RATIONALLY CONNECTED VARIETIES OVER FINITE FIELDS

JÁNOS KOLLÁR AND ENDRE SZABÓ

In this paper we study rationally connected varieties defined over finite fields. Then we lift these results to rationally connected varieties over local fields.

Roughly speaking, a variety X over an algebraically closed field is rationally connected if it contains a rational curve through any number of assigned points $P_1, \dots, P_n$. See [Kollár01a] for an introduction to their theory and for an explanation of the place of rationally connected varieties in the framework of the classification theory of algebraic varieties. There are many equivalent conditions defining this notion, see, for instance, [Kollár96, IV.3]. The definition given below essentially corresponds to the case of 2 points. In positive characteristic we also have to be mindful of some inseparability problems.

Definition 1. A smooth, proper variety X is called *separably rationally connected* or *SRC*, if there is a variety U and a morphism $F : U \times \mathbb{P}^1 \longrightarrow X$ such that the induced map

$$F(\_\,, (0{:}1)) \times F(\_\,, (1{:}0)) : U \longrightarrow X \times X$$

is dominant and separable.

If X is defined over a field k and the P_i are also defined over k, it is of interest to find rational curves defined over k passing through these points. In general this cannot be done. For instance, consider the surface

$$S^0 := (x^2 + y^2 = -\prod_{i=1}^{2m}(z - a_i)) \subset \mathbb{A}^3, \quad \text{where } a_i \in \mathbb{R}.$$

Let $S \supset S^0$ be any smooth compactification. The fibers of the projection to the z-axis $S^0 \longrightarrow \mathbb{A}_z$ are conics, thus S is ruled over $\mathbb{P}^1$. Therefore S is rational over $\mathbb{C}$. As a topological space $S(\mathbb{R})$ has m connected components. If we pick points P_1, P_2 in different connected components of $S(\mathbb{R})$ then there is no rational curve defined over $\mathbb{R}$ passing through both points (since $\mathbb{P}^1(\mathbb{R})$ is connected).

The main technical result of the present paper (Theorem 16) implies that for finite fields one can almost always find such rational curves. More precisely, we have the following:

Theorem 2. *There is a function* $\Phi : \mathbb{N}^3 \longrightarrow \mathbb{N}$ *with the following property:*

Let $X \subset \mathbb{P}^N$ *be a smooth, projective, separably rationally connected variety of dimension* ≥ 3 *over a finite field* k *and* $S \subset X$ *a zero–dimensional smooth subscheme.*

If $|k| > \Phi(\deg X, \dim X, \deg S)$*, then there is a smooth rational curve* $C_S \subset X$ *containing* S*, where* $|k|$ *denotes the cardinality of* k*.*

Remark 3. (1) If $\dim X = 2$ then there are very few smooth rational curves even over $\bar{k}$. For instance, in $\mathbb{P}^2$ the only smooth rational curves are lines and conics. If S consists of 3 points on a line plus another point then there is no smooth rational curve containing S.

For all applications considered in this paper, one can work with $X \times \mathbb{P}^1$ instead of the surface X to obtain the relevant conclusions.

One can also see from the proof that if X is a surface, we obtain an immersed curve $\mathbb{P}^1 \longrightarrow X$ through S.

(2) We can also be more precise about the curve C_S. For instance, with small modifications of the proof one sees that Theorem 2 remains true if we only assume that S has embedding dimension 1.

We can also require that C_S avoid a given codimension 2 subset of X and that it be transversal to a given divisor at all points outside S.

(3) The function Φ is explicitly computable from the proof though it is rather huge. Even if we allow singular curves C_S, some condition on $|k|$ is needed. For instance, the number of k-points in S cannot exceed $|k|+1$, the number of k-points in $\mathbb{P}^1$. We do not know any other necessary condition with singular C_S. In the smooth case, there are also problems with too few k-points. For instance, [Swinnerton-Dyer81, 5.5] contains an example of a cubic surface S over $\mathbb{F}_2$ with only one rational point. Thus $S \times S$ does not contain any smooth rational curves defined over $\mathbb{F}_2$. We do not know what a reasonable Φ should be.

(4) By a recent result of [Esnault02], an SRC variety defined over a finite field always has a point in that field.

We give two applications of this result to the study of the R-equivalence of points and the Chow group of 0-cycles $\mathrm{CH}_0(X)$. The interesting part is the Chow group of degree zero 0-cycles, denoted by $\mathrm{CH}_0^0(X)$.

The Chow group of zero cycles over a finite field is completely described in [Kato-Saito83]. Their result implies that if X is a smooth, projective variety defined over a finite field which is geometrically simply connected then $\mathrm{CH}_0^0(X) = 0$. SRC varieties were recently shown to be geometrically simply connected (see [Kollár01b] or [Debarre01, 3.6]), so $\mathrm{CH}_0^0(X) = 0$ for them. We give a geometric proof of this result for SRC varieties. The advantage of this proof is that it allows one to go from finite fields to local fields. (For our purposes *local fields* are the quotient fields of complete discrete valuation rings with finite residue field.) Lifting results of this type have been known for cubic surfaces [Swinnerton-Dyer81]. Recently this was extended to cubic hypersurfaces [Madore02]. For surfaces both results were established by [Colliot-Thélène83].

Theorem 4. [Kato-Saito83] *Let k be a finite field and X a smooth, projective, separably rationally connected variety over k. Then* $\mathrm{CH}_0^0(X) = 0$.

Theorem 5. *Let K be a local field with residue field k and X a variety over K. Assume that X has a smooth, projective, separably rationally connected reduction over k. Then* $\mathrm{CH}_0^0(X) = 0$.

R-equivalence on cubic surfaces was introduced in [Manin72] in order to study the parametrization of rational points. [Swinnerton-Dyer81] proves that R-equivalence is trivial on smooth cubic surfaces over finite fields. Precise computations on certain algebraic groups and related spaces are done in [Gille98, Gille01, Moret-Bailly01, Moret-Bailly02].

Definition 6. Let X be a proper variety defined over a field k. Two points $x, x' \in X(k)$ are called directly R-equivalent if there is a morphism $f : \mathbb{P}^1 \longrightarrow X$ (defined over k) such that $f(0{:}1) = x$ and $f(1{:}0) = x'$. The equivalence relation generated by this is called *R-equivalence*. Informally speaking, two points are *R-equivalent* if they can be connected by a chain of rational curves over k. If all the points in $X(k)$ are R-equivalent, then we say that *R-equivalence is trivial* on $X(k)$.

Theorem 7. *There is a function $\Psi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ with the following property:*

Let $X \subset \mathbb{P}^N$ be a smooth, projective, separably rationally connected variety over the finite field k. If $|k| > \Psi(\deg X, \dim X)$ then R-equivalence is trivial on $X(k)$.

Theorem 8. *There is a function $\Psi' : \mathbb{N}^2 \longrightarrow \mathbb{N}$ with the following property:*

Let K be a local field with residue-field k and $X \subset \mathbb{P}^N$ a variety over K. Assume that X has a smooth, projective, separably rationally connected reduction over k. If $|k| > \Psi'(\deg X, \dim X)$ then R-equivalence is trivial on $X(K)$.

The following corollary answers a question of Colliot-Thélène which was the starting point of our investigations. Let X be a smooth projective variety over a number field K. For a prime P, let X_P denote the corresponding variety over the P-adic completion K_P. Our results imply that the Chow group of zero cycles and R-equivalence are both trivial for almost every completion. This has been known in several special cases, see [CT-S77, Colliot-Thélène83, CT-S-SD87, Gille01, Moret-Bailly01, Moret-Bailly02].

Corollary 9. *Let X be a smooth projective rationally connected variety over a number field K. Then $\mathrm{CH}_0^0(X_P) = 0$ and R-equivalence is trivial on $X(K_P)$ for all but finitely many primes P.* □

10 (Open problems)**.** There are several unsolved problems closely related to these results. First of all, one may wish to know something about the function $\Psi(\deg X, \dim X)$. Unlike in the case of Φ, the known examples suggest that here the bound is not needed at all:

Question 11. Do Theorems (7) and (8) remain true without the assumption $|k| > \Psi(\deg X, \dim X)$?

More optimistically, one can ask the following question, which could be the strongest version of (2) that does not need any assumption on $|k|$.

Question 12. Let X be a smooth, projective, separably rationally connected variety over a finite field k. Let C be a smooth curve over k and $T \subset C$ a zero–dimensional subscheme. Is it true that every morphism $g_T : T \longrightarrow X$ can be extended to $g : C \longrightarrow X$?

This is easily seen to be true if X is rational over k.

Our current methods say very little about SRC varieties over local fields with bad reduction. The following basic finiteness result for R-equivalence is known:

Theorem 13. [Kollár99] *Let X be a smooth, projective, separably rationally connected variety over a local field K. Then $X(K)$ has finitely many R-equivalence classes.*

The analogous statement about the Chow group is not known, except for surfaces [Colliot-Thélène83].

Conjecture 14. *Let X be a smooth, projective, separably rationally connected variety defined over a local field. Then $\mathrm{CH}_0^0(X)$ is finite.*

The separability condition in (5) is essential. Indeed, there are varieties of general type over $\mathbb{Q}_p$ whose reduction to $\mathbb{F}_p$ is smooth and purely inseparably uniruled. (Some very nice examples are given in [Shioda-Katsura79].) The Chow group CH_0^0 is very large for some of them, see [Mumford68].

On the other hand, one would expect that our results hold for Fano varieties even if they are not separably rationally connected. The first case to investigate would be smooth hypersurfaces $X \subset \mathbb{P}^n$ of degree $\leq n$. These are all rationally connected but not known to be always SRC in low characteristics. (The examples in [Kollár96, V.5] have some mild singularities.)

15. Let X be a smooth, projective, separably rationally connected variety over a field k and $S \subset X$ a zero–dimensional smooth subscheme. Our aim is to find a rational curve $C \subset X$ defined over k, passing through S.

If X is SRC, there are such curves over $\bar{k}$ (cf. [Kollár96, IV.3.9]), thus the space $M \subset \mathrm{Hilb}(X)$ of smooth rational curves containing S is not empty. We want to find a point $[C] \in M(k)$ corresponding to a curve C. For technical reasons it is easier to concentrate on the case when $[C]$ is a smooth point of M. Suppose that there is a smooth k-point $[C]$ and let $M_C \subset M$ denote the unique irreducible component containing $[C]$. Then M_C is geometrically irreducible since it has a smooth k-point.

The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on the irreducible components of $M_{\bar{k}}$ and it fixes M_C. Thus a first step to find k-rational points on M is to find an irreducible subvariety $W \subset M_{\bar{k}}$ which is Galois invariant; that is, defined over k if k is perfect. (16) asserts that this is always possible.

For an arbitrary field, for instance for $k = \mathbb{Q}$, finding W helps very little in finding rational curves defined over k. The situation is, however, much better over finite fields, as we see in (21).

The main technical result of the paper is a statement about SRC varieties over arbitrary perfect fields:

Theorem 16. *Let $X \subset \mathbb{P}^N$ be a smooth, projective, separably rationally connected variety of dimension ≥ 3 over a perfect field k. Let $S \subset X$ be a zero–dimensional smooth subscheme.*

Then there is a nonempty locally closed subset W of the Hilbert scheme $\mathrm{Hilb}(X)$ such that

1. *W parametrizes certain smooth rational curves $C \subset X$ of degree $d(X,S)$ which contain S.*
2. *W is geometrically irreducible and smooth.*
3. *The restriction $T_X|_{C_w}(-S)$ is ample.*

(4) *$d(X,S)$ is bounded from above in terms of* $\dim X$, $\deg X$ *and* $\deg S$.

Remark 17. In contrast with some similar results (for instance [Kollár02, 3]) we no longer have k-points in compactifications of W. We lose this in Step 2 of (22).

Remark 18. Assume that k is algebraically closed. Let $R(X,S) \subset \operatorname{Hilb}(X)$ denote the space of all rational curves in X passing through S. If α denotes a numerical equivalence class of curves, then $R(X)$ decomposes as a disjoint union $R(X,S) = \cup_\alpha R_\alpha(X,S)$. It is reasonable to expect that for "general" α, the spaces $R_\alpha(X,S)$ are irreducible. This would be a much stronger statement than (16). There are, however, very few definitive results in this direction. The case when X is homogeneous and $S = \emptyset$ is treated in [Kim-Pan00]. Lines on low degree hypersurfaces are discussed in [HaMaPa98].

The most straightforward application of (16) is to varieties over pseudo-algebraically closed fields. By definition, a *pseudo-algebraically closed field* or *PAC field* is a field k such that every geometrically irreducible k-variety has a dense set of k-points. (See [Fried-Jarden86] for many examples and properties of such fields.) (16) now implies the following:

Theorem 19. *Let X be a smooth, projective, separably rationally connected variety of dimension ≥ 3 over a perfect PAC field k and $S \subset X$ a zero–dimensional smooth subscheme.*

Then there is a smooth, geometrically rational curve $C_S \subset X$ which is defined over k and contains S.

In particular, R-equivalence is trivial on $X(k)$ and $\mathrm{CH}_0^0(X) = 0$. □

The proof of the implication (19) ⇒ (2) is completely formal using the fact that the ultraproduct of infinitely many finite fields whose orders go to infinity is PAC. See, for instance, [Fried-Jarden86] for such techniques.

A more classical proof, given below, relies on some basic boundedness results and on the Lang-Weil theorem [Lang-Weil54] (which is also at the core of the above ultraproduct result).

For a quasi-projective scheme $U \subset \mathbb{P}^N$ let $\overline{U}$ denote its closure and $\partial U = \overline{U} \setminus U$ its boundary. Define the *degree* of a reduced projective scheme to be the sum of the degrees of its irreducible components (even if they have different dimensions). We call N, $\dim(U)$, $\deg(\overline{U})$ and $\deg(\partial U)$ the *basic projective invariants* of a reduced quasi-projective scheme U.

The set of all reduced quasi-projective schemes with given basic projective invariants is bounded. That is, there is a scheme of finite type S and a locally closed subscheme $U \subset S \times \mathbb{P}^N$ such that every reduced quasi-projective scheme with given basic projective invariants occurs among the fibers of $U \longrightarrow S$. (This is an easy consequence of the classical theorem, going back to Cayley, that the Chow variety of pure dimensional reduced closed subschemes of $\mathbb{P}^N$ of a given degree is bounded. See [Hodge-Pedoe47] for a classical treatment and [Kollár96, I.3–4] for a modern update.)

Conversely, if S is a scheme of finite type and $U \subset S \times \mathbb{P}^N$ is a locally closed subscheme then the basic projective invariants of the fibers form a bounded set.

Using basic existence results of the relative Hilbert scheme (see, for instance, [Kollár96, I.1]), we get the following:

Proposition 20. *Let $X \subset \mathbb{P}^N$ be a projective variety and $S \subset X$ a zero–dimensional subvariety. Let $W_d \subset \mathrm{Hilb}(X)$ be the locally closed subset of the Hilbert scheme parametrizing all smooth curves of degree d containing S. Let $W_d^i \subset W_d$ be any irreducible component.*

Then the basic projective invariants of W_d^i can be bounded in terms of $\dim X, \deg X, \deg S$ *and d.* □

21 (Proof of (16) ⇒ (2))**.** Let us now turn to the situation of (16). Let $\mathrm{Hilb}(X, S)^s$ denote the Hilbert scheme of all smooth rational curves of X which contain S. By the deformation theory of Hilbert schemes containing a given subscheme (see, for instance, [Mori79]), the assumption (16.3) implies that every point of W is a smooth point of $\mathrm{Hilb}(X, S)^s$. (In our case $\mathrm{Hilb}(X, S)^s$ can be identified with an open subset of the Hilbert scheme of $B_S X$, the blow up of X at S, and the more traditional study of $\mathrm{Hilb}(B_S X)$ also gives this result, cf. [Kollár96, I.2.8].)

Since W is geometrically irreducible by assumption (16.2), this implies that W is contained in a unique geometrically irreducible component W' of $\mathrm{Hilb}(X, S)^s$.

By (20), W' is a quasi-projective variety in some $\mathbb{P}^N$, and its basic projective invariants N, $\dim W'$, $\deg \overline{W'}$, $\deg \partial W'$ are bounded in terms of $\dim X$, $\deg X$ and $\deg S$.

The Lang-Weil theorem asserts that an r-dimensional geometrically irreducible projective variety W' has roughly q^r points defined over $\mathbb{F}_q$, and the error term is bounded by $C \cdot q^{r-1/2}$ with C depending only on the basic projective invariants of W'. The same remains true for quasi-projective varieties, because the boundary has smaller dimension, hence its points change only the error term. (The boundary may not be geometrically irreducible, but an *upper* bound for the number of points holds without geometric irreducibility.) So one can easily find a lower bound on q in terms of the basic projective invariants, such that if q is larger than this bound then W' has $\mathbb{F}_q$-points. □

22 (Main steps of the proof of (16))**.** The construction of the geometrically irreducible family of curves W proceeds in four steps.

(1) First, in lemma (23), for every $P \in X(\bar{k})$ we construct geometrically irreducible families (defined over $k(P)$) of maps passing through P
$$V_P \subset \operatorname{Hom}(\mathbb{P}^1, X; (0{:}1) \mapsto P).$$
These families were introduced and used in [Kollár99].

(2) Then in (24) we find some auxiliary curves $C \subset X$ and prove that the space of all maps in V_P whose image intersects C:
$$V_{P,C} := \{f \in V_P : f(1{:}0) \in C\}$$
is geometrically irreducible. This relies on the Lefschetz–type results of [Kollár00, Kollár02] recalled in (25).

(3) Third, starting with (26), given $S = \{P_1, \dots, P_n\}$, we construct a family $V_{S,C}$ of reducible rational curves, called combs, passing through S. Roughly speaking, over $\bar{k}$,
$$V_{S,C} := V_{P_1,C} \times \cdots \times V_{P_n,C},$$
and the curve corresponding to a point of $V_{S,C}$ consists of C and n curves connecting C to the points $P_1, \dots, P_n$. (It is better to think of these as a curve mapping to X, rather than a subscheme of X.) This is the main new idea of the paper.

(4) Finally, in (32), we get the intended W by smoothing the reducible curves in $V_{S,C}$.

The easiest case, Step 1, is handled in the following, essentially proved in [Kollár99, 3.2]. (See also [Araujo-Kollár02, 44] or [Kollár02, Thm.3,Rem.4] where a much stronger result is proved.)

Lemma 23. *Let $X \subset \mathbb{P}^m$ be a smooth, projective, SRC variety over a field k and $P \in X(k)$ a point. Then there is a family of rational curves*
$$F_P : V_P \times \mathbb{P}^1 \longrightarrow X,$$
defined over k, with the following properties:

(1) *$F_P(V_P \times (0{:}1)) = P$.*
(2) *$F_P : V_P \times (1{:}0) \longrightarrow X$ is smooth, hence dominant.*
(3) *V_P is geometrically irreducible and smooth.*
(4) *The pullback of T_X to each curve $\mathbb{P}^1_v := \{v\} \times \mathbb{P}^1$ is ample, even after twisting by $\mathcal{O}(-3)$.*

(5) *The degree of the curves* $F_P(\mathbb{P}^1_v)$ *is bounded in terms of* $\dim X$ *and* $\deg X$. $\square$

24 (Step 2)**.** Next consider the space of rational curves passing through a given point P and meeting a given curve C. We start with the family of morphisms as in (23) and look at the evaluation map

$$f_P : V_P \longrightarrow X$$

which sends $v \in V_P$ to $F_P(v, (1{:}0))$. The inverse image $f_P^{-1}(C)$ can be thought of as a family of curves passing through P and intersecting C. We would like $f_P^{-1}(C)$ to be geometrically irreducible. We cannot provide a single curve C that does the job. For one thing, the curve should work simultaneously for many points P, and for another, the curve must be defined over the base field. On the other hand, one can find a family of curves such that for each point P the general member of the family does work. The precise result we need is the following:

Lemma 25. [Kollár02, Thm.6,Cor.7,Rem.4] *Let* $X \subset \mathbb{P}^m$ *be a smooth, projective variety over a field* k. *Assume that* X *is SRC. Then there is a nonempty family of rational curves, defined over* k,

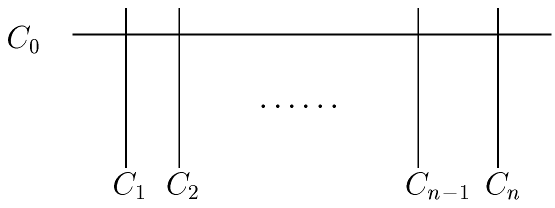
$$F : U \times \mathbb{P}^1 \longrightarrow X$$

with the following properties:

(1) *Let* Y *be a geometrically irreducible variety and* $h : Y \longrightarrow X$ *a dominant morphism. Then there is a dense open set* $U^h \subset U$ *such that for every* $u \in U^h(\bar{k})$ *the fiber product* $Y \times_X \mathbb{P}^1_u$ *is irreducible.*
(2) U *is geometrically irreducible, smooth and open in* $\mathrm{Hom}(\mathbb{P}^1, X)$.
(3) *The pullback of* T_X *to each curve* $\mathbb{P}^1_u$ *is ample.*
(4) *The degree of the curves* $F(\mathbb{P}^1_u)$ *is bounded in terms of* $\dim X$ *and* $\deg X$.
(5) *If* $\dim X \geq 3$ *then* F *is an embedding on every fiber* $\mathbb{P}^1_u$. $\square$

26 (Step 3)**.** Now we move to reducible curves passing through several points. Assume that our curves need to go through the conjugation invariant set of closed points $P_1, \dots, P_n$. For each P_i we choose a family V_{P_i} defined over $k(P_i)$ as in (23) and also choose another family as in (25). Out of these we get configurations of $n+1$ curves: an appropriate rational curve C and for each P_i a rational curve connecting P_i to C.

Definition 27. A genus zero *comb* over k with n *teeth* is a reduced projective curve of genus 0 (i.e. a curve C with $h^1(C, \mathcal{O}_C) = 0$) having $n + 1$ irreducible components over $\bar{k}$. One component, defined over k, is called the *handle*. The other n components, $C_1, \dots, C_n$ are disjoint from each other and intersect C_0 in n distinct points. Every C_i is smooth and rational. The curves $C_1, \dots, C_n$ may not be individually defined over k. A comb can be pictured as below:



Comb with n-teeth

Definition 28. Let X be a smooth projective variety over k and $S \subset X$ a zero–dimensional smooth subset. Set $n = \deg S$. A *k-comb through S* is

(1) a k-comb with n teeth $C = C_0 \cup \cdots \cup C_n$,
(2) a k-morphism $f : C \longrightarrow X$, and
(3) a k-morphism $\sigma : S \longrightarrow C \setminus C_0$ such that $f \circ \sigma = id_S$, and each $C_i, i \geq 1$ contains exactly one point of $S_{\bar{k}}$.

Set $S_{\bar{k}} = \{P_1, \dots, P_n\}$. Over a perfect field, a comb through S can also be given by a k-morphism $g_0 : \mathbb{P}^1 \longrightarrow X$ and a conjugate set of $\bar{k}$-morphisms $g_i : \mathbb{P}^1 \longrightarrow X$, $i = 1, \dots, n$ such that

(1) $g_i(0{:}1) = P_i$, and
(2) $g_i(1{:}0) = g_0(s(P_i))$ for $i = 1, \dots, n$, where $s : S \longrightarrow \mathbb{P}^1$ is some embedding.

A *family of combs through S* parametrized by a scheme T can be identified with $n+1$ morphisms

$$G_i : T \times \mathbb{P}^1 \longrightarrow X \quad i = 0, \dots, n$$

with the expected properties.

29 (Construction of the combs)**.** Let $X \subset \mathbb{P}^N$ be a smooth, projective, separably rationally connected variety over a perfect field k and $S \subset X$ a zero–dimensional smooth subscheme. Set $S_{\bar{k}} = \{P_1, \dots, P_n\}$.

For a given $P_i \in S(\bar{k})$ choose a family of maps

$$F_{P_i} : V_{P_i} \times \mathbb{P}^1 \longrightarrow X,$$

defined over $k(P_i)$, as in (23). For the conjugates $P_{i_1}, \dots, P_{i_s}$ we choose the conjugate families

$$F_{P_{i_j}} : V_{P_{i_j}} \times \mathbb{P}^1 \longrightarrow X.$$

Going through all the points of S, we have a conjugation invariant collection of n families

$$F_{P_i} : V_{P_i} \times \mathbb{P}^1 \longrightarrow X$$

as in (23).

Let $F : U \times \mathbb{P}^1 \longrightarrow X$ denote the family of curves obtained in (25).

For every j, let $U(P_j)$ denote the subset of those

$$(u, v_j) \in U \times V_{P_j}$$

such that

$$F_{P_j}(v_j, (0{:}1)) \in F(\mathbb{P}^1_u).$$

The fiber of the projection

$$\pi_j : U(P_j) \longrightarrow U$$

over a point $u \in U$ consists of those maps in the family V_{P_j} which map $(0{:}1)$ to a point of $F(\mathbb{P}^1_u)$. Thus by (25) there is a nonempty open subset $U^j \subset U_{\bar{k}}$ such that $\pi_j^{-1}(u)$ is geometrically irreducible for $u \in U^j(\bar{k})$. We can further assume that the U^j are conjugates of each other.

As a $\bar{k}$-scheme, let $U(P_1, \ldots, P_n)$ denote the subset of those

$$(u, v_1, \ldots, v_n) \in U \times \prod_{i=1}^{n} V_{P_i}$$

such that

$$F_{P_i}(v_i, (0{:}1)) \in F(\mathbb{P}^1_u) \quad \forall\, i = 1, \ldots, n.$$

By construction

$$U(P_1, \ldots, P_n) = U(P_1) \times_U \cdots \times_U U(P_n).$$

Set $U^S = U^1 \cap \cdots \cap U^n \subset U$. Then the fibers of the projection

$$\Pi : U(P_1, \ldots, P_n) \longrightarrow U$$

are geometrically irreducible over U^S since

$$\Pi^{-1}(u) = \pi_1^{-1}(u) \times \cdots \times \pi_n^{-1}(u)$$

and the product of geometrically irreducible schemes is again geometrically irreducible. Thus we conclude that:

Claim 30. The variety $U(P_1, \ldots, P_n)$ is defined over k and it has a unique geometrically irreducible component $U^0(P_1, \ldots, P_n)$ which dominates U.

By our choices of the families V_{P_i}, the construction of $U(P_1,\dots,P_n)$ is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$. Thus we can henceforth view $U(P_1,\dots,P_n)$ as a k-variety. Not all of the points of $U^0(P_1,\dots,P_n)$ correspond to combs since the curves C_i need not intersect C_0 in distinct points. In the construction, the intersection of C_i and C_0 can be an arbitrary point of the set $C_0 \cap F_{P_i}(V_{P_i} \times (1{:}0))$ which is dense in C_0 for general C_0. Therefore the combs correspond to an open subset $U^c(P_1,\dots,P_n) \subset U^0(P_1,\dots,P_n)$. Combined with (23.4) and (25.3) this implies the following:

Claim 31. $U^c(P_1,\dots,P_n)$ is a geometrically irreducible variety, defined over k. It parametrizes combs through the subset S

$$g : C = C_0 \cup \dots \cup C_n \longrightarrow X$$

with liftings $\sigma : S \longrightarrow C$ such that $g^*T_X(-3\sigma(S))$ is ample on every irreducible component of C.

32 (Step 4 of the proof of (16))**.** First we work in the space of genus zero stable curves $\bar{M}_0(X, S \hookrightarrow X)$ with base point set S as discussed in [Araujo-Kollár02, sec.8], and then we switch to the Hilbert scheme of X.

By the universal property of $\bar{M}_0(X, S \hookrightarrow X)$, there is a moduli map

$$m : U^c(P_1,\dots,P_n) \longrightarrow \bar{M}_0(X, S \hookrightarrow X).$$

We claim that there is a unique irreducible component $U_1 \subset \bar{M}_0(X, S \hookrightarrow X)$ containing $m(U^c(P_1,\dots,P_n))$.

Since $g^*T_X(-3\sigma(S))$ is ample on every irreducible component of C, an easy lemma (cf. [Araujo-Kollár02, 18]) implies that $H^1(C, g^*T_X(-\sigma(S))) = 0$. Thus by [Araujo-Kollár02, 42.1] the space $\bar{M}_0(X, S \hookrightarrow X)$ has a unique irreducible component containing the image of any point of $U^c(P_1,\dots,P_n)$. The latter is irreducible, so this must be the same component, call it U_1, for every point.

Over $\bar{k}$, a general point of U_1 corresponds to an irreducible curve $g_u : D_u \cong \mathbb{P}^1 \longrightarrow X$ with $\sigma_u : S \longrightarrow D_u$ by [Araujo-Kollár02, 42.3]. By semicontinuity, $H^1(D_u, g_u^*T_X(-3\sigma_u(S))) = 0$ for u in an open subset of U_1. The pull back $g_u^*T_X(-\sigma_u(S))$ is a vector bundle on $\mathbb{P}^1$, hence a direct sum of line bundles. The above vanishing implies that every direct summand of $g_u^*T_X(-\sigma_u(S))$ has degree at least $2\deg S - 1 \geq 1$. If $\dim X \geq 3$ then by [Kollár96, II.3.14.3] this implies that there is an open subset $U_2 \subset U_1$ parametrizing embeddings $\mathbb{P}^1 \hookrightarrow X$ passing through S. (If $\dim X = 2$, we get immersions by [Kollár96, II.3.14.2].)

Pick any $u \in U_2(\bar{k})$. Since $g_u : D_u \hookrightarrow X$ is an embedding, there are no automorphisms of D_u commuting with g_u. Thus over U_2 we have a fine moduli problem and there is a universal family $\mathbf{D}_2 \longrightarrow U_2$, see [FultPan97, Thm.2.iii]. The graph of the universal morphism

$$\mathbf{g} : \mathbf{D}_2 \hookrightarrow U_2 \times X$$

defines an embedding $U_2 \longrightarrow \mathrm{Hilb}(X)$ with image W. □

33 (Proof of (7))**.** Let X be a smooth, projective, separably rationally connected variety over the finite field k. Pick $P, Q \in X(k)$. Set $S = P \cup Q$. By (2), if $\dim X \geq 3$ and if $|k| > \Phi(\deg X, \dim X, \deg S)$, then there is a smooth, rational curve $C_S \subset X$ containing S, hence P and Q are R-equivalent. If $\dim X = 2$ then we work with $X \times \mathbb{P}^1$.

Set $\Psi(a, b) = \Phi(a, b, 2)$ for $b \geq 3$ and $\Psi(a, 2) = \Phi(3a, 2, 2)$. ($3 \deg X$ is the degree of $X \times \mathbb{P}^1$ under the Segre embedding when X is a surface.)

□

34 (Proof of (8))**.** Let X be a smooth, projective, separably rationally connected variety over the local field K. Pick $P, Q \in X(K)$. Set $S = P \cup Q$. The base field K is the quotient field of a complete valuation ring R whose residue field is k. By assumption there is a model $X_R \longrightarrow \operatorname{Spec} R$ of X such that the special fibre X_k satisfies the conditions of (7). Since X_R is proper and R is a valuation ring, the points P, Q give rise to sections P_R, Q_R of $X_R \longrightarrow \operatorname{Spec} R$, hence they have well-defined reductions $P_k, Q_k \in X_k$.

Assume first that $P_k \neq Q_k$. By (2), if $|k| > \Phi(\deg X, \dim X, \deg S)$, then there is a smooth, geometrically rational curve $C_S \subset X_k$ containing S_k. Moreover, the twisted restriction $T_X|_{C_{S_k}}(-S_k)$ is ample.

We apply [Araujo-Kollár02, 51] to $X_R \longrightarrow \operatorname{Spec} R$ with base points $P_R \cup Q_R \hookrightarrow X_R$. (The current R corresponds to S of [Araujo-Kollár02], the $P \longrightarrow S$ there is 2 copies of S and $t : P \longrightarrow X_R$ is the imbedding whose image is $P_R \cup Q_R$.) By [Araujo-Kollár02, 51.5] we obtain a smooth R-scheme $U \longrightarrow R$ with a point $u \in U(k)$ and a family of rational curves $U \times \mathbb{P}^1 \longrightarrow X_R$ passing through P_R and Q_R. By the Hensel lemma, $\phi_k : \operatorname{Spec} k \longrightarrow u$ lifts to $\phi_R : R \longrightarrow U$. The image of $\phi_R(\operatorname{Spec} K)$ corresponds to a rational curve $C \subset X_K$ going through P and Q. Thus P and Q are R-equivalent.

If $P_k = Q_k$ then we work with $X \times \mathbb{P}^1$ and consider the points $P \times (0{:}1)$ and $Q \times (1{:}0)$. These have reductions $P_k \times (0{:}1)$ and $Q_k \times (1{:}0)$ which are now different. The previous argument gives a rational curve on $X \times \mathbb{P}^1$ passing through $P \times (0{:}1)$ and $Q \times (1{:}0)$. This again implies that P and Q are R-equivalent.

□

35 (Proof of (4))**.** Let X be a smooth, projective, separably rationally connected variety over the finite field k. Let Z, Z' be two effective zero cycles of the same degree on X. Let S be the support of $Z + Z'$.

By (2), if $|L| > \Phi(\deg X, \dim X, \deg S)$, then there is a smooth rational curve $C_L \subset X_L$ containing S. Thus $Z - Z'$ is trivial in $\mathrm{CH}_0(X_L)$.

The pull back followed by the push forward

$$\mathrm{CH}_0(X_k) \longrightarrow \mathrm{CH}_0(X_L) \longrightarrow \mathrm{CH}_0(X_k)$$

is multiplication by $\deg(L/k)$. This proves that $Z - Z' \in \mathrm{CH}_0(X)$ is torsion whose order divides $\deg(L/k)$.

A finite field has extensions of arbitrary degree. Taking two such extensions of degrees a and $a + 1$ for $a > \Phi(\deg X, \dim X, \deg S)$, we conclude that $Z - Z' \in \mathrm{CH}_0(X)$ is torsion whose order divides $\gcd(a, a + 1) = 1$. $\square$

36 (Proof of (5))**.** Let X be a smooth, projective, separably rationally connected variety over the local field K.

Let $X_R \longrightarrow R$ be a smooth projective model of X_K with special fiber X_k. Every finite extension $k' \supset k$ corresponds to a unique unramified extension $K' \supset K$ (This is essentially the theory of Witt rings as explained in [Serre62, Chap.II].)

Thus there are two extensions K_i/K of relatively prime degree such that $X(K_i) \neq \emptyset$. As in the previous proof, it is enough to prove that $\mathrm{CH}_0^0(X_{K_i}) = 0$. Changing notation, it is enough to prove that $\mathrm{CH}_0^0(X) = 0$ if there is a point $P \in X(K)$.

Let $Q \in X$ be any closed point of degree d. It is enough to prove that $Q - dP \in \mathrm{CH}_0^0(X)$ is zero. Set $L = K(Q)$ and let $Q_1 \in X(L)$ be a point over Q. We have proved that Q_1 and P are R-equivalent in X_L, thus $Q_1 - P$ is zero in $\mathrm{CH}_0^0(X_L)$. Hence its push forward $Q - dP$ is zero in $\mathrm{CH}_0^0(X)$. $\square$

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Princeton University, Princeton NJ 08544-1000
`kollar@math.princeton.edu`

Mathematical Institute of the Hungarian Academy of Sciences,
1364 Budapest, PO Box 127
`endre@renyi.hu`