_{i+1}} \subseteq ins(F_i)$, and let F_{p+1} be a bite with $\frac{1}{2}|E(F_{p+1})| \leq n$ such that $e_{b_{p+1}}, \ldots, e_{k+1} \subseteq ins(F_{p+1})$. Let $\mathcal{F} = \{F_0, F_1, \ldots, F_{p+1}\}$. Then $|\mathcal{F}| = p+2 \leq m+1$, and $I(\mathcal{F})$ includes e_0, e_{k+1}. If X is a component of $S \setminus I(\mathcal{F})$, then there exists h with $1 \leq h \leq p+1$ such that $a_h < i < b_h$ for every $e_i \subseteq X$, and hence $\phi(e_i) \not\geq \omega_h$ for every $e_i \subseteq X$. Hence $\mathcal{F}$ is a feast. Its order is at most $(m+1)n$ since each F_i has $\frac{1}{2}|E(F_i)| \leq n$. ■

A feast $\mathcal{F}$ is *disjoint* if $I(F) = ins(F) \cap (\bar{r} \setminus r)$ for each $F \in \mathcal{F}$, and $ins(F) \cap ins(F') = \{r^*\}$ for all distinct $F, F' \in \mathcal{F}$.

11.3 *Every feast of minimum order with order* $\leq (m+1)n$ *is disjoint.*

Proof. By 7.1, taking $z = r$ and $\kappa = (m+1)n + 2$, we deduce that there is a circuit C of $sk(\Gamma)$, bounding an open disc $\Lambda \subseteq \tilde{\Sigma}$ with $r \subseteq \Lambda$, satisfying (1) and (2) below:

(1) $x \subseteq \Lambda$ *for every* $x \in A(\Gamma)$ *with* $d(r, x) \leq (m+1)n$.

(2) $ins(C^*) \subseteq \bar{\Lambda}$ *for every circuit* C^* *of* Γ^* *with* $U(C^*) \subseteq \bar{\Lambda}$ *and* $|E(C^*)| \leq 2(m+1)n$.

In particular, for every bite F with $\frac{1}{2}|E(F)| \leq (m+1)n$, since $r^* \in V(F)$ it follows that $d(r, x) \leq (m+1)n$ for every $x \in A(\Gamma)$ with $x \cap U(F) \neq \emptyset$, and hence $x \subseteq \Lambda$ for every such x, by (1); and so $F \subseteq \Lambda$, and hence $ins(F) \subseteq \Lambda$, by (2).

Let $\mathcal{F}$ be a feast of minimum order, with order $\leq (m+1)n$, and let $G = \bigcup(F : F \in \mathcal{F})$. Hence $U(G) \subseteq \Lambda$, and since

$$|E(G)| \leq \Sigma(|E(F)| : F \in \mathcal{F}) \leq 2(m+1)n$$

it follows from (2) that $ins(F) \subseteq \Lambda$ for every circuit F of G.

Now G is a connected subdrawing of the planar drawing $\Gamma^* \cap \bar{\Lambda}$, with $r^* \in V(G)$, and every vertex and edge of G is in a circuit, and G has no cutvertex except possibly r^*. It follows that there are circuits $C_1, \ldots, C_p$ of G, bounding closed discs $\Delta_1, \ldots, \Delta_p \subseteq \Lambda$ respectively, such that $\Delta_i \cap \Delta_j = \{r^*\}$ $(1 \leq i < j \leq p)$ and $U(G) \subseteq \Delta_1 \cup \ldots \cup \Delta_p$. Thus $\Delta_i = ins(C_i)$ $(1 \leq i \leq p)$.

Suppose that $r^* \notin V(C_i)$ for some i. Then $p = 1$ (since $\Delta_i \cap \Delta_j = \{r^*\}$ for $j \neq i$, which is impossible) and $i = 1$, and r^* belongs to $ins(C_1) \setminus U(C_1)$; and hence $S \subseteq ins(C_1)$, and $(\Gamma, \phi, \lambda, \mathcal{T})$ is $(m+1)n$-flawed in freedom, a contradiction. This proves that $r^* \in V(C_i)$ for $1 \leq i \leq p$.

Let $\mathcal{F}' = \{C_1, \ldots, C_p\}$. We shall show that $\mathcal{F}'$ is a feast. If $F \in \mathcal{F}$ there exists $C_i \in \mathcal{F}'$ with $ins(F) \subseteq ins(C_i)$ and hence $I(F) \subseteq I(C_i)$. It follows that every component of $S \setminus I(\mathcal{F}')$ is a subset of a component of $S \setminus I(\mathcal{F})$, and $I(\mathcal{F}')$ includes the short sides with a common end with S.

Now for $1 \leq i \leq p$, there is a circuit $C'_i \in \mathcal{F}$ with $E(C'_i) \cap E(C_i) \neq \emptyset$; and hence $U(C''_i) \subseteq \Delta_i$. It follows that $C'_1, \ldots, C'_p \in \mathcal{F}$ are all distinct, and so $p \leq |\mathcal{F}| \leq m+1$.

It follows that $\mathcal{F}'$ is a feast. Its order is $\frac{1}{2}\Sigma|E(C_i)|$, and since $C_1, \ldots, C_p$ are mutually edge-disjoint, it follows that $\mathcal{F}'$ has order $\leq \frac{1}{2}|E(G)|$. But $\mathcal{F}$ has order $\geq \frac{1}{2}|E(G)|$, and $\mathcal{F}$ has minimum order. Consequently we have equality throughout, and in particular $G = C_1 \cup \ldots \cup C_p$, and every edge of G belongs to exactly one member of $\mathcal{F}$, and $F \subseteq G$ for every $F \in \mathcal{F}$. Hence, since $C_1, \ldots, C_p$ are the only circuits of G, it follows that $F \in \{C_1, \ldots, C_p\}$ for every $F \in \mathcal{F}$, and so $\mathcal{F}' = \mathcal{F}$.

This proves that $ins(F) \cap ins(F') = \{r^*\}$ for all distinct $F, F' \in \mathcal{F}$. To complete the proof that $\mathcal{F}$ is disjoint, let $F \in \mathcal{F}$, and suppose that $I(F) \neq ins(F) \cap (\bar{r} \setminus r)$. Then there is an edge e of Γ^* with one end r^*, the other end in $V(F)$, and with $e \not\subseteq ins(F)$. There is a circuit F' with $E(F') \subseteq E(F) \cup \{e\}, ins(F) \subseteq ins(F')$, and $|E(F')| < |E(F)|$. But then $(\mathcal{F} \setminus \{F\}) \cup \{F'\}$ is a feast of smaller order than $\mathcal{F}$, a contradiction.

The result follows. ■

11.4 *Let $\mathcal{F}$ be a feast of minimum order $\leq (m+1)n$ and subject to that with $\Sigma(\lambda(F) : F \in \mathcal{F})$ minimum. Then each $F \in \mathcal{F}$ is λ-linked to $\mathcal{T}$.*

Proof. Let $F \in \mathcal{F}$, and choose F' as in 8.5. Then $r^* \notin ins(F') \setminus U(F')$, because $|E(F')| \leq |E(F)| \leq (m+1)n$ and $(\Gamma, \phi, \lambda, \mathcal{T})$ is not $(m+1)n$-flawed in freedom. Thus $r^* \in V(F')$, and since $ins(F) \subseteq ins(F')$ it follows that $\mathcal{F}' = (\mathcal{F} \setminus \{F\}) \cup \{F'\}$ is a feast. From the minimality of the order of $\mathcal{F}$ we deduce that $|E(F)| = |E(F')|$; but then $\lambda(F') \leq \lambda(F)$, and again it follows that $\lambda(F') = \lambda(F)$. Hence $F' = F$, and so F is λ-linked to $\mathcal{T}$, as required. ■

If $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is $(n, (\omega_1, \ldots, \omega_m))$-flawed on S, it follows from 11.2, 11.3, 11.4 that there is a disjoint feast $\mathcal{F}$ of order $\leq (m+1)n$, such that each of its members is λ-linked to $\mathcal{T}$. We call $\mathcal{F}$ an $((\omega_1, \ldots, \omega_m), S)$-*flaw* for $(\Gamma, \phi, \lambda, \mathcal{T})$.

Proof of 11.1.

As usual, it suffices to prove that if $\mathcal{D}' \subseteq \mathcal{D}$, and each member of $\mathcal{D}'$ has the same $((\omega_1, \ldots, \omega_m), S)$-flaw $\mathcal{F}$, then there is a well-behaved set $\mathcal{C}$ of χ-places, such that for all $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ there is a rooted location which $(m+1)n$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}$. Let the components of $S \setminus I(\mathcal{F})$ be $X_1, \ldots, X_t$. For each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ there exist $\omega'_1, \ldots, \omega'_t \in \{\omega_1, \ldots, \omega_p\}$, such that for $1 \leq i \leq t$, $\phi(e) \not\geq \omega_i$ for each edge e of Γ with $e \subseteq X_i$. Since there are only finitely many such t-tuples $(\omega'_1, \ldots, \omega'_t)$, we may further assume that the same t-tuple $(\omega'_1, \ldots, \omega'_t)$ works for all members of $\mathcal{D}'$. For $1 \leq i \leq t$, let Ω_i be the ideal of $\Omega_\chi(S)$ with element set $\{x \in E(\Omega_\chi(S)) : x \not\geq \omega'_i\}$. Then for $1 \leq i \leq t, \Omega_i \prec \Omega_\chi(S)$, and for all $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}', \phi(e) \in E(\Omega_i)$ for all $e \in E(\Gamma)$ with $e \subseteq X_i$.

Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$. For each $F \in \mathcal{F}$, let $\Delta(F) = ins(F)$; this does not depend on the choice of $(\Gamma, \phi, \lambda, \mathcal{T})$, since $|\mathcal{F}| \geq 2$ and $ins(F)$ meets no member of $\mathcal{F}$ except F. Since $(\Gamma, \phi, \lambda, \mathcal{T})$ is not $((m+1)n+1)$-flawed in distance and $dist(\Gamma) > \frac{1}{2}((m+1)n+1)$, it follows that F meets no cuff except $\bar{r} \setminus r$.

Let Σ' be obtained from Σ_χ by deleting $\Sigma_\chi \cap (\Delta(F) \setminus U(F))$ for each $F \in \mathcal{F}$. Then Σ' is homeomorphic to one of the fragments obtained by cutting Σ_χ along $U(\cup(F : F \in \mathcal{F}))$, and to simplify notation we assume that Σ' is such a fragment, and the associated surjection ψ is the identity when restricted to Σ'. We see that $X_1, \ldots, X_t$ are long sides of $\psi^{-1}(\Phi_\chi) \cap \Sigma'$. Let χ' be the colour scheme with $\Sigma_{\chi'} = \Sigma'$, $\Phi_{\chi'} = \psi^{-1}(\Phi_\chi) \cap \Sigma'$, $\Omega_{\chi'}(k) = \Omega_\chi(k)$ $(k = 2, 3)$, $\Omega_{\chi'}(X_i) = \Omega_i$ $(1 \leq i \leq t)$

and $\Omega_{\chi'}(S') = \Omega_\chi(R)$ for each long side $S' \neq X_1, \ldots, X_t$ of $\Phi_{\chi'}$, where $R \neq S$ is the long side of Φ_χ with $S' \subseteq R$. Let us define $f(X_i) = S (1 \leq i \leq t)$, and $f(S') = R$ for each long side S' of $\Phi_{\chi'}$ with $S' \neq X_1, \ldots, X_t$, where R is the long side of Φ_χ with $S' \subseteq R$. Then f is an embedding of χ' in χ, and so since χ satisfies $\mathbf{S_4}$, χ' is not orientedly bad. But now the result follows as in the proofs of 10.2, 10.4, 10.5. ■

12 Internal label flaws

In this and the next two sections we consider the final kind of flaw. Let $\chi, \mathcal{S}, \mathcal{D}$ be as in section 10, let $\omega_0 \in E(\Omega_\chi(2)) \cup E(\Omega_\chi(3))$, and let $m \geq 1, n \geq 4$. We say that $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is (m, n, ω_0)-*flawed internally* if

- $rep(\mathcal{T}) \geq n \cdot 5^{2m+2}$, and $dist(\Gamma) \geq \frac{1}{2} n \cdot 5^{2m+2} + 1$
- $(\Gamma, \phi, \lambda, \mathcal{T})$ is not $n \cdot 5^{2m+2}$-flawed in distance
- there do not exist edges $e_1, \ldots, e_m$ of Γ such that $\phi(e_i) \geq \omega_0$ $(1 \leq i \leq m)$, $d(e_i, e_j) > n$ $(1 \leq i < j \leq m)$, and $d(e_i, r^*) > n$ $(1 \leq i \leq m)$ for every pole r^*, where d is the metric of $\mathcal{T}$.

Throughout this section and the next, $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is (m, n, ω_0)-flawed internally. Let $W \subseteq \hat{\Sigma}$ be a closed disc with $bd(W) \subseteq U(\Gamma^*)$, let H be a subdrawing of Γ^* with $U(H) \subseteq W \setminus bd(W)$, and let $\rho \geq 1$ be an integer with $rep(\mathcal{T}) \geq 2\rho + 3$. We say that W is a *wheel*, and H is a W-*hub of radius* ρ, if:

- W includes $ins(F)$ for every circuit F of Γ^* with $|E(F)| \leq 4\rho + 4$ and $U(F) \subseteq W$
- W contains every vertex of Γ^* adjacent in Γ^* to two members of $V(\Gamma) \cap bd(W)$, and W includes every edge of Γ^* with both ends in $bd(W)$
- either $|V(H)| = 1$ or H is a $\mathcal{T}$-enclave with $|E(H)| \leq 4\rho + 4$; if $|V(H)| = 1$, then there is no $\mathcal{T}$-enclave H' with $|E(H')| \leq 4\rho + 4$ and $V(H) \subseteq ins(H') \setminus U(H')$; and if H is a $\mathcal{T}$-enclave, then there is no $\mathcal{T}$-enclave $H' \neq H$ with $|E(H')| \leq 4\rho + 4$ and $U(H) \subseteq ins(H')$
- W contains every vertex of Γ joined to $V(H)$ by a path in Γ^* of $\leq \rho$ edges, and
- for every $v \in V(\Gamma) \cap bd(W)$ there is a path of Γ^* from v to $V(H)$ with $\leq \rho$ edges, and no such path with $\leq \rho - 2$ edges.

The *radius* of a wheel W is the minimum radius of all W-hubs.

12.1 *Let $\rho \geq 2$, even, let $rep(\mathcal{T}) \geq 5\rho + 12$, and let H be a subdrawing of Γ^*, satisfying the third condition in the definition of a wheel above. Then there is a wheel W such that H is a W-hub of radius ρ.*

Proof. Let $z^* \in V(H)$, and let z be the atom of Γ with $z^* \subseteq z$. By 7.1 (taking $\kappa = 3\rho + 4$) there is a circuit C of Γ^*, bounding an open disc $\Lambda \subseteq \hat{\Sigma}$, such that

- $x \subseteq \Lambda$ for every atom x of Γ with $d(z, x) \leq 3\rho + 2$, and

- $ins(F) \subseteq \bar{\Lambda}$ for every circuit F of Γ^* with $U(F) \subseteq \bar{\Lambda}$ and $|E(F)| \leq 4\rho + 4$.

(1) *$U(P) \subseteq \Lambda$ for every path P of Γ^* with one end in $V(H)$ and with $|E(P)| \leq \rho$.*

Subproof. Let h be an end of P in $V(H)$, and let $a \in A(\Gamma)$ with $h \subseteq a$. Then $d(a, z) \leq \frac{1}{2}|E(H)|$, and so for every $x \in A(\Gamma)$ with $x \cap U(P) \neq \emptyset$,

$$d(z, x) \leq d(x, a) + d(a, z) \leq |E(P)| + \frac{1}{2}|E(H)| \leq 3\rho + 2$$

and so $x \subseteq \Lambda$ by the first property of C above. Hence $U(P) \subseteq \Lambda$. This proves (1).

From (1), it follows that if $k \leq \frac{1}{2}\rho$ and $r_1, \ldots, r_k$ is a sequence of regions of Γ such that $\bar{r}_1 \cap U(H) \neq \emptyset$ and $\bar{r}_i \cap \bar{r}_{i+1} \neq \emptyset$ for $1 \leq i < k$ then $\bar{r}_k \subseteq \Lambda$. Consequently the same statement holds for $sk(\Gamma)$. Since $\rho \geq 2$, and $U(H)$ is non-empty and arc-wise connected, it follows from theorem 5.2 of [8] that there is a circuit C_1 of $sk(\Gamma)$ with $U(C_1) \subseteq \Lambda$, bounding an open disc in Λ including $U(H)$, such that every edge of C_1 is incident with a region r of $sk(\Gamma)$ satisfying $\bar{r} \cap U(H) \neq \emptyset$. By (1) and $\frac{1}{2}\rho - 1$ further applications of theorem 5.2 of [8], there are circuits $C_1, \ldots, C_{\frac{1}{2}\rho}$ of $sk(\Gamma)$, such that for $2 \leq i \leq \frac{1}{2}\rho, U(C_i)$ bounds an open disc in Λ including $U(C_{i-1})$, and every edge of C_i is incident with a region r of $sk(\Gamma)$ satisfying $\bar{r} \cap U(C_{i-1}) \neq \emptyset$. We deduce

(2) *For each $v \in V(C_{\frac{1}{2}\rho})$ there is a path of Γ^* from $U(H)$ to v with $\leq \rho$ edges, and every such path has $\geq \rho - 1$ edges.*

Subproof. Let $v \in V(C_{\frac{1}{2}\rho})$. We have seen, from the construction of $C_1, \ldots, C_{\frac{1}{2}\rho}$, that there is a sequence $r_1, \ldots, r_{\frac{1}{2}\rho}$ of regions of $sk(\Gamma)$ such that $\bar{r}_1 \cap U(H) \neq \emptyset, v \in \bar{r}_{\frac{1}{2}\rho}$, and $\bar{r}_i \cap \bar{r}_{i+1} \neq \emptyset$ for $1 \leq i < \frac{1}{2}\rho$. But for any two vertices of Γ^*, if there is a region of $sk(\Gamma)$ whose closure contains them both, then there is also a region of Γ whose closure contains them both. Hence we may assume that $r_1, \ldots, r_{\frac{1}{2}\rho}$ are regions of Γ; and so there is a path of Γ^* from $U(H)$ to v with $\leq \rho$ edges. Any such path P meets $V(C_1), \ldots, V(C_{\frac{1}{2}\rho})$, and $|V(P) \cap V(\Gamma)| \leq \frac{1}{2}(|E(P)| + 1)$, and so $\frac{1}{2}(|E(P)| + 1) \geq \frac{1}{2}\rho$, that is, $|E(P)| \geq \rho - 1$. This proves (2).

Let F be a circuit of Γ^* with $U(F) \subseteq \bar{\Lambda}$, bounding a disc $D(F)$ in $\bar{\Lambda}$ including $U(C_{\frac{1}{2}\rho})$, such that $U(F) \cap V(\Gamma) \subseteq V(C_{\frac{1}{2}\rho})$; and subject to that with $D(F)$ maximal. From the maximality of $D(F)$ it follows that $D(F)$ contains every vertex of Γ^* with two neighbours in $V(F) \cap V(C_{\frac{1}{2}\rho})$, and $D(F)$ includes every edge of Γ^* with both ends in $V(F)$. It follows that $D(F)$ is a wheel and H is a $D(F)$-hub of radius ρ. This proves 12.1. ■

We recall that $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ is (m, n, ω_0)-flawed internally. Let $Z = \{e \in E(\Gamma) : \phi(e) \geq \omega_0\}$. Since χ is disjoint, every member of Z is internal. We denote the set of all poles of Γ^* by P^*.

12.2 *There exists $X_1 \subseteq V(\Gamma^*)$ with $|X_1| \leq m + c(\Sigma_\chi)$ and $P^* \subseteq X_1$, such that for every $e \in Z$ there exists $x \in X_1$ with $d(e, x) \leq n$.*

Proof. Choose $e_1, \ldots, e_k \in Z$ with k maximum such that $d(e_i, e_j) > n$ $(1 \leq i < j \leq k)$ and $d(e_i, r^*) > n$ $(1 \leq i \leq k)$ for every pole r^*. Since $(\Gamma, \phi, \lambda, \mathcal{T})$ is (m, n, ω_0)-flawed internally, it follows that $k < m$. For $1 \leq i \leq k$ let $x_i \in \tilde{e}_i$, and let $X_1 = P^* \cup \{x_1, \ldots, x_k\}$. The result follows. ■

12.3 *There exists $X_2 \subseteq V(\Gamma^*)$ with $|X_2| \leq m + c(\Sigma_\chi)$ and $P^* \subseteq X_2$ and an integer t with $n \leq t \leq n \cdot 5^{2m}$, such that*

- *$d(x_1, x_2) > 24t$ for all distinct $x_1, x_2 \in X_2$*
- *for every $e \in Z$ there exists $x \in X_2$ with $d(e, x) \leq t$.*

Proof. Let $k = m + c(\Sigma_\chi)$, and choose X_1 as in 12.2. Choose $X_2 \subseteq X_1$ with $P^* \subseteq X_2$, minimal such that for every $e \in Z$ there exists $x \in X_2$ with

$$d(e, x) \leq n \cdot 5^{2k-2|X_2|}.$$

Let $t = n \cdot 5^{2k-2|X_2|}$. Then $n \leq t \leq n \cdot 5^{2m}$, since $|X_2| \geq c(\Sigma_\chi)$, and so the second assertion of the theorem holds; we must verify the first.

Suppose that $x_1, x_2 \in X_2$ are distinct and $d(x_1, x_2) \leq 24t$. Since $24t < 5^{2m+2}n$ and $\mathcal{T}$ is not $n \cdot 5^{2m+2}$-flawed in distance, it follows that not both x_1, x_2 are poles; thus we may assume that $x_2 \notin P^*$. Let $X_2' = X_2 \setminus \{x_2\}$. For any $e \in Z$ we claim that there exists $x' \in X_2'$ with $d(e, x') \leq n \cdot 5^{2k-2|X_2'|} = 25t$. For certainly there exists $x \in X_2$ with $d(e, x) \leq t$, and if $x \neq x_2$ we may set $x' = x$. If $x = x_2$ we may set $x' = x_1$; for

$$d(e, x_1) \leq d(e, x_2) + d(x_1, x_2) \leq t + 24t = 25t.$$

This proves our claim. But then the minimality of X_2 is contradicted. We deduce that the first assertion holds. This proves 12.3. ■

A set $\mathcal{W}$ of wheels is a *cover* (for $(\Gamma, \phi, \lambda, \mathcal{T})$) if

- the members of $\mathcal{W}$ are mutually disjoint
- $bd(W) \subseteq \Sigma_\chi \setminus bd(\Sigma_\chi)$ for each $W \in \mathcal{W}$
- for each pole r^* there exists $W \in \mathcal{W}$ with $r^* \in W$ and hence with $r \subseteq W$
- for every $e \in Z$ there exists $W \in \mathcal{W}$ with $e \subseteq W$
- for each $W \in \mathcal{W}$, $(W \setminus bd(W)) \cap V(\Gamma) \neq \emptyset$.

12.4 *There is a cover $\mathcal{W}$ with $|\mathcal{W}| \leq m + c(\Sigma_\chi)$, each member of which has radius $\leq 2n \cdot 5^{2m}$.*

Proof. Choose X_2, t as in 12.3. For each $x \in X_2$, let C_x be some circuit of Γ^* with $|E(C_x)| \leq 2t$, with $x \in ins(C_x) \setminus U(C_x)$, chosen with $ins(C_x)$ maximal, if there is such a circuit C_x, and otherwise let C_x be the 1-vertex subgraph of Γ^* with vertex x.

Let $y(x)$ be some vertex of C_x. If there is a $\mathcal{T}$-enclave H with $y(x) \in ins(H) \setminus U(H)$ and with $|E(H)| \leq 8t + 4$, let H_x be such a $\mathcal{T}$-enclave H with $ins(H)$ maximal; and otherwise let H_x be the 1-vertex subgraph of Γ^* with vertex $y(x)$. Let W_x be a wheel such that H_x is a W_x-hub of radius $2t$. (This exists by 12.1.)

(1) *For $x \in X_2, ins(C_x) \subseteq W_x \setminus bd(W_x)$, and in particular, $W_x \setminus bd(W_x)$ contains x and all its neighbours in Γ^*.*

Subproof. Certainly $y(x) \in W_x \setminus bd(W_x)$, and $y(x) \in U(C_x)$, so it suffices for the first claim to show that $C_x \cap bd(W_x)$ is null. Suppose that $v \in V(C_x \cap bd(W_x))$. Then there is a path in Γ^* from $y(x)$ to v of length $\leq t$, and hence a path of Γ^* from $V(H_x)$ to $bd(W_x) \cap V(\Gamma)$ of length $\leq t+1$, contradicting that W_x has radius $2t$. Hence $ins(C_x) \subseteq W_x \setminus bd(W_x)$. Since $x \in ins(C_x)$ it follows that $x \in W_x \setminus bd(W_x)$. If some neighbour of x is in $bd(W_x)$, then $x \in V(C_x)$ and so $x = y(x)$, and there is a path in Γ^* from $V(H_x)$ to $bd(W_x) \cap V(\Gamma)$ of length ≤ 2, contradicting that W_x has radius $2t \geq 6$. This proves (1).

(2) *For each $x \in X_2$, if $v \in bd(W_x) \cap V(\Gamma^*)$ then $d(x, v) \leq 7t + 3$.*

Subproof. There is a path of Γ^* from v to $V(H_x)$ with $\leq 2t + 1$ edges. Then $d(v, y(x)) \leq (2t + 1) + (4t + 2)$ since $\frac{1}{2}|E(H_x)| \leq 4t + 2$; and so $d(v, x) \leq 7t + 3$ since $d(y(x), x) \leq t$. This proves (2).

(3) *For each component r of $\hat{\Sigma}_X = \Sigma_X$, $\bar{r} \subseteq W_{r^*}$ and $\bar{r} \cap bd(W_x) = \emptyset$ for every $x \in X_2$.*

Subproof. By (1), $W_{r^*} \setminus bd(W_{r^*})$ contains r^* and all its neighbours, and so $\bar{r} \subseteq W_{r^*}$. Let $x \in X_2$ and suppose that $\bar{r} \cap bd(W_x) \neq \emptyset$. (Consequently $x \neq r^*$.) Hence $bd(W_x)$ contains a neighbour v of r^*; and therefore $v \in V(\Gamma)$. Since $v \in bd(W_x)$, (2) implies that $d(x, v) \leq 7t+3$, and since $d(r^*, v) \leq 1$ it follows that $d(x, r^*) \leq 7t + 4$, a contradiction since $x \neq r^*$ and $x, r^* \in X_2$. This proves (3).

(4) *For each $e \in Z$ there exists $x \in X_2$ with $e \subseteq W_x$.*

Subproof. Since X_2 satisfies 12.3, there exists $x \in X_2$ with $d(e, x) \leq t$. Let K be a closed walk of Γ^* of length $\leq 2t$ such that x and e both meet $ins(K)$. Suppose first that $U(\Gamma^*|K) \cap ins(C_x) = \emptyset$. Since $x \in ins(C_x)$ and $x \in ins(K)$, there is a circuit C of $\Gamma^*|K$ with $x \in ins(C) \setminus U(C)$; and hence $ins(C_x) \subseteq ins(C) \setminus U(C)$, since $U(K) \cap ins(C_x) = \emptyset$. But K has length $\leq 2t$, and so $|E(C)| \leq 2t$, contrary to the choice of C_x. This proves that $U(\Gamma^*|K) \cap ins(C_x) \neq \emptyset$.

Now suppose that $U(\Gamma^*|K) \cap U(C_x) = \emptyset$. Then $U(\Gamma^*|K) \subseteq ins(C_x) \setminus U(C_x)$, and so

$$e \subseteq ins(K) \subseteq ins(C_x) \subseteq W_x$$

by (1), as required. We may therefore assume that K meets $U(C_x)$. Hence K is a subwalk of a closed walk of length $\leq 4t$ of which $y(x)$ is a vertex. Since W_x has radius $2t$, this subwalk remains within W_x; and so $U(\Gamma^*|K) \subseteq W_x$. Hence $ins(K) \subseteq W_x$, and so $e \in W_x$. This proves (4).

In fact we can prove that the wheels W_x ($x \in X_2$) are mutually disjoint, but that step can be avoided. Let us choose a $\subset \mathcal{W} \subseteq \{W_x : x \in X_2\}$ containing just the maximal wheels. We have seen, by (2), that the sets $bd(W_x)(x \in X_2)$ are mutually disjoint, and consequently that if $W_{x_1} \cap W_{x_2} \neq \emptyset$ then $W_{x_1} \subseteq W_{x_2}$ or $W_{x_2} \subseteq W_{x_1}$, and so the members of $\mathcal{W}$ are mutually disjoint; and the theorem is satisfied. ■

13 The spokes of a wheel

Let $\Gamma, \phi, \mathcal{W}$ etc. be as in 12.4 (and in particular with $rep(\mathcal{T}) \geq n \cdot 5^{2m+2}$), and let $W \in \mathcal{W}$. Our next objective is to examine the structure of the interior of W. Let W have radius ρ, and so $1 \leq \rho \leq 2n \cdot 5^{2m}$. Let H be a W-hub with radius ρ. We recall that $R^* = V(\Gamma^*) \setminus V(\Gamma)$.

13.1 *There is no path in Γ^* from $R^* \cap bd(W)$ to $V(H)$ with $\leq \rho - 1$ edges.*

Proof. Suppose that there is such a path, and let r^* be its end in $R^* \cap bd(W)$. From the fourth condition in the definition of a wheel, W contains every neighbour of r^* in Γ^*. Let e_1, e_2 be the ends of Γ^* incident with r^* with $e_1, e_2 \subseteq bd(W)$, and let their other ends be v_1, v_2 respectively. Now $v_1 \neq v_2$, and indeed $|V(\Gamma^*) \cap bd(W)| \geq 6$, because $bd(W) \cap bd(\Sigma_\chi) = \emptyset$ and $(W \setminus bd(W)) \cap V(\Gamma) \neq \emptyset$ and Γ is internally 3-connected. Since W includes every edge of Γ^* incident with r^*, it follows that there is an edge e of $sk(\Gamma)$ with ends v_1, v_2 and with $e \subseteq \bar{r} \setminus W$. If there is a region $r_1 \neq r$ of Γ in $\hat{\Sigma}$ with $e \subseteq \bar{r}_1$, then r_1^* has two neighbours in $bd(W) \cap V(\Gamma)$ and so $r_1^* \in W$; but then $bd(W) \cap V(\Gamma^*) = \{v_1, v_2, r^*, r_1^*\}$, a contradiction. Thus there is no such r_1. Choose $f \in E(\Gamma)$ with $e \subseteq f$. Then $|\tilde{f}| = 3$, because of the non-existence of r_1. Moreover, $\bar{r} \cap V(\Gamma) = \{v_1, v_2\}$, because $v_1, v_2 \notin bd(\Sigma_\chi)$ and Γ fits Φ_χ (using the fifth condition in the definition of "fit"). Hence the path of Γ^* from r^* to $V(H)$ with $\leq \rho - 1$ edges passes through one of v_1, v_2, a contradiction to the fifth condition in the definition of a wheel. Thus there is no such path, as required. ■

By a *spoke* of W we mean a path P of Γ^* from some $v \in V(\Gamma) \cap bd(W)$ to some $h \in V(H)$, such that

- there is no path of Γ^* from v to $V(H)$ with fewer than $|E(P)|$ edges, and
- there is no path P' of Γ^* from v to $V(H)$ with $|E(P)|$ edges and with $\lambda(P') < \lambda(P)$.

For each $v \in V(\Gamma) \cap bd(W)$ there is thus a unique spoke ending at v, and we denote it by S_v. Clearly no vertex of S_v except its end different from v belongs to $V(H)$. From 13.1, no vertex of S_v except v belongs to $bd(W)$. We call the end of S_v different from v the *inner* end of S_v.

13.2 *Let $v_1, v_2 \in V(\Gamma) \cap bd(\Sigma)$. If S_{v_1} meets S_{v_2} then they have the same inner end, and $S_{v_1} \cap S_{v_2}$ is a path.*

Proof. Let $u \in V(S_{v_1} \cap S_{v_2})$ be the first vertex of S_{v_2} in S_{v_1} (that is, nearest to v_1), and let P_i, Q_i be the subpaths of S_{v_i} from v_i to u and from u to the inner end of S_{v_i} respectively. Since S_{v_1} is a spoke, $E(P_1 \cup Q_2) \geq E(S_{v_1})$, that is, $|E(Q_2)| \geq |E(Q_1)|$, and similarly $|E(Q_1)| \geq |E(Q_2)|$ since S_{v_2} is a spoke. Thus $|E(Q_1)| = |E(Q_2)|$. Again, since S_{v_1} is a spoke, $\lambda(P_1 \cup Q_2) \geq \lambda(S_{v_1})$, that is, $\lambda(Q_2) \geq \lambda(Q_1)$, and similarly $\lambda(Q_1) \geq \lambda(Q_2)$. Thus $\lambda(Q_1) = \lambda(Q_2)$, and so $E(Q_1) = E(Q_2)$. Since Q_1, Q_2 both have one end u, it follows that $Q_1 = Q_2$ and hence that S_{v_1}, S_{v_2} have the same inner end. Since u is the first vertex of S_{v_2} in S_{v_1} we deduce that $V(P_1 \cap S_{v_2}) = \{u\}$, and hence $S_{v_1} \cap S_{v_2} = Q_1$. The result follows. ■

13.3 *Let S_v be a spoke with inner end h, and let F be a $\mathcal{T}$-enclave. If $U(S_v) \cap ins(F) \neq \emptyset$ then $S_v \cap ins(F)$ is a path, and if $U(S_v) \cap ins(F) \not\subseteq U(F)$ then one of $v, h \in ins(F) \setminus U(F)$.*

Proof. Let $\mathcal{T}'$ be a tangle in Γ with $\mathcal{T} \not\subseteq \mathcal{T}' \not\subseteq \mathcal{T}$ such that $(\Gamma \cap ins(F), \Gamma \cap out(F))$ is the $(\mathcal{T}, \mathcal{T}')$-distinction. Suppose that $U(S_v) \cap ins(F) \neq \emptyset$, and $S_v \cap ins(F)$ is not a path. Then there are distinct vertices a, b of S_v, both in $V(F)$, such that no internal vertex or edge of the subpath of S_v between a, b belong to $ins(F)$. Let this subpath be P, and let F_1, F_2 be the two subpaths of F between a and b. Since $P \neq F_1, F_2$ it follows (since S_v is a spoke) that $|E(P)| \leq |E(F_i)|$, and if equality holds then $\lambda(P) < \lambda(F_i)(i = 1, 2)$. But this contradicts 8.5.

Thus $S_v \cap ins(F)$ is a path. Suppose that $v, h \notin ins(F) \setminus U(F)$ and $U(S_v) \cap ins(F) \not\subseteq U(F)$. Then there are distinct vertices a, b of S_v, both in $U(F)$, such that every internal vertex and edge of the subpath P of S_v joining a, b lies in $ins(F) \setminus U(F)$. Let F_1, F_2 be the two subpaths of F between a and b. As before, $|E(P)| \leq |E(F_i)|$ and if equality holds then $\lambda(P) < \lambda(F_i)(i = 1, 2)$. Again, this contradicts 8.5. Thus one of $v, h \in ins(F) \setminus U(F)$, as required. ■

A W-hub H is said to be *optimal* if

- H has the same radius as W, that is, H has radius minimum over all W-hubs, and
- either $|V(H)| = 1$, or there is no 1-vertex W-hub with the same radius as W.

For every wheel W there is an optimal W-hub. For the remainder of this section we assume that H is optimal.

13.4 *Let $r^* \in R^* \cap bd(W)$ with neighbours v_1, v_2 in $\Gamma^* \cap bd(W)$. Let S_{v_1} meet S_{v_2}, and let $S_{v_1} \cap S_{v_2}$ have ends u, h where $h \in V(H)$. Let $\Delta \subseteq W$ be the disc bounded by*

$$(U(S_{v_1} \cup S_{v_2}) \setminus U(S_{v_1} \cap S_{v_2})) \cup \{u\} \cup \bar{f}_1 \cup \bar{f}_2$$

where f_i is the edge of $\Gamma^* \cap bd(W)$ with ends r^*, v_i $(i = 1, 2)$. Then $\Delta \cap U(H) = \emptyset$ unless $u = h$, in which case $\Delta \cap U(H) = \{h\}$; and $\Delta \cap U(S_v) \subseteq bd(\Delta)$ for every $v \in V(\Gamma) \cap bd(W)$.

Proof. Suppose that either $u \neq h$ and $\Delta \cap U(H) \neq \emptyset$, or $u = h$ and $\Delta \cap U(H) \neq \{h\}$. In either case it follows that $U(H) \subseteq \Delta$ and $V(H) \neq \{u\}$. For if $u \neq h$ then $bd(\Delta) \cap U(H) = \emptyset$ and so $U(H) \subseteq \Delta$ and clearly $V(H) \neq \{u\}$; while if $u = h$ then $|V(H)| \neq 1$, and H is a circuit with $U(H) \cap bd(\Delta) = \{h\}$, and so again $U(H) \subseteq \Delta$ and $V(H) \neq \{u\}$. Now u is a vertex of every spoke; for every spoke meets $bd(\Delta)$ since it meets $U(H)$, and hence contains u by 13.2.

Suppose that there is a $\mathcal{T}$-enclave F with $|E(F)| \leq 4\rho + 4$ and with $u \in ins(F) \setminus U(F)$. Since H is a W-hub, it follows that $U(H) \not\subseteq ins(F)$, because $H \neq F$ since $u \in ins(F) \setminus U(F)$. By 8.6 it follows that $U(H) \cap ins(F) \subseteq U(F)$, and in particular $h \notin ins(F) \setminus U(F)$. Thus by 13.3, since $u \in ins(F) \setminus U(F)$, it follows that $v_1, v_2 \in ins(F) \setminus U(F)$, and hence $bd(\Delta) \subseteq ins(F)$. Since the circuit C of Γ^* with $U(C) = bd(\Delta)$ satisfies $|E(C)| \leq 2\rho + 2$, it follows that $\Delta = ins(C)$ because W is a wheel, and $ins(C) \subseteq ins(F)$ since $U(C) \subseteq ins(F)$. Thus $\Delta \subseteq ins(F)$ and in particular, $U(H) \subseteq ins(F)$, a contradiction. Thus there is no such $\mathcal{T}$-enclave F.

Let H' be the 1-vertex graph with vertex u. We claim that H' is a W-hub with radius $\rho - |E(S_{v_1} \cap S_{v_2})|$. For if $v \in V(\Gamma) \cap bd(W)$ then $S_{v_1} \cap S_{v_2}$ is a path of S_v as we have seen, and so the subpath of S_v from v to u has $|E(S_v)| - |E(S_{v_1} \cap S_{v_2})|$ edges, and hence either $\rho - |E(S_{v_1} \cap S_{v_2})|$ or one fewer. For a similar reason, there is no shorter path in Γ^* from v to u, for we could augment it by $S_{v_1} \cap S_{v_2}$ and obtain a path from v to $V(H)$ shorter than S_v. It follows that H' is a W-hub with radius $\rho - |E(S_{v_1} \cap S_{v_2})|$. Since H is an optimal W-hub, we deduce that $E(S_{v_1} \cap S_{v_2}) = \emptyset$, and that

$|V(H)| = 1$. But then $V(H) = \{h\} = \{u\}$, contrary to our supposition. This proves the first claim of the theorem, that $\Delta \cap U(H) = \emptyset$ unless $u = h$, when $\Delta \cap U(H) = \{h\}$. The second claim follows from the first claim and 13.2. ■

Let $v \in R^* \cap bd(W)$, with neighbours v_1, v_2 in $\Gamma^* \cap bd(W)$. Let f_i be the edge of $\Gamma^* \cap bd(W)$ with ends v, v_i $(i = 1, 2)$. If S_{v_1} meets S_{v_2} we define Δ_v to be the disc in W bounded by the closure of

$$(U(S_{v_1} \cup S_{v_2}) \setminus U(S_{v_1} \cap S_{v_2})) \cup f_1 \cup f_2.$$

If S_{v_1} does not meet S_{v_2}, let their inner ends be h_1, h_2 respectively, let P be the path of H joining h_1, h_2 such that the disc in W bounded by

$$U(S_{v_1} \cup S_{v_2} \cup P) \cup \bar{f}_1 \cup \bar{f}_2$$

does not include $ins(H)$, and let Δ_v be this disc. In either case, let D_v be the circuit of Γ^* with $U(D_v) = bd(\Delta_v)$.

13.5 *With notation as above, $|E(D_v)| \leq 4\rho + 4$, and $ins(D_v) = \Delta_v$.*

Proof. Suppose that $|E(D_v)| > 4\rho + 4$. Since $|E(S_{v_1})|, |E(S_{v_2})| \leq \rho$, it follows that $S_{v_1} \cap S_{v_2}$ is null. Let P, Q, R be three paths of Γ^* from h_1 to h_2, where $P \cup Q = D_v$ and $Q \cup R = H$. Now $|E(P)| \leq 2\rho + 2$, and so $|E(Q)| > |E(P)|$. Since $H = Q \cup R$ is a $\mathcal{T}$-enclave, the second assertion of 8.5 implies that $ins(Q \cup R) \cap ins(P \cup R) = U(R)$, which is false. Thus $|E(D_v)| \leq 4\rho + 4$. The second claim of the theorem follows since W is a wheel of radius ρ. ■

13.6 *With notation as above, let $\mathcal{T}'$ be a tangle with $(\Gamma \cap out(D_v), \Gamma \cap ins(D_v)) \in \mathcal{T}'$, and let F be a $\mathcal{T}$-enclave around $\mathcal{T}'$. Then $ins(F) \subseteq \Delta_v$.*

Proof. The proof requires several steps. Let the inner end of S_{v_i} be h_i $(i = 1, 2)$.

(1) $|E(F)| \leq |E(D_v)| \leq 4\rho + 4$.

Subproof. Since $(\Gamma \cap ins(D_v), \Gamma \cap out(D_v)) \in \mathcal{T} \setminus \mathcal{T}'$, it follows that $|E(F)| \leq |E(D_v)|$, and by 13.5, $|E(D_v)| \leq 4\rho + 4$.

(2) *There is an edge of* Γ *included in* $ins(F) \cap ins(D_v)$.

Subproof. $(\Gamma \cap out(F), \Gamma \cap ins(F)), (\Gamma \cap out(D_v), \Gamma \cap ins(D_v)) \in \mathcal{T}'$, and so not every edge belongs to $out(F) \cup out(D_v)$ by theorem 2.3 of [5]. This proves (2).

(3) $ins(F) \cap ins(H) = U(F) \cap U(H)$.

Subproof. Otherwise either $|V(H)| = 1$ and $V(H) \subseteq ins(F) \setminus F$, or H is a circuit and $ins(H) \subseteq ins(F)$, or H is a circuit and $ins(F) \subseteq ins(H)$, by 8.6. The first and second contradict that H is a W-hub (since $H \neq F$ by (2)), and the third contradicts (2). This proves (3).

(4) *There is a closed disc* $\Delta_0 \subseteq \hat{\Sigma}$ *with* $W \cup ins(F) \subseteq \Delta_0$*, such that* Δ_0 *includes* $ins(C)$ *for every circuit* C *of* Γ^* *with* $|E(C)| \leq 4\rho + 4$ *and* $U(C) \subseteq \Delta_0$*.*

Subproof. If $ins(F) \subseteq W$ we may take $\Delta_0 = W$. Otherwise, since $U(H) \not\subseteq ins(F) \setminus U(F)$, it follows that $V(F) \cap bd(W) \neq \emptyset$. Choose $z^* \in V(F) \cap bd(W)$, and let $z \in A(\Gamma)$ with $z^* \in z$. By 7.1 (with $\kappa = 4\rho + 6$), since $rep(\mathcal{T}) \geq 6\rho + 14$, there is a circuit of $sk(\Gamma)$ bounding an open disc $\Lambda \subseteq \hat{\Sigma}_X$ with $z \subseteq \Lambda$, such that

- $x \subseteq \Lambda$ for every $x \in A(\Gamma)$ with $d(z, x) \leq 4\rho + 4$, where d is the metric of $\mathcal{T}$, and
- $ins(C^*) \subseteq \bar{\Lambda}$ for every circuit C^* of Γ^* with $U(C^*) \subseteq \bar{\Lambda}$ and $|E(C^*)| < 4\rho + 4$.

Now if T is a path of Γ^* with $\leq 4\rho + 4$ edges and with one end z^*, then $d(z, x) \leq 4\rho + 4$ for every $x \in A(\Gamma)$ with $x \cap U(T) \neq \emptyset$, and so every such x is a subset of Λ; and hence $U(T) \subseteq \Lambda$. But every edge of H or of $\Gamma^* \cap bd(W)$ or of any spoke of W belongs to some such path T, and hence $U(H) \subseteq \Lambda, bd(W) \subseteq \Lambda$ and $U(S) \subseteq \Lambda$ for every spoke S of W. Let $\Delta_0 = \bar{\Lambda}$. Since H and each $D_{v'}$ have at most $4\rho + 4$ edges, it follows that $ins(H), ins(D_{v'}) \subseteq \Delta_0$ for each v', and so $W \subseteq \Delta_0$. This proves (4).

Since $ins(F), \Delta_v$ are both discs in Δ_0 and their interiors intersect by (2), it follows from (4) that there is a circuit C of Γ^* with $C \subseteq F \cup D_v$, such that $U(C)$ bounds a closed disc in Δ_0 including $ins(F) \cup \Delta_v$.

(5) $|E(C)| \leq |E(D_v)|$*, and if equality holds then* $\lambda(C) \leq \lambda(D_v)$*.*

Subproof. Let $(A, B) = (\Gamma \cap out(F), \Gamma \cap ins(F))$, and $(A', B') = (\Gamma \cap out(D_v), \Gamma \cap ins(D_v))$. Since $(A, B), (A', B') \in \mathcal{T}'$ it follows from the second tangle axiom that $(B \cap B', A \cup A') \notin \mathcal{T}'$. But the latter has order at most the sum of the orders of (A, B) and (A', B'), and hence at most $4\rho + 4$ by (1), and so $(B \cap B', A \cup A') \in \mathcal{T}$ since $(B, A) \in \mathcal{T}$. Thus $(B \cap B', A \cup A') \in \mathcal{T} \setminus \mathcal{T}'$, and it follows that $(A \cup A', B \cap B')$ does not have smaller λ-order than (A, B). By 8.3, $(A \cap A', B \cup B')$ has λ-order at most that of (A', B'). In particular, since every edge of C is split by $(A \cap A', B \cup B')$ and only edges of D_v are split by (A', B'),

$$\frac{1}{2}|E(C)| \leq |V((A \cap A') \cap (B \cup B'))| \leq |V(A' \cap B')| = \frac{1}{2}|E(D_v)|.$$

If equality occurs, then since $(A \cap A', B \cup B')$ has λ-order at most that of (A', B'), it follows that $\lambda(C) \leq \lambda(D_v)$. This proves (5).

Let Δ be the closed disc in Δ_0 bounded by $U(C)$.

(6) *If* $ins(H) \cap \Delta \subseteq U(C)$ *then the theorem holds, that is,* $ins(F) \subseteq \Delta_v$*.*

Subproof. Suppose that $ins(H) \cap \Delta \subseteq U(C)$. If $h_1 \neq h_2$, let $K = D_v \cap H$, and if $h_1 = h_2$ let K be the null graph. Then $K \subseteq C$, because

$$U(K) \subseteq ins(H) \cap \Delta_v \subseteq ins(H) \cap \Delta \subseteq U(C).$$

If $h_1 = h_2$, note that $U(H)$ may not meet $U(\Delta_v)$, and indeed may not meet $U(C)$; let L be a minimal subpath of $S_{v_1} \cap S_{v_2}$ from $V(C)$ to $h_1 = h_2$ (this exists, because $h_1 \notin \Delta \setminus U(C)$ and $\Delta_v \subseteq \Delta$). If $h_1 \neq h_2$, let L be the null graph.

Since $V(K \cup L) \subseteq W \setminus bd(W)$ and some vertex of $K \cup L$ belongs to C, it follows that some vertex of C is in $W \setminus bd(W)$. If $U(C) \not\subseteq W$ let M be a maximal path of C with at least one edge, with no edge in W; and if $U(C) \subseteq W$ let M be the path formed by the two edges f_1, f_2. In either case $|E(M)| \geq 2$, since W is a wheel; and we may number the ends of M as v'_1, v'_2 such that there are two paths P_1, P_2 of C, with P_1, P_2, M, K mutually edge-disjoint and $P_1 \cup P_2 \cup M \cup K = C$, such that $P_i \cup L$ is a path of Γ^* from v'_i to h_i ($i = 1, 2$).

We claim that $|E(P_i \cup L)| \geq |E(S_{v_i})|$ ($i = 1, 2$). For certainly $|E(P_i \cup L)| \geq \rho - 1$ and $|E(S_{v_i})| \leq \rho$, and if equality holds in both then since Γ^* is bipartite it follows that $v'_i \in R^*$, contrary to 13.1. Thus $|E(P_i \cup L)| \geq |E(S_{v_i})|$ ($i = 1, 2$). From (5),

$$\begin{aligned}
|E(D_v)| \geq |E(C)| &= |E(P_1)| + |E(P_2)| + |E(K)| + |E(M)| \\
&\geq |E(S_{v_1})| + |E(S_{v_2})| - 2|E(L)| + |E(K)| + |E(M)| \\
&= |E(D_v)| - 2 + 2|E(S_{v_1} \cap S_{v_2})| - 2|E(L)| + |E(M)|.
\end{aligned}$$

Thus $2|E(S_{v_1} \cap S_{v_2})| + |E(M)| \leq 2|E(L)| + 2$. Since $L \subseteq S_{v_1} \cap S_{v_2}$ and $|E(M)| \geq 2$, we have equality throughout. In particular, $|E(D_v)| = |E(C)|$, $L = S_{v_1} \cap S_{v_2}$, $|E(M)| = 2$, and $|E(P_i \cup L)| = |E(S_{v_i})|$ ($i = 1, 2$). Hence $v'_1, v'_2 \in V(\Gamma)$, since Γ^* is bipartite. Let v' be the middle vertex of M. Since v' has two neighbours v'_1, v'_2 both in $V(\Gamma) \cap bd(W)$, it follows that $v' \in W$ since W is a wheel. Consequently both edges of M belong to W, and so, from the choice of M, $U(C) \subseteq W, v' = v$, $\{v'_1, v'_2\} = \{v_1, v_2\}$, and $E(M) = \{f_1, f_2\}$. Since $\Delta \subseteq \Delta_0$ and $bd(\Delta) \subseteq W$ it follows that $\Delta \subseteq W$. Since S_{v_1}, S_{v_2} are spokes and $|E(P_i \cup L)| = |E(S_{v_i})|$, it follows that $\lambda(P_i) + \lambda(L) = \lambda(S_{v_i})$ ($i = 1, 2$). From (5),

$$\begin{aligned}
\lambda(D_v) \geq \lambda(C) &= \lambda(P_1) + \lambda(P_2) + \lambda(K) + \lambda(M) \\
&= \lambda(S_{v_1}) + \lambda(S_{v_2}) - 2\lambda(L) + \lambda(K) + \lambda(M) \\
&\geq \lambda(S_{v_1}) + \lambda(S_{v_2}) - 2\lambda(S_{v_1} \cap S_{v_2}) + \lambda(K) + \lambda(M) = \lambda(D_v)
\end{aligned}$$

and so again we have equality throughout. In particular, $\lambda(P_i \cup L) = \lambda(S_{v_i})(i = 1, 2)$. Since $E(S_{v_i}) \neq \emptyset$, it follows that $P_i \cup L = S_{v_i}$ ($i = 1, 2$), and so

$$E(D_v) = (E(S_{v_1} \cup S_{v_2}) \setminus E(S_{v_1} \cap S_{v_2})) \cup E(K) \cup E(M) \subseteq E(C).$$

Hence $C = D_v$, and so $ins(F) \subseteq D_v$, as required. This proves (6).

Henceforth, then, we suppose, for a contradiction, that $ins(H) \cap \Delta \not\subseteq U(C)$. In particular, $F \neq C$, by (3). Let $h \in (\Delta \setminus U(C)) \cap ins(H)$, choosing $h \notin U(H)$ if H is a circuit. (h is a point of the surface, but not necessarily a vertex of Γ^*.)

If P is a path and $u, v \in V(P)$, we denote by $P[u, v]$ the subpath of P with ends u, v. An *arc* is a path of F with distinct ends both in $V(D_v)$, and with no internal vertex in $V(D_v)$ and no edge in $E(D_v)$. For every arc P, either $U(P) \subseteq \Delta_v$ or $U(P) \cap \Delta_v$ consists of the ends of P; we call these *inner* and *outer* arcs respectively. Since $F \neq C$ and hence $\Delta_v \not\subseteq ins(F)$, there is an inner arc; and since $D_v \neq C$ (because $h \in \Delta \setminus U(C)$) there is an outer arc.

(7) $h \notin ins(F)$.

Subproof. Suppose that $h \in ins(F)$. By (3), $h \in U(F) \cap U(H)$, and so $V(H) = \{h\}$ and $h \in V(F)$, and $h_1 = h_2 = h$. We claim that every arc has one end v. For let P be an arc with ends a, b say. Since $a \neq b$, we may assume that $a \neq v_1$ and $a \in V(S_{v_1})$ without loss of generality. Since $a, h \in V(F)$, it follows from 13.3 that $S_{v_1}[a, h] \subseteq F$. Since there are ≥ 2 arcs, a and b do not belong to the same component of $F \cap D_v$, and hence nor do h, b. By 13.3, it follows that $b \notin V(S_{v_1}) \cup V(S_{v_2})$, and so $b = v$. This proves our claim that every arc has one end v. Hence there is exactly one outer arc, say P_1, and exactly one inner arc, say P_2.

Let P_i have ends $v, a_i (i = 1, 2)$. Since $h \in V(F)$, if $E(S_{v_1} \cap S_{v_2}) \neq \emptyset$ then $a_1 = h$, and if $E(S_{v_1} \cap S_{v_2}) = \emptyset$ then both $V(S_{v_1})$ and $V(S_{v_2})$ meet $\{a_1, a_2\}$. Thus in either case we may assume that $a_1 \in V(S_{v_1})$ and $a_2 \in V(S_{v_2})$.

Let S_i be the path consisting of S_{v_i} and the vertex v and the edge f_i $(i = 1, 2)$. Let v' be the first vertex of P_1 (that is, closest to v) different from v that belongs to W. Since W is a wheel and the edge of P_1 incident with v is not in W, it follows that $|E(P_1[v, v'])| \geq 2$, and since G^* is bipartite, either $|E(P_1[v, v'])| \geq 3$ or $v' \in R^*$. But

$$P_1[v', a_1] \cup S_1[a_1, h]$$

is a path from a vertex of $bd(W)$ (namely, v') to h, and so it has $\geq \rho - 1$ edges, and at least ρ if $v' \in R^*$, by 13.1. By combining these two paths we deduce that $P_1 \cup S_1[a_1, h]$ has $\geq \rho + 2$ edges, and consequently has strictly more edges than both S_1 and S_2.

Let $S_{v_1} \cap S_{v_2}$ have ends h, h'. Let

$$C_1 = P_2 \cup S_1[v, h'] \cup S_2[h', a_2]$$
$$C_2 = P_2 \cup S_2[v, a_2].$$

Then C_1, C_2 are both circuits, bounding the two discs into which Δ_v is divided by P_2. Since $|E(S_1)| < |E(P_1 \cup S_1[a_1, h])|$ and since $P_1 \cup S_1[a_1, h], P_2 \cup S_2[h', a_2]$ are edge-disjoint paths of F, it follows that $|E(C_1)| < |E(F)|$. Since $|E(S_2)| < |E(P_1 \cup S_1[a_1, h])|$ and $P_1 \cup S_1[a_1, h], P_2$ are edge-disjoint paths of F, it follows that $|E(C_2)| < |E(F)|$. Since

$$(\Gamma|out(D_v), \Gamma|ins(D_v)) \in \mathcal{T}'$$

and $ins(C_1) \cup ins(C_2) = ins(D_v)$, one of

$$(\Gamma|out(C_1), \Gamma|ins(C_1))$$
$$(\Gamma|out(C_2), \Gamma|ins(C_2))$$

belongs to $\mathcal{T}'$, contradicting that $(\Gamma|ins(F), \Gamma|out(F))$ is the $(\mathcal{T}, \mathcal{T}')$- distinction. This proves (7).

(8) *Let P be an outer arc with ends a, b. Then $a, b \in V(S_{v_1}) \cup V(S_{v_2}) \cup \{v\}$, and $V(S_{v_1}), V(S_{v_2})$ each contain at most one of a, b.*

Subproof. If say $a \notin V(S_{v_1}) \cup V(S_{v_2}) \cup \{v\}$ then a is an internal vertex of the path $D_v \cap H$

(and in particular, H is a circuit and $h_1 \neq h_2$). Thus the edge of P incident with a is included in $ins(H) \setminus H$ (since it is not in Δ_v) contrary to (3). Thus, $a, b \in V(S_{v_1}) \cup V(S_{v_2}) \cup \{v\}$. If say $a, b \in V(S_{v_1})$ then by 13.3, $S_{v_1}[a, b] \subseteq F$, and so F has a proper subgraph which is a circuit, a contradiction. This proves (8).

Define $\Delta'_v = \Delta_v$ if $h \notin \Delta_v$, and $\Delta'_v = \Delta_v \setminus \{h\}$ if $h \in \Delta_v$. If $h \in \Delta_v$ then $V(H) = \{h\}$ and $h \in V(D_v)$, and so in both cases Δ'_v is simply-connected. Since $h \in \Delta \setminus U(C), U(C)$ is non-null-homotopic in $\Delta_0 \setminus \{h\}$ (as a closed curve), and hence is not homotopic in $\Delta_0 \setminus \{h\}$ to any closed curve in Δ'_v. But C is the union of some outer arcs and some paths of D_v, and hence there is at least one outer arc that is not homotopic (as a curve with fixed endpoints) in $\Delta_0 \setminus \{h\}$ to any curve in Δ'_v with the same end-points. But by (7), $U(F)$ is null-homotopic in $\Delta_0 \setminus \{h\}$, and so there cannot be exactly one such outer arc, since all outer arcs belong to F. Hence there are at least two outer arcs with this property. Let F_0, F_1 be distinct outer arcs, with ends a_i, b_i ($i = 0, 1$) respectively, such that for $i = 0, 1$, F_i is not homotopic (as a curve with fixed end-points) in $\Delta_0 \setminus \{h\}$ to any curve in Δ'_v with end-points a_i, b_i.

(9) $\{a_0, b_0, a_1, b_1\} \not\subseteq V(S_{v_1}) \cup V(S_{v_2})$.

Subproof. Otherwise, by (8), we may assume that $a_0, a_1 \in V(S_{v_1}) \setminus V(S_{v_2})$ and $b_0, b_1 \in V(S_{v_2}) \setminus V(S_{v_1})$. By 13.3 $F = F_1 \cup F_2 \cup S_{v_1}[a_0, a_1] \cup S_{v_2}[b_0, b_1]$, contradicting that there is an inner arc. This proves (9).

(10) *$F \cap S_{v_1} \cap S_{v_2}$ is null.*

Subproof. We may assume that $a_0, b_0, a_1, b_1 \notin V(S_{v_1} \cap S_{v_2})$. Also we may assume that $b_0, b_1 \neq v$ and hence $b_0, b_1 \in V(S_{v_1} \cup S_{v_2}) \setminus V(S_{v_1} \cap S_{v_2})$, by (8). If b_0, b_1 both lie in $V(S_{v_1})$ then $F \cap S_{v_1}$ is the subpath of S_{v_1} between b_0, b_1 by 13.3 and so $F \cap S_{v_1} \cap S_{v_2}$ is null. Thus we may assume without loss of generality that $b_1 \in V(S_{v_1}) \setminus V(S_{v_2})$ and $b_0 \in V(S_{v_2}) \setminus V(S_{v_1})$. Suppose that $u \in V(F \cap S_{v_1} \cap S_{v_2})$. Then by 13.3, F includes the path of S_{v_1} between b_1 and u, and the path of S_{v_2} between b_0 and u, and so since there is an inner arc, it is not the case that $a_0 = a_1 = v$. Without loss of generality we assume that $a_0 \neq v$. By (8), $a_0 \in V(S_{v_1}) \setminus V(S_{v_2})$, and by 13.3 $F \cap S_{v_1}$ is the path of S_{v_1} between a_0 and b_1, contrary to $u \in F \cap S_{v_1} \cap S_{v_2}$. This proves (10).

From (8), (9) and (10) we may assume that $a_1 = v$ and $b_1 \in V(S_{v_1}) \setminus V(S_{v_2})$, and $b_0 \neq v$. If $a_0, b_0 \notin V(S_{v_2}) \setminus V(S_{v_1})$, then by (8) $a_0 = v$ and $b_0 \in V(S_{v_1}) \setminus V(S_{v_2})$, and by 13.3 F consists of the union of F_0, F_1 and the path of S_{v_1} between b_0 and b_1, contradicting that there is an inner arc. Thus we may assume without loss of generality that $b_0 \in V(S_{v_2}) \setminus V(S_{v_1})$.

Let P_1 be the path from v to b_1 consisting of f_1 and the path of S_{v_1} from v_1 to b_1.

(11) $C = P_1 \cup F_1$.

Subproof. Let Δ' be the disc in Δ_0 bounded by $P_1 \cup F_1$. Since $\Delta' \not\subseteq \Delta_0 \setminus \{h\}$ from the choice of F_1, it follows that $h \in \Delta'$, and so $ins(H) \cap (\Delta' \setminus bd(\Delta')) \neq \emptyset$. Since no point of $bd(\Delta')$ lies in $ins(H) \setminus U(H)$, we deduce that $ins(H) \subseteq \Delta'$. We claim that $\Delta_v \cap (\Delta' \setminus bd(\Delta')) \neq \emptyset$; for if $S_{v_1} \cap S_{v_2}$ is non-null then $U(S_{v_1} \cap S_{v_2})$ is a subset of $\Delta' \setminus bd(\Delta')$ meeting Δ_v (because it is connected and meets

$U(H)$ and not $bd(\Delta')$ by (10)), and if $S_{v_1} \cap S_{v_2}$ is null then any point of $U(D_v \cap H) \setminus \{h_1, h_2\}$ belongs to $\Delta_v \cap (\Delta' \setminus bd(\Delta'))$ (for it belongs to Δ_v and to $ins(H)$ and not to $bd(\Delta')$). This proves our claim that $\Delta_v \cap (\Delta' \setminus bd(\Delta')) \neq \emptyset$. Since $bd(\Delta') \cap (\Delta_v \setminus bd(\Delta_v)) = \emptyset$, it follows that $\Delta_v \subseteq \Delta'$. We claim that $ins(F) \subseteq \Delta'$. For otherwise, there is a path F' of F with both ends in $V(P_1)$ and with no edge or internal vertex in Δ'. Let the ends of F' be a', b', where a' is closer than b' to a_1 on the path P_1. Then $b' \in V(S_{v_1})$, and so by 13.3 F includes $S_{v_1}[b', b_1]$, and so $a' \neq a_1$ (because otherwise F has a proper subgraph which is a circuit). Hence $a' \in V(S_{v_1})$, and by 13.3 F includes the path of S_{v_1} from a' to b', again a contradiction. This proves that $ins(F) \subseteq \Delta'$, and hence $C = P_1 \cup F_1$. This proves (11).

(12) $a_0 \in V(S_{v_1}) \setminus V(S_{v_2})$.

Subproof. If $a_0 = v$ then $F_0 \subseteq C$, from (11) and the symmetry between F_0 and F_1. But then $F_0 \subseteq P_1 \cup F_1$ from (11), a contradiction. Thus $a_0 \neq v$, and the claim follows from (8). This proves (12).

Let F_2 be a minimal path of F from v to $V(S_{v_2})$, edge-disjoint from F_1. Let the end of F_2 in S_{v_2} be b_2. From (10), $b_2 \in V(S_{v_2}) \setminus V(S_{v_1})$. From 13.3,

$$F = F_0 \cup F_1 \cup F_2 \cup S_{v_1}[a_0, b_1] \cup S_{v_2}[b_0, b_2].$$

By 13.3, F_2 is disjoint from S_{v_1}, and so F_2 is an arc. Since there is an inner arc, and F_0, F_1, F_2 are the only arcs, F_2 must be an inner arc. Let P_2 be the path from v to b_2 consisting of f_2 and $S_{v_2}[v_1, b_2]$.

(13) $|E(P_i)| \geq |E(F)| - |E(F_i)|$ *for* $(i = 1, 2)$.

Subproof. We may assume that $|E(P_i \cup F_i)| \leq 4\rho + 4$, since $|E(F)| \leq 4\rho + 4$. Hence $ins(P_i \cup F_i)$ is the disc in Δ_0 bounded by $U(P_i \cup F_i)$. Now $\Delta_v \subseteq ins(P_1 \cup F_1) = \Delta$ by (11), and so

$$(\Gamma \cap ins(P_1 \cup F_1), \Gamma \cap out(P_1 \cup F_1)) \in \mathcal{T} \setminus \mathcal{T}',$$

and $|E(P_1 \cup F_1)| \geq |E(F)|$ from the properties of F. Moreover,

$$(\Gamma \cap ins(P_2 \cup F_2), \Gamma \cap out(P_2 \cup F_2)) \in \mathcal{T} \setminus \mathcal{T}',$$

for it does not belong to $\mathcal{T}'$ by the second tangle axiom, since

$$(\Gamma \cap out(F), \Gamma \cap ins(F)), (\Gamma \cap out(D_v), \Gamma \cap ins(D_v)) \in \mathcal{T}'$$

and

$$ins(P_2 \cup F_2) \cup out(F) \cup out(D_{v_2}) = \hat{\Sigma}_\chi.$$

Again, $|E(P_2 \cup F_2)| \geq |E(F)|$ from the properties of F. This proves (13).

(14) $|E(F_i)| \geq |E(P_i)|$ $(i = 1, 2)$.

Subproof. Let Q_i be the union of F_i with the path of S_{v_i} from b_i to h_i. Then $|E(Q_i)| \geq \rho$ by 13.1, and hence $|E(Q_i)| \geq |E(S_{v_i})|$. But $|E(Q_i)| - |E(S_{v_i})|$ is odd, since Γ^* is bipartite and v, v_i are adjacent, and so $|E(Q_i)| \geq |E(S_{v_i})| + 1$. Hence $|E(F_i)| \geq |E(P_i)|$. This proves (14).

From (13) and (14), $|E(F_i)| \geq |E(F)| - |E(F_i)| (i = 1, 2)$, and so $|E(F_1)| + |E(F_2)| \geq |E(F)|$. But $|E(F_0)| + |E(F_1)| + |E(F_2)| \leq |E(F)|$, and so $E(F_0) = \emptyset$, a contradiction. This completes the proof of 13.6. ■

We shall also need the following lemma.

13.7 *With notation as in 13.6, let S_{v_1}, S_{v_2} have the same inner end. Let the vertices of S_{v_i} in $V(\Gamma)$ be $v_i^1, \ldots, v_i^k$ in order, where $v_i^1 = v_i (i = 1, 2)$. Then there are k paths $P_1, \ldots, P_k$ of $sk(\Gamma)$, mutually vertex-disjoint, such that P_j has ends v_1^j and v_2^j $(1 \leq j \leq k)$, and such that for $1 \leq j \leq k$, either*

- $v_1^j = v_2^j \in V(S_{v_1} \cap S_{v_2})$ *and* $V(P_j) = \{v_1^j\}$, *or*
- $v_1^j, v_2^j \notin V(S_{v_1} \cap S_{v_2})$ *and* $U(P_j) \subseteq \Delta_v$.

Proof. Choose t maximum such that $v_1^1, \ldots, v_1^t \in \Delta_v$ (and hence $v_2^1, \ldots, v_2^t \in \Delta_v$). Let the path $S_{v_1} \cap S_{v_2}$ have ends $u \in \Delta_v$ and $h \in V(H)$. It suffices to show that there are t mutually disjoint paths of $sk(\Gamma \cap \Delta_v)$ between $\{v_1^1, \ldots, v_1^t\}$ and $\{v_2^1, \ldots, v_2^t\}$. Suppose not; then by a form of Menger's theorem for planar graphs (see [2], for example), there exists a path Q in Γ^* with $U(Q) \subseteq \Delta_v$ from v to u, such that $|V(Q) \cap V(\Gamma)| < t$. But then $|E(Q)| \leq 2t - 1$, with equality only if $u \in V(\Gamma)$. Let P_1 be the path of S_{v_1} between v_1 and u. Then $|E(P_1)| \geq 2t - 1$, with equality only if $u \notin V(\Gamma)$; and so $|E(P_1)| \geq |E(Q)| + 1$. Since $|E(S_{v_1})| \leq \rho$, it follows that $|E(Q \cup (S_{v_1} \cap S_{v_2}))| \leq \rho - 1$, contrary to 13.1. ■

14 Removing wheel covers

The object of this section is to prove the following.

14.1 *Let $\chi, \mathcal{S}, \mathcal{D}$ be as in section 10, and let χ satisfy* $\mathbf{S_2}$*; let $\omega_0 \in E(\Omega_\chi(2)) \cup E(\Omega_\chi(3))$, and let $m \geq 1, n \geq 4$. Then there is a well-behaved set $\mathcal{C}_5(m, n, q_0)$ of χ-places with the following property. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ be (m, n, ω_0)-flawed internally. Then there is a rooted location $\mathcal{L}$ which $(4n \cdot 5^{2m} + 3)$-isolates $\mathcal{T}$, and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}_5(m, n, \omega_0)$.*

Proof. Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$ be (m, n, ω_0)-flawed internally. By 12.4 there is a cover $\mathcal{W}$ with $|\mathcal{W}| \leq m + c(\Sigma_\chi)$, each member of which has radius $\leq 2n \cdot 5^{2m}$. For each $W \in \mathcal{W}$, let $H(W)$ be an optimal W-hub. If $|V(H(W))| = 1$, designate some vertex of $R^* \cap bd(W)$ as *singular* , and the other vertices of $R^* \cap bd(W)$ as *regular* . If $|V(H(W))| > 1$, we say $r^* \in R^* \cap bd(W)$ is *singular* if S_{v_1}, S_{v_2} have distinct inner ends, where v_1, v_2 are the neighbours of r^* in $\Gamma^* \cap bd(W)$, and S_{v_1}, S_{v_2} are the corresponding spokes; and r^* is *regular* otherwise. We observe that for each $W \in \mathcal{W}$, there are most $|V(H(W))|$ singular members of $R^* \cap bd(W)$; and there is at least one (for since $H(W)$ is optimal, it follows that if $H(W)$ is a circuit then not all spokes have the same inner end.) Let $S(W)$ be the set of singular vertices in $R^* \cap bd(W)$.

For each $W \in \mathcal{W}$, let $N(W)$ be the set of all $v \in V(\Gamma) \cap bd(W)$ with a singular neighbour in $\Gamma^* \cap bd(W)$. Thus $|N(W)| \leq 2|V(H(W))|$. Let $G(W)$ be the drawing with

$$U(G(W)) = bd(W) \cup U(H(W)) \cup \bigcup(U(S_v) : v \in N(W))$$
$$V(G(W)) = S(W) \cup N(W) \cup V(H(W)) \cup \bigcup(V(S_v) : v \in N(W))$$

where again S_v is the spoke corresponding to v. We see that

$$\begin{aligned}|V(G(W))| &\leq |V(H(W))| + |S(W)| + 2|S(W)|(2n \cdot 5^{2m})\\ &\leq (8n \cdot 5^{2m} + 4)(4n \cdot 5^{2m} + 2)\end{aligned}$$

and $E(G(W))$ is similarly bounded. We call $(\bigcup(G(W) : W \in \mathcal{W}), \mathcal{W})$ a ω_0-*flaw* in $(\Gamma, \phi, \lambda, \mathcal{T})$.

By 5.2, to prove the theorem it suffices to show the following.

(1) *Let $\mathcal{D}' \subseteq \mathcal{D}$, such that each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ is (m, n, ω_0)-flawed internally and they all have the same ω_0-flaw. Then there is a well-behaved set $\mathcal{C}$ of χ-places such that for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ there is a rooted location $\mathcal{L}$ which $(4n \cdot 5^{2m} + 3)$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}$.*

Since each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ has the same ω_0-flaw, it follows that there is a $\mathcal{W}$ such that $\mathcal{W}$ is a cover for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$; and for each $W \in \mathcal{W}$ there exists $H(W)$ and $\rho(W)$ such that $H(W)$ is a W-hub of radius $\rho(W)$ for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$ (they have the same radius in each Γ because the common ω_0-flaw includes a spoke). Moreover, for each $W \in \mathcal{W}$ the sets $S(W), N(W)$ are the same for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, and the spokes $S_v(v \in N(W))$ are also the same for each $(\Gamma, \phi, \lambda, \mathcal{T})$.

Thus we may speak of $S(W), N(W), S_v(v \in N(W))$ and $\Delta_v(v \in S(W))$ without specifying a particular member of $\mathcal{D}'$. (This does not hold for general spokes, of course; they will differ for different members of $\mathcal{D}'$, as indeed will the sets $V(\Gamma) \cap bd(W)$.) For each $W \in \mathcal{W}$, fix a march $\pi(H(W))$ with $\bar{\pi}(H(W)) = V(\Gamma) \cap V(H(W))$ for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, and for each $r^* \in S(W)$ fix a march $\pi(r^*)$ with $\bar{\pi}(r^*) = V(\Gamma) \cap V(S_{v_1} \cup S_{v_2})$ for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, where v_1, v_2 are the neighbours of r^* in $\Gamma^* \cap bd(W)$ (and hence $v_1, v_2 \in N(W)$).

Let Σ' be obtained from Σ_χ by deleting $\Sigma_\chi \cap (W \setminus bd(W))$ for each $W \in \mathcal{W}$. Then Σ' is a surface since $bd(W) \cap bd(\Sigma_\chi) = \emptyset$ for each $W \in \mathcal{W}$; and $\hat{\Sigma}' \cong \hat{\Sigma}_\chi$. Let Φ' be the frame in Σ' with

$$\begin{aligned}U(\Phi') &= bd(\Sigma')\\ V(\Phi') &= \bigcup(N(W) : W \in \mathcal{W})\end{aligned}$$

where the edges of Φ' are directed arbitrarily, and an edge of Φ' is designated as short if it includes a member of $S(W)$ for some $W \in \mathcal{W}$, and long otherwise.

Let $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}'$, and let $W \in \mathcal{W}$. For $v \in V(\Gamma) \cap bd(W)$, let the spoke S_v and disc Δ_v be defined as in 13.5. If $|V(H(W))| > 1$, we define the *hub sector* (of $(\Gamma, \phi, \lambda, \mathcal{T})$ at W) to be the rooted hypergraph $(\Gamma \cap ins(H(W)), \pi(W))$. If $|V(H(W))| = 1$ then there is no hub sector. For each $r^* \in (V(\Gamma^*) \setminus V(\Gamma)) \cap bd(W)$, let r^* have neighbours v_1, v_2 in $\Gamma^* \cap bd(W)$ and let A^- be the union of $\Gamma \cap \Delta_{r^*}$ and the hypergraph with vertex set $V(S_{v_1} \cap S_{v_2}) \cap V(\Gamma)$ and edge set empty. Let A be the rooted hypergraph (A^-, π), where $\pi = \pi(r^*)$ if $r^* \in S(W)$, and otherwise π is an arbitrary march with $\bar{\pi} = V(S_{v_1} \cup S_{v_2}) \cap V(\Gamma)$. We call A the r^*- *sector* (at W). Let $\mathcal{L}(W)$ be the set containing the hub sector at W if there is one and each r^*-sector at W, for all $r^* \in (V(\Gamma^*) \setminus V(\Gamma)) \cap bd(W)$; and let $\mathcal{L} = \bigcup(\mathcal{L}(W) : W \in \mathcal{W})$.

(2) $\mathcal{L}(4n \cdot 5^{2m} + 3)$-*isolates* $\mathcal{T}$.

Subproof. We claim first, that each $\in$ of $\mathcal{L}$ has order at most $4n \cdot 5^{2m} + 2$. For each $|E(H(W))| \leq$

$4\rho(W)+4$ and so hub sectors have order $\leq 2\rho(W)+2$; for r^* singular, each r^*-sector has order

$$\frac{1}{2}|bd(\Delta_{r^*})\cap V(\Gamma)| \leq \frac{1}{2}(4\rho(W)+4),$$

by 13.5; and for r^* regular, each r^*-sector has order $|V(S_{v_1}\cup S_{v_2})\cap V(\Gamma)| \leq \rho(W)+1$ where v_1, v_2 are the neighbours of r^* in $\Gamma^* \cap bd(W)$. Since each $\rho(W) \leq 2n\cdot 5^{2m}$, this proves our claim. Moreover, $\mathcal{L}^- \subseteq \mathcal{T}$ by 13.5. Let $\mathcal{T}'$ be a tangle in Γ of order $\geq 4n\cdot 5^{2m}+23$ such that $(B,A)\in\mathcal{T}'$ for some $(A,B)\in\mathcal{L}^-$ where $(A,B)\in(\mathcal{L}(W))^-$ say; and let (C,D) be the $(\mathcal{T},\mathcal{T}')$-distinction. Let F be a $\mathcal{T}$-enclave around $\mathcal{T}'$. Then $|E(F)|\leq 4\rho(W)+4$. If (A,B) is the hub sector at W, then some edge of A is included in $ins(F)$, and so by 8.6, either $ins(H(W))\subseteq ins(F)$ or $ins(F)\subseteq ins(H(W))$. Now since $|E(F)|\leq 4\rho(W)+4$ and $H(W)$ is a W-hub, we deduce that if $ins(H(W))\subseteq ins(F)$ then $F=H(W)$. Thus if (A,B) is the hub sector then $ins(F)\subseteq ins(H(W))$ and so $C\subseteq A$ and $B\subseteq D$ as required. On the other hand, if (A,B) is an r^*-sector for some $r^*\in(V(\Gamma^*)\setminus V(\Gamma))\cap bd(W)$, then by 13.6 $ins(F)\subseteq\Delta_{r^*}$, and again $C\subseteq A$ and $B\subseteq D$. Thus $\mathcal{L}(4n\cdot 5^{2m}+3)$-isolates $\mathcal{T}$. This proves (2).

Let $\mathcal{C}$ be the set of all $(\Gamma,\phi,\mathcal{L})$ as above such that $(\Gamma,\phi,\lambda,\mathcal{T})\in\mathcal{D}'$ for some $\lambda,\mathcal{T}$. It remains to show that $\mathcal{C}$ is well-behaved. Thus, let Ω be a well-quasi-order, and let $(\Gamma_i,\phi_i,\mathcal{L}_i)(i=1,2)$ be a countable sequence of members of $\mathcal{C}$. For each $i\geq 1$, let $\xi_i:\mathcal{L}_i\to E(\Omega)$ be some function. We must show that there exist $j>i\geq 1$ and an outline $\tau:\mathcal{L}_i\to\mathcal{L}_j$ such that $\xi_i(A)\leq\xi_j(\tau(A))$ for all $A\in\mathcal{L}_i$. Let $(\Gamma_i,\phi_i,\lambda_i,\mathcal{T}_i)\in\mathcal{D}'$ for each $i\geq 1$.

For each W, if $|V(H(W))|\neq 1$, let $A_i(W)$ be the hub sector of $(\Gamma_i,\phi_i,\lambda_i,\mathcal{T}_i)$ at W. There is an infinite set $I\subseteq\{1,2,\ldots\}$ such that for all $i,j\in I$ with $j>i, \xi_i(A_i(W))\leq\xi_j(A_j(W))$ because Ω is a well-quasi-order. To simplify notation, let us replace our initial sequence by this subsequence. We may therefore assume that

(3) *For each $W\in\mathcal{W}$ with $|V(H(W))|\neq 1, \xi_1(A_1(W)\leq\xi_2(A_2(W))\leq\ldots$*

For each $r^*\in(V(\Gamma_i^*)\setminus V(\Gamma_i))\cap bd(W)$, let $A_i(r^*)$ be the sector of $(\Gamma_i,\phi_i,\lambda_i,\mathcal{T}_i)$ at r^*. We may similarly assume that

(4) *For each $W\in\mathcal{W}$ and each $r^*\in S(W),\xi_1(A_1(r^*))\leq\xi_2(A_2(r^*))\leq\ldots$*

Let R be the well-quasi-order with element set all triples (π,π_1,π_2), where π is a march in Σ_χ with $|\bar{\pi}|\leq 4n\cdot 5^{2m}+2$, and π_1,π_2 are marches with $\bar{\pi}_1,\bar{\pi}_2\subseteq\bar{\pi}$. We order R by isomorphism; that is, $(\pi,\pi_1,\pi_2)\leq(\pi',\pi_1',\pi_2')$ if there is a bijection of $\bar{\pi}$ to $\bar{\pi}'$ mapping π to π',π_1 to π_1' and π_2 to π_2'. For each $i\geq 1$, each $W\in\mathcal{W}$, and each regular $r^*\in(V(\Gamma_i^*)\setminus V(\Gamma_i))\cap bd(W)$, let e_{r^*} be the component of $bd(W)\setminus\{v_1,v_2\}$ containing r^*, where v_1,v_2 are the neighbours of r^* in $\Gamma_i^*\cap bd(W)$. Let

$$\phi_i'(e_{r^*})=(\xi(A_i(r^*)),(\pi(A_i(r^*)),\mu_i(v_1),\mu_i(v_2)),$$

where $\mu_i(v_1),\mu_i(v_2)$ are the marches given by enumerating the vertices of $V(S_{v_1})\cap V(\Gamma_i), V(S_{v_2})\cap V(\Gamma_i)$ (respectively) in order, starting with v_1 and v_2. Then $\phi_i'(e_{r^*})\in\Omega\times R$.

Let Σ' be obtained from Σ_χ by deleting $\Sigma_\chi\cap(W\setminus bd(W))$ for each $W\in\mathcal{W}$. Then Σ' is a surface since the W's are disjoint and each $bd(W)$ is disjoint from $bd(\Sigma_\chi)$. Moreover, $bd(\Sigma')\cap bd(\Sigma_\chi)=\emptyset$, and $\hat{\Sigma}\cong\hat{\Sigma}_\chi$. Let Φ' be the frame with $U(\Phi')=bd(\Sigma'), V(\Phi')=\bigcup(N(W):W\in\mathcal{W})$, where the edges of Φ' are directed arbitrarily, and an edge of Φ' is short if it contains a vertex of $S(W)$ for some $W \in \mathcal{W}$, and long otherwise. Let χ' be the colour scheme where $\Sigma_{\chi'} = \Sigma', \Phi_{\chi'} = \Phi', \Omega_{\chi'}(k)$ is an ideal of $\Omega_\chi(k)$ defined by

$$E(\Omega_{\chi'}(k)) = \{x \in E(\Omega_\chi(k)) : x \not\geq \omega_0\} \ (k = 2, 3),$$

and for each long side S of $\Phi', \Omega_{\chi'}(S) = \Omega \times R$.

(5) χ' *is not orientedly bad.*

Subproof. Since $\omega_0 \in E(\Omega_\chi(2)) \cup E(\Omega_\chi(3))$, it follows that either $\Omega_{\chi'}(2) \prec \Omega_\chi(2)$ or $\Omega_{\chi'}(3) \prec \Omega_\chi(3)$. Since χ satisfies $\mathbf{S_2}$, this proves (5).

For each $i \geq 1$, let Γ_i' be the painting in Σ' defined by

$$U(\Gamma_i') = (U(\Gamma_i) \cap \Sigma') \cup bd(\Sigma')$$
$$V(\Gamma_i') = V(\Gamma_i) \cap \Sigma'$$

and for each edge e of $\Gamma_i', \gamma_{\Gamma_i'}(e)$ equals $\gamma_{\Gamma_i}(e)$ if $e \in \Gamma_i$, and $\gamma_{\Gamma_i}(e)$ agrees with the direction of the short side of Φ' containing e otherwise. For each $e \in E(\Gamma_i')$, if $e \in E(\Gamma_i)$ let $\phi_i'(e) = \phi_i(e)$; if e is a short side let $\phi_i'(e) = e$; and otherwise $\phi_i'(e)$ has already been defined. We see that

(6) *For each* $i \geq 1, (\Gamma_i', \phi_i')$ *is a* χ'*- coloured painting, and the set* $\{(\Gamma_i', \phi_i') : i \geq 1\}$ *is similarly oriented.*

By (5), we deduce

(7) *There exist* $j > i \geq 1$ *and a linear inflation* σ' *of* (Γ_i', ϕ_i') *in* (Γ_j', ϕ_j').

For each $A \in \mathcal{L}_i$, if A is a hub sector or an r^*-sector for some singular r^*, let $\tau(A)$ be the corresponding sector of $\mathcal{L}_j$. If A is an r^*-sector of some $W \in \mathcal{W}$ for some regular r^*, let e be the border edge of Γ_i' with $r^* \in e$, and let $s^* \in (V(\Gamma_j^*) \setminus V(\Gamma_j)) \cap \sigma'(e)$. Let $\tau(A)$ be the s^*-sector of $(\Gamma_j, \phi_j, \lambda_j, \mathcal{T}_j)$ at W. We claim that $\tau : \mathcal{L}_i \to \mathcal{L}_j$ is an outline, satisfying $\xi_i(A) \leq \xi_j(\tau(A))$ for all $A \in \mathcal{L}_i$.

(8) *For every* $A \in \mathcal{L}_i, \xi_i(A) \leq \xi_j(\tau(A))$.

Subproof. If A is a hub sector or an r^*-sector for some singular r^*, this follows from (3) and (4). If A is an r^*-sector for some regular r^*, then $\tau(A)$ is an s^*-sector for some regular s^*, and $\xi_i(A) \leq \xi_j(\tau(A))$ since $\phi_i'(e) \leq \phi_j'(\sigma'(e))$, where e is the border edge of Γ_i' with $r^* \in e$. This proves (8).

Let $G' = \bigcup(\sigma'(v) : v \in V(\Gamma_i'))$. Now G' is not necessarily a subgraph of $sk(\Gamma_j)$, because some of its edges are included in $bd(\Sigma')$. Let $E' = \{e \in E(G') : e \subseteq bd(\Sigma')\}$. For each $e \in E'$, choose $v \in V(\Gamma_j^*) \setminus V(\Gamma_j)$ with $v \in e$. Then v is regular, because e is not a short side of Φ'. Let $P(e)$ be the union of the paths given by 13.6. Let G be the union of

$$(G' \cap sk(\Gamma_j)) \cup \bigcup(P(e) : e \in E')$$

with the graph with vertex set $V(M(\Gamma_j, \mathcal{L}_j))$ and no edges.

For each $v \in V(\Gamma_i')$ there is a component K of G with $V(\sigma'(v)) \subseteq V(K)$; because for each $e \in E(\sigma(v)) \setminus E(sk(\Gamma_j)) \subseteq E'$, there is a path of $P(e)$ joining the ends of e. Let $\sigma''(v) = K$. For $v \in V(M(\Gamma_i, \mathcal{L}_i)) \setminus V(\Gamma_i')$, if $v \in V(H(W)) \cap V(\Gamma_i)$ for some $W \in \mathcal{W}$, let $\sigma''(v)$ be the component of G with $v \in V(\sigma''(v))$. If v is the kth term of $\mu_i(u)$ for some $u \in V(\Gamma) \cap bd(W)$, let $\sigma''(v)$ be the component of G containing the kth term of $\mu_j(u')$, where if e is the border edge of Γ_i' with tail u then u' is the tail of $\sigma'(e)$. For each v, let $\sigma(v)$ be the induced subgraph of $sk(\Gamma)$ with vertex set $V(\sigma''(v))$. For each edge e of $M(\Gamma_i, \mathcal{L}_i)$, let $\sigma(e) = \sigma'(e)$. We claim that σ is an inflation of $M(\Gamma_i, \mathcal{L}_i)$ in Γ_j, as in the definition of outline. We omit the verification, because it is lengthy and almost identical with (but easier than) the major part of the proof of theorem 9.1 of [11].

Thus $\tau : \mathcal{L}_i \to \mathcal{L}_j$ is an outline and by (8) the proof is complete. ■

15 Conclusion

The proof will be completed by using one further result. To prove it we need the following lemma.

15.1 *Let Γ be a 2-cell drawing with $E(\Gamma) \neq \emptyset$ in a surface Σ with $bd(\Sigma) = \emptyset$, and let $\mathcal{T}$ be a tangle in Γ with $rep(\mathcal{T}) \geq \theta$. Let r be a region of Γ, and let $X \subseteq V(\Gamma) \cap \bar{r}$ with $|X| \leq \theta$. Then the following are equivalent:*

- *there exists $(A, B) \in \mathcal{T}$ of order $< |X|$ with $X \subseteq V(A)$*
- *there is a circuit C of Γ^* of length $< 2\theta$, such that $|X \cap ins(C)| > \frac{1}{2}|E(C)|$ and $r^* \in ins(C)$.*

Proof. That the second statement implies the first is easy, for if C is as in the second statement, let

$$\begin{aligned} V(A) &= V(\Gamma \cap ins(C)) \cup X \\ E(A) &= E(\Gamma \cap ins(C)) \\ B &= \Gamma \cap out(C). \end{aligned}$$

Then $(A, B) \in \mathcal{T}, (A, B)$ has order $\frac{1}{2}|E(C)| + |X \setminus ins(C)| < |X|$, and $X \subseteq V(A)$; and so the first statement holds.

For the converse, let (A, B) satisfy the first statement. Let $G \subseteq \Gamma^*$ be the subdrawing consisting of all edges of Γ^* split by (A, B), and their ends. Let $C_1, \ldots, C_k$ be all the circuits C of G with $ins(C)$ maximal. By theorem 6.3 of [6], every edge of A is a subset of $ins(C_i)$ for some i and hence so is every edge of G; and by theorem 4.3 of [6], for $1 \leq i \leq k$ every path P of G with distinct ends both in $V(C_i)$ satisfies $U(P) \subseteq ins(C_i)$. Suppose that $r^* \in ins(C_i) \setminus U(C_i)$ for some i. Then $X \subseteq ins(C_i)$, and C_i satisfies the second statement of the theorem as required. We assume therefore that there is no such i.

Now r^* may belong to some of the C_i's, say to $C_1, \ldots, C_b$. Consequently $C_1, \ldots, C_b$ pairwise meet in precisely $\{r^*\}$. For $1 \leq i \leq b$, let $X_i = X \cap ins(C_i)$, and let $X_0 = X \setminus X_1 \cup \ldots \cup X_b$. Suppose that some $x \in X_0$ is not in $V(A \cap B)$. Certainly $x \in V(A)$, and so every edge of Γ incident with x is in $E(A)$. Since Γ is 2-cell and has an edge, x is incident with at least one edge, and so there exists $e \in E(A)$ incident with x. Hence there exists i with $1 \leq i \leq k$ and $e \in ins(C_i)$, and so $x \in ins(C_i)$.

Since $x \in X_0$ it follows that $r^* \notin V(C_i)$, and since x is adjacent to r^* in Γ^* it follows that $x \in V(C_i)$. Consequently $x \in V(C_i) \cap V(\Gamma) \subseteq V(A \cap B)$, a contradiction. This proves that $X_0 \subseteq V(A \cap B)$.

Consequently,

$$\sum_{1 \le i \le b} \frac{1}{2}|E(C_i)| + |X_0| \le |V(A \cap B)| < |X|$$

and so

$$\sum_{1 \le i \le b} \frac{1}{2}|E(C_i)| < |X \setminus X_0| = \sum_{1 \le i \le b} |X_i|.$$

Hence there exists i with $1 \le i \le b$ and $\frac{1}{2}|E(C_i)| < |X_i|$, and then C_i satisfies the second statement of the theorem, as required. ■

We recall that the function $l(e_1, e_2)$ was defined in the start of section 11.

15.2 *Let χ be a colour scheme, and let (Γ_0, ϕ_0) be a χ-coloured painting. Then there exist $n \ge 0$ with the following property. Let (Γ, ϕ) be a χ-coloured painting, such that (Γ_0, ϕ_0) and (Γ, ϕ) are similarly oriented, and there is no inflation of (Γ_0, ϕ_0) in (Γ, ϕ). Let $\mathcal{T}$ be a tangle in Γ with metric d, with $rep(\mathcal{T}) > \frac{3}{2}n$. Then either*

- *$d(r_1^*, r_2^*) \le n$ for two distinct poles r_1^*, r_2^*, or*
- *there is a circuit F of Γ^* with $\frac{1}{2}|E(F)| \le n$ such that $ins(F)$ includes a long side or more than $\frac{1}{2}|E(F)|$ vertices of Φ_χ, or*
- *for some long side S, let the edges of Γ_0 bordering S be $f_1, \ldots, f_k$, in order; then there do not exist $e_1, \ldots, e_k \in E(\Gamma)$, bordering S in order, such that $\phi_0(f_i) \le \phi(e_i)$ $(1 \le i \le k)$, $l(e_i, e_j) > \frac{1}{2}n$ $(1 \le i < j \le k)$ and $l(e_i, s) > \frac{1}{2}n$ $(1 \le i \le k)$ for every short side s in the same cuff as S, or*
- *let $k = |E(\Gamma_0)|$; then for some internal edge f of Γ_0, there do not exist $e_1, \ldots, e_k \in E(\Gamma)$ such that $|\tilde{e}_i| = |\tilde{f}|$ and $\phi_0(f) \le \phi(e_i)$ $(1 \le i \le k)$, $d(e_i, e_j) > n$ $(1 \le i < j \le k)$ and $d(e_i, r^*) > n$ $(1 \le i \le k)$ for every pole r^*.*

Proof. Given χ, Γ_0, ϕ_0, choose $n \ge 4|V(\Gamma_0)| + 8$ and also satisfying another condition in terms of Γ_0 which we shall describe later. Now let $(\Gamma, \phi), \mathcal{T}$ be as in the theorem, and suppose for a contradiction that none of the four outcomes hold.

Let $f_1, \ldots, f_m$ be the internal edges of Γ_0. Choose t maximum with $t \le m$ such that there exist edges $e_1, \ldots, e_t$ of Γ with the properties that

- $|\tilde{e}_i| = |\tilde{f}_i|$ and $\phi(e_i) \ge \phi_0(f_i)$ $(1 \le i \le t)$
- $d(e_i, e_j) > \frac{1}{2}n$ $(1 \le i < j \le t)$
- $d(e_i, r^*) > n$ $(1 \le i \le t)$ for every pole r^*.

We claim that $t = m$. For suppose that $t < m$. Since the fourth outcome of the theorem does not hold, there are $|E(\Gamma_0)|$ internal edges $c_1, \ldots, c_k$ say of Γ, where $k = |E(\Gamma_0)|$, such that $|\tilde{c}_j| = |\tilde{f}_{t+1}|$ and $\phi(c_j) \ge \phi_0(f_{t+1})$ $(1 \le j \le k)$, and $d(c_i, c_j) > n$ $(1 \le i < j \le k)$, and $d(c_i, r^*) > n$ $(1 \le i \le k)$

for every pole r^*. From the maximality of t, for $1 \le j \le k$ there exist i_j with $1 \le i_j \le t$ such that $d(e_{i_j}, c_j) \le \frac{1}{2}n$. Since

$$k = |E(\Gamma_0)| \ge m > t,$$

it follows that $i_j = i_{j'}$ for some $j \neq j'$; say $i_1 = i_2$, and $e_{i_1} = e$. Then $d(e, c_1), d(e, c_2) \le \frac{1}{2}n$, and so $d(c_1, c_2) \le n$, a contradiction. This proves that $t = m$.

It follows, since the third outcome of the theorem does not hold, that

(1) *There is an injection $\beta : E(\Gamma_0) \to E(\Gamma)$ with the following properties:*

- *$\beta(s) = s$ for every short side s of Φ_χ*
- *$\beta(e)$ and e have the same size, and $\phi(\beta(e)) \ge \phi_0(e)$, for all $e \in E(\Gamma_0)$*
- *for each long side S, if the edges of Γ_0 bordering S are $f_1, \ldots, f_k$ in order, then $\beta(f_1), \ldots, \beta(f_k)$ also border S in order, and $l(\beta(f_i), \beta(f_j)) > \frac{1}{2}n$ for $1 \le i < j \le k$, and $l(\beta(f_i), s) > \frac{1}{2}n$ for $1 \le i \le k$ and every short side s bordering the same cuff as S*
- *for every internal $e \in E(\Gamma_0), d(\beta(e), r^*) > n$ for every pole r^*, and for all distinct internal $e_1, e_2 \in E(\Gamma_0)$, $d(\beta(e_1), \beta(e_2)) > \frac{1}{2}n$.*

If F is a line, we denote the set of ends of F by $bd(F)$.

(2) *There is a Φ_χ-preserving homeomorphism $\alpha : \Sigma_\chi \to \Sigma_\chi$ such that for every $e \in E(\Gamma_0)$ except short sides, $\alpha(\beta(e)) \subseteq e \setminus bd(\bar{e})$, and the orientation of $\alpha(\beta(e))$ defined by $\alpha(\gamma_\Gamma(\beta(e)))$ agrees with the orientation of e defined by $\gamma_{\Gamma_0}(e)$.*

Subproof. There is clearly an α satisfying these conditions for all border edges e; for if e_1, e_2 are distinct border edges and not short sides, then $\beta(e_1), \beta(e_2)$ have no common ends and have no end in common with any short side (by the third property of β in (1), since $n > 6$); and if they border the same long side then so do $\beta(e_1), \beta(e_2)$, and the latter are in the proper order. Now for any internal $e \in E(\Gamma_0), \beta(e)$ has no end in common with $\beta(e')$ for any $e' \in E(\Gamma_0) \setminus \{e\}$, and it is easy to arrange that $\alpha(\beta(e)) \subseteq e \setminus bd(\bar{e})$, and that if $|\tilde{e}| = 2$ the orientation given by $\alpha(\gamma_\Gamma(\beta(e)))$ and by $\gamma_{\Gamma_0}(e)$ agree. It remains to arrange this orientation condition when $|\tilde{e}| = 3$. Now if Σ_χ is orientable, this last condition is automatically satisfied, for (Γ_0, ϕ_0) and (Γ, ϕ) are similarly oriented, and so we may assume that Σ_χ is not orientable. Since Σ_χ is connected, there is, for each internal $e \in E(\Gamma_0)$ with $|\tilde{e}| = 3$, a Φ_χ-preserving homeomorphism $\alpha_e : \Sigma_\chi \to \Sigma_\chi$ which maps $\beta(e)$ onto itself with reversed orientation, and fixes $bd(\Sigma)$ and every $\beta(f)$ ($f \in E(\Gamma_0) \setminus \{e\}$) pointwise. By an appropriate combination of the α_e's we may correct every $e \in E(\Gamma_0)$ with $|\tilde{e}| = 3$ for which $\alpha(\gamma_\Gamma(\beta(e)))$ and $\gamma_{\Gamma_0}(e)$ give opposite orientation of $\beta(e)$. This proves (2).

(3) *For each $v \in V(\Gamma_0)$ there is a tree T_v in Σ_χ such that*

- *for distinct $v, v' \in V(\Gamma_0), U(T_v) \cap U(T_{v'}) = \emptyset$*
- *for each $v \in V(\Gamma_0), e \in E(\Gamma_0)$ and $1 \le i \le |\tilde{e}|$, $U(T_v)$ contains the ith term of $\gamma_\Gamma(\beta(e))$ if and only if v is the ith term of $\gamma_{\Gamma_0}(e)$, and*

- *for each $v \in V(\Gamma_0)$ and $e \in E(\Gamma_0), U(T_v) \cap \beta(e) = \emptyset$.*

Subproof. Let S_v be a tree in Σ_χ for each $v \in V(\Gamma_0)$, such that

- for distinct $v, v' \in V(\Gamma_0), U(S_v) \cap U(S_{v'}) = \emptyset$
- for each $v \in V(\Gamma_0), e \in E(\Gamma_0)$ and $1 \le i \le |\tilde{e}|$, $U(S_v)$ contains the ith term of $\alpha(\gamma_\Gamma(\beta(e)))$ if and only if v is the ith term of $\gamma_{\Gamma_0}(e)$, and
- for each $v \in V(\Gamma_0)$ and $e \in E(\Gamma_0), U(S_v) \cap \alpha(\beta(e)) = \emptyset$.

These S_v's clearly exist (let each S_v be a star centered at v, with edges entering all those $e \in E(\Gamma_0)$ with $v \in \tilde{e}$ except short sides). Let $T_v = \alpha^{-1}(S_v)$ for each $v \in V(\Gamma_0)$; then (3) is satisfied. This proves (3).

(4) *Let r^* be a pole, and let $N(r^*) = \bigcup(\tilde{\beta}(e) : e \in E(\Gamma_0), e \subseteq \bar{r} \setminus r)$. There is no circuit F of Γ^* with $\frac{1}{2}|E(F)| < rep(\mathcal{T})$ and with $r^* \in ins(F)$ and $\frac{1}{2}|E(F)| < |ins(F) \cap N(r^*)|$.*

Subproof. Suppose that F is such a circuit. Now $|N(r^*)| \le 2|V(\Gamma_0)|$, and so

$$\frac{1}{2}|E(F)| < 2|V(\Gamma_0)| \le \frac{1}{2}n - 2.$$

Suppose first that $r^* \in ins(F) \setminus U(F)$. Since $\frac{1}{2}|E(F)| \le n$, and $\bar{r} \subseteq ins(F)$, it follows that $\bar{r}$ includes no long side of Φ_χ, because the second outcome of the theorem does not hold. Hence $N(r^*) = V(\Phi_\chi) \cap (\bar{r} \setminus r)$, and so $ins(F)$ contains more than $\frac{1}{2}|E(F)|$ vertices of Φ_χ, and the third outcome of the theorem holds, a contradiction.

Thus $r^* \notin ins(F) \setminus U(F)$, and so $r^* \in V(F)$. Hence F is a bite at r^*. We may assume that $ins(F) \cap (\bar{r} \setminus r)$ is a line, for otherwise there is a circuit F' with $|E(F')| < E(F)$ and $ins(F) \subseteq ins(F')$. Suppose that $\beta(e) \subseteq ins(F)$ for some $e \in E(\Gamma_0)$ bordering $\bar{r} \setminus r$ which is not a short side. Let S be the long side with $e \subseteq S$, and let s_1, s_2 be the short sides with common ends with S. (Possibly $s_1 = s_2$.) Let v_1, v_2 be the corresponding ends of S. If $v_1 \in ins(F)$ then $l(\beta(e), s_1) \le \frac{1}{2}|E(F)| + 2 \le \frac{1}{2}n$, a contradiction to the third assertion of (1). Thus $v_1, v_2 \notin ins(F)$. Since $ins(F) \cap (\bar{r} \setminus r)$ is a line, it follows that $ins(F) \cap (\bar{r} \setminus r) \subseteq S$. By (1) again, $\beta(f) \not\subseteq ins(F)$ for any $f \in E(\Gamma_0) \setminus \{e\}$ which borders S, and so $|ins(F) \cap N(r^*)| = 2$. Hence $\frac{1}{2}|E(F)| \le 1$, which is impossible.

Thus there is no $e \in E(\Gamma_0)$ bordering $\bar{r} \setminus r$, not a short side, with $\beta(e) \subseteq ins(F)$, and so

$$N(r^*) \cap ins(F) = V(\Phi_\chi) \cap ins(F).$$

But then the second outcome of the theorem holds, a contradiction. This proves (4).

Let $\mathcal{T}'$ be the set of all separations (A', B') of $sk(\Gamma)$ of order $\le n$ such that there exists $(A, B) \in \mathcal{T}$ with $V(A) = V(A')$ and $V(B) = V(B')$. By theorem 14.1 of [9], $\mathcal{T}'$ is a tangle in $sk(\Gamma)$ of order $n+1$ and $rep(\mathcal{T}') = n + 1$, since $ord(\mathcal{T}) > \frac{3}{2}n$ and $rep(\mathcal{T}) \ge n + 1$.

(5) *Let $r^*, N(r^*)$ be as in (4). There is no $(A', B') \in \mathcal{T}'$ with $N(r^*) \subseteq V(A')$ and $|N(r^*)| > |V(A' \cap B')|$.*

Subproof. Choose $(sk(\Gamma))^*$ such that $\Gamma^* \subseteq (sk(\Gamma))^*$. Suppose there is such an (A', B'). Then by 15.1, there is a circuit F of $(sk(\Gamma))^*$ of length $\leq 2n$, bounding a closed disc Δ with $r^* \in \Delta$, such that

$$(sk(\Gamma) \cap \Delta, sk(\Gamma) \cap \Delta') \in \mathcal{T}'$$

where Δ' is the closure of $\hat{\Sigma}_\chi \setminus \Delta$, and

$$|N(r^*) \cap \Delta| > \frac{1}{2}|E(F)|.$$

For every vertex v of F which lies in the interior of an edge of Γ, there is a vertex v' of Γ^* adjacent in $(sk(\Gamma))^*$ to both neighbours of v in F; and by replacing v by v', and repeating, we may assume that $F \subseteq \Gamma^*$. But this contradicts (4). This proves (5).

(6) *Let $e \in E(\Gamma_0)$ be internal. There is no disc $\Delta \subseteq \hat{\Sigma}_\chi$ with $bd(\Delta)$ $sk(\Gamma)$-normal, such that $|\tilde{\beta}(e) \cap \Delta| > |V(\Gamma) \cap bd(\Delta)|$ and $(\Gamma \cap \Delta, \Gamma \cap \Delta') \in \mathcal{T}'$, where Δ' is the closure of $\hat{\Sigma}_\chi \setminus \Delta$.*

Subproof. Suppose that Δ is such a disc. Then $|V(\Gamma) \cap bd(\Delta)| \leq 2$, since $|\tilde{\beta}(e) \cap \Delta| \leq 3$. Since $d(\beta(e), r^*) > n$ for every pole r^*, it follows that $\Delta \subseteq \Sigma_\chi \setminus bd(\Sigma_\chi)$. Now

$$|V(\Gamma) \cap \Delta| \geq |\tilde{\beta}(e) \cap \Delta| > |V(\Gamma) \cap bd(\Delta)|$$

and so $\Delta \cap V(\Gamma) \not\subseteq bd(\Delta)$. Since Γ is internally 3-connected and hence so is $sk(\Gamma)$, we deduce that Δ' is a disc and $\Delta' \cap V(\Gamma) \subseteq bd(\Delta')$. But then the third tangle axiom is violated, because $V(\Gamma \cap \Delta) = V(\Gamma)$. This proves (6).

Let n be so large (in terms of $\hat{\Sigma}_\chi$ and Γ_0) that theorem 3.2 of [7] applies (as applied below). From (1), (3), (4), (6), and theorem 3.2 of [7] applied to $\mathcal{T}'$ and $sk(\Gamma)$, the trees T_v in (3) may be chosen to be subgraphs of $sk(\Gamma)$. For each $v \in V(\Gamma_0)$, let $\sigma(v)$ be the induced subgraph of $sk(\Gamma)$ with vertex set $V(T_v)$, and for each $e \in E(\Gamma_0)$ let $\sigma(e) = \beta(e)$. Then σ is an inflation of (Γ_0, ϕ_0) in (Γ, ϕ), a contradiction, as required. ∎

Proof of 4.1.

Suppose that some χ satisfying $\mathbf{S_1}$–$\mathbf{S_4}$ is orientedly bad. Then (by replacing the $\Omega_\chi(k)$'s and $\Omega_\chi(S)$'s by isomorphic well-quasi-orders) we may choose χ to be disjoint. Let $(\Gamma_0, \phi_0), (\Gamma_1, \phi_1), \ldots$ be a similarly oriented bad sequence for χ. By 6.1 we may assume that for all $j > i \geq 0$ there is no inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j). Choose $n \geq 4$ and even such that 15.2 holds (for (Γ_0, ϕ_0)). By 5.4 and 5.5, we may assume (by replacing the sequence (Γ_i, ϕ_i) $(i \geq 1)$ by a subsequence) that $dist(\Gamma_i), rep(\Gamma_i) > 25n \cdot 5^{2|E(\Gamma_0)|}$ for all $i \geq 1$. From 15.2 we obtain

(1) *For each $i \geq 1$, and for every tangle $\mathcal{T}$ in Γ_i, and every tie-breaker λ in Γ_i, either*

- *$rep(\mathcal{T}) \leq 25n \cdot 5^{2|E(\Gamma_0)|}$, or*
- *$(\Gamma, \phi, \lambda, \mathcal{T})$ is $25n \cdot 5^{2|E(\Gamma_0)|}$-flawed in distance, or*
- *$(\Gamma, \phi, \lambda, \mathcal{T})$ is $\frac{1}{2}n(|E(\Gamma_0)| + 1)$-flawed in freedom, or*
- *for some long side S of Φ_χ, let $f_1, \ldots, f_k$ be the edges of Γ_0 bordering S in order; then $(\Gamma, \phi, \lambda, \mathcal{T})$ is $(\frac{1}{2}n, (\phi_0(f_1), \phi_0(f_2), \ldots, \phi_0(f_k)))$-flawed on S, or*

- *for some internal $e \in E(\Gamma_0)$, $(\Gamma, \phi, \lambda, \mathcal{T})$ is $(|E(\Gamma_0)|, n, \phi_0(e))$-flawed internally.*

Let $\mathcal{S}$ be a similarly oriented set of χ-coloured paintings containing (Γ_i, ϕ_i) for each $i \geq 0$, and closed under Φ_χ-preserving homeomorphisms of Σ_χ. This exists since $\{(\Gamma_i, \phi_i) : i \geq 0\}$ is similarly oriented and so are all images of each (Γ_i, ϕ_i) under Φ_χ-preserving homeomorphisms. Let $\mathcal{D}$ be the set of all $(\Gamma, \phi, \lambda, \mathcal{T})$ such that $(\Gamma, \phi) \in \mathcal{S}$, λ is a tie-breaker in Γ, and $\mathcal{T}$ is a tangle in Γ.

By 10.3, 10.4, 10.5, 11.1 and 14.1, there is a well-behaved set $\mathcal{C}$ of χ-places such that for each $(\Gamma, \phi, \lambda, \mathcal{T}) \in \mathcal{D}$, if $ord(\mathcal{T}) > 25n \cdot 5^{2|E(\Gamma_0)|}$ then there is a rooted location $\mathcal{L}$ which $(25n \cdot 5^{2|E(\Gamma_0)|} + 1)$-isolates $\mathcal{T}$ and for which $(\Gamma, \phi, \mathcal{L}) \in \mathcal{C}$. By 9.2 there exist $j > i \geq 1$ such that there is an inflation of (Γ_i, ϕ_i) in (Γ_j, ϕ_j). This is a contradiction, and completes the proof. ■

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