GRAPH MINORS. XVII. TAMING A VORTEX

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May 1987 — Revised April 1999

ABSTRACT

The main result of this series serves to reduce several problems about general graphs to problems about graphs which can "almost" be drawn in surfaces of bounded genus. In applications of the theorem we usually need to encode such a nearly–embedded graph as a hypergraph which can be drawn completely in the surface. The purpose of this paper is to show how to "tidy up" near–embeddings to facilitate the encoding procedure.

1. INTRODUCTION

In [2] we gave a theorem about the structure of graphs with no minor isomorphic to a fixed graph. (Graphs in this paper are finite, and may have loops or multiple edges. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges.) That theorem said that for every graph H, every graph with no minor isomorphic to H can be expressed as a tree–structure of "pieces", where each piece is a graph which can be drawn in a surface in which H cannot be drawn, except for a bounded number of vertices and a bounded number of "local areas of non–planarity" (or *vortices*, as we call them). (The bounds here depend on H, but not on the graph being decomposed.) In applications each piece is usually then encoded as a hypergraph which can be drawn completely on the surface, with edges labelled from an appropriate quasi–order, and in which every edge has two or three ends. To carry out the encoding procedure, however, we need to arrange that the near–embeddings of the pieces have nice connectivity properties. The object of this paper is to show that any graph which can be near–embedded in

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This Research was partially supported by NSF grant DMS 8504054 and was partially performed under a consulting agreement with Bellcore.

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some surface can also be "nicely" near–embedded in some surface, homeomorphic to or simpler than the first, after removing a bounded number of vertices. (Readers are warned that, unless they understand theorem (3.1) of [2], further reading of the present paper is a waste of time.) Most of the paper is concerned with, given a near–embedding, finding a better one, and analyzing its structure when there is no better one. In section 13 we connect this with excluding a minor.

The main result of this paper is therefore rather humdrum; it will not cause the reader any great excitement, and its proof is unfortunately quite lengthy. To stimulate the reader's interest, let us mention that *finding* the result was not humdrum at all. We really needed this to be true, for all the applications of this series of papers, and for a long time the proof eluded us. (The crucial idea was that of "warp", defined in section 3. This may seem unnatural, but nothing simpler works as far as we can see.)

Since for some applications it is needed, we work with near–embeddings of hypergraphs rather than of graphs. Near–embeddings of hypergraphs arise when a minor is excluded from the "1–skeleton" of the hypergraph, a situation which occurs in applications. This is discussed in section 12 and 14. So a second objective of the paper is to convert near–embeddings of 1–skeletons of hypergraphs to near–embeddings of the hypergraphs themselves. This is straightforward, however, and is nowhere near as delicate or difficult as the first objective.

A *surface* is a compact 2–manifold with (possibly empty) boundary. We denote the boundary of a surface Σ by $bd(\Sigma)$. Each component of $bd(\Sigma)$ is homeomorphic to a circle, and we call these components the *cuffs* of Σ. We denote by $\Sigma(a,b,c)$ a surface obtained from a sphere by adding a handles and b crosscaps and deleting the interiors of c mutually disjoint closed discs. Thus $\Sigma(0,0,1)$ is a closed disc (which we shall usually just call a *disc*; when we mean an open disc we shall say so), $\Sigma(0,0,2)$ is a cylinder, $\Sigma(1,0,0)$ is a torus, etc. Surfaces in this paper will usually be connected. It is known that

(1.1) (i) *For every non–null connected surface Σ there are integers a,b,c such that $\Sigma \simeq \Sigma(a,b,c)$.*

(ii) *$\Sigma a', b', c'$ if and only if $c = c', a + 2b = a' + 2b'$, and both or neither of b, b' are zero.*

[$\simeq$ denotes homeomorphism.]

Let Σ, Σ' be connected surfaces, where $\Sigma \simeq \Sigma(a,b,c), \Sigma' \simeq \Sigma a', b', c'$. We say that Σ is *simpler than* Σ' if the following conditions are satisfied:

(i) if $b' = 0$, then $b = 0$

(ii) $2a + b \leqslant 2a' + b'$

(iii) $4a + 2b + c < 4a' + 2b' + c'$.

¿From (1.1), this definition does not depend on the choice of a, b, c, a', b', c'. Equivalently, Σ is simpler than Σ' if there exist a, b, c, a', b', c' with $\Sigma \simeq \Sigma(a, b, c)$, $\Sigma' \simeq \Sigma a', b', c'$, $a \leqslant a', b \leqslant b'$ and $4a + 2b + c < 4a' + 2b' + c'$.

An O–*arc* in a surface Σ is a subset homeomorphic to a circle, and a $[0, 1]$–*arc* is homeomorphic to $[0, 1]$. The *ends* of a $[0, 1]$–arc are defined in the natural way. An I–arc is a $[0, 1]$–arc with both ends in $bd(\Sigma)$ and with no other point in $bd(\Sigma)$.

We shall need to cut along certain O– and I–arcs. Let Σ be a connected surface. If F is an I–arc with ends in different cuffs, then we may cut along F and obtain a new surface with one cuff fewer. If F is an I–arc with its ends in the same cuff we may cut along F and obtain either

(i) a connected surface simpler than Σ, or

(ii) a surface with two components, both simpler than Σ, or

(iii) a surface with two components, one a disc and the other homeomorphic to Σ.

Lastly, if $F \subseteq \Sigma$ is an O–arc with $F \cap bd(\Sigma) = \emptyset$, we may cut along F and obtain either

(i) a connected surface simpler than Σ, or

(ii) a surface with two components, both simpler than Σ, or

(iii) a surface with two components, one a cylinder and the other homeomorphic to Σ, or

(iv) a surface with two components, one a disc.

If $X \subseteq \Sigma$, we denote the closure of X by $\overline{X}$, and we denote $\overline{X} - S$ by $\tilde{X}$. A *painting* Γ in a surface Σ is a pair (U, N), where $U \subseteq \Sigma$ is closed and $N \subseteq U$ is finite, such that

(i) $U - N$ has only finitely many arc–wise connected components (which we call *cells* of Γ)

(ii) for each cell c, its closure $\overline{c}$ is a disc and $\tilde{c}$ is a subset of the boundary of this disc, and $|\tilde{c}| \leqslant 3$

(iii) $bd(\Sigma) \subseteq U$

(iv) for each cell c, if $c \cap bd(\Sigma) \neq \emptyset$ then $|\tilde{c}| = 2$ and $\overline{c} \cap bd(\Sigma)$ is a $[0, 1]$–arc with ends the two members of $\tilde{c}$.

(Thus, a painting is something like a drawing, except that we use discs instead of line segments, and cells can have up to three "ends" instead of two.) We define $U(\Gamma) = U$ and $N(\Gamma) = N$. The members of $N(\Gamma)$ are called the *nodes* of Γ. The set of cells of Γ is denoted by $C(\Gamma)$. A *region* of Γ is a component of $\Sigma - U$. Thus, each region is a connected open set disjoint from $bd(\Sigma)$. A node n is a *border* node if $n \in bd(\Sigma)$, and otherwise it is *internal.* Similarly, a cell c is a *border* cell if $c \cap bd(\Sigma) \neq \emptyset$ and otherwise it is *internal.* (The reader should note that $\tilde{c}$ may contain border nodes even if c is an internal cell.) The members of $\tilde{c}$ are the *ends* of c. A node n and cell c are said to *border* a cuff Θ if $n \in \Theta$ and $c \cap \Theta \neq \emptyset$. A subset $X \subseteq \Sigma$ is Γ–*normal* if $X \cap U(\Gamma) \subseteq N(\Gamma)$.

2. HYPERGRAPHS AND THEIR PORTRAYAL

A *hypergraph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges*, and an incidence relation between them. The vertices incident with an edge e are called the *ends* of e. Thus G is a graph if and only if every edge has one or two ends. We say a hypergraph H is a *subhypergraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and every $e \in E(H)$ has the same ends in H and in G; and we denote this by $H \subseteq G$. A *separation* of a hypergraph G is a pair (A, B) of subhypergraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$ (defining $A \cup B, A \cap B$ in the natural way) and its order is $|V(A \cap B)|$.

Let G be a hypergraph. A *portrayal* $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of G consists of

(i) a non–null connected surface Σ

(ii) a painting Γ in Σ

(iii) a function α which assigns to each cell c of Γ a subhypergraph $\alpha(c)$ of G

(iv) a function β which assigns to each border node n of Γ a subset $\beta(n)$ of $V(G)$

(v) an injection γ from a subset $dom(\gamma)$ of $N(\Gamma)$ into $V(G)$

satisfying the axioms below. For any $X \subseteq N(\Gamma)$ we denote $\{\gamma(n) : n \in X \cap\ dom(\gamma)\}$ by $\gamma(X)$: for each $c_0 \in C(\Gamma)$ we denote $\cup(\alpha(c) : c \in C(\Gamma) - \{c_0\})$ by $\alpha(-c_0)$: and for each border node n we define $\beta(n+) = \beta(n)$ if $\notin\ dom(\gamma)$, and $\beta(n+) \cup \{\gamma(n)\}$ if $n \in dom(\gamma)$. The axioms are as follows:

(P1) $G = \cup(\alpha(c) : c \in C(\Gamma))$, and $E(\alpha(c) \cap \alpha(c')) = \emptyset$ for distinct cells c, c'

(P2) $\gamma(\tilde{c}) \subseteq V(\alpha(c))$ for each cell c

(P3) $\gamma(n) \notin \beta(n)$ for each $n \in dom(\gamma) \cap bd(\Sigma)$, and $\beta(n) \subseteq V(\alpha(c))$ for each border cell c and each $n \in \tilde{c}$

(P4) if n_1, n_2 are nodes bordering different cuffs, then $\beta(n_1) \cap \beta(n_2) = \emptyset$, and if n_1 is a node bordering a cuff Θ and $n_2 \in dom(\gamma)$ does not border Θ, then $\gamma(n_2) \notin \beta(n_1)$

(P5) if c is an internal cell then $V(\alpha(c) \cap \alpha(-c)) \subseteq \gamma(\tilde{c})$

(P6) if c is a border cell with ends n_1, n_2, then

$$V(\alpha(c) \cap \alpha(-c)) \subseteq \beta(n_1+) \cup \beta(n_2+)$$

(P7) if n_1, n_2, n_3, n_4 are nodes bordering the same cuff of Σ and in order, then $\beta(n_1+) \cap \beta(n_3+) \subseteq \beta(n_2+) \cup \beta(n_4+)$.

("In order", here and later in the paper, refers to the order of occurrence around the cuff. We see that the conclusion of **(P7)** holds trivially unless n_1, n_2, n_3, n_4 are all distinct.) For readers having trouble digesting this definition, it may help at this point to jump forward to (13.1) in order to better grasp what portrayals are.

Some remarks:

(1) Permitting $dom(\gamma) \neq N(\Gamma)$ is a helpful but artificial device to assist in removing vertices. We shall eventually be able to restrict ourselves to portrayals with $dom(\gamma) = N(\Gamma)$.

(2) Unfortunately, we need to permit $\gamma(n_1) \in \beta(n_2)$ for distinct nodes n_1, n_2 bordering the same cuff. Eventually this will be eliminated, but for the moment the reader is warned that it can happen.

(3) **(P6)** and **(P7)** tell us, in spirit, that for each cuff Θ the $\alpha(c)$'s for c bordering Θ are arranged in a circle, each overlapping the next in $\beta(n+)$ (where n is the common end). However, there is a possible degeneracy to beware of. If c_1, c_2 border Θ, with ends n_1, n_2 and n_2, n_3, it is possible that $\beta(n_1+) \cap \beta(n_3+) \subseteq \beta(n_2+)$; but any vertex in $\beta(n_1+) \cap \beta(n_3+) - \beta(n_2+)$ lies in $\beta(n_4+)$ for every $n_4 \neq n_2$ bordering Θ. Eventually this degeneracy too will be eliminated.

The reason for interest in portrayals is that they provide a means of "encoding" the hypergraph G by the painting Γ, labelling the cells of Γ appropriately; and the product of the main theorem of [2] can be converted into a portrayal. The reason we are interested in portrayals of hypergraphs and not just of graphs is that, by applying [2] to the "1–skeleton" of a hypergraph, we obtain a portrayal of the hypergraph, as we shall see, and this will be important in proving Nash–Williams' "immersions" conjecture.

First, let us observe the following.

(2.1) *Given a portrayal* $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ *of* G, *let* (I_1, I_2) *be a partition of* $C(\Gamma)$. *For* $i = 1, 2$, *let* $A_i = \bigcup(\alpha(c) : c \in I_i)$. *Then* $E(A_1 \cap A_2) = \emptyset$, *and* $V(A_1 \cap A_2) = \bigcup(\gamma(\tilde{c}_1) \cap \gamma(\tilde{c}_2) : c_1 \in I_1, c_2 \in I_2) \cup \bigcup(\beta(n)$: *there exist border cells* $c_1 \in I_1$ *and* $c_2 \in I_2$ *with* $n \in \tilde{c}_1 \cap \tilde{c}_2)$.

Proof. $E(A_1 \cap A_2) = \emptyset$ from **(P1)**. For $i = 1, 2$, let $X_i = \bigcup(\tilde{c} : c \in I_i)$ and let

$$Y_i = \bigcup(\tilde{c} : c \in I_i \quad \text{is a border cell}).$$

We must show that

$$V(A_1 \cap A_2) = \gamma(X_1 \cap X_2) \cup \bigcup(\beta(n) : n \in Y_1 \cap Y_2)\ .$$

(1) $$\gamma(X_1 \cap X_2) \subseteq V(A_1 \cap A_2)\ .$$

Subproof. Let $n \in X_1 \cap X_2 \cap \mathit{dom}(\gamma)$, and choose $c_1 \in I_1, c_2 \in I_2$ with $n \in \tilde{c}_1 \cap \tilde{c}_2$. By **(P2)**, $\gamma(n) \in V(\alpha(c_1)) \cap V(\alpha(C_2)) \subseteq V(A_1 \cap A_2)$, as required.

(2) $$\bigcup(\beta(n) : n \in Y_1 \cap Y_2) \subseteq V(A_1 \cap A_2)\ .$$

Subproof. Let $n \in Y_1 \cap Y_2$, and choose border cells $c_1 \in I_1, c_2 \in I_2$, with $n \in \tilde{c}_1 \cap \tilde{c}_2$. Then from **(P3)**, $\beta(n) \subseteq V(\alpha(c_1)) \cap V(\alpha(c_2)) \subseteq V(A_1 \cap A_2)$ as required.

(3) $$V(A_1 \cap A_2) \subseteq \gamma(X_1 \cap X_2) \cup \bigcup(\beta(n) : n \in Y_1 \cap Y_2)\ .$$

Subproof. Let $v \in V(A_1 \cap A_2)$. Choose $c_1 \in I_1, c_2 \in I_2$ with $v \in V(\alpha(c_1)), V(\alpha(c_2))$, and choose them, moreover, such that as many of c_1, c_2 as possible are border cells. Now if $v \in \gamma(\tilde{c}_1) \cap \gamma(\tilde{c}_2)$ then since γ is an injection it follows that $v = \gamma(n)$ for some $n \in \tilde{c}_1 \cap \tilde{c}_2 \cap \mathit{dom}(\gamma) \subseteq X_1 \cap X_2 \cap \mathit{dom}(\gamma)$ as required. Thus, from the symmetry, we may assume that $v \notin \gamma(\tilde{c}_1)$. But

$$v \in V(\alpha(c_1) \cap \alpha(c_2)) \subseteq V(\alpha(c_1) \cap \alpha(-c_1))$$

and so from **(P5)**, c_1 is a border cell, and by **(P6)**, there exists $n_1 \in \tilde{c}_1$ with $v \in \beta(n_1)$. Let c_1 border a cuff Θ. Suppose first that c_2 does not border Θ. Then since

$$v \in V(\alpha(c_2) \cap \alpha(-c_2)) \cap \beta(n_1)$$

it follows from **(P5)**, **(P6)** and **(P7)** that $v = \gamma(n)$ for some $n \in \tilde{c}_2 \cap \mathit{dom}(\gamma)$. By **(P4)**, n borders Θ and so c_2 is internal; let c be a cell bordering Θ with $n \in \tilde{c}$.

¿From our initial choice of c_1, c_2 it follows that $c \notin I_2$, and so $c \in I_1$. But then $n \in \tilde{c} \cup \tilde{c}_2 \subseteq X_1 \cap X_2$ as required. We may assume then that c_2 does border Θ. 66666 $v \in V(\alpha(c_2) \cap \alpha(-c_2))$, it follows from **(P6)** that $v \in \beta(n_2+)$ for some $n_2 \in \tilde{c}_2$. Since $c_1 \in I_1$ and $c_2 \in I_2$, there exist nodes n_3, n_4 bordering Θ such that c_1, n_3, c_2, n_4 are in order and n_3 and n_4 are both in $Y_1 \cap Y_2$. Since n_1, n_3, n_2, n_4 are in order and $v \in \beta(n_1) \cap \beta(n_2+)$, it follows from **(P6)** that $v \in \beta(n_3+) \cup \beta(n_4+)$. ¿From the symmetry we assume that $v \in \beta(n_3+)$. But $n_3 \in Y_1 \cap Y_2$, and so either

$$v \in \beta(n_3) \subseteq \bigcup(\beta(n) : n \in Y_1 \cap Y_2)$$

or $n_3 \in dom(\gamma)$ and

$$v = \gamma(n_3) \in \{\gamma(n) : n \in X_1 \cap X_2 \cap \, dom(\gamma)\} \, .$$

This proves (3). From (1–3) the result follows. ■

3. WARP OF A PORTRAYAL

If V is a finite set, K_V denotes the complete graph with vertex set V; that is, the edges of K_V are the 2–element subsets of V, with the natural incidence relation. If G is a hypergraph, its 1–*skeleton* is the subgraph of $K_{V(G)}$ in which distinct $u, v \in V(G)$ are adjacent if there is an edge of G incident with both u and v. We denote the 1–skeleton of G by $G^\star$.

A *linkage* in a hypergraph G is a set $\{P_1, \dots, P_p\}$ of mutually vertex–disjoint paths of $G^\star$. (Paths have no "repeated" vertices, and we recognize the one–vertex path; each path has an *initial* vertex and a *terminal* vertex.) If $\{P_1, \dots, P_p\}$ is a linkage and for $1 \leqslant i \leqslant p, P_i$ has initial vertex s_i and terminal vertex t_i, we say $\{P_1, \dots, P_p\}$ is *from* $\{s_1, \dots, s_p\}$ *to* $\{t_1, \dots, t_p\}$ and it *pairs* s_i with $t_i (1 \leqslant i \leqslant p)$.

Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a portrayal of G, and let $c \in C(\Gamma)$ border a cuff Θ, with ends n_1, n_2. For $p \geqslant 0$ we say that c has *warp* $\leqslant p$ (in π) if

(i) $|\beta(n_1)|, |\beta(n_2)| \leqslant p$, and

(ii) if $|\beta(n_1+)| = |\beta(n_2+)| = p+1$, then every linkage in $\alpha(c)^\star$ from $\beta(n_1+)$ to $\beta(n_2+)$ pairs $\gamma(n_1)$ with $\gamma(n_2)$.

The *warp* of c is the minimum $p \geqslant 0$ such that c has warp $\leqslant p$; and π has *warp* p at Θ if p is the maximum warp of all the cells bordering Θ. The *warp* of π is the maximum of the warp of π at Θ, over all cuffs Θ (or 1 if $bd(\Sigma) = \emptyset$).

4. TANGLES

If $\theta \geqslant 1$ is an integer, a *tangle of order* θ in a hypergraph G is a set $\mathcal{T}$ of separations of G each of order $< \theta$, such that

(i) for every separation (A, B) of G of order $< \theta, \mathcal{T}$ contains one of $(A, B), (B, A)$

(ii) if $(A_i, B_i) \in \mathcal{T}(i = 1, 2, 3)$ then $A_1 \cap A_2 \cup A_3 \neq G$

(iii) if $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

For elementary properties of tangles, see [1]. We define $ord(\mathcal{T}) = \theta$.

We say that $A \subseteq G$ is *small* (*relative to* $\mathcal{T}$) if $(A, B) \in \mathcal{T}$ for some $B \subseteq G$. If A is a subhypergraph of G, an *attachment* of A in G is a vertex of A incident with an edge of G not in $E(A)$. It is easy to see (compare [1, theorems (2.2) and (2.9)]) that

(4.1) (i) *If $A \subseteq G$ is small then $(A, B) \in \mathcal{T}$ for every $B \subseteq G$ such that (A, B) is a separation of G of order $< ord(\mathcal{T})$.*

(ii) *If $A_1, A_2 \subseteq G$ are small and $a \subseteq A_1 \cup A_2$ has $< ord(\mathcal{T})$ attachments in G then A is small.*

If G is a hypergraph and $Z \subseteq V(G)$, we denote by G/Z the hypergraph with vertex set $V(G) - Z$ and edge set $E(G)$, in which $e \in E(G)$ is incident in G/Z with $v \in V(G) - Z$ if and only if e is incident in G with v.

If $Z \subseteq V(G)$ and $\mathcal{T}$ is a tangle in G of order $\theta > |Z|$, we denote by $\mathcal{T}/Z$ the set of all separations (A', B') of G/Z of order $< \theta - |Z|$ such that there exists $(A, B) \in \mathcal{T}$ with $Z \subseteq V(A \cap B), A/Z = A'$ and $B/Z = B'$. It is shown in [1, theorem (6.2)] that

(4.2) *If $\mathcal{T}$ is a tangle in a hypergraph G, and $Z \subseteq V(G)$ with $|Z| < ord(\mathcal{T})$, then $\mathcal{T}/Z$ is a tangle in G/Z of order $ord(\mathcal{T}) - |Z|$.*

If $\mathcal{T}$ is a tangle in a hypergraph G, and $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ is a portrayal of G, π is said to be $\mathcal{T}$*–central* if $\alpha(c)$ is small relative to $\mathcal{T}$ for every $c \in C(\Gamma)$. We shall need the following lemma.

(4.3) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$* be a *$\mathcal{T}$–central portrayal of a hypergraph G, with warp $\leqslant p$, and let $\mathcal{T}$ have order θ. There is no small subhypergraph $A \subseteq G$ with $< \theta - 2p$ attachments, such that $\gamma(n) \in V(A)$ for all $n \in dom(\gamma)$.*

Proof. Suppose that there exists such $A \subseteq G$, and choose A maximal.

(1) *For each internal cell $c, \alpha(c) \subseteq A$.*

Subproof. Since every attachment of $A \cup \alpha(c)$ is an attachment of A, it follows from

(4.1) that $A \cup \alpha(c)$ is small, and so $\alpha(c) \subseteq A$ from the maximality of A.

Now since $A \neq G$, there is some cell c_1 with $\alpha(c_1) \nsubseteq A$. By (1), c_1 borders a cuff Θ. Let the cells and nodes bordering Θ be $n_1, c_1, n_2, c_2, \ldots, n_k, c_k, (n_1)$ in order. For $0 \leqslant j \leqslant k$, let $X_j = A \cup \alpha(c_1) \cup \cdots \cup \alpha(c_j)$.

(2) *For $1 \leqslant j \leqslant k$, every attachment of X_j which is not an attachment of A is in $\beta(n_1) \cup \beta(n_{j+1})$. In particular X_j has fewer than θ attachments.*

Subproof. The second claim follows from the first, since $|\beta(n_k) \cup \beta(n_j)| < 2p$, and A has $< \theta - 2p$ attachments. For the first claim, let v be an attachment of X_j which is not an attachment of A. Then $v \notin V(A)$, and $v \in V(\alpha(c_i))$ for some i with $1 \leqslant i \leqslant j$, and there is an edge e of G, incident with v, with $e \notin E(X_j)$. Let c be a cell of Γ with $e \in E(\alpha(c))$. Then $v \in V(\alpha(c_i) \cap \alpha(c))$. If c does not border Θ, then by **(P5)**, **(P6)** and **(P7)** it follows that $v = \gamma(n)$ for some $n \in \Theta \cap dom(\gamma)$, and hence $v \in V(A)$, a contradiction. Thus c borders Θ, and $c = c_{j'}$, say, where $j < j' \leqslant k$. Since $v \in V(\alpha(c_i) \cap \alpha(-c_i))$ and $v \notin \gamma(\tilde{c_i})$, it follows that $v \in \beta(n_h)$ where $h = i$ or $i+1$; and similarly $v \in \beta(n_{h'})$ where $h' = j'$ or $j'+1$. Now $n_1, n_h, n_{j+1}, n_{h'}$ are in order, and so

$$v \in \beta(n_h) \cap \beta(n_{h'}) \subseteq \beta(n_1) \cup \beta(n_{j+1}) \ .$$

s proves (2).

(3) *For $1 \leqslant j \leqslant k, X_j$ is small.*

Subproof. X_0 is small; and for $1 \leqslant j \leqslant k$ if X_{j-1} is small then so is X_j, from (4.1), because $X_j = X_{j-1} \cup \alpha(c_j)$ and X_j has $< \theta$ attachments and $\alpha(c_j)$ is small. The result follows by induction on j.

In particular, X_k is small. But every attachment of X_k is an attachment of A, as is easily seen, and so X_k has $< \theta - 2p$ attachments. Since $\alpha(c_1) \subseteq X_k$ and $\alpha(c_1) \nsubseteq A$ it follows that $X_k \neq A$; but this contradicts the maximality of A, as required. ■

5. SIMPLIFYING A PORTRAYAL

The main aim of this paper is to show that any $\mathcal{T}$–central portrayal of a hypergraph can be converted to one with nice connectivity properties (possibly by removing a few vertices) if $\mathcal{T}$ has large enough order. In this section we begin the process. Here we are concerned with $\mathcal{T}$–central portrayals such that there is no simpler $\mathcal{T}$–central portrayal of the same hypergraph.

We define "simpler" as follows. Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a portrayal of a hypergraph G, and let $\pi' = (\Sigma', \Gamma', \alpha', \beta', \gamma')$ be a portrayal of a hypergraph G'. We say that π is *simpler* than π' if either

(i) Σ is simpler than Σ' and the warp of π is at most that of π', or

(ii) there is a homeomorphism $\psi : \Sigma \to \Sigma'$ such that for every cuff Θ of Σ, the warp of π at Θ is at most the warp of π' at $\psi(\Theta)$, with strict inequality for some cuff Θ.

We say that π *resembles* π' if there is a homeomorphism $\psi : \Sigma \to \Sigma'$ such that for every cuff Θ of Σ, the warp of π at Θ equals the warp of π' at $\psi(\Theta)$.

Now let $\mathcal{T}$ be a tangle in a hypergraph G. A $\mathcal{T}$–central portrayal π of G is 0*–redundant* (relative to $\mathcal{T}$) if there is no $\mathcal{T}$–central portrayal of G simpler than π. In this section we shall develop some consequences of 0–redundancy.

(5.1) *If $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ is a 0–redundant $\mathcal{T}$–central portrayal of a hypergraph G, then for each cuff Θ of Σ, π has warp $\geqslant 1$ at Θ.*

Proof. If π has warp 0 at Θ, then $\beta(n) = \emptyset$ for each $n \in \Theta \cap N(\Gamma)$. Let Σ' be the surface obtained by pasting a disc on Θ, and let β' be the restriction of β to $bd(\Sigma')$. Then $\pi' = (\Sigma', \Gamma, \alpha, \beta', \gamma)$ is a $\mathcal{T}$–central portrayal of G, simpler than π, a contradiction. ■

¿From (5.1), every 0–redundant $\mathcal{T}$–central portrayal has warp $\geqslant 1$.

Let Γ be a painting in a surface Σ, let $c \in C(\Gamma)$ be internal, and let $n \in \tilde{c}$. Choose a disc $\Delta \subseteq \overline{c}$ with $n \notin \Delta$ and $\tilde{c} - \{n\} \subseteq \Delta$, and define

$$\Gamma' = ((U(\Gamma) - c) \cup \Delta, N(\Gamma)) \ .$$

Then Γ' is a painting in Σ, with $C(\Gamma') = (C(\Gamma) - \{c\}) \cup \{\Delta - (\tilde{c} - \{n\})\}$. We say that Γ' is obtained from Γ by *shrinking c from n*, and its cell $\Delta - (\tilde{c} - \{n\})$ is the *shrunken c*.

If $f : A \to B$ is a function and C is a set, we denote the restriction of f to $A \cap C$ by $f|C$.

(5.2) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a hypergraph G. Let $c \in C(\Gamma)$ be internal, and let $n \in \tilde{c} - dom(\gamma)$. Let Γ' be obtained by shrinking c from n; and let α' be defined by $\alpha'(c') = \alpha(c)$, where c' is the shrunken c, and otherwise $\alpha' = \alpha$. Then $\pi' = (\Sigma, \Gamma', \alpha', \beta, \gamma)$ is a portrayal of G.*

The proof is clear.

Given a portrayal $(\Sigma, \Gamma, \alpha, \beta, \gamma)$, if $\Sigma' \subseteq \Sigma$ we denote $\bigcup\,(\alpha(c) : c \in C(\Gamma), c \subseteq \Sigma')$ by $\alpha(\Sigma')$.

(5.3) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a 0-redundant Γ–central portrayal of a hypergraph G. Let F be a Γ–normal I–arc with ends n_1, n_2 such that $\beta(n_1) = \beta(n_2) = F \cap dom(\gamma) = \emptyset$. Then there is a disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ such that $\alpha(\Delta)$ is small.*

Proof. By (5.2) we may assume that $F \cap N(\Gamma) = \{n_1, n_2\}$ and $\tilde{c} \cap \{n_1, n_2\} = \emptyset$ for every internal cell c of Γ. Let Δ be a "neighbourhood" of F; that is, $\Delta \subseteq \Sigma$ is a disc, $F \subseteq \Delta$, and

$$F \cap bd(\Delta) = \Delta \cap bd(\Sigma) = \Delta \cap U(\Gamma) = \{n_1, n_2\}\ .$$

Let Σ_0 be the surface obtained from Σ by cutting along F. Let Δ', Δ'' be the two discs into which Δ is divided by this cutting, and let $n_i' \in \Delta', n_i'' \in \Delta''$ correspond to n_i $(i = 1, 2)$. Let $c' = \Delta' - \{n_1', n_2'\}$, $c'' = \Delta'' - \{n_1'', n_2''\}$. Let

$$U_0 = (U(\Gamma) - \{n_1, n_2\}) \cup \Delta' \cup \Delta''$$
$$N_0 = (N(\Gamma) - \{n_1, n_2\}) \cup \{n_1', n_2', n_1'', n_2''\}$$

and let $\Gamma_0 = (U_0, N_0)$. Then Γ_0 is a painting in Σ_0, and c', c'' are cells of it.

Suppose first that Σ_0 is connected, whence it is simpler than Σ. We define a portrayal $\pi_0 = (\Sigma_0, \Gamma_0, \alpha_0, \beta_0, \gamma)$ of G as follows:

$\alpha_0(c'), \alpha_0(c'')$ are both null

$\alpha_0(c) = \alpha(c)$ $(c \in C(\Gamma_0) - \{c', c''\})$

$\beta_0(n_1') = \beta_0(n_2') = \beta_0(n_1'') = \beta_0(n_2'') = \emptyset$

$\beta_0(n) = \beta(n)$ $(n \in N_0 \cap bd(\Sigma_0) - \{n_1', n_2', n_1'', n_2''\})$.

It is easy to verify that π_0 is a portrayal of G; it is $\mathcal{T}$–central; and yet it is simpler than π, a contradiction to the 0–redundancy of π.

Thus Σ_0 is not connected. It therefore has exactly two components Σ', Σ'', with $\Delta' \subseteq \Sigma'$, $\Delta'' \subseteq \Sigma''$. ¿From the symmetry we may assume that $(\alpha(\Sigma''), \alpha(\Sigma')) \in \mathcal{T}$, for by (2.1) this is a separation of G of order 0.

Let $\Gamma' = (U_0 \cap \Sigma', N_0 \cap \Sigma')$; then Γ' is a painting in Σ', and c' is a cell of it.

We define a portrayal

$$
\begin{aligned}
\alpha'(c') &= \alpha(\Sigma'') \\
\alpha'(c) &= \alpha(c)\,(c \in C(\Gamma') - \{c'\}) \\
\beta'(n_1') &= \beta'(n_2') = \emptyset \\
\beta'(n) &= \beta(n)\,(n \in N(\Gamma') \cap bd(\Sigma') - \{n_1', n_2'\}) \\
\gamma' &= \gamma | N(\Gamma')\ .
\end{aligned}
$$

Again, it is easy to verify that π' is a portrayal of G, and is $\mathcal{T}$–central, and its warp is at most that of π. Since it is not simpler than π (since π is 0–redundant) it follows that $\Sigma' \simeq \Sigma$ and Σ'' is a disc. This disc satisfies the theorem. ∎

We shall need a second, similar result, the following.

(5.4) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a 0–redundant $\mathcal{T}$–central portrayal of a hypergraph G. Let $F \subseteq \Sigma$ be a Γ–normal O–arc with $F \cap dom(\gamma) = \emptyset$. Then there is a disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$ such that $\alpha(\Delta)$ is small.*

Proof. By (5.2) we may assume that $F \cap N(\Gamma) \subseteq bd(\Sigma)$ and for every internal cell c of $\Gamma, \tilde{c} \cap F = \emptyset$. Since F is Γ–normal, it follows that $F \cap bd(\Sigma) \subseteq N(\Gamma)$, and by "diverting" F around each member of $F \cap bd(\Sigma)$, we may assume that $F \cap bd(\Sigma) = \emptyset$ (for if the result holds for the new O–arc then it also holds for the original). Thus $F \cap U(\Gamma) = \emptyset$.

Let Σ_0 be the surface obtained from Σ by cutting along F, and let $\widehat{\Sigma}_0$ be obtained from Σ_0 by pasting discs on the resulting (one or two) new cuffs in Σ_0. Suppose that $\widehat{\Sigma}_0$ is connected. Then $(\widehat{\Sigma}_0, \Gamma, \alpha, \beta, \gamma)$ is a $\mathcal{T}$–central portrayal of G, simpler than π (because $\widehat{\Sigma}_0$ is simpler than Σ), a contradiction.

Thus Σ_0 has exactly two components Σ', Σ'', and from the symmetry we may assume that $(\alpha(\Sigma''),\ \alpha(\Sigma')) \in \mathcal{T}$, since by (2.1) this separation has order 0. Let $\widehat{\Sigma}'$ be the component of $\widehat{\Sigma}_0$ corresponding to Σ'. Let $\Gamma' = (U(\Gamma) \cap \Sigma',\ N(\Gamma) \cap \Sigma')$; then Γ' is a painting in $\widehat{\Sigma}'$. Let $c' \in C(\Gamma')$ (this exists, since $\alpha(\Sigma'') \neq G$ from one of the tangle axioms). Define $\pi' = (\widehat{\Sigma}', \Gamma',\ \alpha', \beta', \gamma')$ as follows:

$$
\begin{aligned}
\alpha'(c') &= \alpha(c') \cup \alpha(\Sigma'') \\
\alpha'(c) &= \alpha(c)\ (c \in C(\Gamma') - \{c'\}) \\
\beta'(n) &= \beta(n)\,(n \in N(\Gamma') \cap bd(\widehat{\Sigma}')) \\
\gamma' &= \gamma | N(\Gamma')\ .
\end{aligned}
$$

Then it is easy to see that π' is a portrayal of G; and it is $\mathcal{T}$–central since $\alpha'(c')$ is small by (4.1). Its warp is at most that of π, but it is not simpler, and so $\widehat{\Sigma}'$ is not simpler than Σ. Thus Σ'' is a disc, satisfying the theorem. ■

6. REDUNDANCY

Now we make a somewhat stronger hypothesis; we study portrayals such that there is no simpler portrayal even if we remove a few vertices.

Let $\mathcal{T}$ be a tangle in a hypergraph G, and let π be a $\mathcal{T}$–central portrayal of G. We say that π is *z–redundant*, if $z < ord\ (\mathcal{T})$, and for every $Z \subseteq V(G)$ with $|Z| \leqslant z$ there is no $\mathcal{T}/Z$–central portrayal of G/Z simpler than π. In this section we develop some consequences of z–redundancy.

We shall need the following lemma, a relative of [1, theorem (6.3)].

(6.1) *Let $\mathcal{T}$ be a tangle of order θ in a hypergraph G, and let $Z \subseteq V(G)$ with $|Z| < \theta$. Let $A \subseteq G$ such that A/Z' has fewer than $\theta - |Z|$ attachments in G/Z, where $Z' = Z \cap V(A)$. Then A is small relative to $\mathcal{T}$ if and only if A/Z' is small relative to $\mathcal{T}/Z$.*

Proof. Let $C \subseteq G$, where $V(C) = V(A) \cup Z$ and $E(C) = E(A)$, and choose $D \subseteq G$ such that $Z \subseteq V(C \cap D)$ and (C, D) is a separation of G of order $< \theta$. (This exists, from our hypothesis about the attachments of A/Z in G/Z.) Then $A/Z' = C/Z$. Suppose first that $(C, D) \in \mathcal{T}$. Then $(D, A) \notin \mathcal{T}$ by the second tangle axiom, since $C \cup D = G$, and so $(A, D) \in \mathcal{T}$ (for (A, D) is a separation of order $< \theta$) and so A is small relative to $\mathcal{T}$. Moreover, $(C/Z, D/Z) \in \mathcal{T}/Z$, and so $C/Z = A/Z'$ is small relative to $\mathcal{T}/Z$. We may assume therefore that $(C, D) \notin \mathcal{T}$, and so $(D, C) \in \mathcal{T}$. Consequently A is not small with respect to $\mathcal{T}$, for $A \cup D = G$; and since $(D/Z, C/Z) \in \mathcal{T}/Z$, it follows similarly that A/Z' is not small with respect to $\mathcal{T}/Z$, as required. ■

(6.2) *Let $\mathcal{T}$ be a tangle in a hypergraph G and let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a $\mathcal{T}$–central portrayal of G with warp $\leqslant p$. Let $Z \subseteq V(G)$, with $|Z| < ord\ (\mathcal{T}) - 2p - 2$. Define*

$$\begin{aligned}\alpha'(c) &= \alpha(c)/(Z \cap V(\alpha(c)))\ (c \in C(\Gamma))\\ \beta'(n) &= \beta(n) - Z\ (n \in N(\Gamma) \cap bd(\Sigma))\\ \gamma' &= \gamma|\{n : n \in dom(\gamma), \gamma(n) \notin Z\}\ .\end{aligned}$$

Then $\pi' = (\Sigma, \Gamma, \alpha', \beta', \gamma')$ is a $\mathcal{T}/Z$–central portrayal of G/Z, resembling or simpler than π.

Proof. It is easy to verify **(P1)**–**(P7)**, and so π' is a portrayal of G/Z; and clearly the warp in π' of every border cell of Γ is at most its warp in π. Moreover, π' is $\mathcal{T}/Z$–central; for if $c \in C(\Gamma), \alpha'(c)$ has at most $2p+2 < \; ord(\mathcal{T}) - |Z|$ attachments in G/Z, and so $\alpha'(c)$ is small by (6.1). ■

(6.3) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a z–redundant $\mathcal{T}$–central portrayal of G, with warp $\leqslant p$. Let $ord(\mathcal{T}) > z + 2p + 2$, and let $F \subseteq \Sigma$ be a Γ–normal I-arc with ends n_1, n_2 and with $|F \cap \; dom(\gamma)| \leqslant z - 2p$. Then there is a disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ such that $\alpha(\Delta)$ is small.*

Proof. Let $Z = \beta(n_1) \cup \beta(n_2) \cup \{\gamma(n) : n \in F \cap \; dom(\gamma)\}$. Then $|Z| \leqslant z$. Define $\pi' = (\Sigma, \Gamma, \alpha', \beta', \gamma')$ as in (6.2). Then π' is not simpler than π, and so π' resembles π. Moreover, π' is 0–redundant, since π is z–redundant, and so by (5.3) there is a disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ such that $\alpha'(\Delta)$ is small relative to $\mathcal{T}/Z$. Hence, by (6.1), $\alpha(\Delta)$ is small relative to $\mathcal{T}$, as required. ■

(6.4) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a z–redundant $\mathcal{T}$–central portrayal of G, with warp $\leqslant p$. Let $ord(\mathcal{T}) > z + 2p + 2$, and let $F \subseteq \Sigma$ be a Γ–normal O–arc with $|F \cap dom(\gamma)| \leqslant z$. Then there is a disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$ such that $\alpha(\Delta)$ is small.*

Proof. Let $Z = \{\gamma(n) : n \in F \cap \; dom(\gamma)\}$. Then $|Z| \leq z$. Define $\pi' = (\Sigma, \Gamma, \alpha', \beta', \gamma')$ as in (6.2); then π' resembles π, and is 0–redundant. The result follows from (5.4) and (6.1). ■

(6.5) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a z–redundant $\mathcal{T}$–central portrayal of a hypergraph G with warp $\leqslant q$, let $(A, B) \in \mathcal{T}$ have order $\leqslant z$, let Θ be a cuff of Σ, and let π have warp p at Θ. Let $ord(\mathcal{T}) > z + 2q + 2$. Then there exists either*

(i) *a node n* bordering Θ *with $|\beta(n)| = p$ and $\beta(n) \cap V(A) = \emptyset$, or*

(ii) *a cell c bordering Θ with ends $n_1, n_2 \in dom(\gamma)$, such that $\gamma(n_1), \gamma(n_2) \notin V(A)$, there is no path of $(A \cap \alpha(c))^\star$ between $\beta(n_1)$ and $\beta(n_2)$, and*

$$|\beta(n_1) - V(A)| = |\beta(n_2) - V(A)| = p - 1 \geqslant 1 \; .$$

Proof. Let $Z = V(A \cap B);$ *then* $|Z| \leqslant z$. Choose $c' \in C(\Gamma)$, and define a portrayal $\pi' = (\Sigma, \Gamma, \alpha', \beta', \gamma')$ of G/Z as follows:

$$\begin{aligned}
\alpha'(c') &= ((\alpha(c')/(Z \cap V(\alpha(c')))) \cap \; (B/Z)) \cup (A/Z) \\
\alpha'(c) &= (\alpha(c)/(Z \cap V(\alpha(c)))) \cap \; (B/Z) \; (c \in C(\Gamma), c \neq c') \\
\beta'(n) &= \beta(n) - V(A) \; (n \in N(\Gamma) \cap bd\,(\Sigma)) \\
\gamma' &= \gamma | \{n : n \in \; dom(\gamma), \gamma(n) \notin V(A)\} \; .
\end{aligned}$$

We can view this as obtained by, first, applying (6.2) to obtain a $(\mathcal{T}/Z)$–central portrayal of G/Z; second, restricting it to obtain a portrayal of B/Z; and third, adding A/Z back into $\alpha(c')$ for one cell c', to obtain a $(\mathcal{T}/Z)$–central portrayal of G/Z. This third step is possible since $(A/Z, B/Z)$ is a separation G/Z of order 0, and is in $\mathcal{T}/Z$. We hope these remarks make it evident that π' is a $(\mathcal{T}/Z)$–central portrayal of G/Z. Its warp at each cuff is at most that of π, but it is not simpler than π since π is z–redundant, and so its warp at every cuff, in particular at Θ, equals that of π.

We may assume that $\beta'(n)| \leqslant p-1$ for each node n bordering Θ, for otherwise the theorem holds (since $|\beta(n)| \leqslant p$). ¿From the definition of warp, there is a cell c bordering Θ with ends n_1, n_2 such that $|\beta'(n_1+)| = |\beta'(n_2+)| = p$ and there is a linkage $\{P_1, \ldots, P_p\}$ in $\alpha'(c)$ from $\beta'(n_1+)$ to $\beta'(n_2+)$ which does not pair $\gamma'(n_1)$ with $\gamma'(n_2)$. In particular, $p \geqslant 2$, and $n_1, n_2 \in dom(\gamma')$, and so $n_1, n_2 \in dom(\gamma)$ and $\gamma(n_1), \gamma(n_2) \notin V(A)$. Moreover,

$$p = |\beta'(n_1+)| = 1 + |\beta(n_1) - V(A)|$$

and so

$$|\beta(n_1) - V(A)| = |\beta(n_2) - V(A)| = p - 1\ .$$

We claim that for $1 \leqslant i \leqslant p, P_i$ is a path of $(B/Z)^\star$. For it is certainly a path of $(G/Z)^\star$, and it meets $V(B/Z)$ since it meets $\beta'(n_1+) = \beta(n_1+) - V(A)$; but $(A/Z, B/Z)$ is a separation of G/Z of order 0 and P_i is connected, and so $P_i \subseteq (B/Z)^\star$ as claimed.

Suppose that there is a path P_{p+1} of $(\alpha(c) \cap A)^\star$ between $\beta(n_1)$ and $\beta(n_2)$. Then $\{P_1, \ldots, P_{p+1}\}$ is a linkage in $\alpha(c)$ from $\beta(n_1+)$ to $\beta(n_2+)$ violating the warp condition, a contradiction. Thus there is no such P_{p+1}, and the result follows. ■

(6.6) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a z–redundant $\mathcal{T}$–central portrayal of G with warp $\leqslant p$ and $ord(\mathcal{T}) > z + 2p + 2$, and let Θ be a cuff. There is no $X \subseteq V(G)$ with $|X| \leqslant z/3$ such that $X \cap \beta(n+) \neq \emptyset$ for every node n bordering Θ.*

Proof. Let π have warp $q \geqslant 1$ at Θ, and fix an orientation of Θ. From **(P7)** we have immediately:

(1) *For each $x \in X$ there is at most one $n \in N(\Gamma) \cap \Theta$ such that $x \in \beta(n+)$ and $x \notin \beta(n'+)$ where n' is the node of $N(\Gamma)$ bordering Θ immediately following n.*

We call n as in (1) the *terminal* of x. For each $x \in X$, if there is a terminal n of X with $n \in dom(\gamma)$, let $y(x) = \gamma(n)$. (This is unique, by (1).) Otherwise, let

$y(x) = x$. For each $x \in X$, if $x = \gamma(n)$ for some $n \in dom(\gamma)$ with $|\beta(n)| = q$, let $z(x) \in \beta(n)$. Otherwise, let $z(x) = x$. Let

$$W = X \cup \{y(x), z(x) : x \in X\} .$$

(2) *For each $n \in N(\Gamma) \cap \Theta$, if $|\beta(n)| = q$ then $W \cap \beta(n) \neq \emptyset$.*

Subproof. Let $x \in \beta(n+) \cap X$. We may assume that $x \notin \beta(n)$, and so $n \in dom(\gamma)$ and $x = \gamma(n)$. Consequently either $|\beta(n)| < q$ or $z(x) \subseteq \beta(n)$, as required.

(3) *For each cell c bordering Θ with ends $n_1, n_2 \in dom(\gamma)$, if $\gamma(n_1), \gamma(n_2) \notin W$ and $|\beta(n_1) - W| = |\beta(n_2) - W| = q - 1$ then $W \cap \beta(n_1) \cap \beta(n_2) \neq \emptyset$.*

Subproof. We assume that n_2 follows n_1. Let $x \in X \cap \beta(n_1+)$. Since $\gamma(n_1) \notin W$ it follows that $x \neq \gamma(n_1)$ and so $x \in \beta(n_1)$. Since $\gamma(n_1) \notin W$ it also follows that $\gamma(n_1) \neq y(x)$, and so n_1 is not the terminal of x. Consequently $x \in \beta(n_2+)$. Since $x \neq \gamma(n_2)$ it follows that $x \in \beta(n_2)$, and so $x \in W \cap \beta(n_1) \cap \beta(n_2)$ as required.

Let A be the subhypergraph of G with $V(A) = W$ and $E(A) = \emptyset$. Then $(A, G) \in \mathcal{T}$ since $|V(A)| \leqslant 3k < ord(\mathcal{T})$, and so A is small. But (2), (3) contradict (6.5). ■

(6.7) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a redundant $\mathcal{T}$–central portrayal of G with warp $\leqslant p$. Let $ord(\mathcal{T}) > z + 2p + 2$, and let F be a Σ–normal I–arc with ends n_1, n_2 and with $|F \cap dom(\gamma)| \leqslant z - 2p - 1$. Let Δ be as in (6.3). Then $\beta(n_1+) \cap \beta(n_2+) \subseteq \beta(n+)$ for all $n \in N(\Gamma) \cap \Delta \cap bd(\Sigma)$.*

Proof. Let Θ be the cuff with $n_1, n_2 \in \Theta$, and let π have warp q at Θ. Suppose that $n' \in N(\Gamma) \cap \Delta \cap bd(\Sigma)$ and $v \in \beta(n_1+) \cap \beta(n_2+) - \beta(n'+)$. Let $F' = \Delta \cap \Theta$, and let F'' be the other closed line segment in Θ with ends n_1, n_2. From **(P7)**, $v \in \beta(n+)$ for all $n \in N(\Gamma) \cap F''$.

If there exists $n \in N(\Gamma) \cap \Theta$ such that $\gamma(n) = v$ and $\beta(n) \neq \emptyset$, choose $u \in \beta(n)$, and otherwise let $u = v$. Let $A \subseteq G$ be the hypergraph with $E(A) = E(\alpha(\Delta))$, $V(A) = (\alpha(\Delta)) \cup \{u\}$. Now $\alpha(\Delta)$ has at most $|F \cap dom(\gamma)| + 2p < z$ attachments, and so A has at most z attachments. Since $\alpha(\Delta)$ is small it follows from (4.1) (ii) that A is small. Since $ord(\mathcal{T}) > z + 2p + 2$, (6.5) implies that there exists either

(i) a node $n \in N(\Gamma) \cap \Theta$ with $|\beta(n)| = q$ and $\beta(n) \cap V(A) = \emptyset$, or
(ii) a cell c bordering Θ with ends $m_1, m_2 \in dom(\gamma)$, such that $\gamma(m_1), \gamma(m_2) \notin V(A)$ and $V(A) \cap \beta(m_1) \cap \beta(m_2) = \emptyset$.

If (i) occurs, then since

$$|\beta(n) \cap V(A)| = 0 < q = |\beta(n)|$$

by (5.1), it follows that $n \notin F'$, and so $n \in F''$. Consequently, $v \in \beta(n+)$. Since $\beta(n) \cap V(A) = \emptyset$ and $v \in V(A)$, it follows that $v = \gamma(n)$, and so $u \in \beta(n)$ since $\beta(n) \neq \emptyset$ (since γ is an injection). Hence $u \in V(A) \cap \beta(n) = \emptyset$, a contradiction.

If (ii) occurs, then $\gamma(m_1) \notin V(A)$ it follows that $c \cap \Theta \subseteq F''$ and so $m_1, m_2 \in F''$. Consequently $v \in \beta(m_1+) \cap \beta(m_2+)$, and since $v \in V(A)$ and $\gamma(m_1), \gamma(m_2) \notin V(A)$ it follows that $v \neq \gamma(m_1), \gamma(m_2)$, and so $v \in \beta(m_1) \cap \beta(m_2)$. But this contradicts the truth of (ii).

Since in both cases we have obtained a contradiction, it follows that there is no such v, n', as required. ■

7. SOME CONSTRUCTIONS

Now we give three ways of making new portrayals from old ones, which will be of use later.

(7.1) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a $\mathcal{T}$–central portrayal of a hypergraph G. Let $\Delta \subseteq \Sigma$ be a disc with $bd(\Delta)$ Γ–normal and $|bd(\Delta) \cap N(\Gamma)| \leqslant 1$, such that $\alpha(\Delta)$ is small. Let $c' \in C(\Gamma)$ with $c' \subseteq \overline{\Sigma - \Delta}$, chosen with $\tilde{c}' \cap bd(\Delta) \neq \emptyset$ if possible. If $\tilde{c}' \cap bd(\Delta) \neq \emptyset$, let $\Gamma' = (U(\Gamma) \cap \overline{\Sigma - \Delta},\ N(\Gamma) \cap \overline{\Sigma - \Delta})$. If $\tilde{c}' \cap bd(\Delta) = \emptyset$, let $\Gamma' = (U(\Gamma) - \Delta,\ N(\Gamma) - \Delta)$. Define*

$$\begin{aligned} \alpha'(c') &= \alpha(c') \cup \alpha(\Delta) \\ \alpha'(c) &= \alpha(c) \ (c \in C(\Gamma') - \{c'\}) \\ \gamma' &= \gamma | N(\Gamma') \ . \end{aligned}$$

Then $\pi' = (\Sigma, \Gamma', \alpha', \beta, \gamma')$ is a $\mathcal{T}$–central portrayal of G resembling π.

The proof is clear.

(7.2) *Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a $\mathcal{T}$–central portrayal of a hypergraph G. Let $\Delta \subseteq \Sigma$ be a disc, with $bd(\Delta)$ Γ–normal and $|bd(\Delta) \cap N(\Gamma)| \leqslant 3$, such that $\alpha(\Delta)$ is small. Define $\Gamma' = (U(\Gamma) \cup \Delta, N(\Gamma) \cap \overline{\Sigma - \Delta})$; thus $c' = \Delta - (bd(\Delta) \cap N(\Gamma))$ is a cell of Γ'. Let $A \subseteq G$ be minimal such that $\alpha(\Delta) \subseteq A$ and $\gamma(N(\Gamma) \cap bd(\Delta)) \subseteq V(A)$. Define*

$$\begin{aligned} \alpha'(c') &= A \\ \alpha'(c) &= \alpha(c) \ (c \in C(\Gamma) - \{c'\}) \\ \gamma' &= \gamma | N(\Gamma') \ . \end{aligned}$$

Then $\pi' = (\Sigma, \Gamma', \alpha', \beta, \gamma')$ *is a* $\mathcal{T}$*–central portrayal of* G *resembling* π.

Again, the proof is clear.

We shall need a third construction, and to prove that it works we need the following lemma.

(7.3) *Let* $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ *be a portrayal of a hypergraph* G. *Let* n_1, n_2 *be distinct nodes bordering a cuff* Θ, *and let* $F \subseteq \Sigma$ *be a* Γ*–normal* I*–arc with ends* n_1, n_2 *and with* $F \cap N(\Gamma) = \{n_1, n_2\}$. *Let* $\Delta \subseteq \Sigma$ *be a disc with* $F \subseteq bd(\Delta) \subseteq F \cup \Theta$, *and for each* $n \in N(\Gamma) \cap bd(\Delta)$ *let* $\beta(n_1+) \cap \beta(n_2+) \subseteq \beta(n+)$. *Let* π *have warp* $\leqslant p$ *at* Θ, *and let* $|\beta(n_1+)| = |\beta(n_2+)| = p+1$. *Then every linkage in* $\alpha(\Delta)$ *from* $\beta(n_1+)$ *to* $\beta(n_2+)$ *pairs* $\gamma(n_1)$ *with* $\gamma(n_2)$.

Proof. Let $\{P_0, \ldots, P_p\}$ be a linkage in $\alpha(\Delta)$ from $\beta(n_1+)$ to $\beta(n_2+)$. Since $|\beta(n_1+)| = \beta(n_2+)| = p+1$ and $|\beta(n_1)|, |\beta(n_2)| \leqslant p$, it follows that equality holds and $n_1, n_2 \in dom(\gamma)$, and every vertex in $\beta(n_1+)$ is the initial vertex of one of $P_0, \ldots, P_n$. Let $\gamma(n_1)$ be the initial vertex of P_0 say. We must show that $\gamma(n_2)$ is the terminal vertex of P_0.

Let M be the graph $P_0 \cup \ldots \cup P_p$. For each internal cell c with $c \subseteq \Delta$, at most one of $P_0, \ldots, P_p$ has an edge in $\alpha(c)^\star$ since $|V(\alpha(c) \cap \alpha(-c))| \leqslant 3$, and so the graph $M \cap \alpha(c)^\star$ may be drawn in the disc $\overline{c}$ in the natural way. Let H be the union of these drawings (taken over all internal cells $c \subseteq \Delta$). Then H is a drawing in Δ of a subgraph of M denoted by $M \cap \Delta$. Every component of H is a path with both ends in I, where $I = \Theta \cap bd(\Delta)$.

Now H is a drawing in Δ; let R be the region of H in Δ whose closure includes F. Let the members of $N(\Gamma) \cap I$ in the closure of R be $m_1, m_2, \ldots, m_k$, in order in I. Hence $m_1 = n_1$ and $m_k = n_2$.

Choose i with $1 \leqslant i \leqslant k$ maximum so that $m_i \in dom(\gamma)$ and $\gamma(m_i) \in V(P_0)$, and suppose for a contradiction that $i < k$. Let S be the closed line segment with $S \subseteq bd(R)$ and with ends m_i, m_{i+1}, such that $S \cap F = \emptyset$. Then no internal point of S belongs to $N(\Gamma) \cap I$ by the definition of $m_1, \ldots, m_k$. But every $s \in S$ either belongs to the drawing H or belongs to $bd(\Delta)$, since $S \subseteq bd(R)$, and it follows that either $S \subseteq I$ or S is part of H. The latter is impossible since $\gamma(m_{i+1}) \notin V(P_0)$, and so $S \subseteq I$. Since no internal point of S is in $N(\Gamma) \cap I$, it follows that $S \cap N(\Gamma) = \{m_i, m_{i+1}\}$, and there is a border cell c with $S = \overline{c} \cap I$ and $\tilde{c} = \{m_i, m_{i+1}\}$.

Let I_1 be the part of I between n_1 and m_i, and let I_2 be that between m_{i+1}

and n_2. (Thus I_1 and I_2 are either closed line segments or singleton sets.) Let

$$B_h = \bigcap(\beta(n+) : n \in I_h \cap N(\Gamma)) \ (h = 1, 2) \ .$$

Let $0 \leqslant j \leqslant p$. Certainly P_j has initial vertex in B_1 and terminal vertex in B_2, and so there is a minimal subpath Q_j of P_j with initial vertex s_j (say) in B_1 and terminal vertex t_j in B_2. Consequently no internal vertex of Q_j is in $B_1 \cup B_2$. We claim that Q_j is a path of $\alpha(c)^\star$ from $\beta(m_i+)$ to $\beta(m_{i+1}+)$.

To show this, suppose first that $s_j = t_j$. Then $s_j \in \beta(n+) \cap \beta(n'+)$ where $n \in I_1 \cap N(\Gamma)$ and $n' \in I_2 \cap N(\Gamma)$. Suppose that $s_j \notin \beta(m_i+)$. Since n_1, n, m_i, n' are in order it follows that

$$s_j \in \beta(n+) \cap \beta(n'+) \subseteq \beta(n_1+) \cup \beta(m_i+)$$

and so $s_j \in \beta(n_1+)$; and similarly $s_j \in \beta(n_2+)$. Hence

$$s_j \in \beta(n_1+) \cap \beta(n_2+) \subseteq \beta(m_i+)$$

a contradiction. Thus $s_j \in \beta(m_i+)$ and similarly $s_j \in \beta(m_{i+1}+)$, and Q_j is a subpath of $\alpha(c)^\star$ from $\beta(m_i+)$ to $\beta(m_{i+1}+)$ as required.

We may therefore assume that $s_j \neq t_j$. ¿From the minimality of Q_j, no vertex of Q_j is in $B_1 \cap B_2$. Now $Q_j \nsubseteq M$, since every component of M has intersection with $B_1 \cup B_2$ either included in B_1 or in B_2. Since no internal vertex of Q_j is in $B_1 \cup B_2$ it follows that no edge of Q_j is in M. Hence for every edge e of Q_j there is a boundary cell $c' \subseteq \Delta$ with $e \in E(\alpha(c')^\star)$. Since no internal vertex of Q_j belongs to

$$V(\alpha(c') \cup \alpha(-c')) \subseteq B_1 \cup B_2$$

for any boundary cell c', we deduce that some boundary cell $c' \subseteq \Delta$, Q_j is a path of $\alpha(c')^\star$. But $V(Q_j)$ meets $B_1 - B_2$ and $B_2 - B_1$, and so $V(\alpha(c') \cap \alpha(-c'))$ is not a subset of B_1 or B_2. Consequently $c' = c$ and our claim holds.

It follows that $\{Q_0, \dots, Q_p\}$ is a linkage in $\alpha(c)$ from $\beta(m_i+)$ to $\beta(m_{i+1}+)$. Hence $|\beta(m_{i+1}+)| = p+1$ and so $m_{i+1} \in dom(\gamma)$; and from the warp condition for c, the linkage pairs $\gamma(m_i)$ with $\gamma(m_{i+1})$. But $\gamma(m_i)$ is the initial vertex of Q_0, and so $\gamma(m_{i+1})$ is its terminal vertex, and in particular, $\gamma(m_{i+1}) \in V(P_0)$ contrary to the maximality of i.

Hence $i = k$, and the result holds. ■

(7.4) *Let* $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ *be a* $(2p+3)$*–redundant* $\mathcal{T}$*–central portrayal of a hypergraph* G*, with warp* $\leqslant p$*, and let* $ord(\mathcal{T}) > 4p+5$*. Let* $F \subseteq \Sigma$ *be a* Γ*–normal* I*–arc with* $|F \cap N(\Gamma)| = 2$*. Let* $\Delta \subseteq \Sigma$ *be a disc with* $F \subseteq bd(\Delta) \subset F \cup bd(\Sigma)$*, such that* $\alpha(\Delta)$ *is small. Define* $\Gamma' = (U(\Gamma) \cup \Delta, N(\Gamma) \cap \overline{\Sigma - \Delta})$ *and let* c' *be the cell* $\Delta - (F \cap N(\Gamma))$*. Define*

$$\begin{aligned} \alpha'(c') &= \alpha(\Delta) \\ \alpha'(c) &= \alpha(c) \ (c \in C(\Gamma') - \{c'\}) \\ \beta' &= \beta|(N(\Gamma') \cap bd(\Sigma)) \\ \gamma' &= \gamma|N(\Gamma') \ . \end{aligned}$$

Then $\pi' = (\Sigma, \Gamma'\alpha', \beta', \gamma')$ *is a* $\mathcal{T}$*–central portrayal of* G *resembling* π*.*

Proof. **(P1)**–**(P5)** and **(P7)** are easily verified, and we omit them. Let the ends of F be n_1, n_2. For **(P6)**, it suffices to show that

$$V(\alpha'(c') \cap \alpha'(-c')) \subseteq \beta(n_1+) \cup \beta(n_2+) \ .$$

But this follows from (2.1), setting $I_1 = \{c \in C(\Gamma) : c \subseteq \Delta\}$, $I_2 = C(\Gamma) - I_1$.

Thus π' is a portrayal of G. Now $\alpha(\Delta)$ is small, and so π' is $\mathcal{T}$–central. Moreover, the warp in π' of each cell of $C(\Gamma') - \{c'\}$ is the same as in π. To complete the proof we must show that c' has warp in π' at most the warp (q say) of π at Θ, where Θ is the cuff of Σ containing n_1, n_2. (For then it will follow that π' resembles or is simpler than π; and it cannot be simpler, since π is 0–redundant.) By (5.1), $q \geqslant 1$.

Let $F' = bd(\Delta) \cap \Theta$; then F' is a $[0,1]$–arc with ends n_1, n_2. Let $F'' \subseteq \Theta$ be the other $[0,1]$–arc with ends n_1, n_2. ¿From (6.7) we deduce that for every $n \in N(\Gamma) \cap F', \beta(n_1+) \cap \beta(n_2+) \subseteq \beta(n+)$. ¿From (7.3), c' has warp $\leqslant q$ in π'. This completes the proof. ■

8. THE PAINTING OF A TRUE PORTRAYAL

Now we begin to use the results of the last two sections to show that, if G has a $\mathcal{T}$–central portrayal with reasonable redundancy, then it has a resembling $\mathcal{T}$–central portrayal with even more desirable properties.

Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a portrayal of G. Its *truth* is the sequence $(\tau_1, \ldots, \tau_6)$

where

$$\tau_i = |\{c \in C(\Gamma) : c \text{ is internal and } |\tilde{c}| = 4 - i\}| (i = 1, \ldots, 4)$$
$$\tau_5 = |N(\Gamma) - bd(\Sigma)|$$
$$\tau_6 = |N(\Gamma) \cap bd(\Sigma)| .$$

We order truths lexicographically; thus, if π' is another portrayal with truth $(\tau'_1, \ldots, \tau'_6)$, we say that π is *truer* than π' if for some k $(1 \leqslant k \leqslant 6)$, $\tau_i = \tau'_i$ for $1 \leqslant i < k$ and $\tau_k < \tau'_k$.

Let $\mathcal{T}$ be a tangle in a hypergraph G. A 0–redundant $\mathcal{T}$–central portrayal π is *true* (relative to $\mathcal{T}$) if there is no truer $\mathcal{T}$–central portrayal resembling π. (Certainly there is none simpler than π, since π is 0–redundant.) In this section we shall study properties of paintings in true portrayals. Throughout this section, $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ is a true, $(2p{+}3)$–redundant, $\mathcal{T}$–central portrayal of a hypergraph G with warp $\leqslant p$, and $ord(\mathcal{T}) > 4p + 5$.

(8.1) *Every region of* Γ *is an open disc.*

Proof. Otherwise, there is an O–arc $F \subseteq \Sigma$ with $F \cap U(\Gamma) = \emptyset$ such that there is no disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$ and $\Delta \cap U(\Gamma) = \emptyset$. But then (7.1), applied to the disc provided by (6.4), yields a truer $\mathcal{T}$–central portrayal resembling π, a contradiction. ■

(8.2) *The boundary of every region is an O–arc, and in particular,* $|\tilde{c}| \geqslant 2$ *for every cell* c.

Proof. The second claim follows from the first. If the first is false then there is a Γ–normal O–arc F with $|F \cap N(\Gamma)| = 1$, such that there is no disc $\Delta \subseteq F$ with $bd(\Delta) = F$ and $\Delta \cap U(\Gamma) \subseteq F$. Hence (7.1) and (6.4) yield a truer portrayal, a contradiction. ■

(8.3) *For every* Γ*–normal* I*–arc* F *with* $|F \cap N(\Gamma)| = 2$*, there is a disc* $\Delta \subseteq \Sigma$ *with* $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ *such that* $\Delta \cap N(\Gamma) \subseteq F$ *and* Δ *includes exactly one cell of* Γ.

Proof. By (6.3), there is a disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ such that $\alpha(\Delta)$ is small. But (7.4) applied to Δ does not yield a truer portrayal, and so the result holds. ■

(8.4) $dom(\gamma) = N(\Gamma)$.

Proof. Suppose that $n \in N(\Gamma) - dom(\gamma)$. If $n \in \tilde{c}$ and c is internal then shrinking

c from n yields a truer portrayal, a contradiction. Thus, $n \in \tilde{c}$ for only two cells c, both border cells. But this contradicts (8.3). ■

(8.5) *For every* Γ*–normal* O*–arc* $F \subseteq \Sigma$ *with* $|F \cap N(\Gamma)| \leqslant 3$*, there is a disc* $\Delta \subseteq \Sigma$ *with* $F = bd(\Delta)$ *such that* $\alpha(\Delta)$ *is small, and either*

(i) $\Delta \cap N(\Gamma) \subseteq F$ *and* Δ *includes at most one cell of* Γ*, or*
(ii) $|F \cap N(\Gamma)| = 3$*, and* $|\tilde{c}| = 2$ *for every cell* $c \subseteq \Delta$*.*

Proof. By (6.4), there is a disc $\Delta \subseteq \Sigma$ with $F = bd(\Delta)$ such that $\alpha(\Delta)$ is small. We may assume that Δ includes some cell, for otherwise (i) holds, by (8.2). We may therefore assume that there is a cell $c \subseteq \Delta$ with $|\tilde{c}| \geqslant |F \cap N(\Gamma)|$, for otherwise (ii) holds, by (8.2). But (7.2) applied to Δ does not yield a truer portrayal, and so (i) holds, as required. ■

(8.6) *If* c *is an internal cell, then* $|\overline{c} \cap bd(\Sigma)| \leqslant 2$*, and if equality holds then* $|\tilde{c}| = 3$ *and the two members of* $\tilde{c} \cap bd(\Sigma)$ *are consecutive.*

Proof. ¿From (8.3), any two members of $\overline{c} \cap bd(\Sigma)$ lie on the same cuff, and are consecutive nodes in that cuff. It follows that $|\overline{c} \cap bd(\Sigma)| \leqslant 2$ (unless $\overline{c} \cap bd(\Sigma) = N(\Gamma)$ and $|C(\Gamma)| \leqslant 4$, which is easily seen to be impossible). If $|\overline{c} \cap bd(\Sigma)| = 2$, then again there is an I–arc violating (8.3) unless $|\tilde{c}| \geq 3$, as required. ■

We define $C^\star(\Gamma)$ to be the set of all $c \in C(\Gamma)$ such that there is no $c' \in C(\Gamma)$ with $c' \neq c$ and $\tilde{c} \subseteq \tilde{c}'$.

(8.7) *For every node* n *there are at least two cells* $c \in C^\star(\Gamma)$ *with* $n \in \tilde{c}$*, and if there are only two and* $|\tilde{c}| = 2$ *for one of them, then* c *is a border cell.*

Proof. If there is at most one such c, then n is internal and (8.5) is violated. If there are only two, and $|\tilde{c}| = 2$ for one of them, then (8.5) is violated, unless c is a border cell. ■

(8.8) *If* c_1, c_2 *are distinct cells with* $\tilde{c}_1 \subseteq \tilde{c}_2$*, then* c_1 *is a border cell, and* $|\tilde{c}_2| = 3$

Proof. If c_1 is a border cell the claim follows from (8.6). Suppose then that c_1 is internal. If c_2 is a border cell, then $\tilde{c}_1 = \tilde{c}_2$ and (8.6) is violated. Thus c_2 is internal. Define $\Gamma' = (U(\Gamma) - c_1,\ N(\Gamma))$; this is a painting. Define

$$\alpha'(c_2) = \alpha(c_1) \cup \alpha(c_2)$$
$$\alpha'(c) = \alpha(c)\ (c \in C(\Gamma') - c_2)\ .$$

Then $(\Sigma, \Gamma', \alpha', \beta, \gamma)$ is a truer portrayal, a contradiction. ■

9. CELL CONNECTIVITY IN A TRUE PORTRAYAL

Again, throughout this section $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ is a true, $(2p+3)$–redundant, $\mathcal{T}$–central portrayal of a hypergraph G, with warp $\leqslant p$, and $ord(\mathcal{T}) > 4p+5$. Now we study connectivity properties of the $\alpha(c)$'s for individual cells c.

(9.1) *Let c_0 be an internal cell, and let $v_1, v_2 \in \gamma(\tilde{c}_0)$. Then there is a path in $\alpha(c_0)^\star$ from v_1 to v_2 with no other vertex in $\gamma(\tilde{c}_0)$.*

Proof. Otherwise there is a separation (A_1, A_2) of $\alpha(c)$ with $v_i \in V(A_i)$ $(i = 1, 2)$ and $V(A_1 \cap A_2) \subseteq \gamma(\tilde{c}_0) - \{v_1, v_2\}$. Hence $|V(A_1 \cap A_2)| \leqslant 1$. Choose discs $\Delta_1, \Delta_2 \subseteq \overline{c}_0$ such that

$$\Delta_1 \cap \Delta_2 = \{n \in \tilde{c}_0 : \gamma(n) \in V(A_1 \cap A_2)\}$$
$$\Delta_i \supseteq \{n \in \tilde{c}_0 : \gamma(n) \in V(A_i)\} \qquad (i = 1, 2)\ .$$

Let $c_i = \Delta_i - N(\Gamma)$ $(i = 1, 2)$, and let $\Gamma' = ((U(\Gamma) - c_0) \cup \Delta_1 \cup \Delta_2, N(\Gamma))$. Then Γ' is a painting in Σ with $C(\Gamma') = (C(\Gamma) - \{c_0\}) \cup \{c_1, c_2\}$. Define

$$\alpha'(c_1) = A_1, \alpha'(c_2) = A_2$$
$$\alpha'(c) = \alpha(c)\ (c \in C(\Gamma') - \{c_1, c_2\})\ .$$

Then $(\Sigma, \Gamma', \alpha', \beta, \gamma)$ is a $\mathcal{T}$–central portrayal of G resembling π but truer, a contradiction. ■

(9.2) *Let c_0 be an internal cell with $|\tilde{c}_0| = 3$, and $\tilde{c}_0 = \{n_1, n_2, n_3\}$. Then there do not exist subhypergraphs $A_1, A_2, A_3 \subseteq \alpha(c_0)$ and $v \in V(\alpha(c_0))$ such that $\gamma(n_i) \in V(A_i)$ $(i = 1, 2, 3)$, $A_1 \cup A_2 \cup A_3 = \alpha(c_0)$, and for $1 \leqslant i < j \leqslant 3$, $E(A_i \cap A_j) = \emptyset$ and $V(A_i \cap A_j) = \{v\}$.*

Proof. Suppose that such A_1, A_2, A_3 exist. By (9.1), $v \neq \gamma(n_1), \gamma(n_2), \gamma(n_3)$. Choose $n_0 \in c_0$, and discs $\Delta_i \subseteq \overline{c}_0$ with $n_i \in \Delta_i$ $(1 = 1, 2, 3)$ such that for $1 \leqslant i < j \leqslant 3$, $\Delta_i \cap \Delta_j = \{n_0\}$. Let $c_i = \Delta_i - \{n_0, n_i\}$ $(i = 1, 2, 3)$, and let

$$\Gamma' = ((U(\Gamma) - c_0) \cup \Delta_1 \cup \Delta_2 \cup \Delta_3, N(\Gamma) \cup \{n_0\})\ .$$

Define $\alpha'(c_i) = A_i$ $(i = 1, 2, 3)$ and $\alpha'(c) = \alpha(c)$ $(c \in C(\Gamma') - \{c_1, c_2, c_3\})$. Define $\gamma'(n_0) = v$, $\gamma'(n) = \gamma(n)$ $(n \in N(\Gamma') - \{n_0\})$. Then $(\Sigma, \Gamma', \alpha', \beta, \gamma')$ is a $\mathcal{T}$–central portrayal of G resembling π but truer, a contradiction. ■

(9.3) *Let c_0 be a cell bordering a cuff Θ with $\tilde{c}_0 = \{n_1, n_2\}$, and let π have warp q at Θ. Suppose that there is no internal cell c with $\tilde{c}_0 \subseteq \tilde{c}$. Then there are $q+1$*

mutually vertex–disjoint paths of $\alpha(c_0)$ *from* $\beta(n_1+)$ *to* $\beta(n_2+)$*, and in particular* $|\beta(n_i)| = q$ $(i = 1, 2)$.

Proof. Otherwise, there is a separation (A_1, A_2) of $\alpha(c_0)$ of order $\leqslant q$ with $\beta(n_1+) \subseteq V(A_1)$ and $\beta(n_2+) \subseteq V(A_2)$. Let r be the region of Γ with $\overline{r} \cap c_0 \neq \emptyset$. Then $\overline{r} \cap bd(\Sigma) = \{n_1, n_2\}$, for otherwise there would be an I–arc violating (8.3). Since there is no internal cell c with $\tilde{c}_0 \subseteq \tilde{c}$, it follows that there is an internal node n_0 incident with r. Let $\Delta_0 \subseteq r \cup \{n_0\} \cup \overline{c}_0$ be a disc with $\{n_0\} \cap \overline{c}_0 \subseteq \Delta_0$. Then $n_0, n_1, n_2 \in bd(\Delta_0)$; let $F_1 \subseteq bd(\Delta_0)$ be the line segment between n_0 and n_1 not containing n_2, and define F_2 similarly. For $i = 1, 2$, let $\Delta_i \subseteq r \cup \{n_0, n_i\}$ be a disc with $\Delta_0 \cap \Delta_i = F_i$. Let $\Sigma' = \Sigma - (\Delta_0 - (F_1 \cup F_2))$, let

$$\Gamma' = ((U(\Gamma) - c_0) \cup \Delta_1 \cup \Delta_2, N(\Gamma)) \ .$$

Then Γ' is a painting in Σ', and $c_i = \Delta_i - \{n_0, n_i\}$ is a cell of it $(i = 1, 2)$. For $i = 1, 2$, let A_i' be the hypergraph with $E(A_i') = E(A_i)$ and $V(A_i') = V(A_i) \cup \{\gamma(n_0)\}$. Define

$$\begin{aligned} \alpha'(c_i) &= A_i' \ (i = 1, 2) \\ \alpha'(c) &= \alpha(c) \ (c \in C(\Gamma) - \{c_0\}) \\ \beta'(n_0) &= V(A_1 \cap A_2) \\ \beta'(n) &= \beta(n) \ (n \in N(\Gamma) \cap bd(\Sigma)) \ . \end{aligned}$$

It is straightforward to verify that $\pi' = (\Sigma', \Gamma', \alpha', \beta', \gamma)$ is a portrayal of G. Moreover, it has warp $\leqslant q$ at Θ, for $|V(A_1 \cap A_2)| \leqslant q$, and there is no linkage in $\alpha'(c_1)$ between $\beta(n_1+)$ and $\beta(n_0+)$ of cardinality $q+1$ since $\gamma(n_0)$ is an isolated vertex of $\alpha'(c_1)$ and is not in $\beta(n_1+)$ (and similarly for $\alpha'(c_2)$). Since A_i is small and hence A_i' is small $(i = 1, 2)$ by (4.1)(ii), it follows that π' is $\mathcal{T}$–central. But it is truer than π, a contradiction. ■

(9.4) *Let* π *have warp* q *at a cuff* Θ*; then* $|\beta(n)| = q$ *for all* $n \in N(\Gamma) \cap \Theta$.

Proof. Let $n_1 \in N(\Gamma) \cap \Theta$, and suppose that $|\beta(n_1)| < q$. Let c_0 be a border cell with $n_1 \in \tilde{c}_0$. Since $|\beta(n_1)| < q$ there are not $q + 1$ mutually vertex–disjoint paths as in (9.3), and so by (9.3) there is an internal cell c_3 with $\tilde{c}_0 \subseteq \tilde{c}_3$. By (8.8) $|\tilde{c}_3| = 3$ and by (8.6) the third node of $\tilde{c}_3$ is internal. Let $\tilde{c}_3 = \{n_1, n_2, n_3\}$ where $\tilde{c}_0 = \{n_1, n_2\}$ and n_3 is internal.

Let $F_i \subseteq c_3 \cup \{n_i, n_3\}$ be a closed line segment with ends n_i, n_3 and $F_i \cap bd(\overline{c}_3) = \{n_i, n_3\}$ $(i = 1, 2)$, chosen so that $F_1 \cap F_2 = \{n_3\}$. Let $\Delta \subseteq \Sigma$ be the disc bounded by $F_1 \cup F_2 \cup (c_0 \cap \Theta)$ and let $\Sigma' = \Sigma - (\Delta - (F_1 \cup F_2))$. Let F_1' be the closed line segment in $bd(\overline{c}_3)$ between n_3 and n_1 not containing n_2, and define F_2' similarly. For $i = 1, 2$, let Δ_i be the disc bounded by $F_i \cup F_i'$, and let $c_i = \Delta_i - \{n_i, n_3\}$. Let

$$\Gamma' = (U(\Gamma) - (\Delta - (F_1 \cup F_2)),\ N(\Gamma))\ ;$$

then Γ' is a painting in Σ' and c_1, c_2 are cells of it. By (8.3) , $C(\Gamma') = (C(\Gamma) - \{c_0, c_3\}) \cup \{c_1, c_2\}$. Define

$$\begin{aligned} \alpha'(c_1) &= \alpha(c_0) \cup \alpha(c_3) \\ V(\alpha'(c_2)) &= \beta(n_1) \cup \{\gamma(n_1), \gamma(n_3)\},\ E(\alpha'(c_2)) = \emptyset \\ \alpha'(c) &= \alpha(c)\ (c \in C(\Gamma) - \{c_0, c_3\}) \\ \beta'(n_3) &= \beta(n_1+) \\ \beta'(n) &= \beta(n)\ (n \in N(\Gamma) \cup bd(\Sigma))\ . \end{aligned}$$

Let $\pi' = (\Sigma', \Gamma', \alpha', \beta', \gamma)$; we claim that π' is a portrayal of G. **(P1)**–**(P6)** are obvious; let us verify **(P7)**. Let Θ' be the cuff of Σ' corresponding to Θ, and let $m_1, m_2, m_3, m_4 \in N(\Gamma')$ border Θ' in order. We must show that

$$\beta(m_1+) \cap \beta(m_3+) \subseteq \beta(m_2+) \cup \beta(m_4+)\ .$$

We may assume that m_1, m_2, m_3, m_4 are all distinct, and that one of them is n_3; indeed, we may assume that $m_1 = n_3$ or $m_2 = n_3$. If $m_1 = n_3$, then n_1, m_2, m_3, m_4 are in order around Θ (although they are not necessarily distinct), and so

$$\beta(n_1+) \cap \beta(m_3+) \subseteq \beta(m_2+) \cup \beta(m_4+)\ .$$

Since $\beta(n_1+) \cap \beta(m_3+) = \beta'(m_1+) \cap \beta(m_3+)$ since $\gamma(n_3) \notin \beta(m_3+)$, the claim follows. If $m_2 = n_3$ it follows similarly, since now m_1, n_1, m_3, m_4 are in order around Θ.

Thus, π' is a portrayal, and by (4.1)(ii) it is $\mathcal{T}$–central. It remains to check its warp, and it suffices to check the warp of c_1 and c_2 in π'. Since $|\beta'(n_1+)| \leqslant q$, it follows that c_1 has warp $\leqslant q$. Let $\{P_0, \ldots, P_q\}$ be a linkage in $\alpha'(c_2)$ from $\beta'(n_2+) = \beta(n_2+)$ to $\beta'(n_3+) = \beta(n_1+) \cup \{\gamma(n_3)\}$. Then $\gamma(n_3)$ is the terminal vertex of one of these paths, say P_0; and we must show that P_0 has initial vertex $\gamma(n_2)$. But $\gamma(n_1), \gamma(n_2)$ are not internal vertices of P_0, since each of them is an end of one of $P_0, \ldots, P_q$. But $\gamma(n_3) \notin \alpha(c_0)$, and so the edge of P_0 incident with $\gamma(n_3)$

is an edge of $\alpha(c_3)^\star$. Since no internal vertex of P_0 is in $\gamma(c_3)$ it follows that P_0 is a path of $\gamma(c_3)$, and hence its initial vertex is in

$$\beta(n_2+) \cap V(\alpha(c_3)) = \{\gamma(n_2)\} \cup (\{\gamma(n_1)\} \cap \beta(n_2)) .$$

But $\gamma(n_1)$ is the terminal vertex of one of $P_1, \ldots, P_q$ and so is not the initial vertex of P_0; and so $\gamma(n_2)$ is the initial vertex of P_0 as required.

We conclude that π' is a $\mathcal{T}$–central portrayal of G, resembling or simpler than π. But it is truer, a contradiction. ■

We shall need the following standard lemma from network flow theory. If H is a graph and $X \subseteq V(H)$, we denote by $H \backslash X$ the graph obtained by deleting X.

(9.5) *Let H be a graph, let $A, B \subseteq V(H)$, and let $X \subseteq A$ and $Y \subseteq B$ with $|X| = |Y| = q$. Suppose that there is a linkage in $H \backslash (A - X)$ from X to a q–subset of B, and there is a linkage in $H \backslash (B - Y)$ from a q–subset of A to Y. Then either there is a linkage in $H \backslash ((A-X) \cup (B-Y))$ from X to Y, or there are $q+1$ mutually vertex–disjoint paths in H from A to B.*

(9.5) is used to prove the following.

(9.6) *Let c_0 be a cell bordering a cuff Θ with $\tilde{c}_0 = \{n_1, n_2\}$, and let π have a warp q at Θ. Then there are q mutually vertex–disjoint paths of $\alpha(c_0)^\star$ from $\beta(n_1)$ to $\beta(n_2)$, each containing neither $\gamma(n_1)$ nor $\gamma(n_2)$.*

Proof. We claim first that there is a linkage in $\alpha(c_0)^\star \backslash \{\gamma(n_1)\}$ from $\beta(n_1)$ to a subset of $\beta(n_2+)$ of cardinality q. For suppose not; then by Menger's theorem, there is a separation (A_1, A_2) of $\alpha(c_0)$ of order q with $\beta(n_1+) \subseteq V(A_1)$ and $\beta(n_2+) \cup \{\gamma(n_1)\} \subseteq V(A_2)$. Let c_1 be the border cell with $c_1 \neq c_0$ and $n_1 \in \tilde{c}_1$. Define

$$\begin{aligned}
\alpha'(c_0) &= A_2 \\
\alpha'(c_1) &= A_1 \cup \alpha(c_1) \\
\alpha'(c) &= \alpha(c) \ (c \in C(\Gamma) - \{c_0, c_1\}) \\
\beta'(n_1) &= V(A_1 \cap A_2) - \{\gamma(n_1)\} \\
\beta'(n) &= \beta(n) \ (n \in N(\Gamma) \cap bd(\Sigma) - \{n_1\}) .
\end{aligned}$$

Then $\pi' = (\Sigma, \Gamma, \alpha', \beta', \gamma)$ is a Γ–central portrayal resembling or simpler than π. (Both c_1 and c_2 have warp $\leqslant q$ in π' because $|\beta'(n_1)| < q$.) Hence π' resembles π, and (since Γ is unchanged) π' is true; but $|\beta'(n_1)| < q$, contrary to (9.4).

Thus there is a linkage in $\alpha(c_0)\backslash\{\gamma(n_1)\}$ from $\beta(n_1)$ to a q–subset of $\beta(n_2+)$, and similarly there is a linkage in $\alpha(c_0)\backslash\gamma(n_2)$ from $\beta(n_2)$ to a q–subset of $\beta(n_1+)$. ¿From (9.5) we deduce that either there is a linkage in $\alpha(c_0)\backslash\{\gamma(n_1),\gamma(n_2)\}$ from $\beta(n_1)$ to $\beta(n_2)$ (and hence the theorem holds) or there is a linkage $\{P_0,\dots,P_q\}$ of cardinality $q+1$ from $\beta(n_1+)$ to $\beta(n_2+)$. We may therefore assume the latter. By the warp condition it pairs $\gamma(n_1)$ with $\gamma(n_2)$, and so we may assume P_0 has initial vertex $\gamma(n_1)$ and terminal vertex $\gamma(n_2)$; but then $P_1,\dots,P_q$ satisfies the theorem. ■

(9.7) *For every border node $n, \beta(n)\cap\gamma(N(\Gamma))=\emptyset$.*

Proof. Let n border a cuff Θ, and suppose that $\gamma(n')\in\beta(n)$ for some $n'\in dom(\gamma)=N(\Gamma)$. By **(P4)**, n' borders Θ. Let c_1,c_2 be the border cells with $n'\in\tilde{c}_1,\tilde{c}_2$ and let $\tilde{c}_i=\{n',n_i\}$ $(i=1,2)$. Then n,n_1,n',n_2 are in order, and so

$$\gamma(n')\in\beta(n+)\cap\beta(n'+)\subseteq\beta(n_1+)\cup\beta(n_2+)$$

and we may assume that $\gamma(n')\in\beta(n_1+)$. Since $n'\neq n_1$ and so $\gamma(n')\neq\gamma(n_1)$, it follows that $\gamma(n')\in\beta(n_1)$. But by (9.6) there is a linkage from $\beta(n')$ to $\beta(n_1)$ not passing through $\gamma(n')$, a contradiction. ■

We summarize (9.3), (9.4), (9.6), (9.7) in the following.

(9.8) *Let c_0 be a border cell with $c_0=\{n_1,n_2\}$. Then*

(i) $|\beta(n_1)|=|\beta(n_2)|=q$ *and* $\beta(n_1)\cap\gamma(N(\Gamma)),\ \beta(n_2)\cap\gamma(N(\Gamma))=\emptyset$

(ii) *there is a linkage $\{P_1,\dots,P_q\}$ in $\alpha(c_0)\backslash\{\gamma(n_1),\gamma(n_2)\}$ from $\beta(n_1)$ to $\beta(n_2)$ such that either there is a path P_0 such that $\{P_0,P_1,\dots,P_q\}$ is a linkage in $\alpha(c_0)$ from $\beta(n_1+)$ to $\beta(n_2+)$, or there is an internal cell c with $|\tilde{c}|=3$ and $\tilde{c}_0\subseteq\tilde{c}$.*

10. CIRCUMNAVIGATING A VORTEX

Throughout this section, $\pi=(\Sigma,\Gamma,\alpha,\beta,\gamma)$ is a true, $\mathcal{T}$–central portrayal of a hypergraph G, with warp p, and it is z–redundant where $z\geqslant 2p+3$, and $ord(\mathcal{T})>z+2p+2$.

Assign arbitrarily an orientation to each cuff of Σ, called *clockwise*. Then for each cell c bordering a cuff Θ, we may call one member of $\tilde{c}$ the *tail* of c and the other its *head*, so that the head of each cell is the tail of the next as the cuff is traversed clockwise. Let c be a border cell with tail n_1 and head n_2, and let c have warp q in π. By (9.8) we may choose a linkage $\{Q_1,\dots,Q_q\}$ of $\alpha(c)$ from

$\beta(n_1)$ to $\beta(n_2)$ such that $\gamma(n_1), \gamma(n_2) \notin V(Q_1), \ldots, V(Q_q)$, and either there is a linkage $\{Q_0, Q_1, \ldots, Q_q\}$ of $\alpha(c)$ from $\beta(n_1+)$ to $\beta(n_2+)$, or there is an internal cell c' with $\tilde{c} \subseteq \tilde{c}'$. Choose such a linkage $\{Q_1, \ldots, Q_q\}$ arbitrarily; we call it the *standard linkage* in $\alpha(c)$.

Let Θ be a cuff, let π have warp q at Θ, let n_0 be a node bordering Θ, and let $v \in \beta(n_0)$. Let the nodes and cells bordering Θ be $n_0, c_1, n_1, c_2, \ldots, c_k, n_k = n_0$, in clockwise order. Let $v_0 = v$, and inductively, having defined $v_0, \ldots, v_{i-1}$ and $P_1, \ldots, P_{i-1}$ for some $1 \leqslant i \leqslant k$, let P_i be the path of the standard linkage in $\alpha(c_i)$ from $\beta(n_{i-1})$ to $\beta(n_i)$ with initial vertex v_{i-1}, and let v_i be its terminal vertex. From this inductive definition we see that $v_i \in \beta(n_i)$ $(0 \leqslant i \leqslant k)$. We define $L(v, n_0)$ to be $P_1 \cup \ldots \cup P_k$.

(10.1) *With notation as before, if $v_k \neq v$ then $L(v, n_0)$ is a path, and if $v_k = v$ then $L(v, n_0)$ is a circuit; and for distinct $v, v' \in \beta(n_0), L(v, n_0)$ and $L(v', n_0)$ are edge–disjoint, and if some vertex u belongs to both of them then both are paths and u is an initial vertex of one and a terminal vertex of the other.*

The proof (using (6.6) with $|X| = 1$ and **(P6)** and **(P7)**) is clear but lengthy, and we omit it.

(10.2) *There is no $(A, B) \in \mathcal{T}$ of order $\leqslant z$ with $V(L(v, n_0)) \subseteq V(A)$.*

Proof. Suppose that such an (A, B) exists, and choose it with B minimal. By [1, theorem (2.8)], $L(v, n_0) \subseteq A$. Define $P_1, \ldots, P_k$ as before, and then we see that for $1 \leqslant i \leqslant k, P_i$ is a path of $(A \cap \alpha(c_i))^\star$ from $\beta(n_{i-1})$ to $\beta(n_i)$. But this contradicts (6.5). ■

(10.3) *For each cuff Θ, there is no $(A, B) \in \mathcal{T}$ of order $\leqslant z$ with $\gamma(N(\Gamma) \cap \Theta) \subseteq V(A)$.*

Proof. Suppose that there is such an (A, B), and choose it with A maximal and B minimal. By [1, theorem (2.8)] applied to a separation (B_1, B_2) of B with $B_1 \cap B_2 = A \cap B$, we deduce that $G/V(A)$ is connected. By (4.3), since A has $\leqslant z < ord(\mathcal{T}) - 2p$ attachments, there exists $n_0 \in N(\Gamma)$ with $\gamma(n_0) \notin V(A)$. By (5.1) and (10.2), there exists $c_0 \in C(\Gamma)$ bordering Θ with $V(\alpha(c_0)) \nsubseteq V(A)$. Since $G/V(A)$ is connected there is a path P of $(G/V(A))^\star$ with one end $\gamma(n_0)$ and the other in $V(\alpha(c_0))$. But by (2.1), (A_1, A_2) is a separation of G where

$$A_1 = \cup(\alpha(c) : c \in C(\Gamma) \text{ borders } \Theta)$$
$$A_2 = \cup(\alpha(c) : c \in C(\Gamma) \text{ does not border } \Theta)$$

and $V(A_1 \cap A_2) \subseteq \gamma(N(\Gamma) \cap \Theta)$. Since P has one end in $V(A_1)$ and the other in $V(A_2)$ (for $\gamma(n_0) \in V(A_2)$ by (8.7)) it follows that $V(P \cap A_1 \cap A_2) \neq \emptyset$. But $V(P) \subseteq V(G) - V(A)$, and

$$V(A_1 \cap A_2) \subseteq \gamma(N(\Gamma) \cap \Theta) \subseteq V(A) ,$$

a contradiction. ■

11. SURFACE SEPARATIONS IN A TRUE PORTRAYAL

This section is devoted to analyzing separations $(A, B) \in \mathcal{T}$ with $\alpha(c_0) \subseteq A$ of order $\leqslant |V(\alpha(c_0) \cap \alpha(-c_0))|$, for cells c_0 in a true portrayal. Throughout this section $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ is a true, $(2p+4)$–redundant, $\mathcal{T}$–central portrayal of a hypergraph G, with warp $\leqslant p$, and $ord(\mathcal{T}) > 6p + 8$. For $I \subseteq C(\Gamma)$ we define $\gamma(I) = \cup(\gamma(\tilde{c}) : c \in I)$. We begin with the following.

(11.1) *Let I, J be a partition of $C(\Gamma)$ with $\gamma(I), \gamma(J) \neq N(\Gamma)$ and $|\gamma(I) \cap \gamma(J)| \leqslant 3$. Then $|\gamma(I) \cap \gamma(J)| = 3$ and there is a disc $\Delta \subseteq \Sigma$ such that:*

- (i) *$\{c \in C(\Gamma) : c \subseteq \Delta\}$ is one of I, J*
- (ii) *$\alpha(\Delta)$ is small, and*
- (iii) *either $bd(\Delta)$ is a Γ–normal O–arc in Σ and $N(\Gamma) \cap bd(\Delta) = \gamma(I) \cap \gamma(J)$, or $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ and $N(\Gamma) \cap F = \gamma(I) \cap \gamma(J)$ for some Γ–normal I–arc F.*

Proof. Let $\widehat{\Sigma}$ be a surface with $bd(\widehat{\Sigma}) = \emptyset$, obtained from Σ by pasting a disc onto each cuff of Σ. We may regard Γ as a painting in $\widehat{\Sigma}$. For each region r of Γ in $\widehat{\Sigma}$, r is an open disc and $\overline{r} - r$ is an O–arc by (8.1) and (8.2). Choose a representative point from each region of Γ in $\widehat{\Sigma}$, and let H be the simple bipartite graph with vertex set the union of $N(\Gamma)$ and the set of all these representative points, in which the point representing a region r is adjacent to $n \in N(\Gamma)$ if $n \in \overline{r}$. Take a drawing of H in $\widehat{\Sigma}$ in the natural way, and for each circuit C of H let $U(C)$ be the O–arc in $\widehat{\Sigma}$ corresponding to C in the drawing.

(1) *For every circuit C of H of length $\leqslant 6$, either*

- (i) *$U(C)$ is a Γ–normal O–arc in Σ, or*
- (ii) *$U(C) \cap \Sigma$ is a Γ–normal I–arc in Σ, or*
- (iii) *$U(C) \cap N(\Gamma) = \{n_1, n_2, n_3\} \subseteq \Theta$ for some cuff Θ of Σ, and $U(C) \cap \Sigma = F_1 \cup F_2$ where F_1, F_2 are Γ–normal I–arcs with $F_i \cap N(\Gamma) = \{n_i, n_3\}$ $(i = 1, 2)$ and $F_1 \cap F_2 = \{n_3\}$.*

Subproof. Since $|E(C)| \leqslant 6$ it follows that $|U(C) \cap N(\Gamma)| \leqslant 3$ and so $U(C)$ meets at most one component of $\widehat{\Sigma} - \Sigma$. If $U(C) \subseteq \Sigma$ then (i) holds, and so we assume that for some component r of $\widehat{\Sigma} - \Sigma$, the point v representing r belongs to C. Its neighbours in C are both nodes $n_1, n_2 \in N(\Gamma)$ bordering the corresponding cuff Θ. If no other point of $U(C) \cap \Sigma$ belongs to $bd(\Sigma)$ then (ii) holds; and if some point $n_3 \in U(C) \cap \Sigma$ belongs to $bd(\Sigma)$, then $n_3 \in N(\Gamma)$ and $n_3 \in \Theta$ by (6.3), and so (iii) holds. This proves (1).

¿From (1), (6.3) and (6.4) it follows that for every circuit C of H of length $\leqslant 6$, there is a disc $ins(C) \subseteq \widehat{\Sigma}$ bounded by $U(C)$ such that $\alpha(ins(C))$ is small. ¿From (1), (8.3) and (8.4), we deduce that:

(2) *For every circuit C of H with $|E(H)| = 4$, $ins(C)$ is the closure of a region of H in $\widehat{\Sigma}$.*

Now every region of H includes a unique cell of Γ, and every cell of Γ is in a unique region. Let $\delta(I)$ be the subgraph of H consisting of all edges e of H (and their ends) such that one of the regions of H incident with e includes a cell in I, and the other includes a cell in $C(\Gamma) - I$. We see that

(3) *$E(\delta(I)) \neq \emptyset$, and every vertex of $\delta(I)$ has even valency in $\delta(I)$, and every edge of $\delta(I)$ has an end in $V(\delta(I)) \cap N(\Gamma)$; and*

$$V(\delta(I)) \cap N(\Gamma) \subseteq \gamma(I) \cap \gamma(J)$$

and consequently $|V(\delta(I)) \cap N(\Gamma)| \leqslant |\gamma(I) \cap \gamma(J)| \leqslant 3$.

We claim

(4) *$\delta(I)$ is a circuit of length 6.*

Subproof. We prove (4) by induction on $|E(\delta(I))|$. We suppose, for a contradiction, that (4) is false, and hence $\delta(I)$ has a circuit, C say, of length 4, by (3). By (2), $ins(C)$ includes a unique cell c_0 of Γ, and $u, v \in U(C)$ where $\tilde{c}_0 = \{u, v\}$. If $c_0 \in I$ let $I' = I - \{c_0\}$, and if $c_0 \notin I$ let $I' = I \cup \{c_0\}$. Let $J' = C(\Gamma) - I'$. Then $\gamma(I') \subseteq \gamma(I)$ and $\gamma(J') \subseteq \gamma(J)$, since $u, v \in \gamma(I) \cup \gamma(J)$. Consequently, $\gamma(I'), \gamma(J') \neq N(\Gamma)$, and $|\gamma(I') \cap \gamma(J')| \leqslant 3$. Moreover, $E(\delta(I')) = E(\delta(I)) - E(C)$, and so from the inductive hypothesis, $\delta(I')$ is a circuit of length 6. By (3), $V(\delta(I)) \cap N(\Gamma) \subseteq V(\delta(I'))$, and so there is a 2–edge path joining u and v in $\delta(I')$. Consequently there are three 2–edge paths joining u and v in H, contrary to (2). This proves (4).

Let $\delta(I) = D$. Now $U(D)$ partitions $C(\Gamma)$ into two sets, those cells within

$ins(D)$ and the remainder, and by definition of $\delta(I)$, one of these sets is I and the other is J. By applying (1) to D we see there are three cases.

Case 1: $U(D)$ is a Γ–normal O–arc in Σ.

Let $\Delta = ins(D)$; then the theorem holds.

Case 2: $U(D) \cap \Sigma = F$ is a Γ–normal I–arc in Σ.

Let Δ be the disc provided by (6.3); then the theorem holds.

Case 3: $U(D) \cap N(\Gamma) = \{n_1, n_2, n_3\} \subseteq \Theta$ for some cuff Θ, and $U(C) \cap \Sigma = F_1 \cup F_2$ where F_1, F_2 are Γ–normal I–arcs with $F_i \cap N(\Gamma) = \{n_1, n_3\}$ $(i = 1, 2)$ and $F_1 \cap F_2 = \{n_3\}$.

By applying (8.3) to F_1 and to F_2, we see that $ins(D) \cap N(\Gamma) = \{n_1, n_2, n_3\}$ and so one of $\gamma(I), \gamma(J) = N(\Gamma)$, a contradiction.

This completes the proof. ■

(11.2) *Let $(A, B) \in \mathcal{T}$ have order $< ord(\mathcal{T}) - 4p - 3$, let $c_0 \in C(\Gamma)$ with $\alpha(c_0) \subseteq A$, let I, J be a partition of $C(\Gamma)$ so that $\gamma(I) \subseteq V(A)$ and $\gamma(J) \subseteq V(B)$ and let*

$$|\{n \in N(\Gamma) : \gamma(n) \in V(A \cap B)\}| \leqslant |\tilde{c}_0| \ .$$

Then equality holds, and either

(i) *$\{n \in N(\Gamma) : \gamma(n) \in V(A)\} = \tilde{c}_0$ and $\gamma(n) \in V(B)$ for all $n \in N(\Gamma)$, or*

(ii) *$|\tilde{c}_0| = 3$, and there is a Γ–normal I–arc F and a disc $\Delta \subseteq \Sigma$ with $\alpha(\Delta)$ small and with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$, such that $I = \{c \in C(\Gamma) : c \subseteq \Delta\}$ and for all $n \in N(\Gamma)$, $\gamma(n) \in V(A)$ if and only if $n \in \Delta$, and $\gamma(n) \in V(B)$ if and only if $n \in \overline{\Sigma - \Delta}$.*

Proof. Let $X = \{n \in N(\Gamma) : \gamma(n) \in V(A)\}$, and $Y = \{n \in N(\Gamma) : \gamma(n) \in V(B)\}$. Then $X \cup Y = N(\Gamma)$, $\tilde{c}_0 \subseteq X$, $|X \cap Y| \leqslant |\tilde{c}_0|$ and $\tilde{c} \subseteq X$ or $\tilde{c} \subseteq Y$ for every cell c. Moreover, $\gamma(I) \subseteq X$ and $\gamma(J) \subseteq Y$, and so

$$|\gamma(I) \cap \gamma(J)| \leqslant |X \cap Y| \leqslant |c_0| \leqslant 3 \ .$$

Since A is small and has $< ord(\mathcal{T}) - 2p$ attachments, it follows from (4.3) that $\gamma(n) \notin V(A)$ for some $n \in N(\Gamma)$, and consequently $X \neq N(\Gamma)$ and so $\gamma(I) \neq N(\Gamma)$. Suppose first that $\gamma(J) \neq N(\Gamma)$. Then by (11.1), $|\gamma(I) \cap \gamma(J)| = 3$ and there is a disc Δ as in (11.1). Since

$$|\gamma(I) \cap \gamma(J)| \leqslant |X \cap Y| \leqslant |c_0| \leqslant 3$$

it follows that equality holds throughout, and in particular $\gamma(I) \cap \gamma(J) = X \cap Y$, and $|X \cap Y| = |c_0| = 3$. Moreover, $X = \gamma(I)$ and $Y = \gamma(J)$. Since A has $< ord(\mathcal{T}) - 4p - 3$ attachments and $\alpha(\Delta)$ has $\leqslant 2p + 3$ attachments, it follows that $A \cup \alpha(\Delta)$ has $< ord(\mathcal{T}) - 2p$ attachments; and since A and $\alpha(\Delta)$ are small, we deduce from (4.1)(ii) that $A \cup \alpha(\Delta)$ is small. By (4.3) there exists $n \in N(\Gamma)$ with $\gamma(n) \notin V(A \cup \alpha(\Delta))$. Hence $n \notin \Delta$, and $n \notin X$, and so every cell c with $n \in \tilde{c}$ belongs to J. Since there is such a cell, it follows that $\{c \in C(\Gamma) : c \subseteq \Delta\} \neq J$, and so $\{c \in C(\Gamma) : c \subseteq \Delta\} = I$. We deduce that $X \subseteq \Delta \cap N(\Gamma)$ and $Y \subseteq \overline{\Sigma - \Delta} \cap N(\Gamma)$ (because $X = \gamma(I)$ and $Y = \gamma(J)$).

Let us investigate the two cases of (11.1)(iii). Suppose, first, that $bd(\Delta)$ is a Γ–normal O–arc in Σ. Since $c_0 \in I$ and hence $c_0 \subseteq \Delta$, it follows from (8.5) that $\Delta \cap N(\Gamma) \subseteq bd(\Delta)$, and so $Y = N(\Gamma)$ contrary to our assumption. Thus $bd(\Delta)$ is not a Γ–normal O–arc in Σ, and by (11.1)(iii) there is a Γ–normal I–arc F with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ and $N(\Gamma) \cap F = \gamma(I) \cap \gamma(J)$. Then $N(\Gamma) \cap F = X \cap Y$, and (11.2)(iii) holds. We conclude that the theorem is true if $\gamma(J) \neq N(\Gamma)$.

Now let us assume that $\gamma(J) = N(\Gamma)$, and consequently $Y = N(\Gamma)$. Since $|X \cap Y| \leqslant |\tilde{c}_0|$ we deduce that $|X| \leqslant |\tilde{c}_0|$; and since $\tilde{c}_0 \subseteq X$ it follows that $X = \tilde{c}_0$. But then (11.2)(i) holds, as required. ■

(11.3) *Let $F \subseteq \Sigma$ be a Γ–normal I–arc with $F \cap N(\Gamma) = \{n_1, n_2, n_3\}$, where F has ends n_1, n_2, and let $\Delta \subseteq \Sigma$ be the disc with $\alpha(\Delta)$ small and $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$. Suppose that there is a cell $c_0 \subseteq \Delta$ with $|\tilde{c}_0| = 3$. Then either*

- (i) $\tilde{c}_0 = \Delta \cap N(\Gamma)$ *or*
- (ii) *there is no separation (A_1, A_2) of $\alpha(\Delta)$ with $V(A_1 \cap A_2) = \{\gamma(n_1), \gamma(n_2)\}$, $\gamma(n_3) \in V(A_1)$, and $\beta(n_1) \cup \beta(n_2) \subseteq V(A_2)$.*

Proof. Suppose that (ii) is false, and let (A_1, A_2) be a separation as in (ii). Let $F_0 = bd(\Delta) \cap bd(\Sigma)$ and let $F_1, F_2 \subseteq \Delta$ be closed line segments, both with ends n_1, n_2 and with

$$F_1 \cap F_2 = (F_1 \cup F_2) \cap bd(\Delta) = \{n_1, n_2\} ,$$

where F_2 is a subset of the disc in Δ bounded by $F_1 \cup F_0$. Let $\Delta_2 \subseteq \Delta$ be the disc bounded by $F_0 \cup F_2$ and let $\Delta_1 \subseteq \Delta$ be the disc bounded by $(bd(\Delta) - F_0) \cup F_1$. Let

$$\Gamma' = ((U(\Gamma) - \Delta) \cup \Delta_1 \cup \Delta_2, (N(\Gamma) - \Delta) \cup \{n_1, n_2, n_3\}) .$$

Then Γ' is a painting in Σ and $c_1 = \Delta_1 - \{n_1, n_2, n_3\}$, $c_2 = \Delta_2 - \{n_1, n_2\}$ are cells of it, and $C(\Gamma') = \{c_1, c_2\} \cup \{c \in C(\Gamma) : c \not\subseteq \Delta\}$. Define

$$\begin{aligned}
\alpha'(c_1) &= A_1 \\
\alpha'(c_2) &= A_2 \\
\alpha'(c) &= \alpha(c) \ (c \in C(\Gamma') - \{c_1, c_2\}) \\
\beta'(n) &= \beta(n) \ (n \in N(\Gamma') \cap bd(\Sigma)) \\
\gamma'(n) &= \gamma(n) \ (n \in N(\Gamma')) \ .
\end{aligned}$$

Then it is easy to check that $\pi' = (\Sigma, \Gamma', \alpha', \beta', \gamma')$ is a portrayal of G. Since $\alpha(\Delta)$ is small it follows that A_1 and A_2 are small, and so π' is $\mathcal{T}$–central. Let us check its warp. It suffices to check the warp condition for c_2. Now $\beta(n_1+) \cap \beta(n_2+) \subseteq \beta(n+)$ for every $n \in N(\Gamma) \cap F_0$, by (6.7) with $z = 2p+4$, since π is $(2p+4)$–redundant and $ord(\mathcal{T}) > 4p+6$. By applying (7.3) to the "restriction" of π to A_2 we deduce that the warp condition holds for c_2. Consequently, π' is a $\mathcal{T}$–central portrayal of G, resembling π. Since it is not truer, and yet Δ includes a cell c_0 of Γ with $|\tilde{c}_0| = 3$, it follows that (i) holds. ■

We denote by S the union, over all border cells c, of the vertex set of the standard linkage in $\alpha(c)$. For any separation (A, B) of G we define

$$D(A, B) = \{c \in C^\star(\Gamma) : \gamma(\tilde{c}) \not\subseteq V(A) \ \text{ and } \ \gamma(\tilde{c}) \not\subseteq V(B)\} \ .$$

(We remind the reader that $C^\star(\Gamma)$ was defined immediately before (8.7).)

(11.4) *For any separation (A, B) of G, if $d \in D(A, B)$ then*

$$V(A \cap B) \cap (V(\alpha(d)) - (S \cup V(\alpha(-d)))) \neq \emptyset \ .$$

Proof. There exists $u_1 \in \gamma(\tilde{d}) - V(A)$ and $u_2 \in \gamma(\tilde{d}) - V(B)$ since $\gamma(\tilde{d}) \not\subseteq V(A)$ and $\gamma(\tilde{d}) \not\subseteq V(B)$. There is a path P of $\alpha(d)^\star$ from u_1 to u_2 with no internal vertex in $S \cup V(\alpha(-d))$; for if d is internal then this follows from (9.1) since

$$V(\alpha(d)) \cap (S \cup V(\alpha(-d))) = \gamma(\tilde{d}) \ ,$$

while if d is a border cell then since $d \in C^\star(\Gamma), P$ exists by definition of the standard linkage. Since $u_1 \in V(B) - V(A)$ and $u_2 \in V(A) - V(B)$, there is an internal vertex v of P in $V(A \cap B)$. But then

$$v \in V(A \cap B) \cap V(\alpha(d)) - (S \cup V(\alpha(-d)))$$

as required. ■

(11.5) *For any separation* (A,B) *of* G,

$$|D(A,B)| \leqslant |V(A\cap B) - (S \cup \gamma(N(\Gamma)))| \ .$$

Proof. For each $d \in D(A,B)$, there exists

$$v_d \in V(A\cap B) \cap (V(\alpha(d)) - (S \cup V(\alpha(-d))))$$

by (11.4). Since $v_d \in V(\alpha(d)) - V(\alpha(-d))$, it follows that the vertices v_d ($d \in D(A,B)$) are all distinct. Moreover, each v_d belongs to $V(A\cap B)$, and not to $S \cup \gamma(N(\Gamma))$, since $\gamma(N(\Gamma)) \subseteq V(\alpha(-d))$. Hence

$$|D(A,B)| = |\{v_d : d \in D(A,B)\}| \leqslant |V(A\cap B) - (S \cup \gamma(N(\Gamma)))|$$

as required. ■

Throughout the remainder of this section, $c_0 \in C(\Gamma)$ and $(A,B) \in \mathcal{T}$ with $\alpha(c_0) \subseteq A$, satisfying

$$|V(A\cap B)| \leqslant |V(\alpha(c_0) \cap \alpha(-c_0))| \ .$$

(11.6) $|V(A\cap B) - S| \leqslant |\tilde{c}_0|$, *and if equality holds then*

$$|V(A\cap B)\cap S| = |V(\alpha(c_0)\cap\alpha(-c_0))\cap S| \ .$$

Proof. Since $|V(A\cap B)| \leqslant |V(\alpha(c_0)\cap\alpha(-c_0))|$, it follows that

$$|V(A\cap B)\cap S| + |V(A\cap B) - S| \leqslant |V(\alpha(c_0)\cap\alpha(-c_0))\cap S| + |V(\alpha(c_0)\cap\alpha(-c_0)) - S| \ .$$

Since $|V(\alpha(c_0)\cap\alpha(-c_0)) - S| = |\tilde{c}_0|$, it suffices to show that

$$|V(A\cap B)\cap S| \geqslant |V(\alpha(c_0)\cap\alpha(-c_0))\cap S| \ .$$

If c_0 is internal then $V(\alpha(c_0))\cap S = \emptyset$ and the result is trivial. Let c_0 border a cuff Θ with head n_1 and tail n_2. Then

$$V(\alpha(c_0)\cap\alpha(-c_0))\cap S = \beta(n_1)\cup\beta(n_2) \ .$$

Let π have warp q at Θ, and let $\beta(n_1) = \{u_1,\ldots,u_q\}$. For $1 \leqslant i \leqslant q$ let v_i be the terminal vertex of $L(u_i,n_1)$. Thus $\{v_1,\ldots,v_q\} = \{u_1,\ldots,u_q\}$. Let $1 \leqslant i \leqslant q$.

By (10.2) there is a first vertex of $L(u_i, n_1)$ which is not in $V(A)$, and it is not u_i; and so the previous vertex a_i of $L(u_i, n_1)$ is in $V(A \cap B)$. Similarly there is a last vertex of $L(u_i, n_1)$ which is not in $V(A)$, and it is not v_i; and the next vertex b_i is in $V(A \cap B)$.

By (10.1), the paths $L(u_i, n_1) \backslash v_i$ are mutually vertex–disjoint, and so $a_1, \ldots, a_q$ are all distinct, since $a_i \neq v_i$ $(1 \leqslant i \leqslant q)$. Similarly, $b_1, \ldots, b_q$ are all distinct. Let $1 \leqslant i, j \leqslant q$ and suppose that $a_i = b_j$. We claim that $a_i = u_i = v_j = b_j$. For $a_i \neq b_i$ unless $L(u_i, n_1)$ is a circuit and $a_i = u_i = v_i = b_i$ as required. Hence we may assume that $i \neq j$. But $L(u_i, n_1)$ and $L(u_j, n_1)$ are disjoint, except for u_i (if $u_i = v_j$) or u_j (if $u_j = v_i$). Thus either $a_i = b_j = u_i = v_j$ or $a_i = b_j = u_j = v_i$. Since $a_i \neq v_i$ the second is impossible, and the first is our claim.

Now if $v_j = b_j$, the vertex of $L(u_j, n_1)$ before v_j is not in $V(A)$ and so the corresponding path of the standard linkage in $\alpha(c_0)$ has no edges. Hence $v_j \in \beta(n_2)$. We have shown then that if $a_i = b_j$ then $v_j \in \beta(n_2)$. It follows that

$$|\{a_1, \ldots, a_q\} \cap \{b_1, \ldots, b_q\}| \leqslant |\beta(n_1) \cap \beta(n_2)|$$

and so $|\{a_1, \ldots, a_q\} \cup \{b_1, \ldots, b_q\}| \geqslant |\beta(n_1) \cup \beta(n_2)|$. The result follows, since $a_1, \ldots, a_q,\ b_1, \ldots b_q \in V(A \cap B) \cap S$. ■

(11.7) $|V(A \cap B)| = |V(\alpha(c_0) \cap \alpha(-c_0))|$.

Proof. For each $d \in D(A, B)$, since $|\gamma(\tilde{d}| \leqslant 3$, it follows that there exists $n_d \in \tilde{d}$ such that $\gamma(\tilde{d}) \subseteq V(A) \cup \{\gamma(n_d)\}$ or $\gamma(\tilde{d}) \subseteq V(B) \cup \{\gamma(n_d)\}$. Let A' be the graph with vertex set

$$V(A) \cup \{\gamma(n_d) : d \in D(A, B)\}$$

and edge set $E(A)$, and define B' similarly. Then (A', B') is a separation of G of order at most $|D(A, B)|$ more than that of (A, B). Since (A, B) has order $\leqslant 2p+2$ and by (11.5),

$$|D(A, B)| \leqslant |V(A \cap B) - (S \cup \gamma(N(\Gamma)))| \leqslant |V(A \cap B) - S| \leqslant |\tilde{c}_0| \leqslant 3\ ,$$

it follows that (A', B') has order $\leqslant 2p+5 < ord(\mathcal{T})$, and so $(A', B') \in \mathcal{T}$ by (4.1)(ii). Moreover,

$$\{n \in N(\Gamma) : \gamma(n) \in V(A' \cap B')\} = \{n \in N(\Gamma) : \gamma(n) \in V(A \cap B)\} \cup \{n_d : d \in D(A, B)\}$$

and so

$$\begin{aligned} |\{n \in N(\Gamma) : \gamma(n) \in V(A' \cap B')\}| &\leqslant |\{n \in N(\Gamma) : \gamma(n) \in V(A \cap B)\}| + |D(A, B)| \\ &\leqslant |V(A \cap B) \cap \gamma(N(\Gamma))| + |V(A \cap B) - (S \cup \gamma(N(\Gamma)))| \\ &= |V(A \cap B) - S| \leqslant |\tilde{c}_0|\ , \end{aligned}$$

by (11.5), (11.6) and since $\gamma(N(\Gamma)) \cap S = \emptyset$. Furthermore, for every $c \in C(\Gamma)$, either $\gamma(\tilde{c}) \subseteq V(A')$ or $\gamma(\tilde{c}) \subseteq V(B')$; for this is true if $c \in C^\star(\Gamma) - D(A, B)$ since either $\gamma(\tilde{c}) \subseteq V(A) \subseteq V(A')$ or $\gamma(\tilde{c}) \subseteq V(B) \subseteq V(B')$, and it is true if $c \in D(A, B)$ by choice of n_c, and it is true if $c \notin C^\star(\Gamma)$ because there exists $c' \in C^\star(\Gamma)$ with $\tilde{c} \subseteq \tilde{c}'$ (and we have seen it is true for c'). Hence we may apply (11.2), because (A', B') has order $\leqslant 2p + 5 < ord(\mathcal{T}) - 4p - 3$. We deduce that

$$|\{n \in N(\Gamma) : \gamma(n) \in V(A' \cap B')\}| = |\tilde{c}_0|$$

and if $|\tilde{c}_0| = 2$ then $\{n \in N(\Gamma) : \gamma(n) \in V(A')\} = \tilde{c}_0$. Since

$$|\tilde{c}_0| = |\{n \in N(\Gamma) : \gamma(n) \in V(A' \cap B')\}| \leqslant |V(A \cap B) - S| \leqslant |\tilde{c}_0|$$

we deduce that $|V(A \cap B) - S| = |\tilde{c}_0|$. The result follows from (11.6). ■

12. FILLING OUT A PORTRAYAL OF THE 1–SKELETON

Because of a certain excluded minor theorem for graphs that we shall discuss later, we sometimes are provided with a portrayal of the 1–skeleton of a hypergraph G, and what we really want is a portrayal of G itself. In this section we discuss how to convert one to the other.

We proceed with a series of lemmas. A *clique* in a hypergraph G is (for our purposes) a subset of $V(G)$ the members of which are mutually adjacent in $G^\star$.

(12.1) *Let $\mathcal{T}$ be a tangle of order $> 4p + 9$ in a hypergraph G, let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a true, $(2p + 7)$–redundant, $\mathcal{T}$–central portrayal of G with warp $\leqslant p$, and let Θ be a cuff of Σ. For every clique K of G with*

$$K \subseteq \bigcup (V(\alpha(c)) : c \in C(\Gamma) \ \ \textit{bordering} \ \ \Theta)$$

there is a cell c bordering Θ with $K \subseteq V(\alpha(c))$.

Proof. We proceed by induction on $|K|$.

(1) *If $|K| \leqslant 2$ the result is true.*

Subproof. If $|K| \leqslant 1$ it is clear, and so let $|K| = 2, K = (x_1, x_2\}$ say. Let $e \in E(G)$ incident with x_1, x_2 and choose $c \in C(\Gamma)$ with $e \in E(\alpha(c))$. If c borders Θ the result holds, and so we assume it does not. For $i = 1, 2$ there exists $c_i \in c(\Gamma)$ bordering Θ with $x_i \in V(\alpha(c_i))$, and since $x_i \in V(\alpha(c)) \subseteq V(\alpha(-c_i))$ there exists $n_i \in \tilde{c}_i$ with

$x_i \in \beta(n_i+)$. Moreover, $x_1, x_2 \in V(\alpha(c) \cap \alpha(-c))$. If c is a border cell, bordering a cuff $\Theta' \neq \Theta$ say, then by **(P6)**, $\{x_1, x_2\} \subseteq \beta(n_1'+) \cup \beta(n_2'+)$ for some two nodes n_1', n_2' bordering Θ'. It follows from **(P4)** that $x_1 \notin \beta(n_1)$, and so $x_1 = \gamma(n_1)$ since $x_1 \in \beta(n_1+)$; but since $n_1 \neq n_1', n_2'$ it follows that $x_1 \in \beta(n_1') \cup \beta(n_2')$ contrary to **(P4)**. Hence c is an internal cell, and so by **(P5)**, $\{x_1, x_2\} \subseteq \gamma(\tilde{c}) \subseteq \gamma(V(\Gamma))$. By (9.8)(i), $x_i \notin \beta(n_i)$, and so $x_i = \gamma(n_i)$ $(i = 1, 2)$. Since $x_i \in \gamma(\tilde{c})$ it follows that $n_i \in \tilde{c}$ since γ is an injection $(i = 1, 2)$. Hence there is a Γ–normal O–arc F with $F \cap N(\Gamma) = \{n_1, n_2\}$; and so by (8.3), there is a border cell c' with $n_1, n_2 \in \tilde{c}'$. Then $K \subseteq V(\alpha(c'))$ as required. This proves (1).

Thus we may assume that $|K| \geq 3$. Choose $x_1, x_2, x_3 \in K$, distinct. ¿From our inductive hypothesis there are cells c_1, c_2, c_3 bordering Θ with

$$K - \{x_i\} \subseteq V(\alpha(c_i)) \quad (i = 1, 2, 3) ,$$

and we may suppose (for a contradiction) that $x_i \notin V(\alpha(c_i))$ $(1, 2, 3)$ for otherwise the theorem holds. Hence c_1, c_2, c_3 are all distinct.

(2) *$\{x_1, x_2, x_3\} \cap \beta(n+) \neq \emptyset$ for all $n \in N(\Gamma) \cap \Theta$.*

Subproof. We may assume that c_1, n, c_2, c_3 are in order. For $i = 1, 2$, since $x_3 \in V(\alpha(c_i) \cap \alpha(-c_i))$ there is an end n_i of c_i so that $x_3 \in \beta(n_i+)$by **(P6)**. Let n_3 be an end of c_3. By **(P7)**

$$x_3 \in \beta(n_1+) \cap \beta(n_2+) \subseteq \beta(n_3+) \cap \beta(n+) .$$

But $x_3 \notin \beta(n_3+)$ since $x_3 \notin V(\alpha(c_3))$, and so $x_3 \in \beta(n+)$. This proves (2).

But (2) contradicts (6.6) (taking $z = 9$). The result follows. ■

(12.2) *Let $\mathcal{T}$ be a tangle of order $> 4p + 9$ in a hypergraph G, and let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a true, $(2p + 7)$–redundant, $\mathcal{T}$–central portrayal of G, with warp $\leqslant p$. Then for every clique K of G, either:*

- (i) *there is a disc $\Delta \subseteq \Sigma$ such that $\alpha(\Delta)$ is small, $K \subseteq V(\alpha(\Delta))$, and $bd(\Delta)$ is a Γ–normal O–arc with $|bd(\Delta) \cap N(\Gamma)| \leqslant 3$, or*
- (ii) *$K \subseteq V(\alpha(c))$ for some $c \in C(\Gamma)$.*

Proof. By an *arc* of Γ we mean a component of $bd(\overline{c}) - \tilde{c}$ for some cell c. Each arc is homeomorphic to the open interval $(0, 1)$, and the arcs of Γ together with $N(\Gamma)$ yield a drawing of a graph in Σ, which we denote by H.

(1) If for each $x \in K$ there is a node n with $x = \gamma(n)$ then the theorem holds.

Subproof. Let $M = \{n \in N(\Gamma) : \gamma(n) \in K\}$. Then every two members of M are adjacent in H. If $|K| \leqslant 2$ the result is clear, and we may assume therefore that $|M| \geqslant 3$. By (8.5), for every circuit C of H of length 3, there is a disc $\Delta(C) \subseteq \Sigma$ with boundary the drawing of C, such that $\alpha(\Delta(C))$ is small. Choose a circuit C of H of length 3 with $V(C) \subseteq M$ such that $\Delta(C)$ is maximal. We claim that $M \subseteq \Delta(C)$. For suppose that $m_0 \in M - \Delta(C)$. Let $V(C) = \{m_1, m_2, m_3\}$. Let e_i be an edge of H joining m_0 and m_i $(i = 1, 2, 3)$. Let C_i be the circuit of H with vertex set $\{m_0\} \cup (\{m_1, m_2, m_3\} - \{m_i\})$, with one edge in common with C and the other two from $\{e_1, e_2, e_3\}$ appropriately. From our choice of C, none of the discs $\Delta(C_1), \Delta(C_2), \Delta(C_3)$ includes $\Delta(C)$; but then $\Delta(C) \cup \Delta(C_1) \cup \Delta(C_2) \cup \Delta(C_3)$ is a sphere, and hence equals Σ, which is easily seen to be impossible since $\alpha(\Delta(C))$ and each $\alpha(\Delta(C_i))$ are small. We deduce that $M \subseteq \Delta(C)$, and hence the theorem is satisfied. This proves (1).

Next, we claim

(2) If there exists $x \in K$ and a cell c with $x \in V(\alpha(c)) - V(\alpha(-c))$ then the theorem holds.

Subproof. If $y \in K$, there is a cell c' with $x, y \in V(\alpha(c'))$, and hence $c' = c$, since $x \in V(\alpha(-c))$. Thus $K \subseteq V(\alpha(c))$. This proves (2).

¿From (1), we may assume that there exists $x \in K$ such that $\gamma(n) \neq x$ for $n \in N(\Gamma)$. We may choose a cell d with $x \in V(\alpha(d))$; and by (2), we may assume that $x \in V(\alpha(d) \cap \alpha(-d))$. Since $x \notin \gamma(\tilde{d})$ it follows that d is a border cell and $x \in \beta(n)$ for some end n of d. Let d border a cuff Θ. For every $y \in K$ there is a cell c with $x, y \in V(\alpha(c))$; and hence c borders Θ. We deduce that

$$K \subseteq \bigcup (V(\alpha(c)) : c \in C(\Gamma) \quad \text{bordering} \quad \Theta)$$

and the theorem follows from (12.1). ■

If G is a hypergraph and $X \subseteq E(G)$, we denote the subhypergraph $(V(G),\ E(G) - X)$ by $G \backslash X$. If $\mathcal{T}'$ is a tangle in a subhypergraph G' of G, let $\mathcal{T}$ be the set of all separations (A, B) of G of order less then the order of $\mathcal{T}'$, such that $(A \cap G', B \cap G') \in \mathcal{T}'$. Then $\mathcal{T}$ is clearly a tangle in G; we call it the tangle *induced* by $\mathcal{T}'$.

(12.3) *Let e be an edge of a hypergraph G, and let K be the set of ends of e. Let K be a clique of $G \backslash e$. Let $\mathcal{T}$ be a tangle in $G \backslash e$ of order $> 2p + 8$, inducing a tangle $\mathcal{T}'$ in G. Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a 6–redundant, $\mathcal{T}$–central portrayal of $G \backslash e$ with* *warp* $\leqslant p$. *Let* $c_0 \in C(\Gamma)$ *with* $K \subseteq V(\alpha(c_0))$. *Define* α' *by*

$$\alpha'(c_0) = (V(\alpha(c_0)), E(\alpha(c_0)) \cup \{e\})$$
$$\alpha'(c) = \alpha(c) \ (c \in C(\Gamma) - \{c_0\}) \ .$$

Then $\pi' = (\Sigma, \Gamma, \alpha', \beta, \gamma)$ *is a* $\mathcal{T}'$*–central portrayal of* G*, resembling* π.

Proof. Since $K \subseteq V(\alpha(c_0)), \pi'$ is a portrayal of G; and from definition of $\mathcal{T}, \alpha'(c_0)$ is small relative to $\mathcal{T}'$. Thus π' is $\mathcal{T}'$–central. We suppose that π' does not resemble π. Then c_0 borders a cuff Θ, and the warp of c_0 in π' is greater than q, the warp of π at Θ. Let n_1, n_2 be the ends of c_0. Then $|\beta(n_1+)| = |\beta(n_2+)| = q+1$, and there is a linkage $\{P_0, \dots, P_q\}$ in $\alpha'(c_0)$ from $\beta(n_1+)$ to $\beta(n_2+)$ which does not pair $\gamma(n_1)$ with $\gamma(n_2)$. Since the warp of π at Θ is q it follows that $\{P_0, \dots, P_q\}$ is not a linkage in $\alpha(c_0)$, and so there is an edge f of P_0 say with $f \notin E(\alpha(c_0)^\star)$. Let f have ends v_1, v_2. Since $f \in E(\alpha'(c_0)^\star)$ it follows that v_1, v_2 are ends of e, and so $f \in E((G \backslash e)^\star)$. Hence there exists $c \neq c_0$ such that $f \in E(\alpha(c)^\star)$. Since $v_1, v_2 \in V(\alpha(c))$ it follows that

$$v_1, v_2 \in V(\alpha(c_0) \cap \alpha(-c_0)) = \beta(n_1+) \cup \beta(n_2+) \ .$$

But since $P_0, \dots, P_q$ are vertex–disjoint it follows that no internal vertex of P_0 is in $\beta(n_1+) \cup \beta(n_2+)$; and so P_0 has ends v_1, v_2, and $v_i \in \beta(n_i+)$ $(i = 1, 2)$. Since $\{P_0, \dots, P_q\}$ does not pair $\gamma(n_1)$ with $\gamma(n_2)$ and it pairs v_1 with v_2, we may assume that $v_1 \neq \gamma(n_1)$, and so $v_1 \in \beta(n_1)$. Consequently c borders Θ.

(1) $\{v_1, v_2\} \cap \beta(n+) \neq \emptyset$ *for every* $n \in N(\Gamma) \cap \Theta$.

Subproof. Let $n \in (\Gamma) \cap \Theta$; we may assume that n, n_1, n_2, c are in order. Since $v_1 \in V(\alpha(c) \cap \alpha(-c))$, there is an end n_3 of c such that $v_1 \in \beta(n_3+)$. Then from (P7)

$$v_1 \in \beta(n_1+) \cap \beta(n_3+) \subseteq \beta(n_2+) \cup \beta(n+) \ .$$

But $v_1 \notin \beta(n_2+)$, since $v_1 \neq v_2$ and v_2 is the only vertex of P_0 in $\beta(n_2+)$. Thus $v_1 \in \beta(n+)$, as required.

But (1) contradicts (6.6). The result follows. ∎

(12.4) *Let* e *be an edge of a hypergraph* G*, such that the set of ends of* e *is a clique of* $G \backslash e$. *Let* $\mathcal{T}$ *be a tangle in* $G \backslash e$*, of order* $> 4p+9$*, inducing a tangle* $\mathcal{T}'$ *in* G. *Let* π *be a* $(2p+7)$*–redundant,* $\mathcal{T}$*–central portrayal of* $G \backslash e$ *with warp* $\leqslant p$. *Then there is a* $\mathcal{T}'$*–central portrayal of* G *resembling* π.

Proof. There is a $(2p+7)$–redundant $\mathcal{T}$–central portrayal of $G\backslash e$ resembling π which is true; and so we may assume that π is true. Let K be the set of ends of e, and let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$. By (12.2), either there is a disc $\Delta \subseteq \Sigma$ such that $\alpha(\Delta)$ is small, $K \subseteq V(\alpha(\Delta))$, and $bd(\Delta)$ is a Γ–normal O–arc with $|bd(\Delta) \cap N(\Gamma)| \leqslant 3$, or $K \subseteq V(\alpha(c))$ for some $c \in C(\Gamma)$. In the second case the result follows from (12.3), and so we assume that the first case applies. Let π' be the portrayal of $G\backslash e$ obtained by applying (7.2) to Δ. Then π' resembles π, and is $\mathcal{T}$–central. Let $\pi' = (\Sigma, \Gamma', \alpha', \beta', \gamma')$, and let c_0 be the cell of Γ' with $\overline{c}_0 = \Delta$. Then $K \subseteq V(\alpha'(c_0))$, and the result follows from (12.3). ■

(12.5) *Let G be a hypergraph, and let $\mathcal{T}^\star$ be a tangle in $G^\star$ of order θ. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order $< \theta$ such that $A^\star$ is small relative to $\mathcal{T}^\star$. Then $\mathcal{T}$ is a tangle in G of order θ.*

Proof. We verify the three tangle axioms. For the first, if (A, B) is a separation of G of order $< \theta$, there is certainly a separation (A', B') of $G^\star$ with $V(A') = V(A), V(B') = V(B)$ and $A' \subseteq A^\star, B' \subseteq B^\star$. ¿From the symmetry we may assume that $(A', B') \in \mathcal{T}^\star$; but then $A^\star$ is small by [1, theorem (2.9)], and so $(A, B) \in \mathcal{T}$. Thus either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$, and the first tangle axiom is satisfied. But the second and third axioms are clearly satisfied, as required. ■

We call $\mathcal{T}$ in (12.4) the *embodiment* of $\mathcal{T}^\star$ in G.

(12.6) *Let G be a hypergraph, and let $\mathcal{T}^\star$ be a tangle in $G^\star$ of order $> 4p + 9$. Let $\mathcal{T}$ be the embodiment of $\mathcal{T}^\star$ in G. Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ be a $(2p + 7)$– redundant, $\mathcal{T}^\star$–central portrayal of $G^\star$ with warp $\leqslant p$. Then there is a $\mathcal{T}$–central portrayal of G resembling π.*

Proof. We may assume that G and $G^\star$ are both subhypergraphs of some hypergraph, so that $G' = G \cup G^\star$ is defined. Let $\mathcal{T}'$ be the embodiment of $\mathcal{T}^\star$ in G'. If there is a $\mathcal{T}'$–central portrayal of G' resembling π, then there is a $\mathcal{T}$–central portrayal of G resembling π. Thus, it suffices to prove the result for G'; that is, we may assume that $G^\star \subseteq G$.

Let $X = E(G) - E(G^\star)$, so that $G\backslash X = G^\star$. Then $\mathcal{T}$ is the tangle in G induced by $\mathcal{T}^\star$; for if $(A, B) \in \mathcal{T}$ then $A^\star$ is small relative to $\mathcal{T}^\star$, and so certainly $A \cap G^\star$ is small relative to $\mathcal{T}^\star$. Let $X = \{x_1, \ldots, x_n\}$. For $0 \leqslant i \leqslant n$, let $G_i = G\backslash(X - \{x_1, \ldots, x_i\})$, and let $\mathcal{T}_i$ be the embodiment of $\mathcal{T}^\star$ in G_i. Thus $G_0 = G^\star$, and $\mathcal{T}_0 = \mathcal{T}^\star$; and for $1 \leqslant i \leqslant n, G_{i-1} = G_i\backslash x_i$, and $\mathcal{T}_i$ is the tangle induced in G_i by $\mathcal{T}_{i-1}$; and the set of ends of x_i is a clique of G_{i-1}. We claim that for $0 \leqslant i \leqslant n$ there is a $(2p+7)$–redundant, $\mathcal{T}_i$–central portrayal of G_i resembling π. For this holds for $i = 0$. Inductively, suppose that it holds for $i = j$; then by (12.4) there is a $\mathcal{T}_{j+1}$–central portrayal π' of G_{j+1} resembling π. Moreover, π' is $(2p+7)$–redundant, for if $Z \subseteq V(G_{j+1})$ with $|Z| \leqslant 2p+7$ and there is a $(\mathcal{T}/Z)$–central portrayal of G_{j+1}/Z simpler than π', then by deleting $x_1, \ldots, x_{j+1}$ we obtain a $(\mathcal{T}^\star/Z)$–central portrayal of $G^\star/Z$ simpler than π, a contradiction. Thus the claim holds for $i = j+1$, and hence for $0 \leqslant i \leqslant n$. In particular, its truth for $i = n$ yields the theorem. ■

13. EXCLUDING A MINOR

Let G and H be graphs. By an *H–minor* of G we mean a function η with domain $V(H) \cup E(H)$, such that

(i) $\eta(v)$ is a non-null connected subgraph of G for each $v \in V(H)$, and $\eta(u)$ and $\eta(v)$ are disjoint for all distinct $u, v \in V(H)$

(ii) $\eta(e) \in E(G)$ for each $e \in E(H)$, and $\eta(e) \neq \eta(f)$ for all distinct $e, f \in E(H)$

(iii) if $e \in E(H)$ has distinct ends u, v then $\eta(e)$ has one end in $V(\eta(u))$ and the other in $V(\eta(v))$

(iv) if $e \in E(H)$ is a loop with end v, then $\eta(e)$ has both ends in $V(\eta(v))$ and $e \notin E(\eta(v))$.

If $\mathcal{T}$ is a tangle in a graph G and η is an H–minor of G, we say $\mathcal{T}$ *controls* η if for each $v \in V(H)$ there is no $(A, B) \in \mathcal{T}$ of order $< |V(H)|$ such that $V(\eta(v)) \subseteq V(A)$.

If G is a graph and $Z \subseteq V(G)$, we denote by $G \backslash Z$ the graph obtained by deleting Z. If $\mathcal{T}$ is a tangle in G of order $> |Z|$, then $\mathcal{T} \backslash Z$ denotes the set

$$\{(A \backslash Z, B \backslash Z) : (A, B) \in \mathcal{T}, Z \subseteq V(A \cap B)\} .$$

It is shown in [1, theorem (8.5)] that $\mathcal{T} \backslash Z$ is a tangle in $G \backslash Z$ of order $ord(\mathcal{T}) - |Z|$.

(13.1) *For any graph H there are numbers p, q, z and $\theta > z$, such that for every hypergraph G and every tangle $\mathcal{T}$ in G of order $\geqslant \theta$, either*

(i) $\mathcal{T}$ controls an H–minor of $G^\star$, or

(ii) there exists $Z \subseteq V(G)$ with $|Z| \leqslant z$ and a $\mathcal{T} \backslash Z$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of $G \backslash Z$ with warp $\leqslant p$, such that Σ has $\leqslant q$ cuffs and H cannot be drawn in Σ.

Proof. From theorem (3.1) of [2], there are integers p_0, q, z, θ_0 such that for every graph G and every tangle $\mathcal{T}$ in G of order $\geqslant \theta_0$, either $\mathcal{T}$ controls an H–minor of

G, or there exists $Z \subseteq V(G)$ with $|Z| \leqslant z$, and a non–null connected surface Σ_0 with $bd(\Sigma_0) = \emptyset$ in which H cannot be drawn, and a painting Γ_0 in Σ_0, and an injection $\gamma : N(\Gamma_0) \longrightarrow V(G) - Z$, and a function δ assigning to each $c \in C(\Gamma_0)$ a subgraph $\delta(c)$ of G, with the following properties:

(a) $G\backslash Z$ is the union of all the subgraphs $\delta(c)$, and $E(\delta(c) \cap \delta(c')) = \emptyset$ for all distinct $c, c' \in C(\Gamma_0)$

(b) for each cell c and node $n, n \in \tilde{c}$ if and only if $\gamma(n) \in V(\delta(c))$

(c) for distinct cells $c, c', V(\delta(c) \cap \delta(c')) = \{\gamma(n) : n \in \tilde{c} \cap \tilde{c}'\}$

(d) there are at most q cells c with $|\tilde{c}| \geqslant 4$ (so–called *major* cells), and $\overline{c} \cap \overline{c'} = \emptyset$ for all distinct major cells c, c'

(e) for each major cell c there is a function κ which assigns to every node $n \in \tilde{c}$ a subgraph $\kappa(n)$ of $\delta(c)$, such that:

(i) $\delta(c)$ is the union of all the $\kappa(n)$'s for $n \in \tilde{c}$, and $\gamma(n) \in V(\kappa(n))$ for each n, and $E(\kappa(n) \cap \kappa(n')) = \emptyset$ for all distinct $n, n' \in \tilde{c}$

(ii) for distinct nodes $n_1, n_2, n_3, n_4 \in \tilde{c}$ in order, $V(\kappa(n_1) \cap \kappa(n_3)) \subseteq V(\kappa(n_2) \cup \kappa(n_4))$ (actually, what is proved in [2, theorem (3.1)] is somewhat stronger, but this corollary is all we need here)

(iii) for distinct nodes $n, n' \in \tilde{c}, |V(\kappa(n) \cap \kappa(n'))| \leqslant p_0$.

(f) for each cell $c \in C(\Gamma)$, there is no $(A, B) \in \mathcal{T}\backslash Z$ with $B \subseteq \delta(c)$.

Let $p = p_0 + 1$ and $\theta = \max(\theta_0, 2p_0 + z + 4)$; we shall show that p, q, z, θ satisfy the theorem. For let $G, \mathcal{T}$ in the theorem fail to satisfy condition (13.1)(i); then from our choice of p_0, q, z, θ_0, there exists $Z, \Sigma_0, \gamma, \delta, \kappa$ as in (a) ,... ,(f) above.

Let Σ be the surface obtained from Σ_0 by deleting the interior of every major cell. (By (d), this is indeed a surface.) Then H cannot be drawn in Σ. We shall show that there is a $\mathcal{T}\backslash Z$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of $G\backslash Z$ in Σ with warp $\leqslant p$. For each cuff Θ and each component s of $\Theta - N(\Gamma_0)$, choose a disc $\Delta(s) \subseteq \Sigma$ with $\overline{s} \subseteq \Delta(s), \Delta(s) \cap bd(\Sigma) = \overline{s}$, $\Delta(s) \cap U(\Gamma_0) = \overline{s}$, and with $\Delta(s) \cap \Delta(s') = \overline{s} \cap \overline{s}'$ for distinct s , s'. Define

$$\Gamma = ((U(\Gamma_0) \cap \Sigma) \cup \bigcup(\Delta(s)), N(\Gamma_0))$$

the union being taken over all components s of $bd(\Sigma) - N(\Gamma_0)$. Then Γ is a painting in Σ.

To define α and β we proceed as follows. For each internal cell c of Γ we define $\alpha(c) = \delta(c)$. It remains to define $\alpha(c)$ for border cells c and $\beta(n)$ for border nodes n. Let Θ be a cuff of Σ. Enumerate the nodes and cells bordering Θ, in order, as $n_1, c_1, n_2, c_2, \ldots, n_k, c_k, (n_1)$. (Then $k \geqslant 4$.) Let κ be as in (e) above, for the corresponding major cell of Γ_0.

For $1 \leqslant i \leqslant k$, we define $\beta(n_i) = V(\kappa(n_{i-1}) \cap \kappa(n_i)) - \{\gamma(n_i)\}$, where n_0 means n_k. For $1 \leqslant i \leqslant k$ we define $\alpha(c_i) = \kappa(n_i) + \gamma(n_{i+1})$, where n_{k+1} means n_1. (In general, if H is a subhypergraph of a hypergraph G, and $v \in V(G)$, by $H + v$ we mean the hypergraph $(V(H) \cup \{v\}, E(H))$.)

This completes our definition of π. Now we verify that π is a portrayal of $G \backslash Z$, by verifying **(P1)**–**(P7)**.

(P1) From (a) and (e)(i), $G \backslash Z$ is the union of all the $\alpha(c)$'s; and by (a) and (e)(i) again, $E(\alpha(c) \cap \alpha(c')) = \emptyset$ for distinct cells c, c'.

(P2) Let $c \in C(\Gamma)$ and $n \in \tilde{c}$. If $c \in C(\Gamma_0)$ then $\gamma(n) \in V(\delta(c)) = V(\alpha(c))$ by (b) above. If $c \notin C(\Gamma_0)$, then c borders some cuff, and n is an end of c. Let $c = c_i$ and $n \in \{n_1, n_{i+1}\}$, with numbering as before. Then $\gamma(n_i) \in V(\kappa(n_i)) \subseteq V(\alpha((c))$ by (e)(i) and the definition of $\alpha(c_i)$; and $\gamma(n_{i+1}) \in V(\alpha(c_i))$ by definition of $\alpha(c_i)$.

(P3) Both these statements are clear from the definitions of α and β.

(P4) If n_1, n_2 are nodes bordering different cuffs, arising from major cells c_1, c_2 of Γ_0, then $\beta(n_i) \subseteq V(\delta(c_i))$ $(i = 1, 2)$, and by (c) above,

$$V(\delta(c_i) \cap \delta(c_2)) \subseteq \{\gamma(n) : n \in \tilde{c}_1 \cap \tilde{c}_2\}$$

which is null by (d). If n_1 is a node bordering a cuff arising from a major cell c_1 of Γ_0 and n_2 is a node not bordering this cuff, then $\gamma(n_2) \notin V(\delta(c_1))$ by (b), and since $\beta(n_1) \subseteq V(\delta(c_1))$ it follows that $\gamma(n_2) \notin \beta(n_1)$.

(P5) Let c_1 be an internal cell of Γ and let $v \in V(\alpha(c_1) \cap \alpha(-c_1))$. Let $c_2 \in C(\Gamma) - \{c_1\}$ with $v \in V(\alpha(c_2))$. Now $c_1 \in C(\Gamma_0)$, and so if c_2 is an internal cell of Γ then

$$v \in V(\delta(c_1) \cap \delta(c_2)) = \{\gamma(n) : n \in \tilde{c}_1 \cap \tilde{c}_2\} \subseteq \{\gamma(n) : n \in \tilde{c}_1\}$$

by (c), as required. If c_2 borders a cuff arising from a major cell c_0 of Γ_0, then again $v \in V(\delta(c_1) \cap \delta(c_0))$ and so $v = \gamma(n)$ for some $n \in \tilde{c}_1$ from (c), as required.

(P6) Let c_1 be a cell bordering a cuff Θ, arising from a major cell c_0 of Γ_0. Number the nodes and cells bordering Θ as $n_1, c_1, n_2, c_2, \ldots, n_k, c_k, (n_1)$ as before. Let

$v \in V(\alpha(c_1) \cap \alpha(-c_1))$, and choose $d \in C(\Gamma) - \{c_1\}$ with $v \in V(\alpha(d))$, bordering Θ if possible. We must show that $v \in \beta(n_1+) \cup \beta(n_2+)$. Suppose first that d does not border Θ. Then $\alpha(d) \subseteq \delta(d_0)$ for some cell $d_0 \neq c_0$ of Γ_0, and $v \in V(\delta(c_0) \cap \delta(d_0)) \subseteq \{\gamma(n) : n \in \tilde{c}_0 \cap \tilde{d}_0\}$, by (c). Choose $n \in \tilde{c}_0 \cap \tilde{d}_0$ with $\gamma(n) = v$. Since $n \in \tilde{c}_0, n$ borders Θ, and d can be chosen bordering Θ, a contradiction. Thus d borders Θ; $d = c_j$ say where $2 \leqslant j \leqslant k$. We may assume that $v \neq \gamma(n_1), \gamma(n_2)$ since otherwise $v \in \beta(n_1+) \cup \beta(n_2+)$ as required. Since $v \in V(\alpha(c_1))$, we deduce that $v \in V(\kappa(n_1))$. Also, since $v \in V(\alpha(c_j))$, either $v \in V(\kappa(n_j))$ or $v \in V(\kappa(n_{j+1}))$, and the first occurs unless $v = \gamma(n_{j+1})$. Since $v = \gamma(n_{j+1})$ implies $j \neq k$, we may assume (replacing j by $j+1$ if necessary) that $v \in V(\kappa(n_j))$ where $2 \leqslant j \leqslant k$. ¿From (e)(ii), $v \in V(\kappa(n_2) \cup \kappa(n_k))$, since either $n_j = n_2$ or $n_j = n_k$ or n_1, n_2, n_j, n_k are distinct and in order. If $v \in V(\kappa(n_2))$ then $v \in \beta(n_2+)$ (since $v \neq \gamma(n_2)$) and if $v \in V(\kappa(n_k))$ then $v \in \beta(n_1+)$ (since $v \neq \gamma(n_1)$). Thus in either case $v \in \beta(n_1+) \cup \beta(n_2+)$, as required.

(P7) Number the nodes and cells around Θ as $n_1, c_1, \ldots, n_k, c_k, (n_1)$ as before. Let $1 \leqslant e < f < g < h \leqslant k$. Let $v \in \beta(n_e+) \cap \beta(n_g+)$. We must show that $v \in \beta(n_f+) \cup \beta(n_h+)$. Now $v \in V(\kappa(n_e) \cap \kappa(n_g))$, and so $v \in V(\kappa(n_f) \cup \kappa(n_h))$ by (e)(ii). We may assume then that $v \in V(\kappa(n_f))$ without loss of generality. We may also assume that $v \notin \beta(n_f+)$, and so $v \notin V(\kappa(n_{f-1}))$. Hence $f - 1 \neq e$, and so $e, f-1, g, h$ are distinct and in order. Thus

$$v \in V(\kappa(n_e) \cap \kappa(n_g)) \subseteq V(\kappa(n_{j-1}) \cup \kappa(n_h))$$

and so $v \in V(\kappa(n_h))$. We may assume that $v \notin \beta(n_h+)$, and so $v \notin V(\kappa(n_{h-1}))$. Hence $h - 1 \neq g$, and so $e, f-1, g, h-1$ are distinct and in order. But $v \in V(\kappa(n_e) \cap \kappa(n_g))$ and $v \notin V(\kappa(n_{f-1}) \cup \kappa(n_{h-1}))$, a contradiction.

This completes the verification that π is a portrayal of $G \backslash Z$. For its warp, since each $\beta(n)$ has cardinality $\leqslant p_0$ the warp of π is at most $p_0 + 1 = p$, as claimed. Finally, we verify that π is $\mathcal{T} \backslash Z$–central. Let $c \in C(\Gamma)$. Then the separation $(\alpha(c), \alpha(-c))$ of $G \backslash Z$ has order $|\tilde{c}| \leqslant 3$ if c is internal, and has order $\leqslant 2p$ from (P6), if c is a border cell. Thus in either case its order is at most $max(3, 2p)$ and hence less than $\theta - z$. But from $(f), (\alpha(-c),\ \alpha(c)) \notin \mathcal{T} \backslash Z$, and so $(\alpha(c), \alpha(-c)) \in \mathcal{T} \backslash Z$. Thus π is $\mathcal{T} \backslash Z$–central. This completes the proof. ■

Next we would like to convert the portrayal given by (13.1) to a z–redundant one where z is large, so that the theorems of this paper apply to it; our objective now is to prove (13.4) below.

Let $p \geqslant 0$ be an integer. If $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ is a portrayal of a hypergraph G, we define $h_p(\pi) = (4a + 2b + c)^2 p + r$, where Σ is homeomorphic to $\Sigma(a, b, c)$ and r is the sum over all cuffs Θ of Σ of the warp of π at Θ.

(13.2) *Let $p \geqslant 1$, and let π, π' be portrayals of G, G' respectively. If π has warp $\leqslant p$, and π' is simpler than π, then $h_p(\pi') < h_p(\pi)$.*

Proof. Let $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$, and let a, b, c, r be as in the definition of $h_p(\pi)$. Define $\Sigma', \Gamma', \alpha', \beta', \gamma'$,$a', b', c', r'$ similarly, for G', π'. Now π' is simpler than π, and so either

(i) Σ' is simpler than Σ, and π' has warp $\leqslant p$, or

(ii) Σ' is homeomorphic to Σ, and $r' < r$.

In the first case, $4a'+2b'+c' < 4a+2b+c$, and since $r' \leqslant c'p \leqslant (4a+2b+c-1)p, p > 0$ and $4a + 2b + c > 0$, it follows that

$$\begin{aligned} h_p(\pi') &= (4a' + 2b' + c')^2 p + r' \leqslant (4a + 2b + c - 1)^2 p + (4a + 2b + c - 1)p \\ &= (4a + 2b + c - 1)(4a + 2b + c)p = h_p(\pi) - (4a + 2b + c)p - r < h_p(\pi) \end{aligned}$$

as required. In the second case, $4a' + 2b' + c' = 4a + 2b + c$, and so $h_p(\pi') = h_p(\pi) + r' - r < h_p(\pi)$, as required. ■

For each connected surface Σ and all integers $p \geqslant 1$ and $z \geqslant 0$, let $\sigma(\Sigma, p, z) \geqslant 0$ be an integer. We call the function σ a *standard* if $\sigma(\Sigma, p, z) = \sigma(\Sigma', p, z)$ whenever Σ, Σ' are homeomorphic.

(13.3) *For any standard σ and all $p \geqslant 1$ and $h, z_0 \geqslant 0$, there exist θ, z with $\theta > z \geqslant z_0$ and with the following property. Let $\mathcal{T}$ be a tangle of order $\geqslant \theta$ in a graph G, let $Z_0 \subseteq V(G)$ with $|Z_0| \leqslant z_0$, and let π_0 be a $(\mathcal{T} \backslash Z_0)$–central portrayal of $G \backslash Z_0$ with warp $\leqslant p$ and with $h_p(\pi_0) \leqslant h$. Then there exist $Z \subseteq V(G)$ with $Z_0 \subseteq Z$ such that $|Z| \leqslant z$, and a $(\mathcal{T} \backslash Z)$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of $G \backslash Z$, such that π is simpler than or resembles π_0, and π is $\sigma(\Sigma, p, |Z|)$–redundant.*

Proof. For all Σ, z, define $\sigma'(\Sigma, p, z)$ to be the maximum of $\sigma(\Sigma', p, z')$ taken over all connected surfaces Σ' simpler than or homeomorphic to Σ, and over all $z' \leqslant z$. Then σ' is also a standard, and if the result holds for σ' then it holds for σ, since $\sigma(\Sigma, p, z) \leqslant \sigma'(\Sigma, p, z)$. Consequently, by replacing σ by σ', we may assume that

(1) *For all Σ', z', If Σ' is simpler then or homeomorphic to Σ, and $z' \leqslant z$, then $\sigma(\Sigma', p, z') \leqslant \sigma(\Sigma, p, z)$.*

We prove the theorem by induction on h. We first assume that $h = 0$. Let $z = z_0$ and $\theta = \sigma(\Sigma, p, z) + 1$ where Σ is the sphere; we claim that the result holds. For let $\mathcal{T}, G, Z_0, \pi_0 = (\Sigma_0, \Gamma_0, \alpha_0, \beta_0, \gamma_0)$ be as in the theorem, with $h_p(\pi_0) = 0$. Then Σ_0 is a sphere, and so π_0 is z'–redundant for all $z' < ord(\mathcal{T})$. In particular, it is $\sigma(\Sigma_0, p, |Z_0|)$–redundant by (1), as required.

Now we assume that $h > 0$, and that the result holds for $h - 1$ and all z_0. Choose z'_0 so that

$$z'_0 \geqslant z_0 + \sigma(\Sigma(a, b, c), p, z_0)$$

for all $a, b, c \geqslant 0$ such that $p(4a + 2b + c)^2 \leqslant h$. Since $p \geqslant 1$, there are only finitely many such a, b, c, so such a choice is possible.

Let $h' = h - 1$, and choose θ', z' so that the result holds with h, z_0, θ, z replaced by h', z'_0, θ, z. We claim that the theorem holds. For let $\mathcal{T}, G, Z_0$ and $\pi_0 = (\Sigma_0, \Gamma_0, \alpha_0, \beta_0, \gamma_0)$ be as in the theorem. If π_0 is $\sigma(\Sigma_0, p, z_0)$–redundant, the result holds, taking $Z = Z_0, \pi = \pi_0$; for $|Z_0| \leqslant z_0 \leqslant z$, and so $\sigma(\Sigma_0, p, |Z|) \leqslant \sigma(\Sigma_0, p, z_0)$ by (1). We assume then that π_0 is not $\sigma(\Sigma_0, p, z_0)$–redundant. Consequently there exists $Z'_0 \subseteq V(G)$ with $Z_0 \subseteq Z'_0$ and $|Z'_0| \leqslant |Z_0| + \sigma(\Sigma_0, p, z_0) \leqslant z'_0$, such that there is a $\mathcal{T}\backslash Z_0/(Z'_0 - Z_0))$–central portrayal of $G\backslash Z_0/(Z'_0 - Z_0)$ which is simpler than π_0. But $G\backslash Z_0/(Z'_0 - Z_0)$ and $G\backslash Z'_0$ differ only by certain edges with $\leqslant 1$ end. By deleting such edges, we deduce that there is a $(\mathcal{T}\backslash Z'_0)$–central portrayal $\pi'_0 = (\Sigma'_0, \Gamma'_0, \alpha'_0, \beta'_0, \gamma'_0)$ of $G\backslash Z'_0$ which is simpler than π_0. By (13.2), $h_p(\pi'_0) < h_p(\pi'_0) \leqslant h$, and so $h_p(\pi'_0) \leqslant h'$. Since π'_0 is simpler than π_0, it follows that π'_0 has warp $\leqslant p$. ¿From the choice of θ, z, there exists $Z \subseteq V(G)$ with $Z'_0 \subseteq Z$ such that $|Z| \leqslant z$, and there is a $(\mathcal{T}\backslash Z)$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of $G\backslash Z$ which is simpler than or resembles π'_0, such that π is $\sigma(\Sigma, p, |Z|)$–redundant. The result follows, since π'_0 is simpler than π_0. ■

By means of (13.3), we obtain a version of (13.1) with high redundancy, as follows.

(13.4) *For any graph H and standard σ, there are numbers p, q, z and $\theta > z$, such that for every graph G and every tangle $\mathcal{T}$ in G of order $\geqslant \theta$, either*

(i) $\mathcal{T}$ controls an H–minor of G, or

(ii) there exists $Z \subseteq V(G)$ with $|Z| \leqslant z$, and a $(\mathcal{T}\backslash Z)$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of $G\backslash Z$ with warp $\leqslant p$, such that Σ has $\leqslant q$ cuffs, H cannot be drawn in Σ, and π is $\sigma(\Sigma, p, |Z|)$–redundant and true.

Proof. If H is planar, let p, q, z, θ be as in (13.1); then the theorem is satisfied (for (13.1)(ii) cannot hold since H is planar). We assume then that H is non–planar. Choose p, q_0, z_0, θ_0 so that (13.1) is satisfied with p, q, z, θ replaced by p, q_0, z_0, θ_0. Choose $a^\star, b^\star \geqslant 0$ with $2a^\star + b^\star$ maximum so that H cannot be drawn in $\Sigma(a^\star, b^\star, 0)$. Let $q = 4a^\star + 2b^\star + q_0$, and let $h = p(q^2 + q_0)$. Choose θ, z so that (13.3) holds (with the given $\sigma, p, h, z_0, \theta, z$).

We claim that the theorem holds. For let $\mathcal{T}$ be a tangle of order $\geqslant \theta$ in a graph G. We may assume that $\mathcal{T}$ does not control an H–minor of G. By (13.1) there exists $Z_0 \subseteq V(G)$ with $|Z_0| \leqslant z_0$, and a $(\mathcal{T}\backslash Z_0)$–central portrayal $\pi_0 = (\Sigma_0, \Gamma_0, \alpha_0, \beta_0, \gamma_0)$, of $G\backslash Z_0$ with warp $\leqslant p$, such that Σ_0 has $\leqslant q_0$ cuffs and H cannot be drawn in Σ_0. Consequently $h_p(\pi_0) \leqslant h$. By (13.2), there exists $Z \subseteq V(G)$ with $Z_0 \subseteq Z$ such that $|Z| \leqslant z$, and a $(\mathcal{T}\backslash Z)$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of $G\backslash Z$, such that π is simpler than or resembles π_0, and π is $\sigma(\Sigma, p, |Z|)$–redundant. Choose π with maximal truth. We claim that π is true. For if π' is another $(\mathcal{T}\backslash Z)$–central portrayal of $G\backslash Z$ resembling π, then π' is $\sigma(\Sigma, p, |Z|)$–redundant, because π is, and consequently π' is not truer than π. Since π is 0–redundant, it follows that it is true.

Let Σ be homeomorphic to $\Sigma(a, b, c)$, and let Σ_0 be homeomorphic to $\Sigma(a_0, b_0, c_0)$. Then

$$c \leqslant 4a + 2b + c \leqslant 4a_0 + 2b_0 + c_0 \leqslant 4a^\star + 2b^\star + q_0 = q\ .$$

Moreover, H cannot be drawn in Σ, since H cannot be drawn in Σ_0 and Σ is simpler than or homeomorphic to Σ_0. The result follows. ■

14. HYPERGRAPH FORM OF THE EXCLUDED MINOR THEOREM

If $\mathcal{T}$ is a tangle in a hypergraph G, and H is a graph, we say that $\mathcal{T}$ *controls an H–minor of $G^\star$* if there is an H–minor η of $G^\star$ such that for all $v \in V(H)$, there is no $(A, B) \in \mathcal{T}$ of order $< |V(H)|$ with $V(\eta(v)) \subseteq V(A)$.

Our objective now is to obtain a portrayal of a hypergraph, if some tangle $\mathcal{T}$ in it fails to control an H–minor of $G^\star$. For this we need the following lemma.

(14.1) *Let $\mathcal{T}$ be a tangle of order θ in a hypergraph G. Let $\theta^\star \geqslant 1$ with $\theta^\star < \frac{2}{3}\theta + 1$, and let $\mathcal{T}^\star$ be the set of all separations (A', B') of $G^\star$ of order $< \theta^\star$ such that there exists $(A, B) \in \mathcal{T}$ with $V(A) = V(A')$ and $V(B) = V(B')$. Then $\mathcal{T}^\star$ is a tangle in $G^\star$ of order $\theta^\star$.*

Proof. We verify the three tangle axioms. For the first, let (A', B') be a separation of $G^\star$ of order $< \theta^\star$. Then there is a separation (A, B) of G with $V(A) = V(A')$ and $V(B) = V(B')$, and one of $(A, B), (B, A) \in \mathcal{T}$. Hence one of (A', B'), $(B', A') \in \mathcal{T}^\star$, as required.

The third axiom clearly holds, and it remains to verify the second. If $\theta^\star = 1$ the result is easy, and we assume $\theta^\star \geqslant 2$, and hence $\theta \geqslant 2$. We use the following two observations.

(1) *If* $(A'_1, B'_1), (A'_2, B'_2) \in \mathcal{T}^\star$ *then* $B'_1 \nsubseteq A'_2$.

Subproof. Choose $(A_i, B_i) \in \mathcal{T}$ with $V(A_i) = V(A'_i)$, $V(B_i) = V(B'_i)$ $(i = 1, 2)$. If $B'_1 \subseteq A'_2$ then $V(B_1) \subseteq V(A_2)$, and so $(B_2, A_2) \in \mathcal{T}$ by [1, theorem (2.9)], a contradiction.

(2) *There do not exist subhypergraphs* A'_1, A'_2, A'_3 *of* $G^\star$*, mutually edge–disjoint, with* $A'_1 \cup A'_2 \cup A'_3 = G^\star$*, and with* $(A'_1, A'_2 \cup A'_3)$, $(A'_2, A'_3 \cup A'_1)$, $(A'_3, A'_1 \cup A'_2)$ *all in* $\mathcal{T}^\star$.

Subproof. Suppose that such A'_1, A'_2, A'_3 exist. Let

$$\begin{aligned} V(A'_1 \cap A'_2 \cap A'_3) &= W_0 \\ V(A'_2 \cap A'_3) - V(A'_1) &= W_1 \\ V(A'_3 \cap A'_1) - V(A'_2) &= W_2 \\ V(A'_1 \cap A'_2) - V(A'_3) &= W_3 \ . \end{aligned}$$

Then W_0, W_1, W_2, W_3 are mutually disjoint, and

$$\begin{aligned} V(A'_1 \cap (A'_2 \cup A'_3)) &= W_0 \cup W_2 \cup W_3 \\ V(A'_2 \cap (A'_3 \cup A'_1)) &= W_0 \cup W_3 \cup W_1 \\ V(A'_3 \cap (A'_1 \cup A'_2)) &= W_0 \cup W_1 \cup W_2 \ . \end{aligned}$$

Hence $|W_0| + |W_2| + |W_3| \leqslant \theta^\star - 1$, and summing this and two similar inequalities, we obtain

$$3|W_0| + 2(|W_1| + |W_2| + |W_3|) \leqslant 3\theta^\star - 3 \ .$$

Consequently,

$$|W_0| + |W_1| + |W_2| + |W_3| \leqslant \frac{1}{2}(3\theta^\star - 3) < \theta \ .$$

Now there exists $(A_i, B_i) \in \mathcal{T}$ with $V(A_i) = V(A'_i), V(B_i) = V(B'_i)$. Since $|W_0| + |W_1| + |W_2| + |W_3| \leqslant \theta$, it follows by [1, theorem (2.9)] that there exists

$(A_i, B_i) \in \mathcal{T}$ with $V(A_i) = V(A'_i) \cup W_i, V(B_i) = V(B'_i)$. Choose such (A_i, B_i) with $E(A_i)$ maximal. Let e be an edge of G. We claim that $e \in E(A_1 \cup A_2 \cup A_3)$. For if some end v of e is not in $W_0 \cup W_1 \cup W_2 \cup W_3$, say

$$v \in V(A'_1) - V(B'_1) ,$$

then every other end v' of e is in $V(A'_1)$ (since v' is adjacent to v in $G^\star$, and (A'_1, B'_1) is a separation of $G^\star$); but then $e \in E(A_1)$ by [1, theorem (2.9)] and the maximality of $E(A_1)$. On the other hand, if every end of e belongs to $W_0 \cup W_1 \cup W_2 \cup W_3$ then every end of e belongs to $V(A_1)$, and again $e \in E(A_1)$ by [1, theorem (2.9)] and the maximality of A_1. Consequently $E(A_1 \cup A_2 \cup A_3) = E(G)$. Hence $A_1 \cup A_2 \cup A_3 = G$, contrary to the second axiom. This proves (2).

¿From (1), (2) and [1,theorem (4.5)], we deduce that $\mathcal{T}^\star$ satisfies the second tangle axiom, as required. ■

The following is the main result of this section.

(14.2) *For any graph H there are numbers p, q, z and $\theta > z$, such that for every hypergraph G and every tangle $\mathcal{T}$ in G of order $\geqslant \theta$, either*

(i) *$\mathcal{T}$ controls an H–minor of $G^\star$, or*

(ii) *there exists $Z \subseteq V(G)$ with $|Z| \leqslant z$ and a $\mathcal{T}/Z$–central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \gamma)$ of G/Z with warp $\leqslant p$, such that Σ has $\leqslant q$ cuffs and H cannot be drawn in Σ, and π is true and $(2p+7)$–redundant.*

Proof. Let σ be the standard defined by $\sigma(\Sigma, p, z) = 2p{+}7$ for all Σ, z. Choose p, q, z and $\theta_0 > z$ so that (13.4) holds (with θ replaced by θ_0). Let $\theta^\star = \max(|V(H)|, \theta_0, 4p{+}z+10)$ and let $\theta = \lceil 3\theta^\star/2 \rceil$.

We claim that the theorem is satisfied. For let $\mathcal{T}$ be a tangle in a hypergraph G of order $> \theta$, not controlling an H–minor of $G^\star$. Let $\mathcal{T}^\star$ be the tangle in $G^\star$ of order $\theta^\star$ obtained as in (14.1).

(1) *$\mathcal{T}^\star$ controls no H–minor of $G^\star$.*

Subproof. Let η be an H–minor of $G^\star$. Since $\mathcal{T}$ does not control η, there exists $v \in V(H)$ and $(A, B) \in \mathcal{T}$ of order $< |V(H)|$ such that $V(\alpha(v)) \subseteq V(A)$. Let (A', B') be a separation of $G^\star$ with $V(A') = V(A)$, $V(B') = V(B)$. Then (A', B') has order $< |V(H)| \leqslant \theta^\star$, and so $(A', B') \in \mathcal{T}^\star$. Since $V(\alpha(v)) \subseteq V(A')$, it follows that $\mathcal{T}^\star$ does not control η. This proves (1).

By (1) and (13.4), there exist $Z \subseteq V(G)$ with $|Z| < z$ and a $(\mathcal{T}^\star \backslash Z)$–central portrayal $\pi_1 = (\Sigma_1, \Gamma_1, \alpha_1, \beta_1, \gamma_1)$ of $G^\star \backslash Z$ with warp $\leqslant p$, such that Σ_1 has $\leqslant q$ cuffs, H cannot be drawn in Σ_1 and π_1 is $(2p+7)$–redundant.

Now $G^\star \backslash Z = (G/Z)^\star$, and $\mathcal{T}^\star \backslash Z$ is therefore a tangle in $(G/Z)^\star$. Let $\mathcal{T}'$ be the embodiment of $\mathcal{T}^\star \backslash Z$ in G/Z. By (12.6), there is a $2p+7$–redundant T'–central portrayal $\pi_2 = (\Sigma_1, \Gamma_2, \alpha_2, \beta_2, \gamma_2)$ of G/Z.

(2) *π_2 is $\mathcal{T}/Z$–central.*

Subproof. Let $c \in C(\Gamma)$. Since π_2 is $\mathcal{T}'$–central it follows that $(\alpha_2(c), \alpha_2(-c)) \in \mathcal{T}'$ by (4.1)(i), since $(\alpha_2(c), \alpha_2(-c))$ has order $\leqslant \max(2p, 3) < \theta^\star - z \leqslant ord(\mathcal{T}')$. Hence $\alpha_2(c)^\star$ is small relative to $\mathcal{T}' \backslash Z$, since $\mathcal{T}'$ is the embodiment of $\mathcal{T}^\star \backslash Z$. Consequently there exists $B_1 \subseteq \alpha_2(-c)^\star$ with $V(B_1) = V(\alpha_2(-c))$ such that $(\alpha_2(c)^\star, B_1) \in \mathcal{T}^\star \backslash Z$; and so there exists $(A_2, B_2) \in \mathcal{T}^\star$ with $Z \subseteq V(A_2 \cap B_2)$ such that $A_2 \backslash Z = \alpha_2(c)^\star$ and $B_2 \backslash Z = B_1$. By definition of $\mathcal{T}^\star$, there exists $(A_3, B_3) \in \mathcal{T}$ with $V(A_3) = V(A_2)$ and $V(B_3) = V(B_2)$. Let $A_4 = A_3/Z, B_4 = B_4/Z$; then $(A_4, B_4) \in \mathcal{T}/Z$. But

$$V(A_4) = V(A_3) - Z = V(A_2) - Z = V(\alpha_2(c))$$

and so $\alpha_2(c)$ is small relative to $\mathcal{T}/Z$, by [1, theorem (2.9)]. This proves (2).

Since π_2 is a $(2p+7)$–redundant, $(\mathcal{T}/Z)$–central portrayal of G/Z, there is a true, $(2p+7)$–redundant, $(\mathcal{T}/Z)$–central portrayal π of G/Z which resembles π_2. The result follows. ∎

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