Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation

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Abstract. We present a regularity result for weak solutions of the 2D quasi-geostrophic equation with supercritical ($\alpha < 1/2$) dissipation $(-\Delta)^\alpha$: If a Leray-Hopf weak solution is Hölder continuous $\theta \in C^\delta(\mathbb{R}^2)$ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$.

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1 Introduction

We discuss the surface 2D quasi-geostrophic (QG) equation

$$\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2, \ t > 0, \tag{1.1}$$

where $\alpha > 0$ and $\kappa \geq 0$ are parameters, and the 2D velocity field $u = (u_1, u_2)$ is determined from θ by the stream function ψ via the auxiliary relations

$$(u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \qquad (-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{1.2}$$

Using the notation $\Lambda \equiv (-\Delta)^{\frac{1}{2}}$ and $\nabla^\perp \equiv (\partial_{x_2}, -\partial_{x_1})$, the relations in (1.2) can be combined into

$$u = \nabla^\perp \Lambda^{-1} \theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \tag{1.3}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the usual Riesz transforms in $\mathbb{R}^2$. The 2D QG equation with $\kappa > 0$ and $\alpha = \frac{1}{2}$ arises in geophysical studies of strongly rotating fluids (see [5],[15] and references therein) while the inviscid QG equation ((1.1) with $\kappa = 0$) was derived to model frontogenesis in meteorology, a formation of sharp fronts between masses of hot and cold air (see [7],[10],[15]).

The problem at the center of the mathematical theory concerning the 2-D QG equation is whether or not it has a global in time smooth solution for any prescribed smooth initial data. In the subcritical case $\alpha > \frac{1}{2}$, the dissipative QG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [8],[16]). In contrast, when $\alpha \leq \frac{1}{2}$, the issue of global existence and uniqueness is more difficult and has still unanswered aspects. Recently this problem has attracted a significant amount of research ([1],[2],[3],[4],[5],[6],[9], [11],[12],[13],[14],[17],[18],[19],[20], [21],[22],[23]). In Constantin, Córdoba and Wu [6], we proved in the critical case ($\alpha = \frac{1}{2}$) the global existence and uniqueness of classical solutions corresponding to any initial data with L^∞-norm comparable to or less than the diffusion coefficient κ. In a recently posted preprint in arXiv [13], Kiselev, Nazarov and Volberg proved that smooth global solutions exist for any C^∞ periodic initial data, by removing the L^∞-smallness assumption on the initial data of [6]. Caffarelli and Vasseur (arXiv reference [1]) establish the global regularity of the Leray-Hopf type weak solutions (in $L^\infty((0,\infty); L^2) \cap L^2((0,\infty); \mathring{H}^{1/2})$) of the critical 2D QG equation with $\alpha = \frac{1}{2}$ in general $\mathbb{R}^n$.

In this paper we present a regularity result of weak solutions of the dissipative QG equation with $\alpha < \frac{1}{2}$ (the supercritical case). The result asserts that if a Leray-Hopf weak solution θ of (1.1) is in the Hölder class C^δ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$. The proof involves representing the functions in Hölder space in terms of the Littlewood-Paley decomposition and using Besov space techniques. When θ is in C^δ, it also belongs to the Besov space $\mathring{B}^{\delta(1-2/p)}_{p,\infty}$ for any $p \geq 2$. By taking p sufficiently large, we have $\theta \in C^{\delta_1} \cap \mathring{B}^{\delta_1}_{p,\infty}$ for $\delta_1 > 1 - 2\alpha$.

The idea is to show that $\theta \in C^{\delta_2} \cap \mathring{B}^{\delta_2}_{p,\infty}$ with $\delta_2 > \delta_1$. Through iteration, we establish that $\theta \in C^\gamma$ with $\gamma > 1$. Then θ becomes a classical solution.

The results of this paper can be easily extended to a more general form of the quasi-geostrophic equation in which $x \in \mathbb{R}^n$ and u is a divergence-free vector field determined by θ through a singular integral operator.

The rest of this paper is divided into two sections. Section 2 provides the definition of Besov spaces and necessary tools. Section 3 states and proves the main result.

2 Besov spaces and related tools

This section provides the definition of Besov spaces and several related tools. We start with a some notation. Denote by $\mathcal{S}(\mathbb{R}^n)$ the usual Schwarz class and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. $\widehat{f}$ denotes the Fourier transform of f, namely

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

The fractional Laplacian $(-\Delta)^\alpha$ can be defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f} = |\xi|^{2\alpha}\, \widehat{f}(\xi).$$

Let

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^n} \phi(x) x^\gamma dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}.$$

Its dual $\mathcal{S}_0'$ is given by

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ is the space of polynomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}_0'$ if their difference is a polynomial.

It is a classical result that there exists a dyadic decomposition of $\mathbb{R}^n$, namely a sequence $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jn}\Phi_0(2^j x)$$

and

$$\sum_{k=-\infty}^{\infty} \widehat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where

$$A_j = \{\xi \in \mathbb{R}^n :\, 2^{j-1} < |\xi| < 2^{j+1}\}.$$

As a consequence, for any $f \in \mathcal{S}_0'$,

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f. \tag{2.1}$$

For notational convenience, set

$$\Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \cdots. \tag{2.2}$$

Definition 2.1 *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ is defined by*

$$\mathring{B}^s_{p,q} = \left\{ f \in \mathcal{S}_0' : \ \|f\|_{\mathring{B}^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\mathring{B}^s_{p,q}} = \begin{cases} \left(\sum_j \left(2^{js} \|\Delta_j f\|_{L^p} \right)^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_j 2^{js} \|\Delta_j f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

For Δ_j defined in (2.2) and $S_j \equiv \sum_{k<j} \Delta_k$,

$$\Delta_j \Delta_k = 0 \quad \text{if } |j-k| \ge 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \, \Delta_k f) = 0 \quad \text{if } |j-k| \ge 3.$$

The following proposition lists a few simple facts that we will use in the subsequent section.

Proposition 2.2 *Assume that $s \in \mathbb{R}$ and $p, q \in [1, \infty]$.*

1) *If $1 \le q_1 \le q_2 \le \infty$, then $\mathring{B}^s_{p,q_1} \subset \mathring{B}^s_{p,q_2}$.*
2) *(**Besov embedding**) If $1 \le p_1 \le p_2 \le \infty$ and $s_1 = s_2 + n(\frac{1}{p_1} - \frac{1}{p_2})$, then $\mathring{B}^{s_1}_{p_1,q}(\mathbb{R}^n) \subset \mathring{B}^{s_2}_{p_2,q}(\mathbb{R}^n)$.*
3) *If $1 < p < \infty$, then*
$$\mathring{B}^s_{p,\min(p,2)} \subset \mathring{W}^{s,p} \subset \mathring{B}^s_{p,\max(p,2)},$$
where $\mathring{W}^{s,p}$ denotes a standard homogeneous Sobolev space.

We will need a Bernstein type inequality for fractional derivatives.

Proposition 2.3 *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

1) *If f satisfies*
$$\text{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^n : \ |\xi| \le K 2^j\},$$
for some integer j and a constant $K > 0$, then
$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \le C_1 \, 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}.$$

2) If f satisfies

$$\mathrm{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^n : \ K_1 2^j \le |\xi| \le K_2 2^j\} \tag{2.3}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^n)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \le C_2\, 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)},$$

where C_1 and C_2 are constants depending on α, p and q only.

The following proposition provides a lower bound for an integral that originates from the dissipative term in the process of L^p estimates (see [20],[4]).

Proposition 2.4 *Assume either $\alpha \ge 0$ and $p = 2$ or $0 \le \alpha \le 1$ and $2 < p < \infty$. Let j be an integer and $f \in \mathcal{S}'$. Then*

$$\int_{\mathbb{R}^n} |\Delta_j f|^{p-2} \Delta_j f\, \Lambda^{2\alpha} \Delta_j f\, dx \ge C\, 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p$$

for some constant C depending on n, α and p.

3 The main theorem and its proof

Theorem 3.1 *Let θ be a Leray-Hopf weak solution of (1.1), namely*

$$\theta \in L^\infty([0, \infty); L^2(\mathbb{R}^2)) \cap L^2([0, \infty); \mathring{H}^\alpha(\mathbb{R}^2)). \tag{3.1}$$

Let $\delta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If

$$\theta \in L^\infty([t_0, t]; C^\delta(\mathbb{R}^2)), \tag{3.2}$$

then

$$\theta \in C^\infty((t_0, t] \times \mathbb{R}^2).$$

Proof. First, we notice that (3.1) and (3.2) imply that

$$\theta \in L^\infty([t_0, t]; \mathring{B}^{\delta_1}_{p,\infty}(\mathbb{R}^2)),$$

for any $p \ge 2$ and $\delta_1 = \delta(1 - \frac{2}{p})$. In fact, for any $\tau \in [t_0, t]$,

$$\begin{aligned} \|\theta(\cdot, \tau)\|_{\mathring{B}^{\delta_1}_{p,\infty}} &= \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p} \\ &\le \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^\infty}^{1-\frac{2}{p}} \|\Delta_j \theta\|_{L^2}^{\frac{2}{p}} \\ &\le \|\theta(\cdot, \tau)\|_{C^\delta}^{1-\frac{2}{p}} \|\theta(\cdot, \tau)\|_{L^2}^{\frac{2}{p}}. \end{aligned}$$

Since $\delta > 1-2\alpha$, we have $\delta_1 > 1-2\alpha$ when

$$p > p_0 \equiv \frac{2\delta}{\delta-(1-2\alpha)}.$$

Next, we show that

$$\theta \in L^\infty([t_0,t]; \mathring{B}^{\delta_1}_{p,\infty} \cap C^{\delta_1})$$

implies

$$\theta(\cdot,t) \in \mathring{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$ to be specified. Let j be an integer. Applying Δ_j to (1.1), we get

$$\partial_t \Delta_j \theta + \kappa \Lambda^{2\alpha} \Delta_j \theta = -\Delta_j (u\cdot\nabla\theta). \tag{3.3}$$

By Bony's notion of paraproduct,

$$\begin{aligned} \Delta_j(u\cdot\nabla\theta) &= \sum_{|j-k|\le 2} \Delta_j(S_{k-1}u\cdot\nabla\Delta_k\theta) + \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta) \\ &\quad + \sum_{k\ge j-1}\sum_{|k-l|\le 1} \Delta_j(\Delta_k u\cdot\nabla\Delta_l\theta). \end{aligned} \tag{3.4}$$

Multiplying (3.3) by $p|\Delta_j\theta|^{p-2}\Delta_j\theta$, integrating with respect to x, and applying the lower bound

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2}\Delta_j f\, \Lambda^{2\alpha}\Delta_j f\, dx \ge C\, 2^{2\alpha j} \|\Delta_j f\|^p_{L^p}$$

of Proposition 2.4, we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|^p_{L^p} + C\kappa 2^{2\alpha j}\|\Delta_j\theta\|^p_{L^p} \le I_1 + I_2 + I_3, \tag{3.5}$$

where I_1, I_2 and I_3 are given by

$$\begin{aligned} I_1 &= -p \sum_{|j-k|\le 2} \int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot\Delta_j(S_{k-1}u\cdot\nabla\Delta_k\theta)\,dx, \\ I_2 &= -p \sum_{|j-k|\le 2} \int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta)\,dx, \\ I_3 &= -p \sum_{k\ge j-1} \int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot \sum_{|k-l|\le 1}\Delta_j(\Delta_k u\cdot\nabla\Delta_l\theta)\,dx. \end{aligned}$$

We first bound I_2. By Hölder's inequality

$$I_2 \le C\|\Delta_j\theta\|^{p-1}_{L^p} \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^p}\|\nabla S_{k-1}\theta\|_{L^\infty}.$$

Applying Bernstein's inequality, we obtain

$$\begin{aligned} I_2 &\le C\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|j-k|\le 2}\|\Delta_k u\|_{L^p}\sum_{m\le k-1}2^m\|\Delta_m\theta\|_{L^\infty}\\ &\le C\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|j-k|\le 2}\|\Delta_k u\|_{L^p}2^{(1-\delta_1)k}\sum_{m\le k-1}2^{(m-k)(1-\delta_1)}2^{m\delta_1}\|\Delta_m\theta\|_{L^\infty}. \end{aligned}$$

Thus, for $1-\delta_1>0$, we have

$$I_2 \le C\|\Delta_j\theta\|_{L^p}^{p-1}\|\theta\|_{C^{\delta_1}}\sum_{|j-k|\le 2}\|\Delta_k u\|_{L^p}2^{(1-\delta_1)k}.$$

We now estimate I_1. The standard idea is to decompose it into three terms: one with commutator, one that becomes zero due to the divergence-free condition and the rest. That is, we rewrite I_1 as

$$\begin{aligned} I_1 &= -p\sum_{|j-k|\le 2}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta\,dx\\ &\quad -p\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot(S_j u\cdot\nabla\Delta_j\theta)\,dx\\ &\quad -p\sum_{|j-k|\le 2}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot(S_{k-1}u-S_j u)\cdot\nabla\Delta_j\Delta_k\theta\,dx\\ &= I_{11}+I_{12}+I_{13}, \end{aligned}$$

where we have used the simple fact that $\sum_{|k-j|\le 2}\Delta_k\Delta_j\theta=\Delta_j\theta$, and the brackets [] represent the commutator, namely

$$[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta = \Delta_j(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\Delta_j\Delta_k\theta.$$

Since u is divergence free, I_{12} becomes zero. I_{12} can also be handled without resort to the divergence-free condition. In fact, integrating by parts in I_{12} yields

$$I_{12} = \int |\Delta_j\theta|^p\,\nabla\cdot S_j u\,dx \le \|\Delta_j\theta\|_{L^p}^p\|\nabla\cdot S_j u\|_{L^\infty}.$$

By Bernstein's inequality,

$$\begin{aligned} |I_{12}| &\le \|\Delta_j\theta\|_{L^p}^p\sum_{m\le j-1}2^m\|\Delta_m u\|_{L^\infty}\\ &= \|\Delta_j\theta\|_{L^p}^p2^{(1-\delta_1)j}\sum_{m\le j-1}2^{(1-\delta_1)(m-j)}\,2^{m\delta_1}\|\Delta_m u\|_{L^\infty}. \end{aligned}$$

For $1-\delta_1>0$,

$$|I_{12}| \le C\,\|\Delta_j\theta\|_{L^p}^p 2^{(1-\delta_1)j}\|u\|_{C^{\delta_1}} \le C\,\|\Delta_j\theta\|_{L^p}^{p-1}\,2^{(1-2\delta_1)j}\,\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}\|u\|_{C^{\delta_1}}.$$

We now bound I_{11} and I_{13}. By Hölder's inequality,

$$|I_{11}| \le p\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\le 2} \|[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta\|_{L^p}.$$

To bound the the commutator, we have by the definition of Δ_j

$$[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta = \int \Phi_j(x-y)\,(S_{k-1}(u)(x) - S_{k-1}(u)(y))\cdot\nabla\Delta_k\theta(y)\,dy.$$

Using the fact that $\theta \in C^{\delta_1}$ and thus

$$\|S_{k-1}(u)(x) - S_{k-1}(u)(y)\|_{L^\infty} \le \|u\|_{C^{\delta_1}}\,|x-y|^{\delta_1},$$

we obtain

$$\|[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta\|_{L^p} \le 2^{-\delta_1 j}\,\|u\|_{C^{\delta_1}}\,2^k\|\Delta_k\theta\|_{L^p}.$$

Therefore,

$$|I_{11}| \le Cp\,\|\Delta_j\theta\|_{L^p}^{p-1}\,2^{-\delta_1 j}\,\|u\|_{C^{\delta_1}} \sum_{|j-k|\le 2} 2^k\|\Delta_k\theta\|_{L^p}.$$

The estimate for I_{13} is straightforward. By Hölder's inequality,

$$\begin{aligned} |I_{13}| &\le p\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\le 2} \|S_{k-1}u - S_j u\|_{L^p}\|\nabla\Delta_j\theta\|_{L^\infty} \\ &\le Cp\,\|\Delta_j\theta\|_{L^p}^{p-1}\,2^{(1-\delta_1)j}\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^p}. \end{aligned}$$

We now bound I_3. By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned} |I_3| &\le p\|\Delta_j\theta\|_{L^p}^{p-1}\,\|\Delta_j\nabla\cdot\Big(\sum_{k\ge j-1}\sum_{|l-k|\le 1}\Delta_l u\,\Delta_k\theta\Big)\|_{L^p} \\ &\le p\|\Delta_j\theta\|_{L^p}^{p-1}\,2^j\|u\|_{C^{\delta_1}} \sum_{k\ge j-1} 2^{-\delta_1 k}\|\Delta_k\theta\|_{L^p}. \end{aligned} \tag{3.6}$$

Inserting the estimates for I_1, I_2 and I_3 in (3.5) and eliminating $p\|\Delta_j\theta\|_{L^p}^{p-1}$ from both sides, we get

$$\begin{aligned} \frac{d}{dt}\|\Delta_j\theta\|_{L^p} + C\kappa 2^{2\alpha j}\|\Delta_j\theta\|_{L^p} \le\; & C\,2^{(1-2\delta_1)j}\,\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}\|u\|_{C^{\delta_1}} \\ & + C2^{-\delta_1 j}\,\|u\|_{C^{\delta_1}} \sum_{|j-k|\le 2} 2^k\|\Delta_k\theta\|_{L^p} \\ & + C\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} \\ & + C\,2^{(1-\delta_1)j}\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^p} \\ & + C\,2^j\|u\|_{C^{\delta_1}} \sum_{k\ge j-1} 2^{-\delta_1 k}\|\Delta_k\theta\|_{L^p}. \end{aligned} \tag{3.7}$$

The terms on the right can be further bounded as follows.

$$\begin{aligned} C2^{-\delta_1 j}\,\|u\|_{C^{\delta_1}} \sum_{|j-k|\le 2} 2^k\|\Delta_k\theta\|_{L^p} &= C2^{(1-2\delta_1)j}\,\|u\|_{C^{\delta_1}} \sum_{|j-k|\le 2} 2^{\delta_1 k}\|\Delta_k\theta\|_{L^p}\, 2^{(k-j)(1-\delta_1)} \\ &\le C\, 2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}}\, \|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}, \end{aligned}$$

$$\begin{aligned} C\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} &= C2^{(1-2\delta_1)j}\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} 2^{\delta_1 k}\|\Delta_k u\|_{L^p}\, 2^{(k-j)(1-2\delta_1)} \\ &\le C\, 2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\, \|u\|_{\mathring{B}^{\delta_1}_{p,\infty}}, \end{aligned}$$

$$\begin{aligned} C\, 2^{(1-\delta_1)j}\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^p} &= C\, 2^{(1-2\delta_1)j}\,\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\le 2} 2^{\delta_1 k}\|\Delta_k u\|_{L^p}\; 2^{(j-k)\delta_1} \\ &\le C\, 2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\, \|u\|_{\mathring{B}^{\delta_1}_{p,\infty}} \end{aligned}$$

and

$$\begin{aligned} C\, 2^j\|u\|_{C^{\delta_1}} \sum_{k\ge j-1} 2^{-\delta_1 k}\|\Delta_k\theta\|_{L^p} &= C\, 2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}} \sum_{k\ge j-1} 2^{-2\delta_1(k-j)} 2^{\delta_1 k}\|\Delta_k\theta\|_{L^p} \\ &\le C\, 2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}}\, \|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}. \end{aligned}$$

We can write (3.7) in the following integral form

$$\begin{aligned} \|\Delta_j\theta(t)\|_{L^p} &\le e^{-C\kappa\, 2^{2\alpha j}(t-t_0)}\|\Delta_j\theta(t_0)\|_{L^p} \\ &\quad + C\int_{t_0}^t e^{-C\kappa 2^{2\alpha j}\,(t-s)} 2^{(1-2\delta_1)j}(\|\theta\|_{C^{\delta_1}}\, \|u\|_{\mathring{B}^{\delta_1}_{p,\infty}} + \|u\|_{C^{\delta_1}}\, \|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}})\, ds. \end{aligned}$$

Multiplying both sides by $2^{(2\alpha+2\delta_1-1)j}$ and taking the supremum with respect to j, we get

$$\begin{aligned} \|\theta(t)\|_{\mathring{B}^{2\delta_1+2\alpha-1}_{p,\infty}} &\le \sup_j\{e^{-C\kappa\, 2^{2\alpha j}(t-t_0)} 2^{(\delta_1+2\alpha-1)j}\}\, \|\theta(t_0)\|_{\mathring{B}^{\delta_1}_{p,\infty}} \\ &\quad + C\kappa^{-1}\sup_j\{(1-e^{-C\kappa\, 2^{2\alpha j}(t-t_0)})\}\, \max_{s\in[t_0,t]} \|\theta(s)\|_{\mathring{B}^{\delta_1}_{p,\infty}} \|\theta(s)\|_{C^{\delta_1}} \end{aligned}$$

Here we have used the fact that

$$\|u\|_{C^{\delta_1}} \le \|\theta\|_{C^{\delta_1}} \quad \text{and} \quad \|u\|_{\mathring{B}^{\delta_1}_{p,\infty}} \le \|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}$$

Therefore, we conclude that if

$$\theta \in L^\infty([t_0,t]; \mathring{B}^{\delta_1}_{p,\infty} \cap C^{\delta_1}),$$

then

$$\theta(\cdot,t) \in \mathring{B}^{2\delta_1+2\alpha-1}_{p,\infty}. \tag{3.8}$$

Since $\delta_1 > 1-2\alpha$, we have $2\delta_1 + 2\alpha - 1 > \delta_1$ and thus gain regularity. In addition, according to the Besov embedding of Proposition 2.2,

$$\mathring{B}^{2\delta_1+2\alpha-1}_{p,\infty} \subset \mathring{B}^{\delta_2}_{\infty,\infty},$$

where

$$\delta_2 = 2\delta_1 + 2\alpha - 1 - \frac{2}{p} = \delta_1 + \left(\delta_1 - \left(1 - 2\alpha + \frac{2}{p}\right)\right).$$

We have $\delta_2 > \delta_1$ when

$$p > p_1 \equiv \frac{2}{\delta_1 - (1-2\alpha)}.$$

Noting that

$$\mathring{B}^{\delta_2}_{\infty,\infty} \cap L^\infty = C^{\delta_2},$$

we conclude that, for $p > \max\{p_0, p_1\}$,

$$\theta(\cdot,t) \in \mathring{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$. The above process can then be iterated with δ_1 replaced by δ_2. A finite number of iterations allow us to obtain that

$$\theta(\cdot,t) \in C^\gamma$$

for some $\gamma > 1$. The regularity in the spatial variable can then be converted into regularity in time. We have thus established that θ is a classical solution to the supercritical QG equation. Higher regularity can be proved by well-known methods.

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