Hölder continuity of solutions of supercritical dissipative hydrodynamic transport equations

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Abstract. We examine the regularity of weak solutions of quasi-geostrophic (QG) type equations with supercritical ($\alpha < 1/2$) dissipation $(-\Delta)^{\alpha}$. This study is motivated by a recent work of Caffarelli and Vasseur, in which they study the global regularity issue for the critical ($\alpha = 1/2$) QG equation [2]. Their approach successively increases the regularity levels of Leray-Hopf weak solutions: from L^2 to L^∞, from L^∞ to Hölder (C^δ, $\delta > 0$), and from Hölder to classical solutions. In the supercritical case, Leray-Hopf weak solutions can still be shown to be L^∞, but it does not appear that their approach can be easily extended to establish the Hölder continuity of L^∞ solutions. In order for their approach to work, we require the velocity to be in the Hölder space $C^{1-2\alpha}$. Higher regularity starting from C^δ with $\delta > 1 - 2\alpha$ can be established through Besov space techniques and will be presented elsewhere [10].

AMS (MOS) Numbers: 76D03, 35Q35

Keywords: the dissipative quasi-geostrophic equation, regularity, supercritical dissipation, weak solutions.

1 Introduction

This paper studies the regularity of Leray-Hopf weak solutions of the dissipative QG equation of the form

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, & x \in \mathbb{R}^n, \ t > 0, \\ u = \mathcal{R}(\theta), \quad \nabla \cdot u = 0, & x \in \mathbb{R}^n, \ t > 0, \end{cases} \tag{1.1}$$

where $\theta = \theta(x,t)$ is a scalar function, $\kappa > 0$ and $\alpha > 0$ are parameters, and $\mathcal{R}$ is a standard singular integral operator. The fractional Laplace operator $(-\Delta)^\alpha$ is defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n.$$

(1.1) generalizes the 2D dissipative QG equation (see [6],[8],[12],[16] and the references therein). The main mathematical question concerning the 2-D dissipative QG equation is whether or not it has a global in time smooth solution for any prescribed smooth initial data. In the subcritical case $\alpha > \frac{1}{2}$, the dissipative QG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [9],[17]). In contrast, when $\alpha < \frac{1}{2}$, the question of global existence is still open. Recently this problem has attracted a significant amount of research ([2],[3],[4],[5],[6],[7],[11], [13],[14],[15],[18],[19],[20],[21], [22],[23]). In Constantin, Córdoba and Wu [7], we proved in the critical case ($\alpha = \frac{1}{2}$) the global existence and uniqueness of classical solutions corresponding to any initial data with L^∞-norm comparable to or less than the diffusion coefficient κ. In a recent work [14], Kiselev, Nazarov and Volberg proved that smooth global solutions persist for any C^∞ periodic initial data [7], for the critical QG equation. Also recently, Caffarelli and Vasseur [2] proved the global regularity of the Leray-Hopf weak solutions to the critical QG equation in the whole space.

We focus our attention on the supercritical case $\alpha < \frac{1}{2}$. Our study is motivated by the work of Caffarelli and Vasseur in the crtical case. Roughly speaking, the Caffarelli-Vasseur approach consists of three main steps. The first step shows that a Leray-Hopf weak solution emanating from an initial data $\theta_0 \in L^2$ is actually in $L^\infty(\mathbb{R}^n \times (0,\infty))$. The second step proves that the L^∞-solution is C^γ-regular, for some $\gamma > 0$. For this purpose, they represent the diffusion operator $\Lambda \equiv (-\Delta)^{1/2}$ as the normal derivative of the harmonic extension L from $C_0^\infty(\mathbb{R}^n)$ to $C_0^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ and then exploit a version of De Giorgi's isoperimetric inequality to prove the Hölder continuity. The third step improves the Hölder continuity to $C^{1,\beta}$, the regularity level of classical solutions.

We examine the approach of Caffarelli and Vasseur to see if it can be extended to the super-critical case. The first step of their approach can be modified to suit the supercritical case: any Leray-Hopf weak solution can still be shown to be L^∞ for any $x \in \mathbb{R}^n$ and $t > 0$ (see Theorem 2.1). Corresponding to their third step, we can show that any weak solution already in the Hölder class C^δ with $\delta > 1-2\alpha$, is actually a global classical solution. This result is established by representing the Hölder space functions in terms of the Littlewood-Paley decomposition and using Besov space techniques. We will present this result in a separate paper [10]. We do not know if any solution in Hölder space C^γ with arbitrary $\gamma > 0$ is smooth, and therefore there exists a significant potential obstacle to the program: even if all Leray-Hopf solutions are C^γ, $\gamma > 0$, it may still be the case that only those solutions for which $\gamma > 1 - 2\alpha$ are actually smooth. If this would be true, then the critical case would be a fortuitous one, $(1 - 2\alpha = 0)$. If, however, all Leray-Hopf solutions are smooth, then providing a proof of this fact would require a new idea.

The most challenging part is how to establish the Hölder continuity of the L^∞-solutions. It does not appear that the approach of Caffarelli and Vasseur can be easily extended to the supercritical case. In the critical case, Caffarelli and Vasseur lifted θ from $\mathbb{R}^n$ to a harmonic function θ^* in the upper-half space $\mathbb{R}^n \times \mathbb{R}^+$ with boundary data on $\mathbb{R}^n$ being θ. The fractional derivative $(-\Delta)^{\frac{1}{2}}\theta$ is then expressed as the normal derivative of θ^* on the boundary $\mathbb{R}^n$ and the $\mathring{H}^1$-norm of θ^* is then bounded by the natural energy of θ. Taking the advantage of the nice properties of harmonic functions, they were able to obtain a diminishing oscillation result for θ^* in a box near the origin. More precisely, if θ^* satisfying $|\theta^*| \leq 2$ in the box, then θ^* satisfies in a smaller box centered at the origin

$$\sup \theta^* - \inf \theta^* < 4 - \lambda^*$$

for some $\lambda^* > 0$. The proof of this result relies on a local energy inequality, an isoperimetric inequality of De Giorgi and two lengthy technical lemmas. Examining the proof reveals that λ^* depends on the BMO-norm of the velocity u. To show the Hölder continuity at a point, they zoom in at this point by considering a sequence of functions θ_k^* and u_k with (θ_k, u_k) satisfying the critical QG equation. This process is carried out through the natural scaling invariance that $(\theta(\mu x, \mu t), u(\mu x, \mu t))$ solves the critical QG equation if (θ, u) does so. Applying the diminishing oscillation result to this sequence leads to the Hölder continuity of θ^*. An important point is that the BMO-norm of u_k is preserved in this scaling process.

In the supercritical case, the diminishing oscillation result can still be established by following the idea of Caffarelli and Vasseur (see Theorem 3.1). However, the scaling invariance is now represented by $\mu^{2\alpha-1}\theta(\mu x, \mu^{2\alpha}t)$ and $\mu^{2\alpha-1}u(\mu x, \mu^{2\alpha}t)$ and the BMO-norm deteriorates every time the solution is rescaled. This is where the approach of Caffarelli and Vasseur stops working for the supercritical case. If we make the assumption that $u \in C^{1-2\alpha}$, then the scaling process preserves this norm and we can still establish the Hölder continuity of θ. This observation is presented in Theorem 4.1.

2 From L^2 to L^∞

In this section, we show that any Leray-Hopf weak solution of (1.1) is actually in L^∞ for $t > 0$. More precisely, we have the following theorem.

Theorem 2.1 *Let $\theta_0 \in L^2(\mathbb{R}^n)$ and let θ be a corresponding Leray-Hopf weak solution of (1.1). That is, θ satisfies*

$$\theta \in L^\infty([0,\infty), L^2(\mathbb{R}^n)) \cap L^2([0,\infty); \mathring{H}^\alpha(\mathbb{R}^n)). \tag{2.1}$$

Then, for any $t > 0$,

$$\sup_{\mathbb{R}^n} |\theta(x,t)| \le C \, \frac{\|\theta_0\|_{L^2}}{t^{\frac{n}{4\alpha}}}.$$

As a special consequence,

$$\|u(\cdot,t)\|_{BMO(\mathbb{R}^n)} \le C \, \frac{\|\theta_0\|_{L^2}}{t^{\frac{n}{4\alpha}}}$$

for any $t > 0$.

This theorem can be proved by following the approach of Caffarelli and Vasseur [2]. For the sake of completeness, it is provided in the appendix.

3 The diminishing oscillation result

This section presents the diminishing oscillation result. We first recall a theorem of Caffarelli and Silvestre [1]. It states that if $L(\theta)$ solves the following initial and boundary value problem

$$\begin{cases} \nabla \cdot (z^b \nabla L(\theta)) = 0, \quad (x,z) \in \mathbb{R}^n \times (0,\infty), \\ \\ L(\theta)(x,0) = \theta(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

then

$$(-\Delta)^\alpha \theta = \lim_{z\to 0} (-z^b L(\theta)_z) \tag{3.2}$$

where $b = 1 - 2\alpha$. Furthermore, the boundary-value problem (3.1) can be solved through a Poisson formula

$$L(\theta)(x,z) = P(x,z) * \theta \equiv \int_{\mathbb{R}^n} P(x-y,z)\theta(y)\, dy,$$

where the Poisson kernel

$$P(x,z) = C_{n,b} \frac{z^{1-b}}{(|x|^2 + |z|^2)^{\frac{n+1-b}{2}}} = C_{n,\alpha} \frac{z^{2\alpha}}{(|x|^2 + |z|^2)^{\frac{n+2\alpha}{2}}}. \tag{3.3}$$

For notational convenience, we shall write

$$\theta^*(x,z,t) = L(\theta(\cdot,t))(x,z).$$

The following notation will be used throughout the rest of the sections:

$$f_+ = \max(0,f), \quad B_r \equiv [-r,r]^n \subset \mathbb{R}^n, \quad Q_r \equiv B_r \times [0,r] \subset \mathbb{R}^n \times \{t \geq 0\}$$

and

$$B_r^* \equiv B_r \times [0,r] \subset \mathbb{R}^n \times \mathbb{R}^+, \quad Q_r^* \equiv [-r,r]^n \times [0,r] \times [0,r] \subset \mathbb{R}^n \times \mathbb{R}^+ \times \{t \geq 0\}.$$

Theorem 3.1 *Let θ be a weak solution to (1.1) satisfying*

$$\theta \in L^\infty([0,\infty), L^2(\mathbb{R}^n)) \cap L^2([0,\infty); \mathring{H}^\alpha(\mathbb{R}^n))$$

with u satisfying (3.8) below. Assume

$$|\theta^*| \leq 2 \quad \text{in } Q_4^*.$$

Then there exists a $\lambda^* > 0$ such that

$$\sup_{Q_1^*} \theta^* - \inf_{Q_1^*} \theta^* \leq 4 - \lambda^*. \tag{3.4}$$

The proof of this theorem relies on three propositions stated below and will be provided in the appendix. It can be seen from the proofs of this theorem and related propositions that λ^* may depend on $\|u\|_{L^{\frac{n}{\alpha}}}$ in the fashion $\lambda^* \sim \exp(-\|u\|^m_{L^{\frac{n}{\alpha}}})$ for some constant m.

The first proposition derives a local energy inequality which bounds the L^2-norm of the gradient of θ^* in terms of the local L^2-norms of θ and θ^*.

Proposition 3.2 *Let $0 < t_1 < t_2 < \infty$. Let θ be a solution of (1.1) satisfying*

$$\theta \in L^\infty([t_1,t_2]; L^2(\mathbb{R}^n)) \cap L^2([t_1,t_2]; \mathring{H}^\alpha(\mathbb{R}^n)).$$

Assume the velocity u satisfies

$$u \in L^\infty([t_1,t_2]; L^{\frac{n}{\alpha}}(\mathbb{R}^n)). \tag{3.5}$$

Then, for any cutoff function η compactly supported in B_r^* with $r > 0$,

$$\int_{t_1}^{t_2} \int_{B_r^*} z^b \, |\nabla(\eta\theta_+^*)|^2 \, dxdzdt + \int_{B_r} (\eta\theta_+)^2(t_2,x)\, dx \leq \int_{B_r} (\eta\theta_+)^2(t_1,x)\, dx$$

$$+ C_1 \int_{t_1}^{t_2} \int_{B_r} (|\nabla\eta|\theta_+)^2 \, dxdt + \int_{t_1}^{t_2} \int_{B_r^*} z^b (|\nabla\eta|\theta_+^*)^2 \, dxdzdt, \tag{3.6}$$

where

$$C_1 = \|u\|_{L^\infty([t_1,t_2];\, L^{\frac{n}{\alpha}}(\mathbb{R}^n))}. \tag{3.7}$$

If, instead of (3.5), we assume

$$u \in L^\infty([t_1,t_2]; C^{1-2\alpha}(\mathbb{R}^n)) \quad \text{and} \quad \int_{B_r} u(x,t)\,dx = 0, \tag{3.8}$$

then the same local energy inequality (3.6) holds with C_1 in (3.7) replaced by

$$C_2 = \|u\|_{L^\infty([t_1,t_2];\, C^{1-2\alpha}(\mathbb{R}^n))}. \tag{3.9}$$

The following proposition establishes the diminishing oscillation for θ^* under the condition that the local L^2-norms of θ and θ^* are small.

Proposition 3.3 *Let θ be a solution of the supercritical QG equation (1.1) satisfying*

$$\theta \in L^\infty([0,\infty); L^2) \cap L^2([0,\infty); \mathring{H}^\alpha).$$

Assume that u satisfies the condition in (3.8) and

$$\theta^* \le 2 \quad \text{in } B_4^* \times [-4,0].$$

There exist $\epsilon_0 > 0$ and $\lambda > 0$ such that if

$$\int_{-4}^0 \int_{B_4^*} (\theta_+^*)^2\, z^b\, dxdzds + \int_{-4}^0 \int_{B_4} (\theta_+)^2\, dxds \le \epsilon_0, \tag{3.10}$$

then

$$\theta_+ \le 2 - \lambda \quad \text{on } B_1 \times [-1,0]. \tag{3.11}$$

The proof is obtained by following Caffarelli and Vasseur and will be presented in the appendix. The following proposition supplies a condition that guarantees the smallness of the local L^2-norms of θ and θ^*.

Proposition 3.4 *Let θ be a Leray-Hopf weak solution to the supercritical equation (1.1) with u satisfying (3.8). Assume that*

$$\theta^* \le 2 \qquad \text{in } Q_4^*$$

and

$$|\{(x,z,t) \in Q_4^* :\ \theta^* \le 0\}|_w \ge \frac{|Q_4^*|_w}{2},$$

where $|Q_4^*|_w$ denotes the weighted measure of Q_4^* with respect to $z^b\, dxdzdt$. For every $\epsilon_1 > 0$, there exists a constant $\delta_1 > 0$ such that if

$$|\{(x,z,t) \in Q_4^* :\ 0 < \theta^*(x,z,t) < 1\}|_w \le \delta_1,$$

then

$$\int_{Q_1} \theta_+^2\, dxdt + \int_{Q_1^*} (\theta_+^*)^2\, z^b\, dx\, dzdt \le C\, \epsilon_1^\alpha.$$

The proof of this proposition involves a weighted version of De Giorgi's isoperimetric inequality. More details will be given in the appendix. The isoperimetric inequality with no weight was given in Caffarelli and Vasseur [2].

Lemma 3.5 *Let $B_r = [-r,r]^n \subset \mathbb{R}^n$ and $B_r^* = B_r \times [0,r]$. Let $b \in [0,1)$ and let $p > (1+b)/(1-b)$. Let f be a function defined in B_r^* such that*

$$K \equiv \int_{B_r} \int_0^r z^b |\nabla f|^2 \, dz \, dx < \infty.$$

Let

$$\begin{array}{l} \mathcal{A} \equiv \{(x,z) \in B_r^* : \ f(x,z) \leq 0\}, \\ \mathcal{B} \equiv \{(x,z) \in B_r^* : \ f(x,z) \geq 1\}, \\ \mathcal{C} \equiv \{(x,z) \in B_r^* : 0 < \ f(x,z) < 1\} \end{array} \tag{3.12}$$

and let $|\mathcal{A}|_w$, $|\mathcal{B}|_w$ and $|\mathcal{C}|_w$ be the weighted measure of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ with respect to $z^b dxdz$, respectively. Then

$$|\mathcal{A}|_w \, |\mathcal{B}|_w \leq C r^{1+\frac{1}{2}(n+1-\frac{p+1}{p-1}b)(1-\frac{1}{p})} \left(|\mathcal{C}|_w\right)^{\frac{1}{2p}} K^{\frac{1}{2}}$$

where C is a constant independent of r.

Proof. We scale the $z-$variable by

$$\tilde{z} = \frac{1}{b+1} z^{b+1} \quad \text{or} \quad z = ((b+1)\tilde{z})^{\frac{1}{b+1}}.$$

When $(x,z) \in B_r \times [0,r]$, $(x,\tilde{z}) \in B_r \times [0,\tilde{r}]$ with $\tilde{r} = \frac{r}{1+b}$. For notational convenience, we write $E_r = B_r \times [0,\tilde{r}]$. Define

$$g(x,\tilde{z}) = f(x,z) \quad \text{for} \quad (x,\tilde{z}) \in B_r \times [0,\tilde{r}].$$

Let

$$\tilde{\mathcal{A}} \equiv \{(x,\tilde{z}) \in B_r \times [0,\tilde{r}] : \ g(x,\tilde{z}) \leq 0\}$$

and $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ be similarly defined. Therefore,

$$\begin{aligned} |\mathcal{A}|_w \, |\mathcal{B}|_w &\equiv \int_{\mathcal{A}} \int_{\mathcal{B}} z_1^b dx_1 dz_1 \, z_2^b dx_2 dz_2 \\ &= \int_{\mathcal{A}} \int_{\mathcal{B}} (f(x_1,z_1) - f(x_2,z_2) \, z_1^b dx_1 dz_1 \, z_2^b dx_2 dz_2 \\ &= \int_{\tilde{\mathcal{A}}} \int_{\tilde{\mathcal{B}}} (g(x_1,\tilde{z}_1) - g(x_2,\tilde{z}_2)) \, dx_1 d\tilde{z}_1 \, dx_2 d\tilde{z}_2 \\ &= \int_{\tilde{\mathcal{A}}} \int_{\tilde{\mathcal{B}}} (g(\tilde{y}_1) - g(\tilde{y}_2)) \, d\tilde{y}_1 d\tilde{y}_2, \end{aligned} \tag{3.13}$$

where $\tilde{y}_1 = (x_1, \tilde{z}_1)$ and $\tilde{y}_2 = (x_2, \tilde{z}_2)$. This integral now involves no weight and can be handled similarly as in Caffarelli and Vasseur [2].

$$|\mathcal{A}|_w\,|\mathcal{B}|_w \;\le\; C\int_{E_r}\int_{E_r}\frac{|\nabla g(\tilde{y}_1+\tilde{y}_2)|}{|\tilde{y}_2|^n}\,\chi_{\{\tilde{y}_1+\tilde{y}_2)\in\tilde{\mathcal{C}}\}}\,d\tilde{y}_1\,d\tilde{y}_2 \tag{3.14}$$
$$= C\int_{E_r}\int_{E_r+\{\tilde{y}_2\}}|\nabla g(\tilde{y})|\,\chi_{\{\tilde{y}\}}\,d\tilde{y}\,\frac{1}{|\tilde{y}_2|^n}d\tilde{y}_2$$
$$= C\,r\int_{E_r}|\nabla g(\tilde{y})|\,\chi_{\{\tilde{y}\in\tilde{\mathcal{C}}\}}\,d\tilde{y},$$

where χ denotes the characteristic function. By the definition of g,

$$\nabla g(x,\tilde{z}) = (\nabla_x g, \partial_{\tilde{z}} g) = (\nabla_x f, \partial_z f\frac{\partial z}{\partial \tilde{z}}) = (\nabla_x f, \partial_z f\, z^{-b}).$$

By substituting back to the $z-$variable and letting $y=(x,z)$, we have

$$|\mathcal{A}|_w\,|\mathcal{B}|_w \;\le\; C\,r\int_{B_r}\int_0^r \chi_{\{y\in\mathcal{C}\}}\sqrt{|\nabla_x f|^2+(\partial_z f)^2 z^{-2b}}\,z^b\,dz\,dx$$
$$\le\; C\,r\Big(\int_{B_r^*}(|\nabla_x f|^2 z^{2b}+(\partial_z f)^2)\,z^b\,dzdx\Big)^{1/2}\Big(\int_{B_r^*}\chi_{\{y\in\mathcal{C}\}}z^{-b}\,dzdx\Big)^{1/2}.$$

By Hölder's inequality,

$$\int_{B_r^*}\chi_{\{y\in\mathcal{C}\}}z^{-b}\,dzdx \;\le\; \Big(\int_{\mathcal{C}} z^b\,dzdx\Big)^{1/p}\Big(\int_{B_r}\int_0^r z^{-\frac{p+1}{p-1}b}\,dzdx\Big)^{1-1/p}$$
$$= |\mathcal{C}|_w^{1/p}\, r^{(n+1-\frac{p+1}{p-1}b)(1-\frac{1}{p})}.$$

Therefore,

$$|\mathcal{A}|_w\,|\mathcal{B}|_w \le C\,r^{1+\frac{1}{2}(n+1-\frac{p+1}{p-1}b)(1-\frac{1}{p})}\;|\mathcal{C}|_w^{\frac{1}{2p}}\,K^{\frac{1}{2}}.$$

This completes the proof of this lemma.

4 Hölder continuity under the condition $u \in C^{1-2\alpha}$

This section proves the following theorem.

Theorem 4.1 *Let θ be a solution of (1.1) satisfying*

$$\theta \in L^\infty([0,\infty), L^2(\mathbb{R}^n)) \cap L^2([0,\infty); \mathring{H}^\alpha(\mathbb{R}^n)).$$

Let $t_0 > 0$. Assume that

$$\theta \in L^\infty(\mathbb{R}^n \times [t_0,\infty))$$

and

$$u \in L^\infty([t_0,\infty); C^{1-2\alpha}(\mathbb{R}^n)).$$

Then θ is in $C^\delta(\mathbb{R}^n \times [t_0,\infty))$ for some $\delta > 0$.

Proof. Fix $x \in \mathbb{R}^n$ and $t \in [t_0, \infty)$. We show θ is C^δ at (x,t). Define

$$F_0(y,s) = \theta(x + y + x_0(s),\, t + s),$$

where $x_0(s)$ is the solution to

$$\begin{aligned} \mathring{x}_0(s) &= \frac{1}{|B_4|}\int_{x_0(s)+B_4} u(x+y, t+s)\,dy, \\ x_0(0) &= 0. \end{aligned}$$

Note that $x_0(s)$ is uniquely defined from the classical Cauchy-Lipschitz theorem. Since θ is bounded in $\mathbb{R}^n \times [t_0,\infty)$, we can define

$$\begin{aligned} \overline{\theta}_0^* &= \frac{4}{\sup_{Q_4^*} F_0^* - \inf_{Q_4^*} F_0^*}\Big(F_0^* - \frac{\sup_{Q_4^*} F_0^* + \inf_{Q_4^*} F_0^*}{2}\Big), \\ u_0(y,s) &= u(x + y + x_0(s),\, t+s) - \mathring{x}_0(s), \end{aligned}$$

where $F_0^*(y,z,s) = L(F_0(\cdot,s))(y,z)$. Trivially, $|\overline{\theta}_0^*| \le 2$ and thus $|\overline{\theta}_0| \le 2$. To verify that $(\overline{\theta}_0, u_0)$ solves the supercritical QG equation (1.1), it suffices to show that (F_0, u_0) solves (1.1). In fact,

$$\begin{aligned} \partial_s F_0 + u_0 \cdot \nabla_y F_0 &= \mathring{x}_0(s)\cdot\nabla_x\theta + \partial_t\theta + (u - \mathring{x}_0(s))\cdot\nabla_x\theta \\ &= \partial_t\theta + u\cdot\nabla\theta_x = -\Lambda_x^{2\alpha}\theta = -\Lambda_y^{2\alpha}F_0. \end{aligned}$$

In addition, for any $s \ge 0$,

$$\|u_0(\cdot,s)\|_{C^{1-2\alpha}} = \|u(\cdot,t+s)\|_{C^{1-2\alpha}} \quad \text{and} \quad \int_{B_4} u_0(y,s)dy = 0.$$

Let $\mu > 0$ and set for every integer $k > 0$

$$\begin{aligned} F_k(y,s) &= \mu^{2\alpha-1}\,F_{k-1}\left(\mu y + \mu^{2\alpha}\,x_k(s), \mu^{2\alpha}s\right), \\ \overline{\theta}_k^* &= \frac{4}{\sup_{Q_4^*} F_k^* - \inf_{Q_4^*} F_k^*}\Big(F_k^* - \frac{\sup_{Q_4^*} F_k^* + \inf_{Q_4^*} F_k^*}{2}\Big), \\ \mathring{x}_k(s) &= \frac{1}{|B_4|}\int_{B_4+\mu^{2\alpha-1}x_k(s)} u_{k-1}\Big(\mu y, \mu^{2\alpha}s\Big)\,dy, \\ x_k(0) &= 0, \\ u_k(y,s) &= \mu^{2\alpha-1}\,u_{k-1}\Big(\mu y + \mu^{2\alpha}\,x_k(s),\, \mu^{2\alpha}s\Big) - \mu^{2\alpha-1}\,\mathring{x}_k(s). \end{aligned}$$

By the construction, $|\overline{\theta}_k| \le 2$ and

$$\begin{aligned} \|u_k(\cdot,s)\|_{C^{1-2\alpha}} &= \mu^{2\alpha-1}\,\|u_{k-1}(\mu\,\cdot + \mu^{2\alpha}, \mu^{2\alpha}s)\|_{C^{1-2\alpha}} \\ &\le \|u_{k-1}(\cdot,\mu^{2\alpha}s)\|_{C^{1-2\alpha}} \\ &\le \|u_0(\cdot,\mu^{2\alpha k}s)\|_{C^{1-2\alpha}} \\ &= \|u(\cdot,t+\mu^{2\alpha k}s)\|_{C^{1-2\alpha}}. \end{aligned}$$

Furthermore,

$$\int_{B_4} u_k(y,s)dy = 0.$$

We show inductively that $(\overline{\theta}_k, u_k)$ solves (1.1). Assume that $(\overline{\theta}_{k-1}, u_{k-1})$ solves (1.1), we show that $(\overline{\theta}_k, u_k)$ solves (1.1). It suffices to show that (F_k, u_k) solves (1.1). By construction, we have

$$\begin{aligned} \partial_s F_k + u_k \cdot \nabla_y F_k &= \mu^{4\alpha-1}\, \mathring{x}_k(s) \cdot \nabla F_{k-1} + \mu^{4\alpha-1}\, \partial_s F_{k-1} \\ &\quad + \mu^{4\alpha-1}\, (u_{k-1} - \mathring{x}_k(s)) \cdot \nabla F_{k-1} \\ &= \mu^{4\alpha-1}\, (\partial_s F_{k-1} + u_{k-1} \cdot \nabla F_{k-1}) \\ &= -\mu^{4\alpha-1}\, \Lambda^{2\alpha} F_{k-1} \\ &= -\Lambda_y^{2\alpha} F_k. \end{aligned}$$

For every k, we apply the diminishing oscillation result (Theorem 3.1). There exists a λ^* such that

$$\sup_{Q_1^*} \overline{\theta}_k^* - \inf_{Q_1^*} \overline{\theta}_k^* \leq 4 - \lambda^*.$$

λ^* is independent of k since $\|u_k\|_{C^{1-2\alpha}}$ obeys a uniform bound in k. According to the construction of $\overline{\theta}_k^*$, we have

$$\sup_{Q_1^*} \overline{\theta}_k^* - \inf_{Q_1^*} \overline{\theta}_k^* = \frac{4}{\sup_{Q_4^*} F_k^* - \inf_{Q_4^*} F_k^*} (\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^*).$$

Therefore,

$$\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^* \leq \Big(1 - \frac{\lambda^*}{4}\Big) (\sup_{Q_4^*} F_k^* - \inf_{Q_4^*} F_k^*).$$

By the construction of F_k, we have

$$\sup_{(y,s)\in Q_4^*} F_k^*(y,s) - \inf_{(y,s)\in Q_4^*} F_k^*(y,s)$$

$$= \mu^{2\alpha-1} \left(\sup_{(y,s)\in Q_4^*} F_{k-1}^*(\mu y + \mu^{2\alpha} x_k(s), \mu^{2\alpha} s) - \inf_{(y,s)\in Q_4^*} F_{k-1}^*(\mu y + \mu^{2\alpha} x_k(s), \mu^{2\alpha} s) \right).$$

For notational convenience, we have omitted the z-variable. It is easy to see from the construction of $\mathring{x}_k$ that

$$|\mathring{x}_k(s)| \leq \|u_{k-1}(\cdot, \mu^{2\alpha} s)\|_{L^\infty} \leq \|u_{k-1}(\cdot, \mu^{2\alpha} s)\|_{C^{1-2\alpha}} \leq \|u(\cdot, t + \mu^{2\alpha k} s)\|_{C^{1-2\alpha}}. \tag{4.1}$$

For $0 \leq s \leq 1$, we can choose $\mu > 0$ sufficiently small such that

$$|\mu y + \mu^{2\alpha} x_k(s)| \leq 4\mu + C\, \mu^{2\alpha} < 1. \tag{4.2}$$

We then have

$$\sup_{(y,s)\in Q_4^*} F_{k-1}^*(\mu y + \mu^{2\alpha} x_k(s), \mu^{2\alpha} s) - \inf_{(y,s)\in Q_4^*} F_{k-1}^*(\mu y + \mu^{2\alpha} x_k(s), \mu^{2\alpha} s)$$

$$\leq \sup_{(y,s)\in Q_1^*} F_{k-1}^*(y,s) - \inf_{(y,s)\in Q_1^*} F_{k-1}^*(y,s).$$

Consequently,

$$\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^* \leq \mu^{2\alpha-1}\Big(1 - \frac{\lambda^*}{4}\Big)(\sup_{Q_1^*} F_{k-1}^* - \inf_{Q_1^*} F_{k-1}^*).$$

By iteration, for any $k > 0$,

$$\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^* \leq \mu^{(2\alpha-1)k}\Big(1 - \frac{\lambda^*}{4}\Big)^k(\sup_{Q_1^*} F_0^* - \inf_{Q_1^*} F_0^*). \tag{4.3}$$

By construction,

$$\begin{aligned} F_0(y,s) &= \theta(x+y+x_0(s), t+s), \\ F_k(y,s) &= \mu^{(2\alpha-1)k}\,\theta\Big(x + \mu^k y + \mu^{2\alpha+k-1}x_k(s) + \mu^{2\alpha+k-2}x_{k-1}(\mu^{2\alpha}s) \\ &\quad + \cdots + \mu^{2\alpha}x_1(\mu^{2\alpha(k-1)}s) + x_0(\mu^{2\alpha k}s), t + \mu^{2\alpha k}s\Big). \end{aligned}$$

To deduce the Hölder continuity of θ in x, we set $s = 0$. Then (4.3) implies

$$\sup_{y\in B_1} \mu^{(2\alpha-1)k}\,\theta(x+\mu^k y, t) - \inf_{y\in B_1} \mu^{(2\alpha-1)k}\,\theta(x+\mu^k y,t) \leq C\,\mu^{(2\alpha-1)k}\Big(1 - \frac{\lambda^*}{4}\Big)^k.$$

or

$$\sup_{y\in B_1} \theta(x+\mu^k y, t) - \inf_{y\in B_1} \theta(x+\mu^k y,t) \leq C\,\Big(1 - \frac{\lambda^*}{4}\Big)^k. \tag{4.4}$$

To see the Hölder continuity from this inequality, we choose $\delta > 0$ such that

$$1 - \frac{\lambda^*}{4} < \mu^\delta.$$

Then, for any $|y| > 0$, we choose k such that

$$\left(\frac{1 - \frac{\lambda^*}{4}}{\mu^\delta}\right)^k \leq |y|^\delta \quad \text{or} \quad \Big(1 - \frac{\lambda^*}{4}\Big)^k \leq (\mu^k |y|)^\delta.$$

It then follows from (4.4) that

$$\sup_{y\in B_1} \theta(x+\mu^k y, t) - \inf_{y\in B_1} \theta(x+\mu^k y,t) \leq C\,(\mu^k|y|)^\delta.$$

For general $0 \leq s \leq 1$ and $y \in B_1$, we have, according to (4.1),

$$\begin{aligned} r_k &\equiv \mu^{2\alpha+k-1}x_k(s) + \mu^{2\alpha+k-2}x_{k-1}(\mu^{2\alpha}s) + \cdots + \mu^{2\alpha}x_1(\mu^{2\alpha(k-1)}s) + x_0(\mu^{2\alpha k}s) \\ &\leq C\mu^{2\alpha+k-1}|s|\,(1 + \mu^{2\alpha-1} + \cdots + \mu^{(2\alpha-1)k}) \\ &= C\,|s|\,\mu^{2\alpha(k+1)-1}\frac{1-\mu^{(1-2\alpha)(k+1)}}{1-\mu^{1-2\alpha}} \\ &\leq C\,|s|\,\mu^{2\alpha(k+1)-1}. \end{aligned}$$

Without loss of generality, we can assume that $\mu^k|y| > |s|\mu^{2\alpha k}$. Then we can pick up $\delta > 0$ satisfying

$$1 - \frac{\lambda^*}{4} < \mu^{2\alpha\delta}$$

and suitable k such that

$$\sup_{(y,s)\in B_1\times[0,1]} \theta(x + \mu^k y + r_k, t + \mu^{2\alpha k} s) - \inf_{(y,s)\in B_1\times[0,1]} \theta(x + \mu^k y + r_k, t + \mu^{2\alpha k} s)$$

$$\leq C\,(\mu^k|y|)^\delta + C\,(\mu^{2\alpha k}|s|)^\delta.$$

That is, θ is Hölder continuous at (x,t). This completes the proof.

Acknowledgment: PC was partially supported by NSF-DMS 0504213. JW thanks the Department of Mathematics at the University of Chicago for its support and hospitality.

Appendix

The appendix contains the proofs of several theorems and propositions presented in the previous sections. These proofs are obtained by following the ideas of Caffarelli and Vasseur [2]. They are attached here for the sake of completeness.

Proof of Theorem 2.1. We first remark that (2.1) implies that θ satisfiers the level set energy inequality. That is, for every $\lambda > 0$, $\theta_\lambda = (\theta - \lambda)_+$ satisfies

$$\int \theta_\lambda^2(x, t_2)\,dx + 2\int_{t_1}^{t_2}\int |\Lambda^\alpha \theta_\lambda|^2 dx\,dt \leq \int \theta_\lambda^2(x, t_1)\,dx \tag{A.1}$$

for any $0 < t_1 < t_2 < \infty$. This can be verified by using an inequality of A. Córdoba and D. Córdoba [11] for fractional derivatives, namely

$$f'(\theta)(-\Delta)^\alpha\theta \geq (-\Delta)^\alpha f(\theta)$$

for any convex function f. Applying this inequality with

$$f(\theta) = (\theta - \lambda)_+,$$

we have

$$\partial_t\theta_\lambda + u\cdot\nabla\theta_\lambda + \Lambda^{2\alpha}\theta_\lambda \leq 0.$$

Multiplying this equation by θ_λ then leads to (A.1). Let $k \geq 0$ be an integer and let $\lambda = C_k = M(1 - 2^{-k})$ for some M to be determined. It then follows from (A.1) that

$$\theta_k = (\theta - C_k)_+.$$

satisfies

$$\partial_t \int \theta_k^2(x,t)\,dx + \int |\Lambda^\alpha \theta_k|^2 dx \ \le 0. \tag{A.2}$$

Fix any $t_0 > 0$. Let $t_k = t_0(1-2^{-k})$. Consider the quantity U_k,

$$U_k = \sup_{t\ge t_k} \int \theta_k^2(x,t)\,dx + 2\int_{t_k}^\infty \int |\Lambda^\alpha \theta_k|^2 dx\,dt.$$

Now let $s \in [t_{k-1}, t_k]$. We have from (A.2) that for any $s \le t$,

$$\int \theta_k^2(x,t)\,dx + 2\int_s^t \int |\Lambda^\alpha \theta_k|^2 dx\,dt \le \int \theta_k^2(x,s)dx$$

which implies that

$$\sup_{t\ge t_k} \int \theta_k^2(x,t)\,dx \le \int \theta_k^2(x,s)dx, \quad 2\int_s^\infty \int |\Lambda^\alpha \theta_k|^2 dx\,dt \le \int \theta_k^2(x,s)dx$$

Since $s \in (t_{k-1}, t_k)$, we add up these inequalities to get

$$U_k \le 2\int \theta_k^2(x,s)dx.$$

Taking the mean in s over $[t_{k-1}, t_k]$, we get

$$U_k \le \frac{2^{k+1}}{t_0} \int_{t_{k-1}}^\infty \int \theta_k^2(x,t)\,dx\,dt \tag{A.3}$$

By the Gagliardo-Nirenberg inequality,

$$\|\theta_{k-1}\|_{L^q([t_{k-1},\infty)\times\mathbb{R}^n)} \le C\left(\int_{t_{k-1}}^\infty \int_{\mathbb{R}^n} |\Lambda^\alpha \theta_{k-1}|^2\,dx\,dt\right)^{\frac{1}{2}},$$

where q is given by

$$\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{n+1} \quad \text{or} \quad q = 2\frac{n+1}{n+1-2\alpha}. \tag{A.4}$$

Therefore,

$$U_{k-1} \ge C\left(\int_{t_{k-1}}^\infty \int |\theta_{k-1}|^q\,dx\,dt\right)^{2/q}.$$

By the definition of θ_k, $\theta_k \ge 0$. When $\theta_k > 0$,

$$\theta_{k-1} = \theta_k + M2^{-k} \ge M2^{-k}$$

and thus we have

$$\chi_{\{(x,t):\,\theta_k>0\}} \le \left(\frac{2^k\theta_{k-1}}{M}\right)^{q-2},$$

where χ denotes the characteristic function. It then follows from (A.3) that

$$\begin{aligned}
U_k &\le \frac{2^{k+1}}{t_0} \int_{t_{k-1}}^{\infty} \int \theta_k^2(x,t)\, \chi_{\{\theta_k>0\}}\, dx\, dt \\
&\le \frac{2^{k+1}}{t_0} \int_{t_{k-1}}^{\infty} \int \theta_{k-1}^2(x,t)\, \chi_{\{\theta_k>0\}}\, dx\, dt \\
&\le \frac{2^{k+1+(q-2)k}}{t_0 M^{q-2}} \int_{t_{k-1}}^{\infty} \int |\theta_{k-1}|^q\, dx\, dt \\
&\le \frac{2}{t_0 M^{q-2}}\, 2^{(q-1)k}\, U_{k-1}^{\frac{q}{2}}. \qquad \text{(A.5)}
\end{aligned}$$

Since $q > 2$, we rewrite (A.5) as

$$V_k \le V_{k-1}^{\frac{q}{2}}, \qquad \text{(A.6)}$$

where

$$V_k = \frac{2^{\gamma k}\, U_k}{t_0^{2/(q-2)}\, M^2\, 2^{(-\gamma q-2)/(q-2)}} \quad \text{with} \quad \gamma = \frac{2(q-1)}{q-2} > 0.$$

Since $U_0 \le \|u_0\|_{L^2}^2 < \infty$, we can choose sufficiently large M such that $V_0 < 1$ and (A.6) then implies $V_k \to 0$ as $k \to \infty$. Consequently, we conclude that for each fixed $t_0 > 0$ and M sufficiently large, $U_k \to 0$ as $k \to \infty$. That is, $\theta \le M$. Applying this process to $-\theta$ yields a lower bound.

The scaling invariance

$$\theta_\rho(x,t) = \rho^{2\alpha-1}\, \theta(\rho x, \rho^{2\alpha} t)$$

of (1.1) allows us to deduce the following explicit bound

$$\|\theta(\cdot,t)\|_{L^\infty} \le C\, \frac{\|u_0\|_{L^2}}{t^{\frac{n}{4\alpha}}}.$$

This concludes the proof of Theorem 2.1.

Proof of Proposition 3.2. Multiplying the first equation in (3.1) by $\eta^2\theta_+^*$ and integrating over $\mathbb{R}^n \times (0,\infty)$ leads to

$$\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{R}^n} \eta^2 \theta_+^* \nabla \cdot (z^b \nabla \theta^*) dx dz \\
&= \int_0^\infty \int_{\mathbb{R}^n} (\nabla \cdot (\eta^2 \theta_+^*\, z^b \nabla \theta^*) - \nabla(\eta^2 \theta_+^*) \cdot z^b \nabla \theta^*)\, dx dz.
\end{aligned}$$

Since η has compact support on B_r^* and

$$\lim_{z\to 0}(-z^b \partial_z \theta^*) = (-\Delta)^\alpha \theta \equiv \Lambda^{2\alpha}\theta,$$

we have

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n} \eta^2\theta_+\Lambda^{2\alpha}\theta dx - \int_0^\infty \int_{\mathbb{R}^n} z^b \left(2\eta\nabla\eta\theta_+^* \cdot \nabla\theta^* + \eta^2\nabla\theta_+^*\cdot\nabla\theta^*\right) dxdz \\
&= \int_{\mathbb{R}^n} \eta^2\theta_+\Lambda^{2\alpha}\theta dx - \int_0^\infty \int_{\mathbb{R}^n} z^b \left|\nabla(\eta\theta_+^*)\right|^2 dx\, dz \\
&\quad + \int_0^\infty \int_{\mathbb{R}^n} z^b \left|\nabla\eta\right|^2 (\theta_+^*)^2\, dxdz.
\end{aligned}
$$

Multiplying both sides of the QG equation (1.1) by $\eta^2\,\theta_+$, we get

$$
-\int_{\mathbb{R}^n} \eta^2\theta_+\Lambda^{2\alpha}\theta\, dx = \partial_t \int_{\mathbb{R}^n} \eta^2 \frac{\theta_+^2}{2}\, dx - \int_{\mathbb{R}^n} \nabla(\eta^2)\cdot u \frac{\theta_+^2}{2}\, dx.
$$

Combining these two equations, we get

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} z^b \left|\nabla(\eta\theta_+^*)\right|^2 dx\, dz \; + \partial_t \int_{\mathbb{R}^n} \eta^2 \frac{\theta_+^2}{2}\, dx \\
&= \int_0^\infty \int_{\mathbb{R}^n} z^b \left|\nabla\eta\right|^2 (\theta_+^*)^2\, dxdz + \int_{\mathbb{R}^n} \nabla(\eta^2)\cdot u \frac{\theta_+^2}{2}\, dx.
\end{aligned}
$$

Integrating with respect to t over $[t_1, t_2]$, we get

$$
\begin{aligned}
&\int_{t_1}^{t_2} \int_0^\infty \int_{\mathbb{R}^n} z^b \left|\nabla(\eta\theta_+^*)\right|^2 dx\, dz\, dt + \int_{\mathbb{R}^n} \eta^2 \frac{\theta_+^2}{2}(t_2, x)\, dx \\
&= \int_{\mathbb{R}^n} \eta^2 \frac{\theta_+^2}{2}(t_1, x)\, dx + \int_{t_1}^{t_2} \int_0^\infty \int_{\mathbb{R}^n} z^b \left|\nabla\eta\right|^2 (\theta_+^*)^2\, dxdz \\
&\quad + \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \eta\nabla\eta\cdot u\, \theta_+^2\, dx\, dt \right|. \qquad \text{(A.7)}
\end{aligned}
$$

We now bound the last term. By the inequalities of Hölder and Young,

$$
\left| \int_{\mathbb{R}^n} \eta\nabla\eta\cdot u\, \theta_+^2\, dx \right| \le \|\eta\theta_+\|_{L^q} \| |\nabla\eta| u\theta_+\|_{L^{q'}} \le \epsilon\, \|\eta\theta_+\|_{L^q}^2 + \frac{1}{\epsilon} \| |\nabla\eta| u\theta_+\|_{L^{q'}}^2, \qquad \text{(A.8)}
$$

where $\epsilon > 0$ is small, and q and q' satisfies

$$
\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{n}, \qquad \frac{1}{q} + \frac{1}{q'} = 1.
$$

By the Gagliardo-Nirenberg inequality,

$$
\|\eta\theta_+\|_{L^q}^2 \le C\|\eta\theta_+\|_{H^\alpha}^2 = C \int_{\mathbb{R}^n} \eta\theta_+\Lambda^{2\alpha}\eta\theta_+\, dx.
$$

Furthermore, we have

$$\int_{\mathbb{R}^n} \eta\theta\, \Lambda^{2\alpha}\eta\theta\, dx = \int_0^\infty \int_{\mathbb{R}^n} z^b |\nabla(\eta\theta^*)|^2\, dx\, dz, \tag{A.9}$$

which can be established as follows. Since $\eta\theta^*$ is the harmonic extension of $\eta\theta$, i.e.

$$\begin{cases} \nabla \cdot (z^b \nabla(\eta\theta^*)) = 0, & (x,z) \in \mathbb{R}^n \times (0,\infty), \\ \theta^*(x,0,t) = \theta(x,t), & x \in \mathbb{R}^n, \end{cases}$$

we multiply by $\eta\theta^*$ and integrate over $\mathbb{R}^n \times (0,\infty)$ to get

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}^n} \eta\theta^*\, \nabla \cdot (z^b \nabla(\eta\theta^*))\, dxdz \\ &= \int_0^\infty \int_{\mathbb{R}^n} \nabla \cdot (\eta\theta^*\, z^b \nabla(\eta\theta^*))\, dxdz - \int_0^\infty \int_{\mathbb{R}^n} z^b |\nabla(\eta\theta^*)|^2\, dx\, dz \\ &= \int_{\mathbb{R}^n} \eta\theta^*\, (-z^b \partial_z(\eta\theta^*)) dx - \int_0^\infty \int_{\mathbb{R}^n} z^b |\nabla(\eta\theta^*)|^2\, dx\, dz. \end{aligned} \tag{A.10}$$

Inserting the equation

$$\lim_{z\to 0} (-z^b \partial_z(\eta\theta^*)) = \Lambda^{2\alpha}(\eta\theta)$$

in (A.10) yields (A.9). Therefore,

$$\|\eta\theta_+\|_{L^q}^2 \le C \int_0^\infty \int_{\mathbb{R}^n} z^b |\nabla(\eta\theta^*)|^2\, dx\, dz. \tag{A.11}$$

Noticing that $1/q' = 1/2 + \alpha/n$, the second term in (A.8) can be bounded by

$$\| |\nabla\eta| u\theta_+\|_{L^{q'}}^2 \le \|u\|_{L^{n/\alpha}}^2 \| |\nabla\eta| \theta_+\|_{L^2}^2.$$

(3.6) is thus obtained. If we further know that u satisfies (3.8), then

$$\|u\|_{L^{n/\alpha}} = \left(\int_{B_4} \left| u(x,t) - \frac{1}{|B_4|} \int_{B_4} u(y,t)\, dy \right|^{\frac{n}{\alpha}} dx \right)^{\frac{\alpha}{n}} \le C \|u\|_{C^{1-2\alpha}}.$$

This completes the proof of Proposition 3.2.

Proof of Theorem 3.1. It suffices to show that if

$$|\{(x,z,t) \in Q_4^* :\ \theta^* \le 0\}|_w \ge \frac{1}{2} |Q_4^*|_w, \tag{A.12}$$

then there exists a $\lambda^* > 0$ such that

$$\theta^* \le 2 - \lambda^* \quad \text{in } Q_1^*. \tag{A.13}$$

Otherwise, we have

$$|\{(x,z,t)\in Q_4^*:\ -\theta^*\le 0\}|_w \ge \frac{1}{2}|Q_4^*|_w$$

which implies

$$-\theta^* \le 2-\lambda^* \quad \text{or} \quad \theta^* \ge -2+\lambda^* \quad \text{in } Q_1^*.$$

Thus, in either case,

$$\sup_{Q_1^*}\overline{\theta}_k^* - \inf_{Q_1^*}\overline{\theta}_k^* \le 4-\lambda^*.$$

We now show (A.13) under (A.12). Fix ϵ_0 as in (3.10). Choose δ_1 and ϵ_1 as in Proposition 3.4 with $C\,\epsilon_1^\alpha=\epsilon_0$. Let K_+ be the integer

$$K_+ = \left[\frac{|Q_4^*|_w}{2\delta_1}\right]+1. \tag{A.14}$$

For $k\le K_+$, define

$$\overline{\theta}_0=\theta,$$
$$\overline{\theta}_k = 2(\overline{\theta}_{k-1}-1).$$

It is easy to see that $\overline{\theta}_k = 2^k(\theta-2)+2$. Note that for every k, $\overline{\theta}_k$ verifies (1.1), and

$$\overline{\theta}_k\le 2, \quad \text{in } Q_4,$$

$$|\{(x,z,t)\in Q_4^*:\ \overline{\theta}_k^*\le 0\}|_w \ge \frac{1}{2}|Q_4^*|_w.$$

Assume that for all $k\le K_+$, $|\{(x,z,t)\in Q_4^*:\ 0<\overline{\theta}_k^*<1\}|_w\ge\delta_1$. Then, for every k,

$$|\{(x,z,t):\overline{\theta}_k^*\le 0\}|_w = |\{(x,z,t):\overline{\theta}_{k-1}^*<1\}|_w \ge |\{(x,z,t):\overline{\theta}_{k-1}^*\le 0\}|_w+\delta_1$$

Hence,

$$|\{(x,z,t):\overline{\theta}_{K_+}^*\le 0\}|_w \ge K_+\delta_1 + |\{(x,z,t):\theta\le 0\}|_w \ge |Q_4^*|_w.$$

That is, $\overline{\theta}_{K_+}^*\le 0$ almost everywhere, which means

$$2^{K_+}(\theta^*-2)+2\le 0 \quad \text{or} \quad \theta^*\le 2-2^{-K_++1}.$$

(3.4) is then verified by taking $0<\lambda^*<2^{-K_++1}$.

Otherwise, there exists $0\le k_0\le K_+$ such that

$$|\{(x,z,t):0<\overline{\theta}_{k_0}^*<1\}|_w\le\delta_1.$$

Applying Propositions 3.3 and 3.4, we get $\overline{\theta}_{k_0+1}\le 2-\lambda$ which means

$$\theta\le 2-2^{-(k_0+1)}\lambda\le 2-2^{-K_+}\lambda \quad \text{in } Q_2.$$

Consider the function f_3 satisfying

$$\begin{aligned}
&\nabla\cdot(z^b\nabla f_3)=0 \quad \text{in } B_2^*\\
&f_3=2 \quad \text{on the sides of cube except for } z=0\\
&f_3=2-2^{-K_+}\inf(\lambda,1) \quad \text{on } z=0.
\end{aligned}$$

By the maximum principle, $f_3<2-\lambda^*$ in B_1^* and

$$\theta^*(x,z,t)\le f_3(x,z,t)<2-\lambda^* \quad \text{in } Q_1^*.$$

This completes the proof of Theorem 3.1.

Proof of Proposition 3.3. We start with the definition of two barrier functions f_1 and f_2. Here f_1 satisfies

$$\begin{cases}
\nabla\cdot(z^b\nabla f_1)=0, & \text{in } B_4^*,\\
f_1=2 & \text{on the sides of } B_4^* \text{ except for } z=0,\\
f_1=0 & \text{for } z=0.
\end{cases}\tag{A.15}$$

By the maximum principle, for some $\lambda>0$,

$$f_1(x,z)\le 2-4\lambda \quad \text{on } B_2^*.$$

The function f_2 satisfies

$$\begin{cases}
\nabla\cdot(z^b\nabla f_2)=0 & \text{in } [0,\infty)\times[0,1],\\
f_2(0,z)=2 & 0\le z\le 1,\\
f_2(x,0)=f_2(x,1)=0 & 0<x<\infty.
\end{cases}\tag{A.16}$$

By separating variables, we can explicitly solve (A.16) and find that

$$|f_2(x,z)|\le \overline{C}\,e^{-\beta_0 x}$$

for some constants $\overline{C}>0$ and $\beta_0>0$.

It can be verified that there exist $0<\delta\le 1$ and $M>1$ such that for every $k>0$,

$$n\overline{C}\,e^{-\frac{\beta_0}{(2\delta)^k}}\le \lambda 2^{-k-2}, \qquad \frac{\|P(\cdot,1)\|_{L^2}}{M^k\delta^{2\alpha(k+1)}}\le \lambda 2^{-k-2},$$

$$C_{0,k}\,M^{-(k-3)(1+\frac{1}{n+1-2\alpha})}\le M^{-k}, \quad k>12n.$$

where $P(x,z)$ denotes the Poisson kernel defined in (3.3) and $C_{0,k}$ is the constant in (A.25).

(3.11) is established through an inductive procedure, which resembles a local version of the proof for Theorem 2.1. Let k be an integer and set

$$C_k=2-\lambda(1+2^{-k}), \qquad \theta_k=(\theta-C_k)_+.\tag{A.17}$$

and let $\eta_k = \eta_k(x)$ be a cutoff function such that

$$\chi_{B_{1+2^{-k-1}}} \le \eta_k \le \chi_{B_{1+2^{-k}}} \quad \text{and} \quad |\nabla \eta_k| < C2^k, \tag{A.18}$$

where χ denotes the characteristic function. Set

$$A_k = 2\int_{-1-2^{-k}}^{0} \int_0^{\delta^k} \int_{\mathbb{R}^n} z^b \, |\nabla(\eta_k \theta_k^*)|^2 \, dxdzdt + \sup_{[-1-2^{-k},0]} \int_{\mathbb{R}^n} (\eta_k \theta_k)^2 \, dx. \tag{A.19}$$

The goal to prove that

$$A_k \le M^{-k}, \tag{A.20}$$

$$\eta_k \theta_k^* \quad \text{is supported in } 0 \le z \le \delta^k. \tag{A.21}$$

(3.11) then follows as a consequence of (A.20).

We first verify (A.20) for $0 \le k \le 12n$ and (A.21) for $k = 0$. Let

$$T_k = -1 - 2^{-k} \quad \text{and} \quad s \in [T_{k-1}, T_k).$$

Applying (3.6) with $t_1 = s$ and $t_2 = t$, we obtain

$$\int_s^t \int_{B_4^*} z^b \, |\nabla(\eta\theta_+^*)|^2 \, dxdzdt + \int_{B_4} (\eta\theta_+)^2(t,x) \, dx$$

$$\le \int_{B_4} (\eta\theta_+)^2(s,x) \, dx + C_1 \int_s^t \int_{B_4} (|\nabla\eta|\theta_+)^2 \, dxdt + \int_s^t \int_{B_4^*} z^b (|\nabla\eta|\theta_+^*)^2 \, dxdzdt.$$

Taking $\sup_{t\in[T_k,0]}$ for both sides and letting $s = T_k$ on the left gives

$$\int_{T_k}^0 \int_{B_4^*} z^b \, |\nabla(\eta\theta_+^*)|^2 \, dxdzdt + \sup_{t\in[T_k,0]} \int_{B_4} (\eta\theta_+)^2(t,x) \, dx$$

$$\le \int_{B_4} (\eta\theta_+)^2(s,x) \, dx + C_1 \int_s^0 \int_{B_4} (|\nabla\eta|\theta_+)^2 \, dxdt + \int_s^0 \int_{B_4^*} z^b (|\nabla\eta|\theta_+^*)^2 \, dxdzdt$$

$$\le \int_{B_4} (\eta\theta_+)^2(s,x) \, dx + C_1 \int_{T_{k-1}}^0 \int_{B_4} (|\nabla\eta|\theta_+)^2 \, dxdt + \int_{T_{k-1}}^0 \int_{B_4^*} z^b (|\nabla\eta|\theta_+^*)^2 \, dxdzdt.$$

Taking the mean of this inequality in s over $[T_{k-1}, T_k]$ yields

$$\begin{aligned}
&\int_{T_k}^0 \int_{B_4^*} z^b \, |\nabla(\eta\theta_+^*)|^2 \, dxdzdt + \sup_{t\in[T_k,0]} \int_{B_4} (\eta\theta_+)^2(t,x) \, dx \\
&\quad \le 2^k \int_{T_{k-1}}^{T_k} \int_{B_4} (\eta\theta_+)^2(s,x) \, dx \, ds + C_1 \int_{T_{k-1}}^0 \int_{B_4} (|\nabla\eta|\theta_+)^2 \, dxdt \\
&\qquad + \int_{T_{k-1}}^0 \int_{B_4^*} z^b (|\nabla\eta|\theta_+^*)^2. \, dxdzdt.
\end{aligned} \tag{A.22}$$

Letting $\eta = \eta_k(x)\phi_k(z)$ with ϕ_k supported on $[0, \delta^k]$ and using the assumption (3.10), we then verify (A.20) for $0 < k < 12n$ if ϵ_0 satisfies

$$C2^{24n}(1 + C_1)\epsilon_0 \le M^{-12n}.$$

We now show (A.21) for $k = 0$. By the maximum principle,

$$\theta^* \le (\theta_+ 1_{B_4}) * P(\cdot, z) + f_1(x, z)$$

in $B_4^* \times (0, \infty)$. By construction, the function on the right-hand side satisfies

$$\nabla \cdot \left(z^b \nabla((\theta_+ 1_{B_4}) * P(z) + f_1(x, z))\right) = 0$$

and has boundary data greater than or equal to the corresponding ones for θ^*. To obtain an upper bound for θ^*, we first notice that $f_1(x, z) \le 2 - 4\lambda$. In addition,

$$\|(\theta_+ 1_{B_4}) * P(\cdot, z)\|_{L^\infty(\{x \in B_4, z \ge 1\})} \le C\|P(\cdot, 1)\|_{L^2}\sqrt{\epsilon_0} \le C\sqrt{\epsilon_0}.$$

Here we used $\|\theta_+ 1_{B_4}\|_{L^2} \le C\sqrt{\epsilon_0}$, which can be deduced from (3.10) through a simple argument. Choose ϵ_0 small enough to get

$$\theta^* \le 2 - 2\lambda \qquad \text{for } z \ge 1,\ t \le 0 \text{ and } x \in B_4.$$

Therefore,

$$\theta_0^* = (\theta^* - (2 - 2\lambda))_+ \le 0 \qquad \text{for } z \ge 1,\ t \le 0 \text{ and } x \in B_4.$$

Hence, $\eta_0 \theta_0^*$ is supported in $0 \le z \le \delta^0 = 1$.

Now, assuming that (A.20) and (A.21) are verified at k, we show they are also true at $k + 1$. In the process, we will also show for each k,

$$\eta_k \theta_{k+1}^* \le [(\eta_k \theta_k) * P(z)]\, \eta_k \tag{A.23}$$

in the set $\bar{B}_k^* = B_{1+2^{-k}} \times [0, \delta^k]$. First we control θ_k^* in $\bar{B}_k^*$ by a function f satisfying

$$\nabla \cdot (z^b \nabla f)) = 0$$

by considering the contributions on the boundaries. No contributions come from $z = \delta^k$ thanks to the induction property on k. The contribution from $z = 0$ can be controlled by $\eta_k \theta_k * P(\cdot, z)$ since it has the same boundary data as θ_k^* on $B_{1+2^{-k-1}}$. On each of the other sides, the contribution can be controlled by

$$f_2((-x_i + x^+)/\delta^k, z/\delta^k) + f_2((x_i - x^-)/\delta^k, z/\delta^k),$$

where $x^+ = 1 + 2^{-k}$ and $x^- = -x^+$. Recall that f_2 satisfies $\nabla \cdot (z^b \nabla f_2) = 0$ and is no less than 2 on the sides x_i^+ and x_i^-. By the maximum principle,

$$\theta_k^* \le \sum_{i=1}^{n} [f_2((x_i - x^+)/\delta^k, z/\delta^k) + \ f_2((-x_i + x^-)/\delta^k, z/\delta^k)] + (\eta_k \theta_k) * P(\cdot, z).$$

We know that, for any $x \in B_{1+2^{-k}}$,

$$\sum_{i=1}^{n} [f_2((-x_i + x^+)/\delta^k, z/\delta^k) + f_2((x_i - x^-)/\delta^k, z/\delta^k)] \leq n\overline{C} e^{-\frac{\beta_0}{(2\delta)^k}} \leq \lambda 2^{-k-2}.$$

Therefore,

$$\theta_k^* \leq (\eta_k \theta_k) * P(z) + \lambda 2^{-k-2}.$$

Consequently,

$$\theta_{k+1}^* \leq (\theta_k^* - \lambda 2^{-k-1})_+ \leq ((\eta_k \theta_k) * P(z) - \lambda 2^{-k-2})_+$$

Since, for $z = \delta^{k+1}$,

$$|(\eta_k \theta_k) * P(\cdot, z)| \leq A_k \|P(\cdot, z)\|_{L^2} \leq \frac{M^{-k}}{\delta^{2\alpha(k+1)}} \|P(\cdot, 1)\|_{L^2} \leq \lambda 2^{-k-2},$$

we obtain

$$\eta_{k+1} \theta_{k+1}^* \leq 0 \quad \text{on} \quad z = \delta^{k+1}.$$

Let $k > 12n + 1$. Assuming that (A.20) is true for $k-3$, $k-2$ and $k-1$, we show

$$A_k \leq C_{0,k} \, A_{k-3}^{1+\frac{1}{n+1-2\alpha}}, \tag{A.24}$$

where

$$C_{0,k} = \frac{C \, 2^{(1+\frac{4\alpha}{n+1-2\alpha})k}}{\lambda^{\frac{2\alpha}{n+1-2\alpha}}}. \tag{A.25}$$

Since $\eta \theta_+^*$ has the same boundary condition at $z = 0$ as $(\eta \theta_+)^*$,

$$\int_0^\infty \int_{\mathbb{R}^n} z^b \, |\nabla(\eta \theta_+^*)|^2 \, dxdz \geq \int_0^\infty \int_{\mathbb{R}^n} z^b \, |\nabla(\eta \theta_+)^*|^2 \, dxdz = \int_{\mathbb{R}^n} |\Lambda^\alpha(\eta \theta_+)|^2 \, dx.$$

Letting $\eta = \eta_k(x)\phi_k(z)$ with ϕ_k supported on $[0, \delta^k]$ and integrating with respect to t over $[-1 - 2^{-k}, 0]$, we obtain

$$\int_{-1-2^{-k}}^0 \int_0^{\delta^k} \int_{\mathbb{R}^n} z^b \, |\nabla(\eta_k \theta_+^*)|^2 \, dxdzdt \geq \int_{-1-2^{-k}}^0 \int_{\mathbb{R}^n} |\Lambda^\alpha(\eta \theta_+)|^2 \, dxdt.$$

According to the definition of A_k in (A.19),

$$A_{k-3} \geq \int_{-1-2^{-k+3}}^0 \int_{\mathbb{R}^n} |\Lambda^\alpha(\eta_{k-3} \theta_{k-3})|^2 \, dxdt.$$

By the Gagliardo-Nirenberg inequality

$$A_{k-3} \geq C \|\eta_{k-3} \theta_{k-3}\|^2_{L^q([-1-2^{-k+3}, 0] \times \mathbb{R}^n)},$$

where q is defined in (A.4), namely

$$\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{n+1}.$$

It then follows from (A.23) that

$$\|\eta_{k-3}\theta^*_{k-2}\|^2_{L^q} \leq \|P(\cdot,1)\|^2_{L^1}\|\eta_{k-3}\theta_{k-3}\|^2_{L^q}$$

Therefore,

$$\begin{aligned} A_{k-3} &\geq C\|\eta_{k-3}\theta^*_{k-2}\|^2_{L^q} + C\|\eta_{k-3}\theta_{k-3}\|^2_{L^q} \\ &\geq C(\|\eta_{k-1}\theta^*_{k-1}\|^2_{L^q} + \|\eta_{k-1}\theta_{k-1}\|^2_{L^q}) \end{aligned}$$

The second inequality above follows from the simple fact that

$$\theta_{k-3} \geq \theta_{k-1} \quad \text{and} \quad \eta_{k-3} \geq \eta_{k-1}.$$

Letting $\eta = \eta_k(x)\phi_k(z)$ with ϕ_k supported on $[0, \delta^k]$ in (A.22) yields

$$A_k \leq C2^k(C_1+2)\left(\int \eta^2_{k-1}\theta^2_k\,dx + \int \eta^2_{k-1}(\theta^*_k)^2\,dxdz\right).$$

The same trick as in the proof of Theorem 2.1 can then be played here. If $\theta_k > 0$, then $\theta_{k-1} \geq 2^{-k}\lambda$ and thus

$$\chi_{\{\theta_k>0\}} \leq \left(\frac{2^k\theta_{k-1}}{\lambda}\right)^{q-2} \quad \text{and} \quad \chi_{\{\theta^*_k>0\}} \leq \left(\frac{2^k\theta^*_{k-1}}{\lambda}\right)^{q-2}.$$

Then,

$$A_k \leq \frac{C\,2^{(q-1)k}}{\lambda^{q-2}}A^{\frac{q}{2}}_{k-3} = \frac{C\,2^{(1+\frac{4\alpha}{n+1-2\alpha})k}}{\lambda^{\frac{2\alpha}{n+1-2\alpha}}}\,A^{1+\frac{1}{n+1-2\alpha}}_{k-3} = C_{0,k}\,A^{1+\frac{1}{n+1-2\alpha}}_{k-3}.$$

This completes the proof of Proposition 3.3.

Proof of Proposition 3.4. It suffices to show

$$\int_{Q_1}(\theta-1)^2_+\,dxdt + \int_{Q^*_1}(\theta^*-1)^2_+\,z^b\,dxdzdt \leq C\,\epsilon_1^\alpha.$$

From the fundamental local energy inequality (3.6), we have

$$\int_{-4}^0\int_{B^*_4}|\nabla\theta^*_+|^2\,z^b\,dxdzdt \leq C.$$

Take $\epsilon_1 << 1$ and set

$$K = \frac{4\int_{-4}^0\int_{B^*_4}|\nabla\theta^*_+|^2 z^b\,dxdzdt}{\epsilon_1}.$$

We further write

$$I_1 = \left\{ t \in [-4,0] : \int_{B_4^*} |\nabla \theta_+^*|^2(t)\, z^b\, dxdz \leq K \right\}.$$

It follows from the Chebyshev inequality that

$$|[-4,0] \setminus I_1| \leq \frac{\epsilon_1}{4}. \tag{A.26}$$

For all $t \in I_1$, the De Giorgi inequality in Lemma 3.5 gives

$$|\mathcal{A}(t)|_w |\mathcal{B}(t)|_w \leq C\, |\mathcal{C}(t)|_w^{\frac{1}{2p}}\, K^{\frac{1}{2}},$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined in (3.12) with $r = 4$. Set

$$\delta_1 = \epsilon_1^{2p(1+\frac{1}{\alpha})+2}, \quad I_2 = \{ t \in [-4,0] : |\mathcal{C}(t)|_w^{\frac{1}{2p}} \leq \epsilon_1^{1+\frac{1}{\alpha}} \}.$$

Again by the Chebyshev inequality,

$$|[-4,0] \setminus I_2| \leq \frac{|\{(x,z,t) : 0 < \theta^* < 1\}|_w}{\epsilon_1^{2p(1+\frac{1}{\alpha})}} \leq \frac{\delta_1}{\epsilon_1^{2p(1+\frac{1}{\alpha})}} \leq \epsilon_1^2 \leq \frac{\epsilon_1}{4}. \tag{A.27}$$

Now, set $I = I_1 \cap I_2$. According to (A.26) and (A.27),

$$|[-4,0] \setminus I| \leq \frac{\epsilon_1}{4} + \frac{\epsilon_1}{4} = \frac{\epsilon_1}{2}.$$

In addition, if $t \in I$ satisfying $|\mathcal{A}(t)|_w \geq \frac{1}{4}$, then

$$|\mathcal{B}(t)|_w \leq \frac{C\, |\mathcal{C}(t)|_w^{\frac{1}{2p}}\, K^{\frac{1}{2}}}{|\mathcal{A}(t)|_w} \leq 4C\epsilon_1^{\frac{1}{2}+\frac{1}{\alpha}}. \tag{A.28}$$

Therefore,

$$\int_{B_4^*} (\theta_+^*)^2(t)\, z^b\, dxdz \leq 4 \int_{\mathcal{B}\cup\mathcal{C}} z^b\, dxdz \leq 4(|\mathcal{B}|_w + |\mathcal{C}|_w) \leq 16C\, \epsilon_1^{\frac{1}{2}+\frac{1}{\alpha}}.$$

Let

$$p_1 > \frac{2(1+b)}{1-b} = \frac{2-2\alpha}{\alpha} \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}. \tag{A.29}$$

Then $1 - (\frac{1}{2} + \frac{1}{p_1}) b q_1 > 0$ and by Hölder's inequality,

$$\begin{aligned} \int_{B_4} \theta_+^2(t) dx &\leq \int_{B_4} (\max_z \theta_+^*(x,z))^2\, dx \\ &\leq 2 \int_{B_4} \int_0^4 |\theta^*|\, |\partial_z \theta^*|\, dz\, dx \end{aligned}$$

$$
\begin{aligned}
&= \; 2\int_{B_4}\int_0^4 z^{\frac{b}{p_1}}|\theta^*|\; z^{\frac{b}{2}}|\partial_z\theta^*|\, z^{-(\frac12+\frac{1}{p_1})b}\,dz\,dx \\
&\le \; 2\int_{B_4}\Big(\int_0^4 z^b|\theta^*|^{p_1}\,dz\Big)^{\frac{1}{p_1}}\Big(\int_0^4 z^b|\partial_z\theta^*|^2\,dz\Big)^{\frac12}\Big(\int_0^4 z^{-(\frac12+\frac{1}{p_1})bq_1}\,dz\Big)^{\frac{1}{q_1}}\,dx \\
&\le \; C\Big(\int_{B_4^*} z^b|\theta^*|^{p_1}\,dxdz\Big)^{\frac{1}{p_1}}\Big(\int_{B_4^*} z^b|\nabla\theta^*|^2\,dxdz\Big)^{\frac12} \\
&\le \; C\,K^{\frac12}\Big(\int_{B_4^*} z^b\,|\theta^*|^2\,dxdz\Big)^{\frac{1}{p_1}} \\
&\le \; C\,\epsilon^{(\frac12+\frac{1}{\alpha})\frac{1}{p_1}-\frac12}\equiv C\,\epsilon_1^{\nu}. \qquad \text{(A.30)}
\end{aligned}
$$

where, thanks to (A.29),

$$
\nu=\Big(\frac12+\frac1\alpha\Big)\frac{1}{p_1}-\frac12>0.
$$

The next major part proves that $|\mathcal{A}(t)|_w\ge\frac14$ for every $t\in I\cap[-1,0]$. Since

$$
|\{(x,z,t):\;\theta^*\le 0\}|_w\ge\frac{|Q_4^*|_w}{2},
$$

there exists a $t_0\le -1$ such that $|\mathcal{A}(t_0)|_w\ge\frac14$. Thus, for this t_0,

$$
\int_{B_4^*}\theta_+(t_0)^2\,dx\le C\epsilon_1^{\nu}.
$$

Using the local energy inequality (3.6), we have for all $t\ge t_0$,

$$
\int_{B_4^*}\theta_+^2(t)dx\le\int_{B_4^*}\theta_+^2(t_0)\,dx+C(t-t_0).
$$

For $t-t_0\le\delta^*=\frac{1}{64C}$, we have

$$
\int_{B_4^*}\theta_+^2(t)\,dx\le\frac{1}{64}.
$$

Since δ^* does not depend on ϵ_1, we can assume that $\epsilon_1<<\delta^*$. By

$$
\theta_+^*(x,z,t)\le\theta_+(x,t)+\int_0^z\partial_z\theta_+^*dz,
$$

we have

$$
\begin{aligned}
z^b\,(\theta_+^*)^2(x,z,t) \;&\le\; 2z^b\theta_+^2(x,t)+2z^b\Big(\int_0^z\partial_z\theta_+^*dz\Big)^2 \\
&\le\; 2z^b\theta_+^2(x,t)+2z\int_0^z z^b|\nabla\theta^*|^2dz.
\end{aligned}
$$

For $t - t_0 \le \delta^*$, $t \in I$ and $z \le \epsilon_1^2$,

$$\begin{aligned}\int_0^{\epsilon_1^2}\int_{B_4} z^b\,(\theta_+^*)^2\,dxdz &\le \frac{2}{b+1}\epsilon_1^{4-4\alpha}\int_{B_4}\theta_+^2(x,t)\,dx + \epsilon_1^4\int_0^{\epsilon_1^2}\int_{B_4} z^b|\nabla\theta^*|^2\,dxdz \\ &\le \frac{1}{64}\epsilon_1^2 + C\,\epsilon_1^3 \le \frac{1}{4}\epsilon_1^2.\end{aligned}$$

By Chebyshev inequality,

$$|\{(x,z):\ z\le\epsilon_1^2, x\in B_4,\ \theta_+^*(t)\ge 1\}|_w \le \frac{\epsilon_1^2}{4}.$$

Since $|\mathcal{C}(t)|_w \le \epsilon_1^{2p(1+\frac{1}{\alpha})}$, this gives

$$|\mathcal{A}(t)|_w \ge \epsilon_1^2 - \frac{1}{4}\epsilon_1^2 - \epsilon_1^{2p(1+\frac{1}{\alpha})} \ge \frac{1}{2}\epsilon^2.$$

Combining this bound with (A.28) leads to

$$|\mathcal{B}(t)|_w \le 4C\,\sqrt{\epsilon_1}.$$

In turn, this bound leads to

$$|\mathcal{A}(t)|_w \ge 1 - |\mathcal{B}(t)|_w - |\mathcal{C}(t)|_w \ge 1 - 4C\,\sqrt{\epsilon_1} - \epsilon_1^{2p(1+\frac{1}{\alpha})} \ge \frac{1}{4}.$$

Hence, for every $t \in [t_0, t_0+\delta^*]\cap I$, we have $|\mathcal{A}(t)|_w \ge \frac{1}{4}$. On $[t_0+\frac{\delta^*}{2}, t_0+\delta^*]$, there is $t_1 \in I$. The reason is that $|[-4,0]\setminus I| \le \frac{\epsilon}{2}$ and $|[t_0+\frac{\delta^*}{2}, t_0+\delta^*]| = \frac{\delta^*}{2} > \frac{\epsilon}{2}$.

This process allows us to construct an increasing sequence t_n, $0 \ge t_n \ge t_0 + \frac{\delta^*}{2}$ such that $|\mathcal{A}(t)|_w \ge \frac{1}{4}$ on $[t_n, t_n+\delta^*]\cap I$. Since δ^* is independent of t_n, we have

$$|\mathcal{A}(t)|_w \ge \frac{1}{4} \quad \text{for } t \in I\cap[-1,0].$$

According to (A.28), this gives

$$|\mathcal{B}(t)|_w \le 4C\epsilon_1^{\frac{1}{2}+\frac{1}{\alpha}} \le \frac{\epsilon_1}{16} \quad \text{for } t\in I\cap[-1,0].$$

Therefore,

$$\begin{aligned}|\{(x,z,t):\ \theta^*\ge 1\}|_w &= |\{(x,z,t): t\in I\cap[-1,0],\ \theta^*\ge 1\}|_w \\ &\quad + |\{(x,z,t): t\in[-1,0]\setminus I,\ \theta^*\ge 1\}|_w \\ &\le \frac{\epsilon_1}{16} + \frac{\epsilon_1}{2} \le \epsilon_1.\end{aligned}$$

Since $(\theta^*-1)_+ \le 1$,

$$\int_{Q_1^*} z^b\,(\theta^*-1)_+^2\,dxdzdt \le \epsilon_1. \tag{A.31}$$

For fixed x and t,

$$\theta(x,t) - \theta(z)^*(x,z,t) = -\int_0^z \partial_z \theta^* dz.$$

Thus,

$$\begin{aligned} z^b\,(\theta-1)_+^2 &\le 2z^b(\theta^*(z)-1)_+^2 + z^b\left(\int_0^z |\nabla\theta^*|\,dz\right)^2 \\ &\le 2z^b(\theta^*(z)-1)_+^2 + z\int_0^z z^b|\nabla\theta^*|^2\,dz. \end{aligned}$$

Taking the average in z over $[0,\sqrt{\epsilon_1}]$, we get

$$\epsilon_1^{\frac{b}{2}}\,(\theta-1)_+^2 \le \frac{2}{\sqrt{\epsilon_1}}\int_0^{\sqrt{\epsilon_1}} z^b\,(\theta^*-1)_+^2 dz + \sqrt{\epsilon_1}\int_0^{\sqrt{\epsilon_1}} z^b|\nabla\theta^*|^2\,dz.$$

Integrating with respect to $(x,t)\in B_1\times[0,1]$ and invoking (A.31) lead to

$$\int_{Q_1}(\theta-1)_+^2 dxds \le C\epsilon_1^{\alpha}.$$

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