Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds

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Abstract. We introduce complex singularity exponents of plurisubharmonic functions and prove a general semi-continuity result for them. This concept contains as a special case several similar concepts which have been considered e.g. by Arnold and Varchenko, mostly for the study of hypersurface singularities. The plurisubharmonic version is somehow based on a reduction to the algebraic case, but it also takes into account more quantitative informations of great interest for complex analysis and complex differential geometry. We give as an application a new derivation of criteria for the existence of Kähler-Einstein metrics on certain Fano orbifolds, following Nadel's original ideas (but with a drastic simplication in the technique, once the semi-continuity result is taken for granted). In this way, 3 new examples of rigid Kähler-Einstein Del Pezzo surfaces with quotient singularities are obtained.

Contents

Key words: complex singularity exponent, Arnold multiplicity, semicontinuity property, log resolution, log canonical singularities, effective divisor, coherent ideal sheaf, adjunction theory, inversion of adjunction, plurisubharmonic function, multiplier ideal sheaf, closed positive current, Lelong number, Monge-Ampère equation, Aubin-Calabi-Yau theorem, Fano variety, orbifold

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§0. Introduction

The purpose of this work is to show how complex analytic methods (and more specifically L^2 estimates for $\overline{\partial}$) can provide effective forms of results related to the study of complex singularities. We prove in particular a strong form of the semi-continuity theorem for "complex singularity exponents" of plurisubharmonic (psh) functions. An application to the existence of Kähler-Einstein metrics on certain Fano orbifolds will finally be given as an illustration of this result.

We introduce the following definition as a quantitative way of measuring singularities of a psh function φ (the basic definition even makes sense for an arbitrary measurable function φ, though it is unlikely to have any good properties in that case). Our approach is to look at the L^1 integrability of $\exp(-2c\varphi)$ in terms of the Lebesgue measure in some local coordinates. Several other types of analytic or algebraic objects (holomorphic functions, coherent ideal sheaves, divisors, currents, etc) can be treated as special cases of this formalism.

0.1. Definition. *Let X be a complex manifold and φ be a plurisubharmonic (psh) function on X. For any compact set $K \subset X$, we introduce the "complex singularity exponent" of φ on K to be the nonnegative number*

$$c_K(\varphi) = \sup\big\{c \geqslant 0 \,:\, \exp(-2c\varphi) \ \textit{is } L^1 \textit{ on a neighborhood of } K\big\},$$

and we define the "Arnold multiplicity" to be $\lambda_K(\varphi) = c_K(\varphi)^{-1}$:

$$\lambda_K(\varphi) = \inf\big\{\lambda > 0 \,:\, \exp(-2\lambda^{-1}\varphi) \ \textit{is } L^1 \textit{ on a neighborhood of } K\big\}.$$

If $\varphi \equiv -\infty$ near some connected component of K, we put of course $c_K(\varphi) = 0$, $\lambda_K(\varphi) = +\infty$.

The singularity exponent $c_K(\varphi)$ only depends on the singularities of φ, namely on the behavior of φ near its $-\infty$ poles. Let T be a closed positive current of bidegree $(1,1)$ on X. Since $c_K(\varphi)$ remains unchanged if we replace φ with ψ such that $\psi - \varphi$ is bounded, we see that it is legitimate to define

$$c_K(T) = c_K(\varphi), \qquad \lambda_K(T) = \lambda_K(\varphi) \tag{0.1.1}$$

whenever φ is a (local) potential of T, i.e. a psh function φ such that $dd^c\varphi = T$, where $d^c = (2\pi \mathrm{i})^{-1}(\partial - \overline{\partial})$. In particular, if D is an effective integral divisor, we have $c_K([D]) = c_K(\log|g|)$ where $[D]$ is the current of integration over D and g is a (local) generator of $\mathcal{O}(-D)$. When f is a holomorphic function, we write simply $c_K(f)$, $\lambda_K(f)$ instead of $c_K(\log|f|)$, $\lambda_K(\log|f|)$. For a coherent ideal sheaf $\mathcal{I} = (g_1, \ldots, g_N)$ we define in a similar way $c_K = c_K(\log(|g_1| + \cdots + |g_N|))$. It is well known that $c_K(f)$ is a rational number, equal to the largest root of the Bernstein-Sato polynomial of $|f|^{2s}$ on a neighborhood of K ([Lin89], see also [Kol97]); similarly $c_K(\mathcal{I}) \in \mathbb{Q}_+$ for any coherent ideal sheaf. Our main result consists in the following semi-continuity theorem.

0.2. Main Theorem. *Let X be a complex manifold. Let $\mathcal{Z}_+^{1,1}(X)$ denote the space of closed positive currents of type $(1,1)$ on X, equipped with the weak topology, and let $\mathcal{P}(X)$ be the set of locally L^1 psh functions on X, equipped with the topology of L^1 convergence on compact subsets (= topology induced by the weak topology). Then*

(1) *The map $\varphi \mapsto c_K(\varphi)$ is lower semi-continuous on $\mathcal{P}(X)$, and the map $T \mapsto c_K(T)$ is lower semi-continuous on $\mathcal{Z}_+^{1,1}(X)$.*

(2) ("Effective version"). *Let $\varphi \in \mathcal{P}(X)$ be given. If $c < c_K(\varphi)$ and ψ converges to φ in $\mathcal{P}(X)$, then $e^{-2c\psi}$ converges to $e^{-2c\varphi}$ in L^1 norm over some neighborhood U of K.*

As a special case, one gets:

(3) *The map $\mathcal{O}(X) \ni f \mapsto c_K(f)$ is lower semi-continuous with respect to the topology of uniform convergence on compact sets (uniform convergence on a fixed neighborhood of K is of course enough). Moreover, if $c < c_K(f)$ and g converges to f in $\mathcal{O}(X)$, then $|g|^{-2c}$ converges to $|f|^{-2c}$ in L^1 on some neighborhood U of K.*

In spite of their apparent simplicity, the above statements reflect rather strong semi-continuity properties of complex singularities under "variation of parameters". Such properties have been used e.g. by Angehrn-Siu [AnSi95] in their approach of the Fujita conjecture, and our arguments will borrow some of their techniques in section § 3.

Theorem 0.2 is by nature a purely local result, which is easily seen to be equivalent to the special case when $K = \{x\}$ is a single point and X is a small ball centered at x. The proof is made in several steps. The "analytic part" consists in a reduction of (1) and (2) to (3), and in the proof of the effective estimates leading to the convergence statements in (2) and (3) [by contrast, the qualitative part of (3) can be obtained in a purely algebraic way]. The reduction to the holomorphic case (3) is based on the fact that plurisubharmonic functions can be very accurately approximated (both from the point of view of singularities and of L^1_{loc} topology) by special functions of the form

$$\alpha \log(|g_1| + \cdots + |g_N|), \qquad \alpha \geqslant 0, \tag{0.2.4}$$

where the g_j are holomorphic functions. The existence of approximations as in (0.2.4) depends in an essential way on the Ohsawa-Takegoshi L^2 extension theorem ([OhT87], [Ohs88]), see [Dem92, 93] and §2, §4. One is then reduced to the proof for a single holomorphic function (that is, to a psh function of the form $\log|f|$), by taking a suitable generic linear combination $f = \sum \alpha_j g_j$. Another essential idea is to truncate the Taylor expansion of f at x at some order k. It can then be shown that this affects $c_x(f)$ only by a perturbation that is under uniform control. In fact, the singularity exponent $c_x(f)$ is subadditive on holomorphic functions:

$$c_x(f+g) \leqslant c_x(f) + c_x(g), \qquad \forall f, g \in \mathcal{O}_{X,x}. \tag{0.2.5}$$

If p_k is the truncation at order k of the Taylor series, one deduces immediately from (0.2.5) that

$$|c_x(f) - c_x(p_k)| \leqslant \frac{n}{k+1}. \tag{0.2.6}$$

In this way, the proof is reduced to the case of polynomials of given degree. Such polynomials only depend on finitely many coefficients, thus the remaining lower semi-continuity property to be proved is that of the function $t \mapsto c_x(P_t)$ when P_t is a family of polynomials depending holomorphically on some parameters $t = (t_1, \ldots, t_N)$. This is indeed true, as was already observed by Varchenko [Var82, 83]. An algebraic proof can be given by using a log resolution of singularities with parameters. Here, however, a special attention to effective estimates must be paid to prove the convergence statements in (2) and (3). For instance, it is necessary to get as well an effective version of (0.2.6); the Ohsawa-Takegoshi L^2 extension theorem is again crucial in that respect.

As a consequence of our main theorem, we give a more natural proof of the results of Siu [Siu87, 88], Tian [Tia87] and Nadel [Nad89, 90] on the existence of Kähler-Einstein metric on Fano manifolds admitting a sufficiently big group of symmetries. The main point is to have sufficient control on the "multiplier ideal sheaves" which do appear in case the Kähler-Einstein metric fails to exist. This can be dealt with much more easily through our semi-continuity theorem, along the lines suggested in Nadel's note [Nad89] (possibly because of the lack of such semi-continuity results, the detailed version [Nad90] relies instead on a rather complicated process based on a use of "uniform" L^2 estimates for sequences of Koszul complexes; all this disappears here, thus providing a substantially shorter proof). We take the opportunity to adapt Nadel's result to Fano orbifolds. This is mostly a straightforward extension, except that we apply intersection inequalities for currents rather than the existence of a big finite group of automorphisms to derive sufficient criteria for the existence of Kähler-Einstein metrics. In this way, we produce 3 new "exotic examples" of rigid Del Pezzo surfaces with quotient singularities which admit a Kähler-Einstein orbifold metric.

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§1. Complex singularity exponent and Arnold multiplicity

Let X be a complex manifold and φ a psh function of x. The concepts of "complex singularity exponent" $c_K(\varphi)$ and "Arnold multiplicity" $\lambda_K(\varphi)$ of φ along a compact set $K \subset X$ have been defined in 0.1. An equivalent definition can be given in terms of asymptotic estimates for the volume of sublevel sets $\{\varphi < \log r\}$.

1.1. Variant of the definition. *Let $K \subset X$ be a compact set, $U \Subset X$ a relatively compact neighborhood of K, and let μ_U be the Riemannian measure on*

U associated with some choice of hermitian metric ω on X. Then

$$c_K(\varphi) = \sup\big\{c \geqslant 0\,;\ r^{-2c}\mu_U(\{\varphi < \log r\}) \ \textit{is bounded as } r \to 0, \textit{ for some } U \supset K\big\}.$$

The equivalence with the earlier Definition 0.1 follows immediately from the elementary inequalities

$$r^{-2c}\mu_U(\{\varphi < \log r\}) \leqslant \int_U e^{-2c\varphi}\,dV_\omega \leqslant \mu_U(U) + \int_0^1 2c\,r^{-2c}\mu_U(\{\varphi < \log r\})\,\frac{dr}{r},$$

A first important observation is that $c_K(\varphi)$ and $\lambda_K(\varphi)$ depend only on the local behavior of φ:

1.2. Proposition. *Given a point $x \in X$, we write $c_x(\varphi)$ instead of $c_{\{x\}}(\varphi)$. Then*

$$c_K(\varphi) = \inf_{x\in K} c_x(\varphi), \qquad \lambda_K(\varphi) = \sup_{x\in K} \lambda_x(\varphi).$$

The statement is clear from the Borel-Lebesgue Lemma. When x is a pole, that is, when $\varphi(x) = -\infty$, the Arnold multiplicity $\lambda_x(\varphi)$ actually measures the "strength" of the singularity of φ in a neighborhood of x. (It actually "increases" with the singularity, and if x is not a pole, we have $c_x(\varphi) = +\infty$, $\lambda_x(\varphi) = 0$; see Prop. 1.4 below.) We now deal with various interesting special cases:

1.3. Notation.

(1) *If f is a holomorphic function on X, we set $c_K(f) = c_K(\log|f|)$.*

(2) *If $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal sheaf, generated by functions $(g_1, \ldots, g_N)$ on a neighborhood of K, we put*

$$c_K(\mathcal{I}) = c_K\big(\log(|g_1| + \cdots + |g_N|)\big).$$

(3) *If T is a closed positive current of bidegree $(1,1)$ on X which can be written as $T = dd^c\varphi$ on a neighborhood of K, we set $c_K(T) = c_K(\varphi)$.*

(If no global generators exist in (2) or no global potential φ exists in (3), we just split K in finitely many pieces and take the infimum, according to Prop. (1.2)).

(4) *If D is an effective divisor with rational or real coefficients, we set*

$$c_K(D) = c_K([D]) = c_K(\mathcal{O}(-D)) = c_K(g) = c_K(\log|g|)$$

where D is the current of integration over D and g is a local generator of the principal ideal sheaf $\mathcal{O}(-D)$.

No confusion should arise from the above definitions, especially since $c_K(\mathcal{I})$ does not depend on the choice of generators of $\mathcal{I}$. We use similar conventions of notation for $\lambda_K(\varphi)$. The number

$$c_x(f) = \sup\big\{c\,;\ |f|^{-2c} \text{ is } L^1 \text{ on a neighborhood of } x\big\} = \lambda_x(f)^{-1}$$

is clearly a measure of the singularities of the hypersurface $\{f = 0\}$ at point x. This number came up in the literature many times under different names. By [Lin89], $c_x(f)$ is the largest root of the Bernstein-Sato polynomial associated to the germ of f around p. If x is an isolated singularity of $\{f = 0\}$, then $c_x(f) = \min\{1, \beta_{\mathbb{C}}(f_x)\}$ where $\beta_{\mathbb{C}}(f_x)$ is the complex singular index as defined in [ArGV84], vol.II, Sec. 13.1.5; the same thing is called "complex singularity exponent" in [Var92]. See [Kol97] for a discussion of these questions and for related results.

1.4. Elementary properties. *Let $\mathcal{I}$, $\mathcal{J}$ be coherent ideals on X and let φ, ψ be psh functions. Denote by x a point in X and let $K \subset X$ be a compact subset.*

(1) *The function $x \mapsto c_x(\varphi)$ is lower semi-continuous for the holomorphic Zariski topology;*

(2) *If $\varphi \leqslant \psi$, then $c_K(\varphi) \leqslant c_K(\psi)$;*
If $\mathcal{I} \subset \mathcal{J}$, then $c_K(\mathcal{I}) \leqslant c_K(\mathcal{J})$.

(3) $\lambda_K(\varphi + \psi) \leqslant \lambda_K(\varphi) + \lambda_K(\psi)$;
$\lambda_K(\mathcal{I}\mathcal{J}) \leqslant \lambda_K(\mathcal{I}) + \lambda_K(\mathcal{J})$.

(4) $\lambda_K(\alpha\varphi) = \alpha\,\lambda_K(\varphi)$ *for all* $\alpha \in \mathbb{R}_+$;
$\lambda_K(\mathcal{I}^m) = m\lambda_K(\mathcal{I})$ *for all integers* $m \in \mathbb{N}$.

(5) *Let $\mathcal{I} = (g_1, \ldots, g_N)$ and let*

$$\overline{\mathcal{I}} = \big\{f \in \mathcal{O}_{\Omega,x}\,,\ x \in \Omega\,;\ \exists C \geqslant 0,\ |f| \leqslant C \max|g_j| \ \text{near } x\big\}$$

be the integral closure of $\mathcal{I}$. Then $c_K(\overline{\mathcal{I}}) = c_K(\mathcal{I})$.

(6) *If the zero variety germ $V(\mathcal{I}_x)$ contains a p-codimensional irreducible component, then $c_x(\mathcal{I}) \leqslant p$, i.e. $\lambda_x(\mathcal{I}) \geqslant 1/p$.*

(7) *If $\mathcal{I}_Y$ is the ideal sheaf of a p-codimensional subvariety $Y \subset \Omega$, then $c_x(\mathcal{I}_Y) = p$ at every nonsingular point of Y.*

(8) *Define the vanishing order* $\mathrm{ord}_x(\mathcal{I})$ *of $\mathcal{I}$ at x to be the supremum of all integers k such that $\mathcal{I}_x \subset \mathfrak{m}_x^k$, where $\mathfrak{m}_x \subset \mathcal{O}_x$ is the maximal ideal. Then*

$$\frac{1}{n}\,\mathrm{ord}_x(\mathcal{I}) \leqslant \lambda_x(\mathcal{I}) \leqslant \mathrm{ord}_x(\mathcal{I}).$$

More generally, if $\nu_x(\varphi)$ is the Lelong number of φ at x, then

$$\frac{1}{n}\nu_x(\varphi) \leqslant \lambda_x(\varphi) \leqslant \nu_x(\varphi).$$

Proof. (1) Fix a point x_0 and a relatively compact coordinate ball $B := B(x_0, r) \Subset X$. For every $c \geqslant 0$, let $\mathcal{H}_{c\varphi}(B)$ be the Hilbert space of holomorphic functions on B with finite weighted L^2 norm

$$\|f\|_c^2 = \int_B |f|^2 e^{-2c\varphi} dV$$

where dV is the Lebesgue volume element in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} X$. A fundamental consequence of Hörmander's L^2 estimates (Hörmander-Bombieri-Skoda theorem [Hör66], [Bom70], [Sko75]) states that there is an element $f \in \mathcal{H}_{c\varphi}(B)$ with $f(x) = 1$ whenever $e^{-2c\varphi}$ is L^1 on a neighborhood of x. Hence

$$\big\{x \in B\,;\, c_x(\varphi) \leqslant c_0\big\} \cap B = \bigcap_{f \in \bigcup_{c > c_0} \mathcal{H}_{c\varphi}(B)} f^{-1}(0)$$

is an analytic set. This proves the holomorphic Zariski lower semi-continuity.

All other properties are direct consequences of the definitions and do not require "hard" analysis: (2), (4), (5) are immediate; (3) is a consequence of the Hölder inequality; (6,7) follow from the fact that the function $(\sum_{j \leqslant p} |z_j|^2)^{-c}$ is locally integrable along $z_1 = \cdots = z_p = 0$ if and only if $c < p$; Finally, (8) is a well-known result of Skoda [Sko72], depending on the basic properties of Lelong numbers and a use of standard kernel techniques. □

In the case of an ideal sheaf, the following lemma reduces the computation of $c_x(\mathcal{I})$ to the case of a principal ideal (possibly after raising $\mathcal{I}$ to some power $\mathcal{I}^m$).

1.5. Proposition. *Let $(g_1, \ldots, g_p)$ be holomorphic functions defined on an open set $\Omega \subset \mathbb{C}^n$ and let $x \in V(g_1, \ldots, g_p)$. Then*

$$c_x(\alpha_1 g_1 + \cdots + \alpha_p g_p) \leqslant \min\big\{c_x(g_1, \ldots, g_p)\,,\, 1\big\}$$

for all coefficients $(\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$. Moreover, the equality occurs for all $(\alpha_1, \ldots, \alpha_p)$ in the complement of a set of measure zero in $\mathbb{C}^p$. In particular, if $\mathcal{I}$ is an arbitrary ideal and $c_x(\mathcal{I}) \leqslant 1$, there is a principal ideal $(f) \subset \mathcal{I}$ such that $c_x(f) = c_x(\mathcal{I})$.

Proof. The inequality is obvious, since $c_x(\alpha_1 g_1 + \cdots + \alpha_p g_p) \leqslant 1$ by (1.4.6) on the one hand, and

$$\big|\alpha_1 g_1 + \cdots + \alpha_p g_p\big|^{-2c} \geqslant \Big(\sum |\alpha_j|^2\Big)^{-c} \Big(\sum |g_j|^2\Big)^{-c}$$

on the other hand. Now, fix $c < \min\{c_x(g_1, \ldots, g_p)\,,\, 1\}$. There is a neighborhood U_c of x on which

$$\int_{|\alpha|=1} d\sigma(\alpha) \int_{U_c} \big|\alpha_1 g_1(z) + \cdots + \alpha_p g_p(z)\big|^{-2c} dV(z)$$

$$= A_c \int_{U_c} \Big(\sum |g_j(z)|^2\Big)^{-c} dV(z) < +\infty, \tag{1.5.1}$$

where $d\sigma$ is the euclidean area measure on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ and $A_c > 0$ is a constant. The above identity follows from the formula

$$\int_{|\alpha|=1} |\alpha \cdot w|^{-2c} d\sigma(\alpha) = A_c |w|^{-2c},$$

which is obvious by homogeneity, and we have $A_c < +\infty$ for $c < 1$. The finiteness of the right hand side of (1.5.1) implies that the left hand side is finite for all values α in the complement $\mathbb{C}^p \smallsetminus N_c$ of a negligible set. Therefore $c_x(\alpha_1 g_1 + \cdots + \alpha_p g_p) \geqslant c$, and by taking the supremum over an increasing sequence of values c_ν converging to $\min\{c_x(g_1, \ldots, g_p)\,,\, 1\}$, we conclude that the equality holds in Proposition 1.5 for all $\alpha \in \mathbb{C}^p \smallsetminus \bigcup N_{c_\nu}$. $\square$

1.6. Remark. It follows from Theorem 3.1 below that the exceptional set of values $(\alpha_1, \ldots, \alpha_p)$ occurring in Prop. 1.5 is in fact a closed algebraic cone in $\mathbb{C}^p$. $\square$

The singularity exponent $c_K(\mathcal{I})$ of a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ can be computed by means of a "log resolution" of $\mathcal{I}$, that is, a composition $\mu : \widetilde{X} \to X$ of blow-ups with smooth centers such that $\mu^\star \mathcal{I} = \mathcal{O}_{\widetilde{X}}(-D)$ is an invertible sheaf associated with a normal crossing divisor D in $\widetilde{X}$ (such a log resolution always exists by Hironaka [Hir64]). The following proposition is essentially well known (see e.g. [Kol95a] 10.7).

1.7. Proposition. *Let X be a complex manifold, $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf, and let $\mu : \widetilde{X} \to X$ be a modification (= proper bimeromorphic morphism) such that $\mu^\star \mathcal{I} = \mathcal{O}_{\widetilde{X}}(-D)$ is an invertible sheaf. Assume that $\widetilde{X}$ is normal and let $E_i \subset \widetilde{X}$ denote either an exceptional divisor of μ or an irreducible component of D. Write*

$$K_{\widetilde{X}} = \mu^\star K_X + \sum a_i E_i \quad \text{and} \quad D = \sum b_i E_i,$$

where $a_i = 0$ if E_i is not a component of the exceptional divisor of μ (resp. $b_i = 0$ if E_i is not a component of D). Then:

(1) $c_K(\mathcal{I}) \leqslant \min\limits_{i\,:\,\mu(E_i)\cap K \neq \emptyset} \left\{ \dfrac{a_i + 1}{b_i} \right\}.$

(2) *Equality holds if $\widetilde{X}$ is smooth and $\sum E_i$ is a divisor with normal crossings.*

(3) *If $g = (g_1, \ldots, g_N)$ are generators of $\mathcal{I}$ in a neighborhood of K, then for any sufficiently small neighborhood U of K there is a volume estimate*

$$C_1 r^{2c} \leqslant \mu_U(\{|g| < r\}) \leqslant C_2 r^{2c} |\log r|^{n-1}, \qquad \forall r < r_0$$

with $n = \dim_{\mathbb{C}} X$, $c = c_K(\mathcal{I})$ and C_1, C_2, $r_0 > 0$.

Proof. Since the question is local, we may assume that $\mathcal{I}$ is generated by holomorphic functions $g_1, \ldots, g_N \in \mathcal{O}(X)$. Then (1) and (2) are straightforward consequences of the Jacobian formula for a change of variable: if U is an open set in X, the change $z = \mu(\zeta)$ yields

$$\int_{z\in U} |g(z)|^{-2c} dV(z) = \int_{\zeta\in\mu^{-1}(U)} |g\circ\mu(\zeta)|^{-2c} |J_\mu(\zeta)|^2 d\widetilde{V}(\zeta)$$

where J_μ is the Jacobian of μ, and dV, $d\widetilde{V}$ are volume elements of X, $\widetilde{X}$ respectively (embed $\widetilde{X}$ in some smooth ambient space if necessary). Now, if h_i is a generator of $\mathcal{O}(-E_i)$ at a smooth point $\widetilde{x} \in \widetilde{X}$, the divisor of J_μ is by definition $\sum a_i E_i$ and $\mu^\star\mathcal{I} = \mathcal{O}(-\sum b_i E_i)$. Hence, up to multiplicative bounded factors,

$$|J_\mu|^2 \sim \prod |h_i|^{2a_i}, \qquad |g\circ\mu|^2 \sim \prod |h_i|^{2b_i} \qquad \text{near } \widetilde{x},$$

and $|g\circ\mu|^{-2c}|J_\mu|^2$ is L^1 near $\widetilde{x}$ if and only if $\prod |h_i|^{-2(cb_i - a_i)}$ is L^1. A necessary condition is that $cb_i - a_i < 1$ whenever $E_i \ni \widetilde{x}$. We therefore get the necessary condition $c < \min_{i\,:\,\mu(E_i)\cap K\neq\emptyset}\{(a_i+1)/b_i\}$, and this condition is necessary and sufficient if $\sum E_i$ is a normal crossing divisor.

For (3), we choose $(\widetilde{X}, \mathcal{O}(-D))$ to be a (nonsingular) log resolution of $\mathcal{I}$. The volume $\mu_U(\{|g| < r\})$ is then given by integrals of the form

$$\int_{\mu^{-1}(U)\cap\{\zeta\in\widetilde{U}_\alpha,\, \prod|h_i|^{b_i}<r\}} \prod |h_i(\zeta)|^{2a_i} dV(\zeta) \tag{1.7.4}$$

over suitable coordinate charts $\widetilde{U}_\alpha \subset \widetilde{X}$. An appropriate change of variable $\zeta \mapsto w$, $w_i = h_i^{b_i}(\zeta)$, $w_j = \zeta_{k_j}$ (where i runs over the set of indices such that $b_i > 0$ and j over a disjoint set of indices) and a use of a partition of unity leads to estimate (1.7.4) by a linear combination of integrals of the form

$$\int_{P(r)} \prod |w_i|^{2(a_i+1)/b_i - 2} dV(w) \qquad \text{where} \quad P(r) = \{\max|w_i| < 1,\, \textstyle\prod |w_i| < r\}$$

(we assume here that a partial integration with respect to the w_j's has already been performed). The lower bound $C_1 r^{2c}$ is obtained by restricting the domain of integration to a neighborhood of a point in the unit polydisk such that only one coordinate w_i vanishes, precisely for i equal to the index achieving the minimum of $(a_i+1)/b_i$. The upper bound $C_2 r^{2c}|\log r|^{n-1}$, $c = \min(a_i+1)/b_i$, is obtained by using the inequalities

$$\prod |w_i|^{2(a_i+1)/b_i - 2} \leqslant \Big(\prod |w_i|\Big)^{2c-2} \leqslant r^{2c-2}, \qquad \forall w \in P(r),$$

$$\begin{aligned}
\mu(P(r)) &= \int_{\{\max(|w_1|,\ldots,|w_{n-1}|)<1\}} \pi \min\Big(\frac{r^2}{|w_1|^2\cdots|w_{n-1}|^2}, 1\Big) \prod_{i=1}^{n-1} dV(w_i) \\
&\leqslant \pi \int_{\{\exists i\,;\,|w_i|<r\}} \prod_{i=1}^{n-1} dV(w_i) + \pi r^2 \int_{\{\forall i\,;\, r\leqslant|w_i|<1\}} \prod_{i=1}^{n-1} \frac{dV(w_i)}{|w_i|^2} \\
&\leqslant C_2 r^2 |\log r|^{n-1}.
\end{aligned}$$

It should be observed that much finer estimates are known to exist; in fact, one can derive rather explicit asymptotic expansions of integrals obtained by integration along the fibers of a holomorphic function (see [Bar82]). □

§2. L^2 extension theorem and inversion of adjunction

Our starting point is the following special case of the fundamental L^2 extension theorem due to Ohsawa-Takegoshi ([OhT87], [Ohs88], see also [Man93]).

2.1. Theorem ([OhT87], [Ohs88], [Man93]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let L be an affine linear subspace of $\mathbb{C}^n$ of codimension $p \geqslant 1$ given by an orthonormal system s of affine linear equations $s_1 = \cdots = s_p = 0$. For every $\beta < p$, there exists a constant $C_{\beta,n,\Omega}$ depending only on β, n and the diameter of Ω, satisfying the following property. For every $\varphi \in \mathcal{P}(\Omega)$ and $f \in \mathcal{O}(\Omega \cap L)$ with $\int_{\Omega\cap L} |f|^2 e^{-\varphi} dV_L < +\infty$, there exists an extension $F \in \mathcal{O}(\Omega)$ of f such that*

$$\int_{\Omega} |F|^2 |s|^{-2\beta} e^{-\varphi} dV_{\mathbb{C}^n} \leqslant C_{\beta,n,\Omega} \int_{\Omega\cap L} |f|^2 e^{-\varphi} dV_L,$$

where $dV_{\mathbb{C}^n}$ and dV_L are the Lebesgue volume elements in $\mathbb{C}^n$ and L respectively.

In the sequel, we use in an essential way the fact that β can be taken arbitrarily close to p. It should be observed, however, that the case $\beta = 0$ is sufficient to imply the general case. In fact, supposing $L = \{z_1 = \cdots = z_p = 0\}$, a substitution $(\varphi, \Omega) \mapsto (\varphi_k, \Omega_k)$ with

$$\begin{aligned} &\varphi_k(z_1, \ldots, z_n) = \varphi(z_1^k, \ldots, z_p^k, z_{p+1}, \ldots, z_n), \\ &\Omega_k = \big\{z \in \mathbb{C}^n \,;\, (z_1^k, \ldots, z_p^k, z_{p+1}, \ldots, z_n) \in \Omega\big\} \end{aligned}$$

shows that the estimate with $\beta = 0$ implies the estimate with $\beta = p(1 - 1/k)$ (use the change of variable $\zeta_1 = z_1^k, \ldots, \zeta_p = z_p^k$, $\zeta_j = z_j$ for $j > p$, together with the Jacobian formula

$$dV(z) = \frac{\mathrm{Const}}{|\zeta_1|^{2(1-1/k)} \ldots |\zeta_p|^{2(1-1/k)}} dV(\zeta),$$

and take the "trace" of the solution F_k on Ω_k to get the solution F on Ω). The L^2 extension theorem readily implies the following important monotonicity result.

2.2. Proposition. *Let $\varphi \in \mathcal{P}(X)$ be a psh function on a complex manifold X, and let $Y \subset X$ be a complex submanifold such that $\varphi_{|Y} \not\equiv -\infty$ on every connected component of Y. Then, if K is a compact subset of Y, we have*

$$c_K(\varphi_{|Y}) \leqslant c_K(\varphi).$$

(Here, of course, $c_K(\varphi)$ is computed on X, i.e., by means of neighborhoods of K in X).

Proof. By Prop. 1.2, we may assume that $K = \{y\}$ is a single point in Y. Hence, after a change of coordinates, we can suppose that X is an open set in $\mathbb{C}^n$ and that Y is an affine linear subspace. Let $c < c_y(\varphi_{|Y})$ be given. There is a small ball $B = B(y,r)$ such that $\int_{B\cap Y} e^{-2c\varphi}dV_Y < +\infty$. By the L^2 extension theorem applied with $\beta = 0$, $\Omega = B$, $L = Y$ and $f(z) = 1$, we can find a holomorphic function F on B such that $F(z) = 1$ on $B \cap Y$ and $\int_B |F|^2 e^{-2c\varphi}dV_B < +\infty$. As $F(y) = 1$, we infer $c_y(\varphi) \geqslant c$ and the conclusion follows. It should be observed that an algebraic proof exists when φ is of the form $\log|g|$, $g \in \mathcal{O}(X)$; however that proof is rather involved. This is already a good indication of the considerable strength of the L^2 extension theorem (which will be crucial in several respects in the sequel). □

We now show that the inequality given by Proposition 2.2 can somehow be reversed (Theorem 2.5 below). For this, we need to restrict ourselves to a class of psh functions which admit a "sufficiently good local behavior" (such restrictions were already made in [Dem87], [Dem93a] to accommodate similar difficulties).

2.3. Definition. *Let X be a complex manifold. We denote by $\mathcal{P}_h(X)$ the class of all plurisubharmonic functions φ on X such that e^φ is locally Hölder continuous on X, namely such that for every compact set $K \subset X$ there are constants $C = C_K \geqslant 0$, $\alpha = \alpha_K > 0$ with*

$$|e^{\varphi(x)} - e^{\varphi(y)}| \leqslant C\,d(x,y)^\alpha, \qquad \forall x, y \in K,$$

where d is some Riemannian metric on X. We say for simplicity that such a function is a Hölder psh function.

2.4. Example. We are mostly interested in the case of functions of the form

$$\varphi = \max_j \log\Big(\sum_k \prod_l |f_{j,k,l}|^{\alpha_{j,k,l}}\Big)$$

with $f_{j,k,l} \in \mathcal{O}(X)$ and $\alpha_{j,k,l} > 0$. Such functions are easily seen to be Hölder psh. Especially, if $D = \sum \alpha_j D_j$ is an effective real divisor, the potential $\varphi = \sum \alpha_j \log|g_j|$ associated with $[D]$ is a Hölder psh function.

2.5. Theorem. *Let H be a smooth hypersurface of X and let T be a closed positive current of type $(1,1)$ on X such that its local potential functions φ are Hölder psh functions with $\varphi_{|H} \not\equiv -\infty$. We set in this case (somewhat abusively) $T_{|H} = dd^c\varphi_{|H}$. Then for any compact set $K \subset H$, we have*

$$c_K([H] + T) \geqslant 1 \quad \Leftrightarrow \quad c_K(T_{|H}) \geqslant 1.$$

In the algebraic setting (that is, when $T = [D]$ is defined by an effective divisor $D = \sum \alpha_j D_j$), the above result is known as "inversion of adjunction", see

Kollár et al. [K&al92], 17.7. One says that the pair (X,D) is lc (= log canonical) if $c_K(D)\geqslant 1$ for every compact set $K\subset X$, i.e., if the product $\prod|g_j|^{-2c\alpha_j}$ associated with the generators g_j of $\mathcal{O}(-D_j)$ is locally L^1 for every $c<1$. The result can then be rephrased as

$$(X,H+D)\text{ is lc}\quad\Leftrightarrow\quad(H,D_{|H})\text{ is lc.}\tag{2.5.1}$$

Proof of Theorem 2.5. Since the result is purely local, we may assume that $X=D(0,r)^n$ is a polydisk in $\mathbb{C}^n$, that H is the hyperplane $z_n=0$ and $K=\{0\}$. We must then prove the equivalence

$$\begin{aligned}&\forall c<1,\ \exists U\ni 0,\ \ \exp\big(-2c(\log|z_n|+\varphi(z))\big)\in L^1(U)\\&\qquad\qquad\Leftrightarrow\quad\forall c'<1,\ \exists U'\ni 0,\ \ \exp\big(-2c'\varphi(z',0)\big)\in L^1(U'),\end{aligned}$$

where $z=(z',z_n)\in\mathbb{C}^n$ and U, U' are neighborhoods of 0 in $\mathbb{C}^n$, $\mathbb{C}^{n-1}$ respectively.

First assume that $(|z_n|e^{\varphi(z)})^{-2c}\in L^1(U)$. As e^φ is Hölder continuous, we get

$$e^{2c\varphi(z)}\leqslant(e^{\varphi(z',0)}+C_1|z_n|^\alpha)^{2c}\leqslant C_2(e^{2c\varphi(z',0)}+|z_n|^{2c\alpha})$$

on a neighborhood of 0, for some constants C_1, C_2, $\alpha>0$. Therefore the function

$$\frac{1}{|z_n|^{2c}\big(|z_n|^{2c\alpha}+e^{2c\varphi(z',0)}\big)}\leqslant C_2^{-1}(|z_n|e^{\varphi(z)})^{-2c}$$

is in $L^1(U)$. Suppose that $U=U'\times D(0,r_n)$ is a small polydisk. A partial integration with respect to z_n on a family of disks $|z_n|<\rho(z')$ with $\rho(z')=\varepsilon\exp(\alpha^{-1}\varphi(z',0))$ (and $\varepsilon>0$ so small that $\rho(z')\leqslant r_n$ for all $z'\in U'$) shows that

$$\int_U\frac{dV(z)}{|z_n|^{2c}\big(|z_n|^{2c\alpha}+e^{2c\varphi(z',0)}\big)}\geqslant\mathrm{Const}\int_{U'}\frac{dV(z')}{e^{(2c-2(1-c)\alpha^{-1})\varphi(z',0)}}.$$

Hence $\exp(-2c'\varphi(z',0))\in L^1(U')$ with $c'=c-(1-c)\alpha^{-1}$ arbitrarily close to 1. Conversely, if the latter condition holds, we apply the Ohsawa-Takegoshi extension theorem to the function $f(z')=1$ on $L=H=\{z_n=0\}$, with the weight $\psi=2c'\varphi$ and $\beta=c'<1$. Since $F(z',0)=1$, the L^2 condition implies the desired conclusion. $\square$

2.6. Remark. As the final part of the proof shows, the implication

$$c_K([H]+T)\geqslant 1\quad\Leftarrow\quad c_K(T_{|H})\geqslant 1.$$

is still true for an arbitrary (not necessarily Hölder) psh function φ. The implication $\Rightarrow$, however, is no longer true. A simple counterexample is provided in dimension 2 by $H=\{z_2=0\}$ and $T=dd^c\varphi$ with

$$\varphi(z_1,z_2)=\max\big(\lambda\log|z_1|,\,-\sqrt{-\log|z_2|}\big),\qquad\lambda>1$$

on the unit bidisk $D(0,1)^2 \subset \mathbb{C}^2$. Then $c_0([H]+T) = c_0([H]) = 1$ but $c_0(T_{|H}) = c_0(\lambda\log|z_1|) = 1/\lambda$.

2.7. Proposition. *Let X, Y be complex manifolds of respective dimensions n, m, let $\mathcal{I} \subset \mathcal{O}_X$, $\mathcal{J} \subset \mathcal{O}_Y$ be coherent ideals, and let $K \subset X$, $L \subset Y$ be compact sets. Put $\mathcal{I} \oplus \mathcal{J} := \mathrm{pr}_1^\star \mathcal{I} + \mathrm{pr}_2^\star \mathcal{J} \subset \mathcal{O}_{X\times Y}$. Then*

$$c_{K\times L}(\mathcal{I} \oplus \mathcal{J}) = c_K(\mathcal{I}) + c_L(\mathcal{J}).$$

Proof. By Prop. 1.2, it is enough to show that $c_{(x,y)}(\mathcal{I}\oplus\mathcal{J}) = c_x(\mathcal{I}) + c_y(\mathcal{J})$ at every point $(x,y) \in X \times Y$. Without loss of generality, we may assume that $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ are open sets and $(x,y) = (0,0)$. Let $g = (g_1, \ldots, g_p)$, resp. $h = (h_1, \ldots, h_q)$, be systems of generators of $\mathcal{I}$ (resp. $\mathcal{J}$) on a neighborhood of 0. Set

$$\varphi = \log\sum|g_j|, \qquad \psi = \log\sum|h_k|.$$

Then $\mathcal{I}\oplus\mathcal{J}$ is generated by the $p+q$-tuple of functions

$$g \oplus h = (g_1(x), \ldots\, g_p(x), h_1(y), \ldots, h_q(y))$$

and the corresponding psh function $\Phi(x,y) = \log\big(\sum|g_j(x)| + \sum|h_k(y)|\big)$ has the same behavior along the poles as $\Phi'(x,y) = \max(\varphi(x),\psi(y))$ (up to a term $O(1) \leqslant \log 2$). Now, for sufficiently small neighborhoods U, V of 0, we have

$$\mu_{U\times V}\big(\{\max(\varphi(x),\psi(y)) < \log r\}\big) = \mu_U\big(\{\varphi < \log r\} \times \mu_V(\{\psi < \log r\}\big),$$

hence Prop. 1.7 (3) implies
(2.7.1)

$$C_1 r^{2(c+c')} \leqslant \mu_{U\times V}\big(\{\max(\varphi(x),\psi(y)) < \log r\}\big) \leqslant C_2 r^{2(c+c')}\,|\log r|^{n-1+m-1}$$

with $c = c_0(\varphi) = c_0(\mathcal{I})$ and $c' = c_0(\psi) = c_0(\mathcal{J})$. From this, we infer

$$c_{(0,0)}(\mathcal{I}\oplus\mathcal{J}) = c + c' = c_0(\mathcal{I}) + c_0(\mathcal{J}). \qquad \square$$

2.8. Example. As $c_0(z_1^m) = 1/m$, an application of Proposition 2.7 to a quasi-homogeneous ideal $\mathcal{I} = (z_1^{m_1}, \ldots, z_p^{m_p}) \subset \mathcal{O}_{\mathbb{C}^n,0}$ yields the value

$$c_0(\mathcal{I}) = \frac{1}{m_1} + \cdots + \frac{1}{m_p}. \qquad \square$$

Using Proposition 2.7 and the monotonicity property, we can now prove the fundamental subadditivity property of the singularity exponent.

2.9. Theorem. *Let f, g be holomorphic on a complex manifold X. Then, for every $x \in X$,*

$$c_x(f+g) \le c_x(f) + c_x(g).$$

More generally, if $\mathcal{I}$ and $\mathcal{J}$ are coherent ideals, then

$$c_x(\mathcal{I}+\mathcal{J}) \le c_x(\mathcal{I}) + c_x(\mathcal{J}).$$

Proof. Let Δ be the diagonal in $X \times X$. Then $\mathcal{I}+\mathcal{J}$ can be seen as the restriction of $\mathcal{I} \oplus \mathcal{J}$ to Δ. Hence Prop. 2.2 combined with 2.7 implies

$$c_x(\mathcal{I}+\mathcal{J}) = c_{(x,x)}((\mathcal{I} \oplus \mathcal{J})_{|\Delta}) \leqslant c_{(x,x)}(\mathcal{I} \oplus \mathcal{J}) = c_x(\mathcal{I}) + c_x(\mathcal{J}).$$

Since $(f+g) \subset (f)+(g)$, inequality 1.4 (2) also shows that

$$c_x(f+g) \leqslant c_x((f)+(g)) \leqslant c_x(f) + c_x(g). \qquad \square$$

2.10. Remark. If $f(x_1,\ldots,x_n)$, resp. $g(y_1,\ldots,y_n)$, are holomorphic near $0 \in \mathbb{C}^n$, resp. $0 \in \mathbb{C}^m$, and such that $f(0) = g(0) = 0$, we have the equality

$$c_0(f(x_1,\ldots,x_n) + g(y_1,\ldots,y_m)) = \min\{1, c_0(f) + c_0(g)\}.$$

This result is proved in [AGV84], vol. II, sec. 13.3.5 in the case of isolated singularities. Another proof, using the computation of c_0 via a resolution as in Prop. 1.7, is given in [Kol97]. It can also be reduced to Proposition 2.7 through a log resolution of either f or g. $\square$

§3. Semi-continuity of holomorphic singularity exponents

We first give a new proof (in the spirit of this work) of the semi-continuity theorem of Varchenko [Var82] concerning leading zeroes of Bernstein-Sato polynomials attached to singularities of holomorphic functions.

3.1. Theorem ([Var82])**.** *Let X be a complex manifold and S a reduced complex space. Let $f(x,s)$ be a holomorphic function on $X \times S$. Then for any $x_0 \in X$, the function $s \mapsto c_{x_0}(f_{|X\times\{s\}})$ is lower semi-continuous for the holomorphic Zariski topology on S. It even satisfies the following much stronger property: for any $s_0 \in S$, one has*

$$c_{x_0}(f_{|X\times\{s\}}) \geqslant c_{x_0}(f_{|X\times\{s_0\}}) \tag{3.1.1}$$

on a holomorphic Zariski neighborhood of s_0 (i.e. the complement in S of an analytic subset of S disjoint from s_0).

Proof. Observe that if $f_{|X\times\{s_0\}}$ is identically zero, then $c_{x_0}(f_{|X\times\{s_0\}}) = 0$ and there is nothing to prove; thus we only need to consider those s such that $f_{|X\times\{s\}} \not\equiv 0$.

We may of course assume that $X = B$ is a ball in $\mathbb{C}^n$ and $x_0 = 0$. Let $Y = B \times S$, $D = \operatorname{div} f$ and $\mu : \widetilde{Y} \to Y$ a log resolution of (Y, D). After possibly shrinking B a little bit, there is a Zariski dense open set $S_1 \subset S$ such that if $s \in S_1$, the corresponding fiber

$$\mu_s : \widetilde{Y}_s \to B \times \{s\}$$

is a log resolution of $(B, \operatorname{div} f_{|B\times\{s\}})$. Moreover, we may assume that the numerical invariants a_i, b_i attached to $\mu_s : \widetilde{Y}_s \to B$ as in Prop. 1.7 also do not depend on s. In particular, by (1.7.2), $c_0(f_{|B\times\{s\}})$ is independent of $s \in S_1$.

By induction on the dimension of S, we obtain a stratification $S = \bigcup S_i$ (where each S_i is a Zariski dense open subset of a closed complex subspace of S) such that $c_0(f_{|B\times\{s\}})$ only depends on the stratum containing s. Thus (3.1.1) reduces to semi-continuity with respect to the classical topology (considering a 1-dimensional base is enough, so we may assume the base to be nonsingular as well). If we put $\varphi = \log|f|$, this is a special case of the following Lemma, which is essentially equivalent to the Main Theorem of [PS99]. Here, we would like to point out that this result (which we knew as early as end of 1995) can be obtained as a direct consequence of the Ohsawa-Takegoshi theorem [OhT87].

3.2. Lemma. *Let $\Omega \subset \mathbb{C}^n$ and $S \subset \mathbb{C}^p$ be bounded pseudoconvex open sets. Let $\varphi(x, s)$ be a Hölder psh function on $\Omega \times S$ and let $K \subset \Omega$ be a compact set. Then*

(1) *$s \mapsto c_K(\varphi(\bullet, s))$ is lower semi-continuous for the classical topology on S.*

(2) *If $s_0 \in S$ and $c < c_K(\varphi(\bullet, s_0))$, there exists a neighborhood U of K and a uniform bound*

$$\int_U e^{-2c\varphi(x,s)}dV(x) \leqslant M(c)$$

for s in a neighborhood of s_0.

Proof. We use the L^2 extension theorem of [OhT87], following an idea of Angehrn-Siu [AnSi95]. However, the "effective" part (2) requires additional considerations. Notice that it is enough to prove (2), since (1) is a trivial consequence. By shrinking Ω and S, we may suppose that e^{φ} is Hölder continuous of exponent α on the whole of $\Omega \times S$ and that

$$\int_\Omega e^{-2c\varphi(x,s_0)}dV(x) < +\infty.$$

Let k be a positive integer. We set

$$\psi_{k,s}(x, t) = 2c\,\varphi(x, s + (kt)^k(s_0 - s)) \qquad \text{on } \Omega \times D,$$

where $D \subset \mathbb{C}$ is the unit disk. Then ψ is well defined on $\Omega \times D$ if s is close enough to s_0. Since $\psi(x, 1/k) = \varphi(x, s_0)$, we obtain by Theorem 2.1 the existence of a holomorphic function $F_{k,s}(x, t)$ on $\Omega \times D$ such that $F_{k,s}(x, 1/k) = 1$ and

$$\text{(3.2.3)} \qquad \int_{\Omega\times D} |F_{k,s}(x, t)|^2 e^{-\psi_{k,s}(x,t)}dV(x)dV(t) \leqslant C_1$$

with C_1 independent of k, s for $|s-s_0|<\delta k^{-k}$. As $\psi_{k,s}$ admits a global upper bound independent of k, s, the family $(F_{k,s})$ is a normal family. It follows from the equality $F_{k,s}(x,1/k)=1$ that there is a neighborhood U of K and a neighborhood $D(0,\varepsilon)$ of 0 in $\mathbb{C}$ such that $|F_{k,s}|\geqslant 1/2$ on $U\times D(0,\varepsilon)$ if k is large enough. A change of variable $t=k^{-1}\tau^{1/k}$ in (3.2.3) then yields

$$\int_{U\times D(0,(k\varepsilon)^k)}\frac{e^{-2c\varphi(x,s+\tau(s_0-s))}}{|\tau|^{2(1-1/k)}}\,dV(x)dV(\tau)\leqslant 4k^4C_1.$$

As in the proof of Theorem 2.5, we get by the Hölder continuity of e^{φ} an upper bound

$$e^{2c\varphi(x,s+\tau(s_0-s))}\leqslant C_2(e^{2c\varphi(x,s)}+|\tau|^{2c\alpha})$$

with a constant C_2 independent of s. Hence, for $k\geqslant 1/\varepsilon$, we find

$$\int_{U\times D}\frac{1}{\big(e^{2c\varphi(x,s)}+|\tau|^{2c\alpha}\big)|\tau|^{2(1-1/k)}}\,dV(x)dV(\tau)\leqslant C_3(k).$$

By restricting the integration to a family of disks $|\tau|<C_4e^{\alpha^{-1}\varphi(x,s)}$ (with C_4 so small that the radius is $\leqslant 1$), we infer

$$\int_U e^{-2(c-1/k\alpha)\varphi(x,s)}dV(x)\leqslant C_5(k).$$

Since $c-1/k\alpha$ can be taken arbitrarily close to $c_K(\varphi)$, this concludes the proof. □

We can now prove the qualitative part of the semi-continuity theorem, in the holomorphic case.

3.3. Theorem. *Let X be a complex manifold and $K\subset X$ a compact subset. Then $f\mapsto c_K(f)$ is lower semi-continuous on $\mathcal{O}(X)$ with respect to the topology of uniform convergence on compact subsets. More explicitly, for every nonzero holomorphic function f, for every compact set L containing K in its interior and every $\varepsilon>0$, there is a number $\delta=\delta(f,\epsilon,K,L)>0$ such that*

$$\sup_L|g-f|<\delta\quad\Rightarrow\quad c_K(g)\geqslant c_K(f)-\varepsilon. \tag{3.3.1}$$

Proof. As a first step we reduce (3.3.1) to the special case when K is a single point. Assume that (3.3.1) fails. Then there is a sequence of holomorphic functions $f_i\in\mathcal{O}(X)$ converging uniformly to f on L, such that

$$c_K(f_i)<c_K(f)-\varepsilon.$$

By Prop. 1.2 we can choose for each i a point $a_i\in K$ such that $c_{a_i}(f_i)<c_K(f)-\varepsilon$. By passing to a subsequence we may assume that the points a_i converge to a point

$a \in K$. Take a local coordinate system on X in a neighborhood of a. Consider the functions F_i defined by

$$F_i(x) = f_i(x + a_i - a)$$

on a small coordinate ball $\overline{B}(a, r) \subset L^\circ$. These functions are actually well defined for i large enough (choose ε so that $\overline{B}(a, r+\varepsilon) \subset L$ and i so large that $|a_i - a| < \varepsilon$). Then F_i converges to f on $\overline{B}(a, r)$, but

$$c_a(F_i) = c_{a_i}(f_i) < c_K(f) - \varepsilon \leqslant c_a(f) - \varepsilon.$$

Therefore, to get a contradiction, we only need proving Theorem 3.3 in case $K = \{a\}$ is a single point. Again we can change notation and assume that X is the unit ball and that our point is the origin 0.

In the second step we reduce the lower semi-continuity of $c_0(f)$ to polynomials of bounded degree. For a given holomorphic function f let P_k denote the degree $\leqslant k$ part of its Taylor series. The subbaditivity property of Theorem 2.9 implies $|c_0(f) - c_0(p_k)| \leqslant c_0(f - p_k)$. As $|f(z) - p_k(z)| = O(|z|^{k+1})$, the function $|f - p_k|^{-2c}$ is not integrable for $c \geqslant n/(k+1)$. From this, it follows that $c_0(f - p_k) \leqslant n/(k+1)$, hence

$$|c_0(f) - c_0(p_k)| \leqslant \frac{n}{k+1}. \tag{3.3.2}$$

Now, if (f_i) converges uniformly to f on a given neighborhood $U \subset \mathbb{C}^n$ of 0, the degree $\leqslant k$ part $p_{i,k}$ converges to p_k in the finite dimension space $\mathbb{C}[z_1, \ldots, z_n]_k$ of polynomials of total degree $\leqslant k$. Let us view polynomials

$$P(z, s) = \sum_{|\alpha| \leqslant k} s_\alpha z^\alpha \in \mathbb{C}[z_1, \ldots, z_n]_k$$

as functions of their coefficients $s = (s_\alpha)$. By Theorem 3.1, we know that the function $s \mapsto c_0(P(\bullet, s))$ is lower semi-continuous. Hence we get

$$c_0(p_{i,k}) > c_0(p_k) - \frac{\varepsilon}{2} \quad \text{for } i > i(k, \varepsilon) \text{ large enough},$$

and thanks to (3.3.2) this implies

$$c_0(f_i) > c_0(f) - \frac{\varepsilon}{2} - \frac{2n}{k+1} > c_0(f) - \varepsilon$$

by choosing $k \geqslant 4n/\varepsilon$. □

In fact, we would like to propose the following much stronger lower semi-continuity conjecture:

3.4. Conjecture. *Notation as in Theorem 3.3. For every nonzero holomorphic function f, there is a number $\delta = \delta(f, K, L) > 0$ such that*

$$\sup_L |g - f| < \delta \quad \Rightarrow \quad c_K(g) \geqslant c_K(f).$$

3.5. Remark. There is an even more striking conjecture about the numbers $c_K(f)$, namely, that the set

$$\mathcal{C} = \{c_0(f) | f \in \mathcal{O}_{\mathbb{C}^n,0}\} \subset \mathbb{R}$$

satisfies the ascending chain condition (cf. [Sho92]; [K&al92], 18.16): any convergent increasing sequence in $\mathcal{C}$ should be stationary. This conjecture and Theorem 3.3 together would imply the stronger form 3.4. Notice on the other hand that there do exist non stationary decreasing sequences in $\mathcal{C}$ by (1.4.8)*.

§4. Multiplier ideal sheaves and holomorphic approximations of psh singularities

The most important concept relating psh functions to holomorphic objects is the concept of *multiplier ideal sheaf*, which was already considered implicitly in the work of Bombieri [Bom70], Skoda [Sko72] and Siu [Siu74]. The precise final formalization has been fixed by Nadel [Nad89].

4.1. Theorem and definition ([Nad89, 90], see also [Dem89, 93a])**.** *If $\varphi \in \mathcal{P}(X)$ is a psh function on a complex manifold X, the multiplier ideal sheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_X$ is defined by*

$$\Gamma(U, \mathcal{I}(\varphi)) = \{f \in \mathcal{O}_X(U)\,;\, |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$. Then $\mathcal{I}(\varphi)$ is a coherent ideal sheaf in $\mathcal{O}_X$.

The proof that $\mathcal{I}(\varphi)$ is coherent is a rather simple consequence of Hörmander's L^2 estimates, together with the strong Noetherian property of coherent sheaves and the Krull lemma. When the psh function φ is defined from holomorphic functions as in 2.4, it is easy to see that $\mathcal{I}(\varphi)$ can be computed in a purely algebraic way by means of log resolutions. The concept of multiplier ideal sheaf plays a very important role in algebraic geometry, e.g. in Nadel's version of the Kawamata-Viehweg vanishing theorem or in Siu's proof [Siu93] of the big Matsusaka theorem.

We now recall the technique employed in [Dem92] and [Dem93b] to produce effective bounds for the approximation of psh functions by logarithms of holomorphic functions. The same technique produces useful comparison inequalities

* It has been recently observed by Phong and Sturm [PS00], in their study of integrals of the form $\int |f|^{-s}$, that the ascending chain condition holds in complex dimension 2. Algebraic geometers seem to have been aware for some time of the corresponding algebraic geometric statement.

for the singularity exponents of a psh function and its associated multiplier ideal sheaves.

4.2. Theorem. *Let φ be a plurisubharmonic function on a bounded open set $\Omega \subset \mathbb{C}^n$. For every real number $m > 0$, let $\mathcal{H}_{m\varphi}(\Omega)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty$ and let $\psi_m = \frac{1}{2m} \log \sum |g_{m,k}|^2$ where $(g_{m,k})$ is an orthonormal basis of $\mathcal{H}_{m\varphi}(\Omega)$. Then:*

(1) *There are constants $C_1, C_2 > 0$ independent of m and φ such that*

$$\varphi(z) - \frac{C_1}{m} \leqslant \psi_m(z) \leqslant \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, ψ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

(2) *The Lelong numbers of φ and ψ_m are related by*

$$\nu(\varphi, z) - \frac{n}{m} \leqslant \nu(\psi_m, z) \leqslant \nu(\varphi, z) \qquad \textit{for every } z \in \Omega.$$

(3) *For every compact set $K \subset \Omega$, the Arnold multiplicity of φ, ψ_m and of the multiplier ideal sheaves $\mathcal{I}(m\varphi)$ are related by*

$$\lambda_K(\varphi) - \frac{1}{m} \leqslant \lambda_K(\psi_m) = \frac{1}{m} \lambda_K(\mathcal{I}(m\varphi)) \leqslant \lambda_K(\varphi).$$

Proof. (1) Note that $\sum |g_{m,k}(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_{m\varphi}(\Omega)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |g_{m,k}|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\psi_m(z) = \sup_{f \in B(1)} \frac{1}{m} \log |f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_{m\varphi}(\Omega)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned} |f(z)|^2 &\leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{|\zeta - z| < r} |f(\zeta)|^2 d\lambda(\zeta) \\ &\leqslant \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2m \sup_{|\zeta - z| < r} \varphi(\zeta)\Big) \int_\Omega |f|^2 e^{-2m\varphi} d\lambda. \end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\psi_m(z) \leqslant \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the right hand inequality in (1) is proved. Conversely, the Ohsawa-Takegoshi extension theorem applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi} d\lambda \leqslant C_3 |a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. This gives the left hand inequality

$$\psi_m(z) \geqslant \frac{1}{m}\log|a| = \varphi(z) - \frac{\log C_3}{2m}. \tag{4.2.4}$$

(2) The above inequality (4.2.4) implies $\nu(\psi_m, z) \leqslant \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r} \psi_m(x) \leqslant \sup_{|\zeta - z|<2r} \varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\psi_m, x) \geqslant \nu(\varphi, x) - \frac{n}{m}.$$

(3) Inequality (4.2.4) already yields $\lambda_K(\psi_m) \leqslant \lambda_K(\varphi)$. Moreover, the multiplier ideal sheaf $\mathcal{I}(m\varphi)$ is generated by the sections in $\mathcal{H}_{m\varphi}(\Omega)$ (as follows from the proof that $\mathcal{I}(m\varphi)$ is coherent), and by the strong Noetherian property, it is generated by finitely many functions $(g_{m,k})_{0\leqslant k\leqslant k_0(m)}$ on every relatively compact open set $\Omega' \Subset \Omega$. It follows that we have a lower bound of the form

$$\psi_m(z) - C_4 \leqslant \frac{1}{2m}\log \sum_{0\leqslant k\leqslant k_0(m)} |g_{m,k}|^2 \leqslant \psi_m(z) \quad \text{on } \Omega'. \tag{4.2.5}$$

By choosing $\Omega' \supset K$, we infer $\lambda_K(\psi_m) = \frac{1}{m}\lambda_K(\mathcal{I}(m\varphi)$. If $\lambda > \lambda_K(\psi_m)$, i.e., $1/m\lambda < c_K(\mathcal{I}(m\varphi))$, and if $U \subset \Omega'$ is a sufficiently small open neighborhood of K, the Hölder inequality for the conjugate exponents $p = 1 + m\lambda$ and $q = 1 + (m\lambda)^{-1}$ yields

$$\int_U e^{-2mp^{-1}\varphi} dV = \int_U \Big(\sum_{0\leqslant k\leqslant k_0(m)} |g_{m,k}|^2 e^{-2m\varphi}\Big)^{1/p} \Big(\sum_{0\leqslant k\leqslant k_0(m)} |g_{m,k}|^2\Big)^{-1/qm\lambda} dV$$

$$\leqslant (k_0(m)+1)^{1/p} \left(\int_U \Big(\sum_{0\leqslant k\leqslant k_0(m)} |g_{m,k}|^2\Big)^{-1/m\lambda} dV\right)^{1/q} < +\infty. \tag{4.2.6}$$

(The estimate in the last line uses the fact that

$$\int_U |g_{m,k}|^2 e^{-2m\varphi} dV \leqslant \int_\Omega |g_{m,k}|^2 e^{-2m\varphi} dV = 1.\ )$$

This implies $c_K(\varphi) \geqslant mp^{-1}$, i.e., $\lambda_K(\varphi) \leqslant p/m = \lambda + 1/m$. As $\lambda > \lambda_K(\psi_m)$ was arbitrary, we get $\lambda_K(\varphi) \leqslant \lambda_K(\psi_m) + 1/m$ and (3) follows. □

The "approximation theorem" 4.2 allows to extend some results proved for holomorphic functions to the case of psh functions. For instance, we have:

4.3. Proposition. *Let $\varphi \in \mathcal{P}(X)$, $\psi \in \mathcal{P}(Y)$ be psh functions on complex manifolds X, Y, and let $K \subset X$, $L \subset Y$ be compact subsets. Then*:

(1) *For all positive real numbers c', c'' with $c' > c_K(\varphi) > c''$ (if any) and every sufficiently small neighborhood U of K, there is an estimate*

$$C_1 r^{2c'} \leqslant \mu_U(\{\varphi < \log r\}) \leqslant C_2 r^{2c''}, \qquad \forall r < r_0$$

for some $r_0 > 0$ and $C_1 = C_1(c')$, $C_2 = C_2(c'')$.

(2) $c_{K\times L}\big(\max(\varphi(x), \psi(y)\big) = c_K(\varphi) + c_L(\psi)$.

(3) *If $X = Y$, then $c_x(\max(\varphi, \psi)) \leqslant c_x(\varphi) + c_x(\psi)$ for all $x \in X$.*

Proof. (1) The upper estimate is clear, since

$$r^{-2c''}\mu_U(\{\varphi < \log r\}) \leqslant \int_U e^{-2c''\varphi} dV < +\infty$$

for $U \subset K$ sufficiently small. In the other direction, we have an estimate

$$\mu_U(\{\psi_m < \log r\}) \geqslant C_{1,m} r^{2c_K(\psi_m)}$$

by Proposition 1.7 (3) and (4.2.5). As $\varphi \leqslant \psi_m + C_{2,m}$ for some constant $C_{2,m} > 0$, we get

$$\{\varphi < \log r\} \supset \{\psi_m < \log r - C_{2,m}\},$$

and as $c_K(\psi_m)$ converges to $c_K(\varphi)$ by 4.2 (3), the lower estimate of $\mu_U(\{\varphi < \log r\})$ follows.

(2), (3) can be derived from (1) exactly as for the holomorphic case in Prop. 2.7 and Theorem 2.9. It should be observed that 4.3 (1) expresses a highly non trivial "regularity property" of the growth of volumes $\mu_U(\{\varphi < \log r\})$ when φ is a psh function (when φ is an arbitrary measurable function, $v(r) = \mu_U(\{\varphi < \log r\})$ is just an arbitrary increasing function with $\lim_{r\to 0} v(r) = 0$). □

4.4. Remark. In contrast with the holomorphic case 1.7 (3), the upper estimate $\mu_U(\{\varphi < \log r\}) \leqslant C_2 r^{2c''}$ does not hold with $c'' = c_K(\varphi)$, when φ is an arbitrary psh function. A simple example is given by $\varphi(z) = \chi \circ \log|z|$ where $\chi : \mathbb{R} \to \mathbb{R}$ is a convex increasing function such that $\chi(t) \sim t$ as $t \to -\infty$, but $e^{\chi(r)} \not\sim r$ as $r \to 0$, e.g. such that $\chi(t) = t - \log|t|)$ when $t < 0$. On the other hand, the lower estimate $\mu_U(\{\varphi < \log r\}) \geqslant C_1 r^{2c'}$ seems to be still true with $c' = c_K(\varphi)$, although we cannot prove it.

§5. Semi-continuity of psh singularity exponents

We are now in a position to prove our main semi-continuity theorem.

5.1. *Proof of Theorem 0.2.* Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex open set and let $\varphi_j \in \mathcal{P}(\Omega)$ be a sequence of psh functions converging to a limit $\varphi \in \mathcal{P}(\Omega)$ in the weak topology of distributions. In fact, this already implies that $\varphi_j \to \varphi$ almost everywhere and in L^1_{loc} topology; to see this, we observe that the coefficients of $T_j := dd^c\varphi_j$ are measures converging to those of $T = dd^c\varphi$ in the weak topology of measures; moreover φ_j and φ can be recovered from T_j and T by an integral formula involving the Green kernel; we then use the well known fact that integral operators involving a L^1 kernel define continuous (and even compact) operators from the space of positive measures equipped with the weak topology, towards the space of L^1 functions with the strong L^1 topology.

Fix a compact set $K \subset \Omega$. By the process described in Theorem 4.2, we get for each $m \in \mathbb{N}^\star$ an orthonormal basis $(g_{j,m,k})_{k\in\mathbb{N}}$ of $\mathcal{H}_{m\varphi_j}(\Omega)$, such that

$$\varphi_j(z) - \frac{C_1}{m} \leqslant \frac{1}{2m}\log\sum_{k\in\mathbb{N}}|g_{j,m,k}|^2 \leqslant \sup_{|\zeta-z|<r}\varphi_j(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n} \tag{5.1.1}$$

for every $z \in \Omega$ and $r < d(z,\partial\Omega)$. In particular, all sequences $(g_{j,m,k})_{j\in\mathbb{N}}$ are uniformly bounded from above on every compact subset of Ω. After possibly extracting a subsequence, we may assume that all $g_{j,m,k}$ converge to a limit $g_{m,k} \in \mathcal{O}(\Omega)$ when $j \to +\infty$. Thanks to (5.1.1) we find in the limit

$$\varphi(z) - \frac{C_1}{m} \leqslant \frac{1}{2m}\log\sum_{k\in\mathbb{N}}|g_{m,k}|^2 \leqslant \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Fix a relatively compact open subset $\Omega' \Subset \Omega$ containing K. By the strong Noetherian property already used for (4.2.5), there exist an integer $k_0(m)$ and a constant $C_4(m) > 0$ such that

$$\varphi(z) - C_4(m) \leqslant \frac{1}{2m}\log\sum_{0\leqslant k\leqslant k_0(m)}|g_{m,k}|^2 \quad \text{on } \Omega'.$$

Now, for $c < c_K(\varphi)$, there is a neighborhood U of K on which

$$\int_U\Big(\sum_{0\leqslant k\leqslant k_0(m)}|g_{m,k}|^2\Big)^{-c/m}dV \leqslant e^{2cC_4(m)}\int_U e^{-2c\varphi}dV < +\infty.$$

Take (without loss of generality) $m \geqslant 2c_K(\varphi)$. Then $c/m < 1/2$ and Formula 1.5.1 shows that there is a linear combination $\sum_{0\leqslant k\leqslant k_0(m)}\alpha_{m,k}g_{m,k}$ with $\alpha = (\alpha_{m,k})$ in the unit sphere of $\mathbb{C}^{k_0(m)+1}$, such that

$$\int_U\Big|\sum_{0\leqslant k\leqslant k_0(m)}\alpha_{m,k}g_{m,k}\Big|^{-2c/m}dV \leqslant C_5(m)\int_U e^{-2c\varphi}dV < +\infty,$$

where $C_5(m)$ is a constant depending possibly on m. By construction,

$$f_{j,m} = \sum_{0\leqslant k\leqslant k_0(m)} \alpha_{k,m} g_{j,m,k}$$

is an element of the unit sphere in $\mathcal{H}_{m\varphi_j}(\Omega)$, and $f_{j,m}$ converges uniformly on Ω to $f_m = \sum \alpha_{m,k} g_{m,k}$ such that $\int_U |f_m|^{-2c/m} dV < +\infty$. By Lemma 5.2 below, for any $c' < c$ and $K \subset U' \Subset U$, we have a uniform bound $\int_{U'} |f_{j,m}|^{-2c'/m} dV \leqslant C_6(m)$ for $j \geqslant j_0$ large enough. Since $\int_\Omega |f_{j,m}|^2 e^{-2m\varphi_j} dV = 1$, the Hölder inequality for conjugate exponents $p = 1 + m/c'$, $q = 1 + c'/m$ yields

$$\begin{aligned}\int_{U'} e^{-2mc'/(m+c')\varphi_j} dV &= \int_{U'} \big(|f_{j,m}|^2 e^{-2m\varphi_j}\big)^{c'/(m+c')} |f_{j,m}|^{-2c'/(m+c')} dV \\ &\leqslant \Big(\int_{U'} |f_{j,m}|^{-2c'/m} dV\Big)^{m/(m+c')} \leqslant C_7(m)\end{aligned}$$

for $j \geqslant j_0$. Since c, c' are arbitrary with $c' < c < c_K(\varphi)$, the exponent $mc'/(m+c')$ can be taken to approach c as closely as we want as m gets large. Hence $c_K(\varphi_j) > c_K(\varphi) - \varepsilon$ for $j \geqslant j_0(\varepsilon)$ large enough. Moreover, by what we have seen above, if $c < c_K(\varphi)$ is fixed and $0 < \delta < c_K(\varphi)/c - 1$, there exists $j_1(\delta)$ such that the sequence $(e^{-2c\varphi_j})_{j\geqslant j_1(\delta)}$ is contained in a bounded set of $L^{1+\delta}(U)$, where U is a small neighborhood of K. Therefore

$$\int_{U\cap\{e^{-2c\varphi_j} > M\}} e^{-2c\varphi_j} dV \leqslant C_8 M^{-\delta}$$

for $j \geqslant j_1(\delta)$, with a constant C_8 independent of j. Since $e^{-2c\varphi_j}$ converges pointwise to $e^{-2c\varphi}$ on Ω, an elementary argument based on Lebesgue's bounded convergence theorem shows that $e^{-2c\varphi_j}$ converges to $e^{-2c\varphi}$ in $L^1(U)$. □

To complete the proof, we need only proving the following effective estimate for holomorphic functions, which is a special case of part (3) in the Main Theorem.

5.2. Lemma. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex open set, and let $f_i \in \mathcal{O}(\Omega)$ be a sequence of holomorphic functions converging uniformly to $f \in \mathcal{O}(\Omega)$ on every compact subset. Fix a compact set $K \subset \Omega$ and $c < c_K(f)$. Then there is a neighborhood U of K and a uniform bound $C > 0$ such that*

$$\int_U |f_i|^{-2c} dV \leqslant C$$

for $i \geqslant i_0$ sufficiently large.

Proof. We already know by Theorem 3.3 that $\int_U |f_i|^{-2c} dV < +\infty$ for U small enough and i large. Unfortunately, the proof given in Theorem 3.3 is not effective because it depends (through the use of Hironaka's theorem in the proof of estimates

1.7 (3) and (2.7.1)) on the use of a sequence of log resolutions on which we have absolutely no control. We must in fact produce an effective version of inequality (3.3.2).

The result of Lemma 5.2 is clearly local. Fix a point $x_0 \in K$ (which we assume to be 0 for simplicity), real numbers c', c'' with $c < c'' < c' < c_K(f) \leqslant c_0(f)$ and an integer k so large that

$$c < c'' - \frac{n}{k+1} < c'' < c' < c_0(f) - \frac{n}{k+1}.$$

Let p_k be the truncation at order k of the Taylor series of f at the origin. As $c_0(p_k) \geqslant c_0(f) - \frac{n}{k+1} > c'$ by (3.3.2), there is a small ball $B' = B(0, r'_0)$ such that

$$\int_{B'} |p_k|^{-2c'} dV < +\infty.$$

Since the truncations $p_{i,k}$ of $f_{i,k}$ converge uniformly to p_k on $\mathbb{C}^n$ as $i \to +\infty$, Lemma 3.2 applied to the universal family of polynomials $P(z,s) = \sum_{|\alpha| \leqslant k} s_\alpha z^\alpha$ shows that for any ball $B'' \Subset B'$, there is a constant $M \geqslant 0$ and an integer i_0 such that

$$\int_{B''} |p_{i,k}|^{-2c''} dV \leqslant M \qquad \text{for } i \geqslant i_0.$$

Let us write $p_{i,k} = f_i - g_{i,k}$ where $g_{i,k}$ consists of the sum of terms of degree $> k$ in the Taylor expansion of f_i at the origin. By the Ohsawa-Takegoshi theorem applied with the weight function $\psi(x,y) = 2c \log |f_i(x) - g_{i,k}(y)|$ on $B'' \times B''$ and $L =$ diagonal of $\mathbb{C}^n \times \mathbb{C}^n$, there is a holomorphic function F_i on $B'' \times B''$ such that $F_i(x,x) = 1$ and

$$\int_{B'' \times B''} |F_i(x,y)|^2 |f_i(x) - g_{i,k}(y)|^{-2c''} dV(x)\, dV(y) \leqslant C_1$$

with a constant C_1 independent of i. The above L^2 estimate shows that (F_i) is bounded in L^2 norm on $B'' \times B''$. Hence, there is a small ball $B = B(0, r_0) \Subset B''$ such that $|F_i(x,y)| \geqslant 1/2$ on $B \times B$ for all $i \geqslant i_0$, and

$$\text{(5.2.1)} \qquad \int_{B \times B} |f_i(x) - g_{i,k}(y)|^{-2c''} dV(x)\, dV(y) \leqslant 4C_1.$$

Moreover, we have a uniform estimate $|g_{i,k}(y)| \leqslant C_2 |y|^{k+1}$ on B with a constant C_2 independent of i. By integrating (5.2.1) with respect to y on the family of balls $|y| < (|f_i(x)|/2C_2)^{1/(k+1)}$, we find an estimate

$$\text{(5.2.2)} \qquad \int_B |f_i(x)|^{2n/(k+1) - 2c''} dV(x) \leqslant C_3.$$

As $c'' - n/(k+1) > c$, this is the desired estimate. It is interesting to observe that the proof of the Main Theorem can now be made entirely independent of

Hironaka's desingularization theorem. In fact, the only point where we used it is in the inequality $c_0(p_k) \geqslant c_0(f) - \frac{n}{k+1}$, which we derived from Proposition 2.7. The latter inequality can however be derived directly from the Ohsawa-Takegoshi theorem through estimates for $\int_{B\times B} |p_k(x) + g_k(y)|^{-2c} dV(x)\, dV(y)$. □

5.3. Remark. It follows from the proof of Prop. 1.7 that the set of positive exponents c such that $|f|^{-2c}$ is summable on a neighborhood of a compact set K is always an *open interval*, namely $]0, c_K(f)[$. We conjecture that the same property holds true more generally for an arbitrary psh function φ ("openness conjecture"); the openness conjecture is indeed true in dimension 1, since we have the well known necessary and sufficient criterion

$$e^{-2\varphi} \in L^1_{\mathrm{loc}}(V(x_0)) \quad \Leftrightarrow \quad \nu(\varphi, x_0) < 1$$

(as follows e.g. from [Sko72]). By using the Main Theorem, the openness conjecture would imply the following stronger statement:

5.4. Strong openness conjecture. *Let $U' \Subset U \Subset X$ be relatively compact open sets in a complex manifold X. Let φ be a psh function on X such that $\int_U e^{-\varphi} dV < +\infty$. Then there exists $\varepsilon = \varepsilon(\varphi, U, U')$ such that for every ψ psh on X*

$$\|\psi - \varphi\|_{L^1(U)} < \varepsilon \quad \Rightarrow \quad \int_{U'} e^{-\psi} dV < +\infty.$$

In other words, the integrability of $e^{-\varphi}$ near a given compact set K should be an open property for the L^1_{loc} (= weak) topology on $\mathcal{P}(X)$.

The main theorem only yields the weaker conclusion $\int_{U'} e^{-(1-\delta)\psi} dV < +\infty$ for $\|\psi - \varphi\|_{L^1(U)} < \varepsilon = \varepsilon(\varphi, U, U', \delta)$.

§6. Existence of Kähler-Einstein metrics on Fano orbifolds

An *orbifold* is a complex variety X possessing only quotient singularities, namely, every point $x_0 \in X$ has a neighborhood U isomorphic to a quotient Ω/Φ where $\Phi = \Phi_{x_0}$ is a finite group acting holomorphically on a smooth open set $\Omega \subset \mathbb{C}^n$. Such an action can always be linearized, so we may assume that Φ is a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ and Ω a Φ-invariant neighborhood of 0 (with x_0 being the image of 0). We may also assume that the elements of G distinct from identity have a set of fixed points of codimension $\geqslant 2$ (otherwise, the subgroup generated by these is a normal subgroup N of Φ, Ω/N is again smooth, and $\Omega/\Phi = (\Omega/N)/(\Phi/N)$). The structure sheaf $\mathcal{O}_X$ (resp. the m-fold canonical sheaf $K_X^{\otimes m}$) is then defined locally as the direct image by $\pi : \Omega \to U \simeq \Omega/\Phi$ of the subsheaf of Φ-invariant sections of the corresponding sheaf on Ω :

$$\Gamma(V, \mathcal{O}_X) = \Gamma(\pi^{-1}(V), \mathcal{O}_\Omega)^\Phi, \qquad \Gamma(V, K_X^{\otimes m}) = \Gamma(\pi^{-1}(V), K_\Omega^{\otimes m})^\Phi,$$

for all open subsets $V \subset U$. There is always an integer m_0 (e.g. $m_0 = \#\,\Phi$) such that $K_\Omega^{\otimes m_0}$ has Φ-invariant local generating sections, and then clearly $K_X^{\otimes m}$ is an invertible $\mathcal{O}_X$-module whenever m is divisible by the lowest common multiple μ of the integers m_0 occurring in the various quotients. Similarly, one can define on U (and thus on X) the concepts of Kähler metrics, Ricci curvature form, etc, by looking at corresponding Φ-invariant objects on Ω. We say that a compact orbifold X is a *Fano orbifold* if $K_X^{-\mu}$ is ample, which is the same as requiring that $K_X^{-\mu}$ admits a smooth hermitian metric with positive definite curvature. In that case, we define the curvature of K_X^{-1} to be $1/\mu$ times the curvature of $K_X^{-\mu}$. The integral of a differential form on X (say defined at least on X_{reg}) is always computed upstairs, i.e. $\int_{\Omega/\Phi}\alpha = \frac{1}{\#\Phi}\int_\Omega \pi^\star\alpha$.

6.1. Definition. *A compact orbifold X is said to be Kähler-Einstein if it possesses a Kähler form $\omega = \frac{i}{2\pi}\sum \omega_{jk}dz_j\wedge d\overline{z}_k$ satisfying the Einstein condition*

$$\mathrm{Ricci}(\omega) = \lambda\omega$$

for some real constant λ, where where $\mathrm{Ricci}(\omega)$ *is the closed* $(1,1)$*-form defined in every coordinate patch by* $\mathrm{Ricci}(\omega) = -\frac{i}{2\pi}\partial\overline{\partial}\log\det(\omega_{jk})$.

Since $\mathrm{Ricci}(\omega)$ is the curvature form of $K_X^{-1} = \det T_X$ equipped with the metric $\det\omega$, a necessary condition for the existence of a Kähler-Einstein metric with constant $\lambda > 0$ is that K_X^{-1} is ample, i.e., that X is Fano. On the other hand, it is well known that not all Fano orbifolds are Kähler-Einstein, even when they are smooth; further necessary conditions are required, e.g. that the group of automorphisms $\mathrm{Aut}(X)^\circ$ is reductive ([Mat57], [Lic57]), and that the *Futaki invariants* vanish [Fut83]; for instance $\mathbb{P}^2$ blown up in 2 points has a non reductive group of automorphisms and therefore is not Kähler-Einstein.

It is usually much harder to prove that a concretely given Fano orbifold is Kähler-Einstein. Siu [Siu87, 88], and slightly later Tian [Tia87] and Nadel [Nad89, 90], gave nice sufficient conditions ensuring the existence of a Kähler-Einstein metric; these conditions always involve the existence of a sufficiently big group of automorphisms. Our goal here is to reprove Nadel's main result in a more direct and conceptual way.

6.2. Technical setting. We first briefly recall the main technical tools and notation involved (see e.g. [Siu87] for more details). The anticanonical line bundle K_X^{-1} is assumed to be ample. Therefore it admits a smooth hermitian metric h_0 whose $(1,1)$-curvature form $\theta_0 = \frac{i}{2\pi}D^2_{h_0}$ is positive definite. Since $\theta_0\in c_1(X)$, the Aubin-Calabi-Yau theorem shows that there exists a Kähler metric $\omega_0\in c_1(X)$ such that $\mathrm{Ricci}(\omega_0) = \theta_0$. [The Aubin-Calabi-Yau is still valid in the orbifold case, because the proof depends only on local regularity arguments which can be recovered by passing to a finite cover, and global integral estimates which still make sense by the remark preceding Def. 6.1]. Since both θ_0 and ω_0 are in $c_1(X)$, we have

$$\omega_0 = \theta_0 + \frac{i}{2\pi}\partial\overline{\partial}f \qquad \text{for some } f\in C^\infty(X). \tag{6.2.1}$$

We look for a new Kähler form $\omega = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ in the same Kähler class as ω_0, such that $\mathrm{Ricci}\,\omega = \omega$. Since $\mathrm{Ricci}(\omega_0) = \theta_0$, this is equivalent to

$$-\frac{i}{2\pi}\partial\overline{\partial}\log(\det\omega) = \omega = \theta_0 + \frac{i}{2\pi}\partial\overline{\partial}(\varphi + f) = -\frac{i}{2\pi}\partial\overline{\partial}\log(\det\omega_0) + \frac{i}{2\pi}\partial\overline{\partial}(\varphi + f),$$

that is,

$$\partial\overline{\partial}\Big(\log\frac{\det\omega}{\det\omega_0} + \varphi + f\Big) = 0,$$

which in its turn is equivalent to the Monge-Ampère equation

$$\log\frac{(\omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi)^n}{\omega_0^n} + \varphi + f + C = 0 \tag{6.2.2}$$

where C is a constant. Here, one can normalize φ so that φ is orthogonal to the 1-dimensional space of constant functions in $L^2(X, \omega_0)$, i.e., $\int_X \varphi\,\omega_0^n = 0$. The usual technique employed to solve (6.2.2) is the so-called "continuity method". The continuity method amounts to introducing an extra parameter $t \in [0, 1]$ and looking for a solution (φ_t, C_t) of the equation

$$\log\frac{(\omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_t)^n}{\omega_0^n} + t(\varphi_t + f) + C_t = 0, \qquad \int_X \varphi_t\omega_0^n = 0 \tag{6.2.3}$$

as t varies from 0 to 1. Clearly $\varphi_0 = 0$, $C_0 = 0$ is a solution for $t = 0$ and $(\varphi, C) = (\varphi_1, C_1)$ provides a solution of our initial equation (6.2.2). Moreover, the linearization of the (nonlinear) elliptic differential operator occuring in (6.2.3) is the operator

$$(\psi, C) \longmapsto \frac{1}{2\pi}\Delta_{\omega_t}\psi + t\psi + C \tag{6.2.4}$$

where ω_t is the Kähler metric $\omega_t = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_t$ and Δ_{ω_t} is the associated Laplace operator (with negative eigenvalues). The equation (6.2.3) is easily seen to be equivalent to $\mathrm{Ricci}(\omega_t) = t\omega_t + (1-t)\theta_0$. From this we infer $\mathrm{Ricci}(\omega_t) > t\omega_t$ for all $t < 1$, and it then follows from the Bochner-Kodaira-Nakano identity that all nonzero eigenvalues of $-\frac{1}{2\pi}\Delta_{\omega_t}$ are $> t$ (this is clear directly for $-\frac{1}{2\pi}\Delta_{\omega_t}$ acting on $(0,1)$-forms, and one uses the fact that $\overline{\partial}$ maps the λ-eigenspace $E^{p,q}(\lambda)$ of $-\frac{1}{2\pi}\Delta_{\omega_t}$ in bidegree (p, q) into the corresponding eigenspace $E^{p,q+1}(\lambda)$). Then, thanks to Schauder's estimates, (6.2.4) induces an isomorphism $\mathcal{C}^{s+2}_\perp(X)\oplus\mathbb{R} \to \mathcal{C}^s(X)$ where $s \in \mathbb{R}_+ \smallsetminus \mathbb{N}$ and $\mathcal{C}^s(X)$ (resp. $\mathcal{C}^s_\perp(X)$) is the space of real functions (resp. real functions orthogonal to constants) of class $\mathcal{C}^s$ on X. Let $\mathcal{T} \subset [0, 1]$ be the set of parameters t for which (6.2.3) has a smooth solution. By elliptic regularity for (nonlinear) PDE equations, the existence of a smooth solution is equivalent to the existence of a solution in $\mathcal{C}^s(X)$ for some $s > 2$. It then follows by a standard application of the implicit function theorem that $\mathcal{T}\cap[0, 1[$ is an open subset of the interval $[0, 1[$.

6.3. Sufficient condition for closedness. In order to obtain a solution for all times $t \in [0,1]$, one still has to prove that $\mathcal{T}$ is *closed.* By the well-known theory of complex Monge-Ampère equations ([Aub78], [Yau78]), a sufficient condition for closedness is the existence of a uniform a priori $\mathcal{C}^0$-estimate $\|\widetilde{\varphi}_t\|_{\mathcal{C}^0} \leqslant \mathrm{Const}$ for the family of functions $\widetilde{\varphi}_t = t\varphi_t + C_t$, $t \in \mathcal{T}$, occuring in the right hand side of (6.2.3). A first observation is that

$$\sup_X \varphi_t \leqslant \mathrm{Const}, \qquad \text{hence} \quad \sup_X \widetilde{\varphi}_t \leqslant C_t + \mathrm{Const}, \tag{6.3.1}$$

as follows from the conditions $\int_X \varphi_t \omega_0^n = 0$ and $\frac{i}{2\pi}\partial\overline{\partial}\varphi_t \geqslant -\omega_0$, by simple considerations of potential theory. On the other hand, by [Siu88, Prop. 2.1] or [Tia87, Prop. 2.3], we have the Harnack-type inequality

$$\sup_X(-\widetilde{\varphi}_t) \leqslant (n+\varepsilon)\sup_X \widetilde{\varphi}_t + A_\varepsilon, \tag{6.3.2}$$

where $\varepsilon > 0$ and A_ε is a constant depending only on ε. Hence $\sup_X(-\widetilde{\varphi}_t) \leqslant (n+\varepsilon)C_t + A'_\varepsilon$ and we thus only need controlling the constants C_t from above. Now, equation (6.2.3) implies

$$\int_X \omega_0^n = \int_X \Big(\omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_t\Big)^n = \int_X e^{-\widetilde{\varphi}_t - tf}\omega_0^n.$$

For $\gamma \in \,]0,1[$, we easily infer from this and (6.3.2) that

$$\begin{aligned}
\int_X \omega_0^n &\leqslant \mathrm{Const}\ \exp\big((1-\gamma)\sup_X(-\widetilde{\varphi}_t)\big)\int_X e^{-\gamma\widetilde{\varphi}_t}\omega_0^n \\
&\leqslant \mathrm{Const}_\varepsilon\, e^{(1-\gamma)(n+\varepsilon)C_t}\int_X e^{-\gamma\widetilde{\varphi}_t}\omega_0^n \\
&\leqslant \mathrm{Const}_\varepsilon\, e^{-(\gamma-(1-\gamma)(n+\varepsilon))C_t}\int_X e^{-\gamma t\varphi_t}\omega_0^n.
\end{aligned}$$

If $\gamma \in \,]\frac{n}{n+1}, 1[$ and ε is small enough, we conclude that C_t admits an upper bound of the form

$$C_t \leqslant B'_\gamma \log\int_X e^{-\gamma t\varphi_t}\omega_0^n + B''_\gamma$$

where B'_γ and B''_γ depend only on γ. Hence closedness of $\mathcal{T}$ is equivalent to the uniform boundedness of the integrals

$$\int_X e^{-\gamma t\varphi_t}\omega_0^n, \qquad t \in \mathcal{T}, \tag{6.3.3}$$

for any choice of $\gamma \in \,]\frac{n}{n+1}, 1[$.

This yields the following basic existence criterion due to Nadel [Nad89, 90].

6.4. Existence criterion for Kähler-Einstein metrics. *Let X be a Fano orbifold of dimension n. Let G be a compact subgroup of the group of complex automorphisms of X. Then X admits a G-invariant Kähler-Einstein metric, unless K_X^{-1} possesses a G-invariant singular hermitian metric $h = h_0 e^{-\varphi}$ (h_0 being a smooth G-invariant metric and φ a G-invariant function in $L^1_{\rm loc}(X)$), such that the following properties occur.*

(1) *h has a semipositive curvature current*

$$\Theta_h = -\frac{i}{2\pi}\partial\overline{\partial}\log h = \Theta_{h_0} + \frac{i}{2\pi}\partial\overline{\partial}\varphi \geqslant 0.$$

(2) *For every $\gamma \in \,]\frac{n}{n+1}, 1[$, the multiplier ideal sheaf $\mathcal{I}(\gamma\varphi)$ is nontrivial, (i.e. $0 \neq \mathcal{I}(\gamma\varphi) \neq \mathcal{O}_X$).*

According to the general philosophy of orbifolds, the orbifold concept of a multiplier ideal sheaf $\mathcal{I}(\gamma\varphi)$ is that the ideal sheaf is to be computed upstairs on a smooth local cover and take the direct image of the subsheaf of invariant functions by the local isotropy subgroup; this ideal coincides with the multiplier ideal sheaf computed downstairs only if we take downstairs the volume form which is the push forward of an invariant volume form upstairs (which is in general definitely larger than the volume form induced by a local smooth embedding of the orbifold).

Proof. Let us start with a G-invariant Kähler metric $\omega_0 = \frac{i}{2\pi}\partial\overline{\partial}\log h_0^{-1}$, where h_0 and ω_0 have the same meaning as in 6.2; indeed, if h_0 is not G-invariant, we can average it by using the G-action, that is, we define a new metric $(h_0^G)^{-1}$ on K_X by putting

$$(h_0^G)^{-1} = \int_{g\in G} g^\star h_0^{-1} d\mu(g),$$

and we again have $\omega_0^G := \frac{i}{2\pi}\partial\overline{\partial}\log(h_0^G)^{-1} > 0$. Now, all φ_t can be taken to be G-invariant. If the continuity process ceases to produce a solution φ_t at $t = t_0 \in [0,1]$ (thus, if $t_0 \in \overline{\mathcal{T}} \smallsetminus \mathcal{T}$), there exists a sequence $t_\nu \in \mathcal{T}$ converging to t_0 and (6.3.3) implies $\lim_{\nu\to+\infty}\int_X e^{-\gamma t_\nu \varphi_{t_\nu}}\omega_0^n = +\infty$ for every $\gamma \in \,]\frac{n}{n+1}, 1[$. As the space of closed positive currents contained in a given cohomology class is compact for the weak topology, one can extract a subsequence $\Theta_{(p)} = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_{t_{\nu(p)}}$ converging weakly to a limit $\Theta = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi \geqslant 0$. The potential φ can be recovered from $\mathrm{Tr}\,\Theta$ by means of the Green kernel, and therefore, by the well-known properties of the Green kernel, we infer that $\varphi_{t_{\nu(p)}}$ converges to φ in $L^1(X)$. The semicontinuity theorem in its effective version 0.2.2 shows that

$$\int_X e^{-\gamma t_0\varphi}\omega_0^n = +\infty \qquad \text{for all } \gamma \in \,]\tfrac{n}{n+1}, 1[,$$

and therefore $\int_X e^{-\gamma\varphi}\omega_0^n = +\infty$ for all $\gamma \in \,]\frac{n}{n+1}, 1[$. From this we conclude that $\mathcal{I}(\gamma\varphi) \neq \mathcal{O}_X$. The fact that $\mathcal{I}(\gamma\varphi) \neq 0$ is clear since $\varphi \not\equiv -\infty$. □

Before going further, we need Nadel's vanishing theorem (a generalized version of the well-known Kawamata-Viehweg vanishing theorem. It is known to be a rather simple consequence of Hörmander's L^2 estimates, see e.g. [Dem89], [Nad89], [Dem93b] or [Dem94]).

6.5. Nadel vanishing theorem. *Let (X,ω) be a Kähler orbifold and let L be a holomorphic orbifold line bundle over X equipped with a singular hermitian metric h of weight φ with respect to a smooth metric h_0 (i.e. $h = h_0 e^{-\varphi}$). Assume that the curvature form $\Theta_h(L) = \frac{i}{2\pi} D_h^2$ is positive definite in the sense of currents, i.e. $\Theta_h(L) \geqslant \varepsilon\omega$ for some $\varepsilon > 0$. If $K_X \otimes L$ is an ivertible sheaf on X, we have*

$$H^q\big(X, K_X \otimes L \otimes \mathcal{I}(\varphi)\big) = 0 \qquad \textit{for all } q \geqslant 1.$$

Recall that an "orbifold line bundle" L, is a rank 1 sheaf which is locally an invariant direct image of an invertible sheaf on Ω by the local quotient maps $\Omega \to \Omega/\Phi$; L itself need not be invertible; similarly, $\otimes$ is meant to be the orbifold tensor product, i.e., we take the tensor product upstairs on Ω and take the direct image of the subsheaf of invariants. The proof is obtained by the standard L^2 estimates applied on X_{reg} with respect to an orbifold Kähler metric on X. It is crucial that $K_X \otimes L$ be invertible on X, otherwise the set of holomorphic sections of $K_X \otimes L$ satisfying the L^2 estimate with respect to the weight $e^{-\varphi}$ might differ from the orbifold tensor product $K_X \otimes L \otimes \mathcal{I}(\varphi)$ [and also, that tensor product might be equal to $K_X \otimes L$ even though $\mathcal{I}(\varphi)$ is non trivial].

6.6. Corollary. *Let X, G, h and φ be as in Criterion 6.4. Then, for all $\gamma \in \,]\frac{n}{n+1}, 1[$,*

(1) *the multiplier ideal sheaf $\mathcal{I}(\gamma\varphi)$ satisfies*

$$H^q(X, \mathcal{I}(\gamma\varphi)) = 0 \qquad \textit{for all } q \geqslant 1.$$

(2) *the associated subscheme V_γ of structure sheaf $\mathcal{O}_{V_\gamma} = \mathcal{O}_X/\mathcal{I}(\gamma\varphi)$ is nonempty, distinct from X, G-invariant and satisfies*

$$H^q(V_\gamma, \mathcal{O}_{V_\gamma}) = \begin{cases} \mathbb{C} & \textit{for } q = 0, \\ 0 & \textit{for } q \geqslant 1. \end{cases}$$

Proof. Apply Nadel's vanishing theorem to $L = K_X^{-1}$ equipped with the singular hermitian metric $h_\gamma = h_0 e^{-\gamma\varphi}$. Then $\Theta_{h_\gamma} = \gamma\Theta_h + (1-\gamma)\Theta_{h_0} \geqslant (1-\gamma)\omega_0 > 0$, and (1) follows. Finally, since X is Fano, we get

$$H^q(X, \mathcal{O}_X) = 0 \qquad \text{for all } q \geqslant 1,$$

by Kodaira vanishing for $L = K_X^{-1}$. The exact sequence

$$0 \to \mathcal{I}(\gamma\varphi) \to \mathcal{O}_X \to \mathcal{O}_{V_\gamma} \to 0$$

immediately implies (2). $\square$

The strategy employed by Nadel [Nad90] to construct Kähler-Einstein metrics is to rule out the existence of any G-invariant subscheme with the properties described in 6.6 (2). Of course, this is easier to achieve if G is large. One uses the following observations (assuming that the closedness property fails, so that all subschemes V_γ are nontrivial).

6.7. Proposition. *All subschemes V_γ are connected. Therefore, if G has no fixed points, V_γ cannot be 0-dimensional.*

Proof. The connectedness of V_γ is a straightforward consequence of the equality $H^0(V_\gamma, \mathcal{O}_{V_\gamma}) = \mathbb{C}$. □

6.8. Proposition. *If V_γ contains irreducible components Z_j of codimension 1, then the corresponding divisor $Z = \sum m_j Z_j$ satisfies the numerical inequality $[Z] \leqslant \gamma[K_X^{-1}]$ in the sense that $\gamma[K_X^{-1}] - [Z]$ can be represented by a closed positive current. In particular, one always has the inequality*

$$(-K_X)^{n-1} \cdot Z \leqslant \gamma(-K_X)^n.$$

If K_X^{-1} generates the group $W(X)$ of Weil divisors of X modulo numerical equivalence, then V_γ must have codimension $\geqslant 2$.

In the smooth case we have of course $W(X) = \mathrm{Pic}(X)$, but in general $\mathrm{Pic}(X)$ is a subgroup of finite index in $W(X)$.

Proof. Consider the closed positive $(1,1)$ current $\Theta_h = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ (which belongs to the first Chern class $c_1(K_X^{-1})$), and let

$$\Theta_h = \sum \lambda_j [Z_j] + R, \qquad \lambda_j > 0, \quad R \geqslant 0$$

be the Siu decomposition of Θ_h (namely, the $[Z_j]$'s are currents of integration over irreducible divisors and R is a closed $(1,1)$-current which has nonzero Lelong numbers only in codimension 2). It is then easy to see that the subscheme V_γ defined by $\mathcal{I}(\gamma\varphi)$ precisely has $[Z] = \sum \lfloor \gamma\lambda_j \rfloor [Z_j]$ as its 1-codimensional part (here, $\lfloor \ \rfloor$ denotes the integral part). Hence $\gamma\Theta_h - [Z] \geqslant 0$ as asserted. If K_X^{-1} generates $\mathrm{Pic}(X)$, this implies $Z = 0$, since there cannot exist any nonzero effective integral divisor numerically smaller than $[K_X^{-1}]$. □

When $\dim X = 3$, G has no fixed points and K_X^{-1} generates $W(X)$, we are only left with the case V_γ is of pure dimension 1. This case can sometimes be ruled out by observing that certain groups cannot act effectively on the curve V_γ (As $H^1(V_\gamma, \mathcal{O}_{V_\gamma}) = 0$, V_γ is a tree of rational curves; see Nadel [Nad90, Th. 4.1, 4.2 and Cor. 4.1]).

Further a priori inequalities can be derived for certain components of the multiplier ideal subschemes V_γ. Especially, for components of codimension 2, we have the following simple bound, based on a use of a self-intersection inequality for the current $\Theta = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi$.

6.9. Proposition. *Assume that $W(X)$ is generated by K_X^{-1} and that a is a nonnegative number such that the orbifold vector bundle $T_X \otimes \mathcal{O}(-aK_X)$ is numerically effective. Then the codimension 2 components Z_j of V_γ satisfy the inequality*

$$\sum \frac{1}{\delta_j}\nu_j(\nu_j-1)(-K_X)^{n-2}\cdot Z_j \leqslant (1+a)(-K_X)^n$$

where $\nu_j \geqslant 1/\gamma$ is the generic Lelong number of $\Theta = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ along Z_j, and δ_j is the order of the local isotropy group of the orbifold at a generic point in Z_j. Especially, if γ is taken to be sufficiently close to $\frac{n}{n+1}$, we have

$$\sum (-K_X)^{n-2}\cdot Z_j \leqslant \frac{n^2}{n+1}\delta(1+a)(-K_X)^n$$

where δ is the maximum of the the orders of the isotropy groups.

Proof. Since V_γ is of codimension 2 for γ arbitrarily close to 1, the generic Lelong number of φ must be $\leqslant 1$ along all components of codimension 1 in the Lelong sublevel sets $E_c(\varphi) = E_c(\Theta)$ [again, Lelong numbers and Lelong sublevel sets are to be interpreted upstairs, on a smooth finite cover]. If a codimension 2 component Z_j occurs in $\mathcal{I}(\gamma\varphi)$, the generic Lelong number $\gamma\nu_j$ of $\gamma\varphi$ along that component must be $\geqslant 1$, hence $\nu_j \geqslant 1/\gamma$. We now apply the regularization theorem for closed $(1,1)$-currents ([Dem92], Main Theorem). For every $c > 1$ we obtain a current $\Theta_{h,c}$ cohomologous to Θ (hence in the class $c_1(K_X^{-1})$), which is smooth on $X \smallsetminus E_c(\Theta)$, thus smooth except on an analytic set of codimension $\geqslant 2$, such that $\Theta_{h,c} \geqslant -(ca+\varepsilon)\omega_0$ and such that the Lelong numbers of $\Theta_{h,c}$ are shifted by c, i.e. $\nu_x(\Theta_{h,c}) = (\nu_x(\Theta)-c)_+$. The intersection product $\Theta \wedge (\Theta_c + (ca+\varepsilon)\omega_0)$ is well defined, belongs to the cohomology class $(1+ca+\varepsilon)(-K_X)^2$ and is larger than $\sum \frac{1}{\delta_j}\nu_j(\nu_j-c)[Z_j]$ as a current. Hence, by taking the intersection with the class $(-K_X)^{n-2}$ we get

$$\sum \frac{1}{\delta_j}\nu_j(\nu_j-c)(-K_X)^{n-2}\cdot Z_j \leqslant (1+ca+\varepsilon)(-K_X)^n.$$

[The extra factor $1/\delta_j$ occurs because we have to divide by δ_j to convert an integral on a finite cover Ω to an integral on the quotient Ω/Φ]. As c tends to $1+0$ and ε tends to $0+$, we get the desired inequality. The last observation comes from the fact that $\mathcal{I}(V_\gamma)$ must be constant on some interval $]\frac{n}{n+1}, \frac{n}{n+1}+\delta[$, by the Noetherian property of coherent sheaves. □

6.10. Example. Let $\mathbb{P}_a = \mathbb{P}^3(a_0,a_1,a_2,a_3)$ be the weighted projective 3-space with weights $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$ such that the components a_i are relatively prime 3 by 3. It is equipped with an orbifold line bundle $\mathcal{O}_X(1)$ which, in general, is not locally free. Let $t = a_0+a_1+a_2+a_3$ and

$$X = \big\{P(x_0,x_1,x_2,x_3) = 0\big\}$$

be a generic surface of weighted degree d in $\mathbb{P}_a$. It is known (see Fletcher [Fle89]) that X has an orbifold structure (i.e., is quasi-smooth in the terminology of Dolgachev [Dol82]), if and only if the following conditions are satisfied:

(i) For all j, there exists a monomial $x_j^m x_{k(j)}$ of degree d;

(ii) For all distinct j, k, either there exists a monomial $x_j^m x_k^p$ of degree d, or there exist monomials $x_j^{m_1} x_k^{p_1} x_{\ell_1}$, $x_j^{m_2} x_k^{p_2} x_{\ell_2}$ of degree d with $\ell_1 \neq \ell_2$;

(iii) For all j, there exists a monomial of degree d which does not involve x_j.

Moreover, $-K_X = \mathcal{O}_X(t-d)$ (and hence $(-K_X)^2 = d(t-d)^2/(a_0 a_1 a_2 a_3)$) if and only if the following condition also holds:

(iv) For every j,k such that a_j and a_k are not relatively prime, there exists a monomial $x_j^m x_k^p$ of degree d.

We would like to use the conditions of Propositions 6.8 and 6.9 to show that X carries a Kähler-Einstein metric.

Proposition 6.8 clearly applies if we can prove that $(-K_X)\cdot Z > \frac{2}{3}(-K_X)^2$ for every effective curve on X. This is not a priori trivial in the examples below since the Picard numbers will always be bigger than 1. Using the torus action, every curve on a weighted projective space can be degenerated to a sum of lines of the form $(x_i = x_j = 0)$. Thus $(-K_X)\cdot Z$ is bounded from below by $(t-d)/(a_2 a_3)$. Thus $(-K_X)\cdot Z > \frac{2}{3}(-K_X)^2$ holds if

$$\frac{t-d}{a_2 a_3} > \frac{2}{3}\frac{d(t-d)^2}{a_0 a_1 a_2 a_3}, \qquad \text{i.e.} \quad a_0 a_1 > \frac{2}{3} d(t-d).$$

In the examples we give at the end, which all concern the case $d = t-1$, this is always satisfied.

In order to apply Proposition 6.9, we need to determine T_X. We have exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}_a} \to \bigoplus \mathcal{O}_{\mathbb{P}_a}(a_i) \to T_{\mathbb{P}_a} \to 0,$$
$$0 \to T_X \to T_{\mathbb{P}_a|X} \to \mathcal{O}_X(d) \to 0,$$

and we get from there a surjective arrow

$$\bigoplus \mathcal{O}_X(a_i) \to \mathcal{O}_X(d)$$

given explicitly by the matrix $(\partial P/\partial x_i)$. From the above exact sequences, we find a sequence of surjective arrows

$$\bigoplus_{i<j} \mathcal{O}_X(a_i + a_j) \to \mathcal{O}_X(\Lambda^2 T_{\mathbb{P}_a|X}) \to T_X \otimes \mathcal{O}_X(d).$$

(Of course, formally speaking, we are dealing with orbifold vector bundles, which can be considered as locally free sheaves only when we pass to a finite Galois cover). Moreover,

$$\bigoplus_{i\neq k\neq j} \mathcal{O}_X(a_i + a_j) \to T_X \otimes \mathcal{O}_X(d)$$

is surjective over the open set where $x_k \neq 0$. This proves that, as an orbifold vector bundle, $T_X \otimes \mathcal{O}_X(d-a_0-a_2)$ is nef if the line $(x_0 = x_1 = 0)$ is not contained in X.

The maximal order δ of the isotropy groups is less than a_3 – which is indeed the maximum for $\mathbb{P}_a$ itself – resp. a_2 if a_3 divides d, since in that case a generic surface of degree d does not pass through the point $[0:0:0:1]$. This shows that we can take $a = (d-a_0-a_2)/(t-d)$ in Proposition 6.9, and as the Z_j are points and $n = 2$, we find the condition

$$1 \leqslant \frac{4}{3} a_3 \Big(1 + \frac{d-a_0-a_2}{t-d}\Big) \frac{d(t-d)^2}{a_0 a_1 a_2 a_3},$$

with the initial a_3 being replaced by a_2 if a_3 divides d. We thus compute the ratio

$$\rho_a = \frac{4}{3} \frac{d(t-d)(t-a_0-a_2)}{a_0 a_1 a_2} \quad \text{if } a_3 \not| d,$$

$$\rho_a = \frac{4}{3} \frac{d(t-d)(t-a_0-a_2)}{a_0 a_1 a_3} \quad \text{if } a_3 | d,$$

and when $\rho_a < 1$ we can conclude that the Del Pezzo surface is Kähler-Einstein. Clearly, this is easier to reach when $t - d$ is small, and we concentrated ourselves on the case $d = t - 1$. It is then easy to check that ρ_a is never less than 1 when $a_0 = a_1 = 1$. On the other hand, a computer check seems to indicate that there is only a finite list of weights with $a_0 > 2$ satisfying the Fletcher conditions, which all satisfy $a_0 \leqslant 14$. Among these, 2 cases lead to $\rho_a < 1$, namely

$$a = (11, 49, 69, 128), \quad d = 256, \ \rho_a \simeq 0.875696, \quad x_0^{17} x_2 + x_0 x_1^5 + x_1 x_2^3 + x_3^2 = 0,$$
$$a = (13, 35, 81, 128), \quad d = 256, \ \rho_a \simeq 0.955311, \quad x_0^{17} x_1 + x_0 x_2^3 + x_1^5 x_2 + x_3^2 = 0.$$

It turns out that there are no other monomials of degree d than those occurring in the above equations. As a result, the above Kähler-Einstein Del Pezzo surfaces are *rigid* as weighted hypersurfaces.

There are several ways to improve the estimates. For instance,

$$T_X \otimes \mathcal{O}_X(d - a_1 - a_2)$$

is nef except possibly along the irreducible components of the curve $(x_0 = 0) \subset X$. The restriction of the tangent bundle to these curves can be computed by hand. This improvement is sufficient to conclude that Propositions 6.8 and 6.9 also apply in one further case:

$$a = (9, 15, 17, 20), \quad d = 60, \quad x_0^5 x_1 + x_0 x_2^3 + x_1^4 + x_3^3 = 0.$$

This is again a rigid weighted hypersurface. We would like to thank P. Boyer and K. Galicki for pointing out a numerical error which had been committed in an earlier version of this work, where a further (incorrect) example $a = (11, 29, 39, 49)$, $d = 127$ was claimed. In [BG00], it is shown that the three above examples lead to the construction of non regular Sasakian-Einstein 5-manifolds.

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