EXCLUDING A GROUP-LABELLED GRAPH

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ABSTRACT. This paper contains a first step towards extending the Graph Minors Project of Robertson and Seymour to group-labelled graphs. For a finite abelian group Γ and Γ-labelled graph G, we describe the class of Γ-labelled graphs that do not contain a minor isomorphic to G.

1. INTRODUCTION

Group-labelled graphs are a generalization of signed graphs. For a group Γ, a Γ-labelled graph is an oriented graph with edges labelled by elements of Γ. Here we are primarily interested in abelian groups, so we will use additive notation. A *minor* of a group-labelled graph G is any group-labelled graph obtained from G by any sequence of the following operations: vertex deletion, edge deletion, contracting zero-labelled edges, and *shifting at a vertex*, which, for a given vertex v and group element γ, amounts to adding γ to the label of each edge entering v and subtracting γ from the label of each edge leaving v.

We hope that the main results of the Graph Minors Project of Robertson and Seymour will extend to group-labelled graphs over any fixed finite abelian group. In particular, the following two conjectures, if true, would generalize the two main results of the Graph Minors Project; see [4] and [6].

Conjecture 1.1. *For any finite abelian group Γ and any infinite sequence $G_1, G_2, \ldots$ of Γ-labelled graphs, there exist integers $i < j$ such that G_i is isomorphic to a minor of G_j.*

Conjecture 1.2. *For any finite abelian group Γ and any Γ-labelled graph H, there is a polynomial-time algorithm to determine whether or not a Γ-labelled graph G contains an H-minor.*

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To avoid algorithmic complications in Conjecture 1.2, we assume that the group Γ is given by its addition table. In the above conjectures the finiteness of the group Γ is certainly necessary, however, there may be extensions to non-abelian groups.

Group-labelled graphs (also known as *biased graphs*) are closely related with several interesting classes of matroids; see Zaslavsky [7,8]. Proving Conjecture 1.1 would prove that two interesting classes of matroids are well-quasi-ordered with respect to taking minors and would go a long way towards well-quasi-ordering binary matroids.

The workhorse of the Graph Minors Project is the Graph Minors Structure Theorem [5], and the main result of this paper is an extension of that result to group-labelled graphs over finite abelian groups. The complete Γ-labelled graph on n vertices is denoted $K(\Gamma, n)$. For a subgroup Γ' of an abelian group Γ, we say that G is Γ'-*balanced* if it is shifting-equivalent to a Γ'-labelled graph.

Theorem 1.3. *Let Γ be a finite abelian group, let Γ' be a subgroup of Γ, let $n \in \mathbb{N}$, and let $t = 8n|\Gamma|^2$. If G is a Γ-labelled graph and H is a minor of G isomorphic to $K(\Gamma', 4t)$, then either*

- *there is a set $X \subseteq V(G)$ with $|X| < t$ such that the unique block of $G - X$ that contains most of $E(H)$ is Γ'-balanced, or*
- *there is a subgroup Γ'' of Γ properly containing Γ' and a minor H' of G with $E(H') \subseteq E(H)$ such that H' is isomorphic to $K(\Gamma'', n)$.*

The specialization of Theorem 1.3 to signed graphs is implicit in [2] and the proof of Theorem 1.3 is a routine extension of the proof given in that paper.

We also prove the following easy result that is complementary to Theorem 1.3. The graph that is obtained from a group-labelled graph G by ignoring the orientation and the group-labels is denoted $\widetilde{G}$.

Theorem 1.4. *Let Γ be a finite group and let $n \in \mathbb{N}$. Then there exists $l \in \mathbb{N}$ such that if $\widetilde{G}$ has a K_l-minor, then G has a $K(\{0\}, n)$-minor.*

Theorems 1.3 and 1.4 are particularly useful when applied in conjunction with "tangles". For the rest of the introduction we assume that the reader is familiar with the definitions in Graph Minors X [3].

Suppose that $\mathcal{T}$ is a tangle of order k in a graph (or group-labelled graph) G. If $X \subseteq V(G)$ with $|X| \leq k - 2$, then it is straightforward to show that there is a unique block B of $G - X$ such that $V(B) \cup X$ is not contained in the $\mathcal{T}$-small side of any separation of order $< k$; we call B the $\mathcal{T}$-large block of $G - X$.

Let Γ be a finite abelian group and let $n \in \mathbb{N}$. Theorems 1.3 and 1.4 imply that there exist $l, t \in \mathbb{N}$ such that if G is a Γ-labelled graph and $\mathcal{T}$ is a tangle in G of order $\geq t+2$, then either:

(1) $\mathcal{T}$ does not control a K_l-minor in $\widetilde{G}$,
(2) there exists $X \subseteq V(G)$ with $|X| \leq t$ such that the $\mathcal{T}$-large block of $G - X$ is Γ'-balanced for some proper subgroup Γ' of Γ, or
(3) $\mathcal{T}$ controls a $K(\Gamma, n)$-minor in G.

In the first case, we can use the Graph Minors Structure Theorem to describe the structure of $\widetilde{G}$.

2. Group-labelled graphs

Let Γ be an abelian group. A Γ-*labelled graph* is an oriented graph with edges labelled by elements of Γ. More formally, if G is a Γ-labelled graph, then G has a *vertex set* $V(G)$ and an *edge set* $E(G)$, and each edge $e \in E(G)$ is assigned a *head*, denoted $\mathrm{head}_G(e)$, in $V(G)$, a *tail*, denoted $\mathrm{tail}_G(e)$, in $V(G)$, and a group-label, denoted $\gamma_G(e)$, in Γ. The head and tail of an edge are referred to as its *ends*.

Let G be a Γ-labelled graph. The graph obtained from G by ignoring the orientation and the group labels is denoted by $\widetilde{G}$. By a *walk* in G, we mean a walk in $\widetilde{G}$. If $e \in E(G)$ and v is an end of e, then we let $\gamma_G(e, v) = \gamma_G(e)$ if $v = \mathrm{head}_G(e)$ and $\gamma_G(e, v) = -\gamma_G(e)$ if $v = \mathrm{tail}_G(e)$. Let $W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ be a walk of G. We let $\gamma_G(W) = \gamma_G(e_1, v_1) + \cdots + \gamma_G(e_k, v_k)$. The *length* of W is k; the *ends* of W are v_0 and v_k; W is *closed* if $v_0 = v_k$; W is a *path* if $v_0, v_1, \ldots, v_k$ are distinct vertices; and W is a *circuit* if $v_0, \ldots, v_{k-1}$ are distinct, $e_1, \ldots, e_k$ are distinct, and $v_0 = v_k$. We let $E(W) = \{e_1, \ldots, e_k\}$ and $V(W) = \{v_0, \ldots, v_k\}$.

Shifting. Let $v \in V(G)$ and let $\delta \in \Gamma$. We obtain a new Γ-labelled graph G' from G by adding δ to the label of each edge with head v and subtracting δ from the label of each edge with tail v. We say that G' is obtained from G by *shifting* at v. Any Γ-labelled graph that is obtained from G by a sequence of shiting operations is said to be *shifting-equivalent* to G. Note that, if G' is shifting-equivalent to G and W is a closed walk of G, then $\gamma_{G'}(W) = \gamma_G(W)$. (Here we do require that Γ is abelian, and this is one reason that we are restricting our attention to abelian groups.)

We omit the elementary proof of the following result.

Lemma 2.1. *If G is a Γ-labelled graph, for some abelian group Γ, and T is a spanning tree of $\widetilde{G}$, then G is switching-equivalent to some Γ-labelled graph G' with $\gamma_{G'}(e) = 0$ for each $e \in E(T)$.*

Balanced labellings. Let Γ' be a subgroup of Γ. We say that G is Γ'-*balanced* if $\gamma_G(C) \in \Gamma'$ for each circuit C of G. Note that, if Γ is abelian and C_1 and C_2 are circuits of G with $E(C_1) = E(C_2)$, then $\gamma_G(C_1) = \pm\gamma_G(C_2)$.

Lemma 2.2. *Let Γ be an abelian group and let G be a Γ-labelled graph. If G is Γ'-balanced for some subgroup Γ' of Γ, then G is switching-equivalent to a Γ'-labelled graph.*

Proof. By possibly adding edges, we may assume that $\widetilde{G}$ is connected; let T be a spanning tree of $\widetilde{G}$. By Lemma 2.1, G is switching-equivalent to some Γ-labelled graph G' with $\gamma_{G'}(e) = 0$ for each $e \in E(T)$. Since Γ is abelian, G' is Γ'-balanced. Consider $e \in E(G') - E(T)$ and a circuit C with $E(C) \subseteq E(T) \cup \{e\}$. We have $\gamma_{G'}(e) = \pm\gamma_{G'}(C) \in \Gamma'$. Hence G' is Γ'-labelled. □

Lemma 2.3. *Let Γ be an abelian group and let G be a Γ-labelled graph. If G is Γ'-balanced for some subgroup Γ' of Γ, then for any closed walk W of G we have $\gamma_G(W) \in \Gamma'$.*

Proof. By Lemma 2.2, G is switching-equivalent to a Γ'-labelled graph G'. Now, for any closed walk W, we have $\gamma_G(W) = \gamma_{G'}(W) \in \Gamma'$. □

We call G *balanced* if it is $\{0\}$-balanced. A set $F \subseteq E(G)$ is *balanced* if the subgraph of G induced by F is balanced; that is, $\gamma_G(C) = 0$ for each circuit C of G with $E(C) \subseteq F$.

Minors. A Γ-labelled graph H is a *minor* of G if it can be obtained from G via any sequence of the following operations: edge deletion, contraction of a zero-labelled edge, shifting at a vertex, and vertex deletion. It is straightforward to see that if F is the set of edges that are contracted in obtaining a minor H of G, then F is balanced in G. Conversely, if $F' \subseteq E(G)$ is balanced, then, by Lemma 2.2, we can shift so that each edge in F' is zero-labelled and then contract these edges. For any set A of zero-labelled edges of G we let G/A denote the minor of G obtained by contracting A. For a set $S \subseteq V(G)$, we let $G[S]$ denote the Γ-labelled subgraph of G induced by S and let $G - S = G[V(G) - S]$.

We will need to view minors in a more global way, as given by the following result; we omit the easy proof.

Lemma 2.4. *Let G be a Γ-labelled graph, for some abelian group Γ, and let H be a minor of G. Then there is a Γ-labelled graph G' that is switching-equivalent to G and there exist vertex-disjoint trees $(T(v) : v \in V(H))$ in G such that:*

- *$\gamma_{G'}(e) = 0$ for each $v \in V(H)$ and $e \in E(T(v))$,*
- *$\gamma_{G'}(e) = \gamma_H(e)$ for each $e \in E(H)$, and*
- *$head_{G'}(e) \in V(T(head_H(e)))$ and $tail_{G'}(e) \in V(T(tail_H(e)))$ for each $e \in E(H)$.*

A-paths. Let $A \subseteq V(G)$. An *A-path* in G is a path of length at least one whose ends are both in A. The following result was proved by Chudnovsky et al. [1].

Theorem 2.5. *Let Γ be a abelian group, let G be a Γ-labelled graph, and let $A \subseteq V(G)$. Then for any $k \in \mathbb{N}$ either*

- *there exist vertex-disjoint A-paths $P_1, \ldots, P_k$ with $\gamma_G(P_i) \neq 0$ for each $i \in \{1, \ldots, k\}$, or*
- *there exists $X \subseteq V(G)$ with $|X| \leq 2(k-1)$ such that $\gamma_G(P) = 0$ for each A-path P in G disjoint from X.*

We require the following elementary corollary.

Corollary 2.6. *Let Γ' be a subgroup of an abelian group Γ, let G be a Γ-labelled graph, and let $A \subseteq V(G)$. Then for any $k \in \mathbb{N}$ either*

- *there exist vertex-disjoint A-paths $P_1, \ldots, P_k$ with $\gamma_G(P_i) \notin \Gamma'$ for each $i \in \{1, \ldots, k\}$, or*
- *there exists $X \subseteq V(G)$ with $|X| \leq 2(k-1)$ such that $\gamma_G(P) \in \Gamma'$ for each A-path P in G disjoint from X.*

Proof. Apply Theorem 2.5 to the quotient group Γ/Γ'. □

Blocks. A *separation* of G is an pair (G_1, G_2) of subgraphs of G such that $E(G) = E(G_1) \cup E(G_2)$ and $V(G) = V(G_1) \cup V(G_2)$; the *order* of the separation is $|V(G_1) \cap V(G_2)|$. We say that G is 2-*connected* if G is connected and for each separation (G_1, G_2) of G of order 1 either $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$. Note that if G is 2-connected and $|V(G)| \geq 2$, then G has no loops. A *block* of G is a maximal 2-connected subgraph of G.

Lemma 2.7. *Let G be a 2-connected Γ-labelled graph, for some abelian group Γ, and let Γ' be a subgroup of Γ. If u and v are distinct vertices of G such that $\gamma_G(P) \in \Gamma'$ for each (u, v)-path P in G, then G is Γ'-balanced.*

Proof. Let C be a circuit of G. Since G is 2-connected, there exist two vertex-disjoint paths P and P' from $\{u, v\}$ to $V(C)$. We may assume that P connects u to $a \in V(C)$ and P' connects v to $b \in V(C)$. Furthermore, we may assume that P and P' each only meet C in one vertex. Let C' be a circuit of G starting at a with $E(C') = E(C)$. Let Q be the (u, v)-path obtained by following P from u to a then following C' to b, and then following P' backward to v. Now let Q' be the (v,u)-path obtained by following P' from v to b then following C' to a, and then following P backward to u. Note that

$$\gamma_G(C) = \pm\gamma_G(C') = \pm(\gamma_G(Q) + \gamma_G(Q')) \in \Gamma'.$$

Hence G is Γ'-balanced. □

Complete graphs. The *complete* Γ-labelled graph on n vertices, denoted $K(\Gamma, n)$, has vertex set $\{1, \dots, n\}$ and edge set $\{e(i, j, \sigma) : i, j \in V(K(\Gamma, n)),\ i \neq j,\ \sigma \in \Gamma\}$ where each edge $e(i, j, \sigma)$ has tail i, head j, and label σ.

We can now prove Theorem 1.4, which we restate here for convenience.

Theorem 2.8. *Let Γ be a finite abelian group and let $n \in \mathbb{N}$. Then there exists $l \in \mathbb{N}$ such that if $\widetilde{G}$ has a K_l-minor, then G has a $K(\{0\}, n)$-minor.*

Proof. By Ramsey's Theorem there exists an l such that, if we colour the edges of a clique on l vertices with $2|\Gamma|$ colours, then there is a monochromatic subclique on $2n$ vertices. We may assume that $\widetilde{G}$ is isomorphic to K_l and that $V(G) = \{1, \dots, l\}$. We partition $E(G)$ into $2|\Gamma|$ sets according to the label $\gamma_G(e)$ of the edge e and to the sign of $\mathrm{head}_G(e) - \mathrm{tail}_G(e)$. By our choice of l, there exists a set $X \subseteq \{1, \dots, l\}$ with $|X| = 2n$, an element $\gamma \in \Gamma$, and a sign $\sigma \in \{-1, 1\}$ such that for each edge e of $G[X]$ we have $\gamma_G(e) = \gamma$ and $\mathrm{head}_G(e) - \mathrm{tail}_G(e) = \sigma$. By symmetry, we may assume that $X = \{1, \dots, 2n\}$ and that $\sigma = 1$. Let H be the subgraph of $G[X]$ with all edges of G having tail in $\{1, \dots, n\}$ and head in $\{n+1, \dots, 2n\}$. Now we obtain a $K(\{0\}, n)$-minor of H by shifting at each of $\{n+1, \dots, 2n\}$ so that all labels in H become zero, and then contracting a perfect matching. □

3. The main theorem

We need one more preliminary result.

Lemma 3.1. *Let Γ be a finite abelian group, let Γ' be a subgroup of Γ, let $n \in \mathbb{N}$, and let $l = n|\Gamma|^2$. If G is a Γ-labelled graph and M is a matching of size l in $\widetilde{G}$ such that $G - M$ is isomorphic to $K(\Gamma', 2l)$ and for each $e \in M$ we have $\gamma_G(e) \notin \Gamma'$, then there is a subgroup Γ'' of Γ properly containing Γ' and a minor H' of G with $E(H') \subseteq E(G - M)$ such that H' is isomorphic to $K(\Gamma'', n)$.*

Proof. There exists $M' \subseteq M$ with $|M'| = n|\Gamma|$ and an element $\sigma \in \Gamma - \Gamma'$ such that $\gamma_G(e) = \sigma$ for each $e \in M'$. Let Γ'' be the subgroup of Γ generated by Γ' and σ and let g be the order of σ.

Now we can easily find a minor G' of G with $V(G') = \{v_k^i : 1 \leq i \leq n,\ 0 \leq k \leq g\}$ and with the following edges:

- For each $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, g\}$, we have an edge $e_k^i \in M'$ with head v_k^i, tail v_{k-1}^i, and label σ.
- For each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, $k \in \{1, \ldots, g\}$, and $\gamma' \in \Gamma'$, we have an edge $e \in E(G) - M$ with tail v_0^j, head v_k^i, and label γ'.

Now we construct a minor H' of G' by shifting each vertex v_k^i by $-k\sigma$ (so that each e_k^i is zero-labelled) and then contracting the edges $\{e_k^i : 1 \leq i \leq n,\ 1 \leq k \leq g\}$. It is straightforward to verify the H' is isomorphic to $K(\Gamma'', n)$ and that $E(H') \subseteq E(H)$. $\square$

We are now ready to prove the main result which we restate here for convenience.

Theorem 3.2. *Let Γ be a finite abelian group, let Γ' be a subgroup of Γ, let $n \in \mathbb{N}$, and let $t = 8n|\Gamma|^2$. Then if G is a Γ-labelled graph and H is a minor of G isomorphic to $K(\Gamma', 4t)$, then either*

- *there is a set $X \subseteq V(G)$ with $|X| < t$ such that the unique block of $G - X$ that contains most of $E(H)$ is Γ'-balanced, or*
- *there is a subgroup Γ'' of Γ properly containing Γ' and a minor H' of G with $E(H') \subseteq E(H)$ such that H' is isomorphic to $K(\Gamma'', n)$.*

Proof. Let $l = n|\Gamma|^2$ and $m = 4t$.

We assume that:

3.2.1. *there is no set $X \subseteq V(G)$ with $|X| < t$ such that the block of $G - X$ that contains most of $E(H)$ is Γ'-balanced.*

By possibly shifting we may assume that there exist vertex-disjoint trees $(T(v) : v \in V(H))$ in G such that:

- $\gamma_G(e) = 0$ for each $v \in V(H)$ and $e \in E(T(v))$,
- $\gamma_G(e) = \gamma_H(e)$ for each $e \in E(H)$, and
- $\text{head}_G(e) \in V(T(\text{head}_H(e)))$ and $\text{tail}_G(e) \in V(T(\text{tail}_H(e)))$ for each $e \in E(H)$.

Consider any $v \in V(H)$. For each $u \in V(H) - \{v\}$ we choose an edge $e \in E(H)$ with $u = \text{tail}_H(e)$ and $v = \text{head}_H(e)$ and we let $f_v(u) = \text{head}_G(e)$; thus $f_v(u) \in V(T(v))$. For each $X \subseteq V(T(v))$ we let $f_v^{-1}(X) = \{u \in V(H) - \{v\} : f_v(u) \in X\}$.

We leave it to the reader to verify that we can choose a vertex $a_v \in V(T(v))$ satisfying:

3.2.2. *for each edge $e \in E(T(v))$ we have $|f_v^{-1}(X)| \le \frac{m-1}{2}$ where X is the vertex set of the component of $T(v) - e$ that does not contain a_v.*

Now we let $A = \{a_v : v \in V(H)\}$. Our particular choice of A is motivated by the following claim.

3.2.3. *Let $S \subseteq V(H)$ with $|S| > \frac{m+1}{2}$, let $L \subseteq \cup(E(T(u)) : u \in S)$, let G' is the Γ-labelled subgraph of G induced by $\cup(V(T(v)) : v \in S)$, and let B be the block of G'/L that contains $E(H[S])$. If $v \in S$ and $L \cap E(T(v)) = \emptyset$, then $a_v \in V(B)$.*

Subproof. Let $G'' = G'/L$. If $a_v \notin V(B)$, then there is a separation (G_1, G_2) of G'' of order 1 with $E(B) \subseteq E(G_2)$ and $a_v \in V(G_1) - V(G_2)$. Let w be the vertex in $V(G_1) \cap V(G_2)$. Since $E(H[S]) \subseteq E(B) \subseteq E(G_2)$ and since some edge in $E(H[S])$ has an end in $V(T(v))$, we have $w \in V(T(v))$. Let e be the edge on the (a_v, w)-path in $T(v)$ that is incident with w and let X be the vertex set of the component of $T(v) - e$ that contains w. Note that, for each $u \in S - \{v\}$, we have $f_v(u) \in X$. Thus $|f_v^{-1}(X)| \ge |S| - 1 > \frac{m-1}{2}$, contradicting our choice of a_v. $\square$

3.2.4. *There exist vertex-disjoint A-paths $P_1, \ldots, P_{4l}$ such that $\gamma_G(P_i) \notin \Gamma'$ for each $i \in \{1, \ldots, 4l\}$.*

Subproof. Suppose otherwise; then, by Corollary 2.6, there is a set $X \subseteq V(G)$ with $|X| < 8l$ such that $\gamma_G(P) \in \Gamma'$ for each A-path P in G that is disjoint from X. Let $S = \{v \in V(H) : V(T(v)) \cap X = \emptyset\}$ and let $H' = H[S]$. Now let B be the block of $G - X$ that contains $E(H')$ and let x and y be distinct vertices in $\{a_v : v \in S\}$. Note that $|S| \ge \frac{m+1}{2}$. So, by 3.2.3, we have $x, y \in V(B)$, and, by our choice of X, $\gamma_B(P) \in \Gamma'$ for each (x, y)-path P in B. Then, by Lemma 2.7, B is Γ'-balanced. However $|S| > \frac{3}{4}|V(H)|$ so $|E(H')| > \frac{1}{2}|E(H)|$ and, hence, B is the unique block of $G - X$ that contains most of $E(H)$, which contradicts 3.2.1. $\square$

Let $F = \cup(E(T(v)) : v \in V(H))$. We choose vertex-disjoint A-paths $P_1, \ldots, P_{4l}$

- minimizing $|E(P_1) - F| + \cdots + |E(P_{4l}) - F|$,
- subject to $\gamma_G(P_i) \notin \Gamma'$ for each $i \in \{1, \ldots, 4l\}$.

Let $A_0 \subseteq A$ be the set of ends of the paths $P_1, \ldots, P_{4l}$, let $W = \{v \in V(H) : A_0 \cap V(T(v)) \neq \emptyset\}$, and let $A_1 = V(H) - W$.

3.2.5. *For each $v \in A_1$ and $i \in \{1, \ldots, 4l\}$, we have $V(T(v)) \cap V(P_i) = \emptyset$.*

Subproof. Suppose that $T(v)$ meets one or more of $P_1, \ldots, P_{4l}$. Then, for some path P_i and $w \in V(P_i) \cap V(T(v))$, the (a_v, w)-path Q in $T(v)$ is internally disjoint from each of $P_1, \ldots, P_{4l}$. Suppose that P_i is an (a_x, a_y)-path. Let Q_x denote the (a_v, a_x)-path with $E(Q_x) \subseteq E(Q) \cup E(P_i)$ and let Q_y denote the (a_v, a_y)-path with $E(Q_y) \subseteq E(Q) \cup E(P_i)$. Note that $\gamma_G(P_i) - \gamma_G(Q_y) + \gamma_G(Q_x) = 0$. Then, since $\gamma_G(P_i) \notin \Gamma'$, either $\gamma_G(Q_x) \notin \Gamma'$ or $\gamma_G(Q_y) \notin \Gamma'$. Moreover, $|E(Q_x) - F| < |E(P) - F|$ and $|E(Q_y) - F| < |E(P) - F|$; this contradicts our choice of $P_1, \ldots, P_{4l}$. □

Let G_1 be the minor of G obtained by contracting each of $(E(T(v)) : v \in A_1)$; we may assume that the vertices of G_1 are labelled so that $H[A_1]$ is a subgraph of G_1; thus $G_1[A_1]$ is isomorphic to $K(\Gamma', 16l)$.

3.2.6. *If $X \subseteq V(G_1)$ and $\gamma_G(P) \in \Gamma'$ for each A_1-path P in $G_1 - X$, then $|X| \geq 2l$.*

Subproof. Suppose that $|X| < 2l$. For $i \in \{1, \ldots, 4l\}$ let $W_i = V(T(x)) \cup V(T(y))$ where $T(x)$ and $T(y)$ are the trees that contain the ends of P_i. Since $|X| < 2l$, there is a path P_i such that $X \cap (V(P_i) \cup W_i) = \emptyset$. Suppose that $W_i = V(T(x)) \cup V(T(y))$ and that P_i is an (a_x, a_y)-path. There is an (a_y, a_x)-path P' in G_1 such that $E(P') \subseteq E(T(x)) \cup E(T(y)) \cup E(H)$; thus $\gamma_{G_1}(P') \in \Gamma'$. Let B be the block of $G_1 - X$ that contains $G_1[(A_1 \cup \{x, y\}) - X]$. By Lemma 2.7, B is Γ'-balanced. By 3.2.3, B contains a_x and a_y. Hence, P_i and P' are both contained in B. Let W' be the closed walk obtained by appending P' to P_i. Thus $\gamma_B(W) = \gamma_B(P') + \gamma_B(P_i) \notin \Gamma'$, contradicting Lemma 2.3. □

By the above claim and Corollary 2.6, there exist vertex-disjoint A_1-paths $Q_1, \ldots, Q_l$ in G_1 such that $\gamma_{G_1}(Q_i) \notin \Gamma'$ for each $i \in \{1, \ldots, l\}$. Now the result follows immediately from Lemma 3.1. □

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