Smoluchowski Navier-Stokes Systems

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ABSTRACT. We discuss equilibria, dynamics and regularity for Smoluchowski equations coupled to Navier-Stokes equations.

Introduction

We consider mixtures of fluids and microscopic inclusions. The microscopic inclusions are characterized by state variables $m \in M$, where M is a compact smooth Riemannian manifold without boundary. The simplest example is that of microscopic rods with directors $m \in \mathbb{S}^2$. The microscopic inclusions evolve stochastically: they are carried by the ambient fluid, agitated by thermal noise and interact with each other. This behavior is modeled in this paper by a Smoluchowski equation for the probability distribution of particles. The inclusions add stresses to the fluid, and thus the system is coupled.

When the coupling is negligible, and the inclusions are in statistical equilibrium, then the system is governed by a single time-independent equation, derived by Onsager for colloidal suspensions of rod-like particles. This equation is variational in nature, nonlinear and nonlocal. The free energy has an entropic part and a microscopic selfinteraction part. The selfinteractions are quadratic but nonlocal and indefinite. In the particular case of a specific microscopic model given by a Maier-Saupe potential, Onsager's equations reduce to few transcendental implicit equations. These can be analyzed, and the limit of strong microscopic interactions can be shown to have a nematic character, which means in this context that the probability distributions concentrate to singular sets in M. High intensity asymptotics for Onsager's equations have been studied in ([**4**]). Qualitative properties of solutions were obtained in ([**11**], [**21**], [**22**], [**12**]).

The transition to nematic states as the intensity of the selfinteractions increases, as well as the fact that the infinite dimensional nonlinear nonlocal Onsager equation reduces to few transcendental equations with a variational structure are not isolated features, due to the fact that the Maier-Saupe interactions are particularly limited. In fact, for generic interactions, Onsager's equation can be written as a sequence of transcendental equations, and the high interaction limit is generically a delta function on M.

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The next level of complexity involves time-dependent evolution of the probability distributions, still in the absence of coupling to fluids. In this case the equation is a quadratically nonlinear nonlocal parabolic equation, a Smoluchowski equation. The dissipative nature of the equation and its relationship to the Onsager equation are both beautiful and subtle. For general interaction potentials it can be shown that the solutions exist for all time, and that the long time limit of the solutions is given by steady states, solving the corresponding Onsager equation. The detailed dynamics, even for Maier-Saupe potentials, and even in two dimensions, is surprisingly tricky to pin down. One can show that the solutions evolve to steady state, and that the destination is picked by one parameter, a critical value of the potential. The time derivative of this function is square integrable in time. Many open questions remain, even for Maier-Saupe potentials, chief among them is whether a global inertial manifold exists to parameterize long time dynamics.

The dynamical issues become very complex when a given fluid carries passively the inclusions, even if the fluid is laminar. This is because the fluid introduces a non-variational element to the dynamics, in a frame of reference moving with the fluid. Nevertheless, a generalized relative entropy calculation shows again that the solutions can be determined by the time evolution of the potential. The long time effects of shear in Doi-Smoluchowski equations have been investigated in ([**13**]), ([**29**]).

Once the added stresses produced by the inclusions are not negligible or small, the system becomes active, and then even the very basic questions of existence of solutions are nontrivial. With the exception of ([**28**]) most of the progress is rather recent. Global existence of weak solutions, with mollified velocities and linear Fokker-Planck equations with additional boundary conditions has been obt ained in ([**1**]). Global existence for shear flow Hookean dumbell models was proved in ([**16**]). The local existence of various systems has been obtained ([**10**], [**17**], [**23**]). Global existence for small data for linear Fokker-Planck coupled with Navier-Stokes equations was obtained in [**20**]. Global regularity for large data in the case of Stokes equations coupled to Smoluchowski equations was proved recently ([**2**], [**26**]). The long time asymptotics of coupled systems using entropy methods has been studied in ([**18**]). We describe results ([**6**]) concerning regularity of the system obtained by coupling Smoluchowski equations to time dependent Stokes systems and to two-dimensional Navier-Stokes equations.

1. Onsager's equation

When the stress added by the microscopic inclusions is negligible then the fluid may be considered at rest. In that case, if the inclusions are in equilibrium, the probability distribution of particles is obtained as a critical point (minimum) of a free energy functional. The free energy functional is:

$$\mathcal{E} = \int_M (f\log f - \frac{1}{2} f\mathcal{K}f)dm. \tag{1.1}$$

Here $f(m)dm$ is the probability distribution of inclusions and dm is Riemannian volume element ([**14**]). We require the normalization

$$\int_M f(m)dm = 1 \tag{1.2}$$

and $f(m) > 0$. The mean field interaction potential

$$(\mathcal{K}f)(m) = \int\limits_M K(m,q)f(q)dq \tag{1.3}$$

is given by a smooth, real valued, symmetric kernel $K(m,q) = K(q,m)$. The critical points of the functional obey Onsager's equation ([**25**])

$$\log f(m) = \int_M K(m,p)f(p)dp - \log Z \tag{1.4}$$

or, equivalently,

$$f = Z^{-1}e^{\mathcal{K}f} \tag{1.5}$$

with Z a constant, determined by the condition (1.2). A special example is provided by $M = \mathbb{S}^{n-1}$, Maier-Saupe potential

$$K(p,q) = b\left((p\cdot q)^2 - \frac{1}{3}\right) \tag{1.6}$$

with $p\cdot q$ the scalar product in $\mathbb{R}^n$ and $b \geq 0$ a parameter. The following was proved in ([**5**]):

Theorem 1.1. *Let, for any real, symmetric, traceless matrix S and positive b:*

$$(S,b) \mapsto Z(S,b) \tag{1.7}$$

$$Z(S,b) = \int\limits_{\mathbb{S}^{n-1}} e^{b(S^{ij}m_im_j)}dm, \tag{1.8}$$

$$\psi_{S,b}(m) = (Z(S,b))^{-1}e^{b(S^{ij}m_im_j)}, \tag{1.9}$$

and

$$\sigma(S)_{ij} = \int\limits_{\mathbb{S}^{n-1}} \left(m_im_j - \frac{\delta_{ij}}{n}\right)\psi_{S,b}(m)dm. \tag{1.10}$$

Then the solutions of (1.5) for $M = \mathbb{S}^2$ and K given by (1.6) are in one-to-one correspondence with the solutions of the implicit integral trancendental matrix equation

$$\sigma(S) = S. \tag{1.11}$$

$Z(S,b)$ depends only on the conjugacy class OSO^{-1}, $O \in O(n)$: if $S_1 = OSO^{-1}$ then $Z(S,b) = Z(S_1,b)$ and $\psi_{S,b}(\phi) = \psi_{S_1,b}(T\phi)$ where $T\phi$ is the angle translation associated to the rotation O, $Ox(\phi) = x(T\phi)$. The rotation invariance implies then that $\widehat{S}(S_1,b) = O\left(\widehat{S}(S,b)\right)O^{-1}$. In the special case of $n = 2$ one has ([**8**]):

Theorem 1.2. *Let $n = 2$, $M = \mathbb{S}^1$. Let $N(b)$ denote the number of distinct steady solutions of (1.5) with (1.6) modulo the $O(2)$ conjugacy. Then, if $b \leq 4$ then $N(b) = 1$. If $b > 4$ then $N(b) = 2$. The non-trivial steady state converges, as $b \to \infty$, to a delta function concentrated on the unit circle.*

In $n = 3$ the situation is more complicated. Because of rotation invariance one may consider, without loss of generality,

$$S^{ij} = \lambda_i \delta_{ij} \tag{1.12}$$

with real eigenvalues $\lambda_i \in [-\frac{1}{3}, \frac{2}{3}]$,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{1.13}$$

Let

$$v_1 = \frac{1}{2}(\lambda_1 + \lambda_2), \quad v_2 = \frac{1}{2}(\lambda_1 - \lambda_2). \tag{1.14}$$

$$y_1(p) = 1 - 3p^2 \tag{1.15}$$

and

$$y_2(p,t) = (1 - p^2)\cos t \tag{1.16}$$

defined for $(p,t) \in K = [-1,1] \times [0, 2\pi]$.

$$y = y(p,t) = (y_1(p), y_2(p,t)), \;\; v = (v_1, v_2).$$

The system can be described by a two dimensional free energy ([**4**]):

Theorem 1.3. *Let*

$$Z_2(v) = \int_K e^{bv\cdot y(p,t)} dp dt \tag{1.17}$$

$$\mathcal{F}(v) = \log(Z_2(v)) - b\left(3v_1^2 + v_2^2\right). \tag{1.18}$$

Then the solutions of (1.5) with (1.6) are in one-to-one correspondence with the critical points $v \in [-\frac{1}{3}, \frac{2}{3}] \times [0, \frac{1}{2}]$ *of* $\mathcal{F}$.

(**i**). *If* $0 < b < 1/2$ *the function* $\mathcal{F}$ *is strictly concave and has a unique critical point at* $v = 0$. *The corresponding unique solution of (1.5) is the uniform state* $f_0 = \frac{1}{4\pi}$.

(**ii**). *If* $b \geq 8$ *then* $v = 0$ *is an isolated critical point. Consequently, no bifurcations from the uniform state* f_0 *can occur in (1.5) for* $b \geq 8$.

Introducing

$$[f] = \int_{\mathbb{S}^2} f(m)\psi_{S,b}(m) dm.$$

the asymptotics as $b \to \infty$ are ([**4**]):

Isotropic: $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

$$\lim_{b\to\infty} [f] = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(p) dp$$

Oblate:

$$\lambda_1 = \frac{1}{6}, \quad \lambda_2 = \frac{1}{6}, \quad \lambda_3 = -\frac{1}{3}.$$

$$\lim_{b\to\infty} [f] = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\varphi, \sin\varphi, 0) d\varphi$$

or

Prolate:

$$\lambda_1 = \frac{2}{3}, \quad \lambda_2 = -\frac{1}{3}, \quad \lambda_3 = -\frac{1}{3}.$$

$$\lim_{b\to\infty}[f] = f(e_1).$$

The solutions are axisymmetric ([**11**], [**12**], [**20**], [**21**]).

1.1. General Onsager Equation. We consider here a smooth kernel K given by

$$K(m,p) = \sum_{j=1}^{\infty} \mu_j \phi_j(m)\phi_j(p)$$

with ϕ_j real, complete, orthonormal in $L^2(M)$,

$$\mathcal{K}\phi_j = \mu_j\phi_j.$$

We denote the Fourier coefficients of a function f as:

$$v_j(f) = \int_M f(p)\phi_j(p)dp.$$

we define a partition function

$$Z(v,b) = \int_M e^{bI(v,m)}dm$$

with

$$I(v,m) = \sum_{j=1}^{\infty} \mu_j v_j \phi_j(m)$$

Note that

$$I(v(f),m) = (\mathcal{K}f)(m).$$

We define a transform of a function $\phi(m)$ by

$$[\phi](v,b) = (Z(v,b))^{-1}\int_M \phi(m)e^{bI(v,m)}dm.$$

Note that the transform maps functions $\phi \in L^2(M)$ to functions $[\phi] : l_2(\mathbb{N}) \to R$. Using $v_j = \int_M f\phi_j$, Onsager's equation

$$f = Z^{-1}e^{b\mathcal{K}f}$$

is equivalent to the countable system of transcendental equations

$$v_j = [\phi_j](v,b). \tag{1.19}$$

This system has a free energy. Indeed,

$$b\mu_j v_j = \frac{\partial \log Z(v,b)}{\partial v_j}$$

and consequently, the Onsager solution is a critical point of the free energy

$$\mathcal{F}(v,b) = \log Z(v,b) - b\sum_{j=1}^{\infty}\mu_j\frac{v_j^2}{2}$$

Differentiating the transform: For any function $\phi(p)$, a simple calculation shows that

$$\frac{\partial[\phi]}{\partial v_i} = b\mu_i\left\{[\phi\phi_i] - [\phi][\phi_i]\right\}. \tag{1.20}$$

Therefore the Hessian $\frac{\partial^2\mathcal{F}}{\partial v_i\partial v_j}$ is

$$\mathcal{H}_{ij} = b^2\mu_i\mu_j[\xi_i\xi_j] - b\mu_i\delta_{ij}$$

with $\xi_j = \phi_j - [\phi_j]$. This implies that for b small the isotropic state $v = 0$ is energetically stable. If the function $I(v,p)$ attains uniquely its maximum on M, say at $p = p(v)$, if $I(v,p)$ is smooth, and if the maximum is nondegenerate then, using a Morse lemma, it is possible to see that

$$\lim_{b\to\infty}[\phi](v,b) = \phi(p(v))$$

for any continuous ϕ. This shows that the prolate nematic state concentrated at one configuration point is generically the high intesnity limit.

2. Smoluchowski Equations

In this section we consider dynamics of distributions $f(m,t)dm$ of inclusions that interact among themselves, diffuse thermally, but do not influence the ambient fluid, which remains at rest. The equations are

$$\partial_t f = \Delta_g f - \mathrm{div}_g(f\nabla_g(\mathcal{K}f)) \tag{2.1}$$

with $\Delta_g, \mathrm{div}_g, \nabla_g$ Laplace-Beltrami, divergence and gradient in M ([**14**]) and $\mathcal{K}f$ as in (1.3). Note that the equation can be written as

$$\partial_t f = \mathrm{div}_g(f\nabla_g(\log f - \mathcal{K}f))$$

The solutions are smooth, positive and normalized (1.2), if the initial data are smooth, positive and normalized. The free energy (1.1) is a Lyapunov functional:

$$\frac{d}{dt}\mathcal{E} = -\int_M f\left|\nabla_g(\log f - \mathcal{K}f)\right|^2 dp$$

The only possible steady solutions are solutions of Onsager's equation (1.5). If we write

$$f(m,t) = Z(t)^{-1}e^{W(m,t)}$$

then the dissipation of the free energy

$$\mathcal{D}(f) = \int_m |\nabla_g(W - \mathcal{K}f)|^2 f dm$$

is an integrable function of time because the free energy is uniformly bounded below.

We describe the situation in the case of the Maier-Saupe potential (1.6) where it is better understood. The dynamical system is dissipative: the solutions are bounded after an initial transient time. The bounds, in very strong norms, are independent of the initial data ([**7**]). The global attractor is compact, finite dimensional and is formed with solutions of Onsager's equations and their unstable manifolds. The case of $n = 2$ can be described by a sequence of ODEs

$$\frac{d}{dt}y_j = -4j^2 y_j + bjy_1\left(y_{j-1} - y_{j+1}\right) \tag{2.2}$$

where

$$f(\theta,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{j=1}^{\infty} y_j \cos(2j\theta) \tag{2.3}$$

$$y_j(t) = \int_0^{2\pi} f(\theta,t)\cos(2j\theta)d\theta, \quad y_0(t) = 1, \tag{2.4}$$

is the Fourier expansion of the function f assumed to be even. The assumption pins the solution and removes the $O(2)$ degree of freedom. The property is preserved in time. A discrete translation by multiples of $\frac{\pi}{2}$ is still an allowed symmetry. The Maier-Saupe potential is

$$\mathcal{K}f = \frac{b}{2} y_1 \cos(2\theta).$$

The steady solutions are of the form

$$g(r) = (Z(r))^{-1} e^{r\cos(2\theta)} \tag{2.5}$$

with

$$Z(r) = \int_0^{2\pi} e^{r\cos(2\theta)} d\theta \tag{2.6}$$

Let us write also

$$g(r)(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{j=1}^{\infty} g_j(r)\cos(2j\theta) \tag{2.7}$$

with

$$g_j(r) = \int_0^{2\pi} g(r)(\theta)\cos(2j\theta)d\theta, \quad g_0(r) = 1. \tag{2.8}$$

Onsager's equation is equivalent to the implicit transcendental equation

$$g_1(r)) = \frac{2r}{b}. \tag{2.9}$$

If $b \leq 4$ this equation has one solution, namely $r = 0$. If $b > 4$ there is exactly one more solution for $r > 0$ at $r = r(b)$. There is exactly one solution for $r < 0$, at $r = -r(b)$ corresponding to a translation of $\frac{\pi}{2}$ of the solution with $r > 0$.

From basic trigonometric identities we have by integration by parts in the definition of $g_j(r)$, for any r,

$$g_j(r) = \frac{r}{2j}\left(g_{j-1}(r) - g_{j+1}(r)\right) \tag{2.10}$$

Let us write the solution f of (2.1) as $f = g + z$, i.e

$$y_j(t) = z_j(t) + g_j. \tag{2.11}$$

(We drop the dependence on r in g_j; also, we make tha tacit convention that the Fourier coefficients of a function z will be denoted by the same letter, indexed by j, i.e. z_j). The the full equation (2.1) in $n = 2$ becomes

$$\frac{d}{dt} z_j = -4j^2 z_j + bjy_1(t)(z_{j-1} - z_{j+1}) + E_j(t) \tag{2.12}$$

for $j \geq 1$, with $z_0(t) = 0$ and

$$E_j(t) = 2j^2 g_j\left(\frac{b}{r} y_1(t) - 2\right) = 2j^2 g_j \frac{b}{r} z_1(t) + 2j^2 g_j\left(\frac{b}{r} g_1 - 2\right) \tag{2.13}$$

We used (2.10) in (2.2) in order to obtain (2.12, 2.13). Note that, indeed, a steady solution $z = 0$ is obtained by demanding (2.9). In view of the fact that $\log g - \mathcal{K}g$ is a constant, it follows that the full nonlinear equation (2.1) for $f = z + g$ can be written as

$$\partial_t z = \partial_\theta\left((g + z)\partial_\theta\left(\log(1 + \frac{z}{g}) - \mathcal{K}z\right)\right) \tag{2.14}$$

We observe that if $E_j(t)$ tend (appropriately) to zero in time, then we have nonlinear stability. Indeed, from the identity

$$\frac{d}{2dt}\sum_{j=1}^{\infty}\frac{1}{j}\left|z_j(t)\right|^2 = -4\sum_{j=1}^{\infty}j\left|z_j(t)\right|^2 + \sum_{j=1}^{\infty}\frac{1}{j}E_j(t)z_j(t) \tag{2.15}$$

we obtain

$$\frac{d}{dt}\|z(t)\|^2_{(-\frac{1}{2})} \le -7\|z(t)\|^2_{(\frac{1}{2})} + \|E(t)\|^2_{(-\frac{3}{2})} \tag{2.16}$$

with

$$\|z(t)\|^2_{(s)} = \sum_{j=1}^{\infty} j^{2s}\left|z_j(t)\right|^2.$$

Because $\|z\|_{(s)}$ is increasing in s we have the exponential bound

$$\|z(t)\|^2_{(-\frac{1}{2})} \le \|z(0\|^2_{(-\frac{1}{2})}e^{-7t} + \int_0^t e^{-7(t-s)}\|E(s)\|^2_{(-\frac{3}{2})}ds \tag{2.17}$$

which implies decay if, for instance $\int_0^\infty \|E(t)\|^2_{(-\frac{3}{2})}dt < \infty$. The expression (2.13) and the rapid decay of g_j as functions of j show that this happens if $\int_0^\infty z_1(t)^2dt$ is finite. In view of the fact that

$$\frac{dy_1}{dt} = (b - 4 - by_2)y_1$$

the sign of y_1 does not change during the time evolution. It was proved in ([**5**]) that if the y_1-s of two solutions converge to each other, then the solutions converge to each other. The dissipation of enenergy and the uniform lower bound on the energy can be used to show that every solution converges ultimately to a steady solution. Rates of convergence are hard to pin down, but the dissipation of the free energy gives important information about the time derivatives. We start by observing that

$$y_j(t) = \frac{1}{2j}\int_0^{2\pi}\sin(2j\theta)W_\theta f d\theta.$$

Therefore, in view of the fact that

$$\mathcal{D}(f) = \int_0^{2\pi} f\left|W_\theta - (\mathcal{K}f)_\theta\right|^2 d\theta$$

we have by using $(\mathcal{K}f)_\theta = by_1\sin(2\theta)$ and trigonometric identities,

$$y_j - \frac{1}{4j}by_1(y_{j-1} - y_{j+1}) = \frac{1}{2j}\int_0^{2\pi}\sin(2j\theta)\left(W_\theta - (\mathcal{K}f)_\theta\right)fd\theta.$$

Multiplying by j and using (2.2) we have the remarkable fact that

$$\left|\frac{dy_j}{dt}\right| \le 2j\sqrt{\mathcal{D}(f)} \tag{2.18}$$

which means that every solution has $\frac{dy_j}{dt} \in L^2(dt)$.

Let us also note that the linearization of (2.1) at $g = g(r)(\theta)$ is

$$\partial_t\zeta = L\zeta \tag{2.19}$$

with

$$L\zeta = \partial_\theta\left(g\partial_\theta\left(\frac{\zeta}{g} - \mathcal{K}\zeta\right)\right) \tag{2.20}$$

with

$$\mathcal{K}\zeta = \frac{b}{2}\zeta_1\cos(2\theta).$$

For fixed $b > 4$, $g(r)$ is an analytic function of θ, bounded away from zero, and therefore L is a sectorial operator, with discrete spectrum. The linearization has the Lyapunov functional

$$e = \frac{1}{2}\int_0^{2\pi}\frac{\zeta^2}{g}d\theta - \frac{b}{4}\zeta_1^2 \tag{2.21}$$

and this decays with a dissipation rate $\delta(\zeta)$,

$$\frac{de}{dt} = -\delta$$

with

$$\delta(\zeta) = \int_0^{2\pi} g\left|\partial_\theta\left(\frac{\zeta}{g} - \mathcal{K}\zeta\right)\right|^2 d\theta \tag{2.22}$$

Because $\frac{d\zeta_j}{dt} = \int_0^{2\pi}\cos(2j\theta)\partial_t\zeta d\theta$, it follows using the equation (2.19) and integrating by parts that

$$\left|\frac{d\zeta_j}{dt}\right| \le 2j\sqrt{\delta(\zeta)} \tag{2.23}$$

This implies linear stability if e is bounded below.

3. Generalized relative entropies

The next level of complexity of systems involve fluids that carry passively inclusions. In this situation the dynamical system has rich behavior, but in some sense it still is low dimensional.

We consider a linear operator

$$\mathcal{D}\rho = \Delta_g\rho + \mathrm{div}_g(U\rho) + V\rho \tag{3.1}$$

where

$$\Delta_g\rho = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\rho) \tag{3.2}$$

is a Laplace-Beltrami operator in an N-dimensional Riemannian manifold X without boundary,

$$U = (U_j)_{j=1,\dots N} \tag{3.3}$$

is a $(0,1)$ tensor,

$$\mathrm{div}_g(U\rho) = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}U_j\rho\right) \tag{3.4}$$

and V is a scalar potential. We will write $\mathcal{D}$ also in the form

$$\mathcal{D}\rho = \Delta_g\rho + U\cdot_g\nabla_g\rho + W\rho \tag{3.5}$$

where

$$U\cdot_g\nabla_g\rho = g^{ij}U_i\partial_j\rho \tag{3.6}$$

and

$$W = V + \mathrm{div}_g(U) \tag{3.7}$$

We associate to $\mathcal{D}$ and to the scalar positive function ρ the operator $\mathcal{D}_\rho$ defined by

$$\mathcal{D}_\rho h = \frac{1}{\rho^2\sqrt{g}}\partial_i\left(\rho^2\sqrt{g}g^{ij}\partial_j h\right) + U\cdot_g \nabla_g h. \tag{3.8}$$

The formal adjoint of the operator $\mathcal{D}$ in $L^2(X)$ (with $dv = \sqrt{g}dx$) is

$$\mathcal{D}^*\phi = \Delta_g\phi - \mathrm{div}_g(U\phi) + W\phi. \tag{3.9}$$

The following is a reformulation of a result of Michel, Mischler and Perthame:

THEOREM 3.1. *(*[**24**],[**27**]*) Let f be a solution of*

$$\partial_t f = \mathcal{D}f \tag{3.10}$$

and let $\rho > 0$ be a positive solution of the same equation,

$$\partial_t\rho = \mathcal{D}\rho. \tag{3.11}$$

Let H be a smooth convex function of one variable and let ϕ be a non-negative function obeying pointwise

$$\partial_t\phi + \mathcal{D}^*\phi = 0. \tag{3.12}$$

Then

$$\frac{d}{dt}\int_X H\left(\frac{f}{\rho}\right)\phi\rho dv \le 0 \tag{3.13}$$

Proof. The proof, a calculation, has two ingredients of interest. The first is a pointwise inequality:

LEMMA 3.2. *Let $h = H\left(\frac{f}{\rho}\right)$ with H convex, let f solve (3.10) and let $\rho > 0$ solve (3.11). Then*

$$\partial_t h - \mathcal{D}_\rho h \le 0 \tag{3.14}$$

The lemma is easily verified. In fact, the identity

$$\partial_t h - \mathcal{D}_\rho h =$$

$$-H''\left(\frac{f}{\rho}\right)\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 + \left\{\frac{1}{\rho}\left(\partial_t f - \mathcal{D}f\right) - \frac{f}{\rho^2}\left(\partial_t\rho - \mathcal{D}\rho\right)\right\}H'\left(\frac{f}{\rho}\right) \tag{3.15}$$

holds for all smooth functions f, ρ where $\rho \neq 0$. The second ingredient concerns the formal adjoint of $\mathcal{D}_\rho$:

LEMMA 3.3. *If $\mathcal{D}_\rho$ is associated to $\rho > 0$, then*

$$\mathcal{D}_\rho^*(\phi\rho) = \rho\mathcal{D}^*\phi - \phi\mathcal{D}\rho \tag{3.16}$$

holds pointwise for any smooth function ϕ.

Indeed,

$$\mathcal{D}_\rho^*\left(\rho\phi\right) = \frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}g^{ij}\rho^2\partial_i\left(\frac{\phi}{\rho}\right)\right) - \mathrm{div}_g\left(U\phi\rho\right) \tag{3.17}$$

and therefore we obtain using (3.9) and (3.5))

$$\mathcal{D}_\rho^*\left(\rho\phi\right) = \rho\left(\Delta_g\phi - \mathrm{div}_g(U\phi)\right) - \phi\left(\Delta_g\rho + U\cdot_g\nabla_g\rho\right) =$$

$$= \rho\mathcal{D}^*\phi - \phi\mathcal{D}\rho.$$

The proof of the theorem follows now from the two lemmas. Using the notation $h = H\left(\frac{f}{\rho}\right)$ as above we have

$$\frac{d}{dt}\int_X \phi\rho h dv = \int_X \{(\phi\partial_t\rho + \rho\partial_t\phi)h + \phi\rho\,[\mathcal{D}_\rho h]\}\,dv + \int_X \phi\rho(\partial_t h - \mathcal{D}_\rho h)dv$$

Using the first lemma we have

$$\frac{d}{dt}\int_X \phi\rho h dv \le \int_X \left[\phi\partial_t\rho + \rho\partial_t\phi + \mathcal{D}_\rho^*(\phi\rho)\right] h dv$$

and using the second lemma we conclude

$$\frac{d}{dt}\int_X \phi\rho h dv \le \int_X (\partial_t\phi + \mathcal{D}^*\phi)\rho h dv + \int_X (\partial_t\rho - \mathcal{D}\rho)\phi h dv \le 0.$$

Note that if $H \ge 0$ then $\phi_t + \mathcal{D}^*\phi \le 0$ is sufficient.

3.1. Extension for nonlinear equations. We consider now a positive function ρ as above and assume it solves a nonlinear forced Fokker-Planck equation

$$\partial_t\rho = \mathcal{D}\rho + R. \tag{3.18}$$

where $\mathcal{D}$ is defined as above (3.1) and (3.5) but now U and V are related to ρ, in a yet to be specified fashion. The forcing R is nonnegative, but otherwise arbitrary, and in particular it can be related to ρ. Let us consider also a function f that solves a similar nonlinear equation, with different coefficients

$$\partial_t f = \Delta_g f + \widetilde{U}\cdot_g \nabla_g f + \widetilde{W} f + F \tag{3.19}$$

with

$$\widetilde{W} = \widetilde{V} + \mathrm{div}_g(\widetilde{U}) \tag{3.20}$$

and where $\widetilde{U}$, $\widetilde{V}$ and F are coefficients related to f just as U, V, R are related to ρ. Writing

$$U' = \widetilde{U} - U \tag{3.21}$$

and

$$V' = \widetilde{V} - V, \quad W' = V' + \mathrm{div}_g(U') \tag{3.22}$$

we express the equation obeyed by f perturbatively as

$$\partial_t f - \mathcal{D}f = U'\cdot_g \nabla_g f + W' f + F. \tag{3.23}$$

Then using the same definition (3.8) for $\mathcal{D}_\rho$ and the identity (3.15) we deduce the analogue of the first lemma:

LEMMA 3.4. *Let H be a convex function of one variable, let $\rho > 0$ solve (3.18) and let f solve (3.19). Let $h = H\left(\frac{f}{\rho}\right)$. Then*

$$\partial_t h - \mathcal{D}_\rho h =$$

$$-H''\left(\frac{f}{\rho}\right)\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 + \frac{1}{\rho}\left\{\left(F - \frac{f}{\rho}R\right) + W' f + U'\cdot_g \nabla_g f\right\} H'\left(\frac{f}{\rho}\right) \tag{3.24}$$

holds.

The identity (3.16) is valid and therefore we obtain

THEOREM 3.5. *Let $\rho > 0$ solve (3.18), f solve (3.19) and $\phi \geq 0$ solve (3.12). Let H be a smooth convex function of one variable. Then*

$$\frac{d}{dt}\int_X \phi\rho H\left(\frac{f}{\rho}\right)dv = -\int_X \phi\rho H''\left(\frac{f}{\rho}\right)\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 dv +$$

$$+\int_X \phi\left\{\left(F-\frac{f}{\rho}R\right)+W'f+U'\cdot_g\nabla_g f\right\}H'\left(\frac{f}{\rho}\right)dv$$

$$+\int_X \phi R H\left(\frac{f}{\rho}\right)dv \tag{3.25}$$

holds.

Indeed, in view of (3.16),

$$\frac{d}{dt}\int_X \phi\rho h dv = \int_X \left\{\phi\rho\left(\partial_t h - \mathcal{D}_\rho h\right) + \rho h\left(\partial_t\phi + \mathcal{D}^*\phi\right) + \phi h\left(\partial_t\rho - \mathcal{D}\rho\right)\right\}dv$$

holds for any three functions f, h, ρ. We then substitute (3.18), (3.19) and (3.24) in the identity above.

Remarks.

(a) Let us note that if $V = 0$ then $\phi = 1$ solves (3.12), because $\mathcal{D}^*\phi = \Delta_g\phi - U\cdot_g \nabla_g\phi + V\phi$.

(b) In the right hand side of (3.25) one has

$$W'f + U'\cdot_g\nabla_g f = V'f + \operatorname{div}_g(U'f).$$

(c) Nowhere did we use that g^{ij} is invertible, nor the fact that g is the determinant of its inverse. The calculations work for degenerate non-negative quadratic forms, and operators associated to them.

3.2. Nonlinear Advective Smoluchowski Equations. We consider here the case of

$$\partial_t f + u(x,t)\cdot\nabla_x f = \epsilon\Delta_g f + \operatorname{div}_g(Gf) \tag{3.26}$$

with $f(x,m,t)$ a scalar function, $X = \mathbb{R}^3\times M$, M a compact smooth Riemannian manifold without boundary, $u(x,t)$ a given smooth divergence-free velocity in $\mathbb{R}^3$, $G = G_u + \nabla_g\mathcal{K}f$ where $G_u(x,m,t)$ is a smooth tensor associated to u. (The typical example is $M = \mathbb{S}^2$ and $G_u(x,m,t) = (\nabla_x u)m - [(\nabla_x u)m\cdot m]m$ where m represents in a mild abuse of language both the point on $\mathbb{S}^2$ and the normal vector to the tangent plane to $\mathbb{S}^2$ at m.) The operator $\mathcal{K}$ is linear and given by a smooth kernel $K(m,p)$

$$\mathcal{K}f(x,m,t) = \int_M K(m,p)f(x,p,t)dp \tag{3.27}$$

This corresponds to the case $U = -u + G_u + \nabla_g(\mathcal{K}f)$, $V = F = 0$. Because $V = 0$ we may take $\phi = 1$. We have therefore from (3.25) and remark (b)

$$\frac{d}{dt}\int_X \rho H\left(\frac{f}{\rho}\right)dv = -\epsilon\int_X \rho H''\left(\frac{f}{\rho}\right)\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 dv$$

$$+\int_X \operatorname{div}_g(U'f)H'\left(\frac{f}{\rho}\right)dv.$$

Now

$$U' = \nabla_g\left(\mathcal{K}(f-\rho)\right)$$

Integrating by parts we obtain

$$\frac{d}{dt}\int_X \rho H\left(\frac{f}{\rho}\right)dv \le -\frac{\epsilon}{2}\int_X \rho H''\left(\frac{f}{\rho}\right)\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 dv$$

$$+\frac{1}{2\epsilon}\int_X \rho\left(\frac{f}{\rho}\right)^2 H''\left(\frac{f}{\rho}\right)\left|\nabla_g(\mathcal{K}(f-\rho))\right|^2 dv \tag{3.28}$$

This is verified for any two solutions , with $\rho > 0$. Choosing

$$H(x) = x\log x - x + 1$$

we obtain

$$\frac{d}{dt}\int_X \rho H\left(\frac{f}{\rho}\right)dv \le -\frac{\epsilon}{2}\int_X \rho H''\left(\frac{f}{\rho}\right)\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 dv$$

$$+\frac{1}{2\epsilon}\int_X \left|\nabla_g(\mathcal{K}(f-\rho))\right|^2 f dv \tag{3.29}$$

and thus

$$\frac{d}{dt}\int_X \rho H\left(\frac{f}{\rho}\right)dv \le -\frac{\epsilon}{2}\int_X \rho\left(\frac{\rho}{f}\right)^2\left|\nabla_g\left(\frac{f}{\rho}\right)\right|^2 dv$$

$$+\frac{1}{2\epsilon}\sup_X\left|\nabla_g(\mathcal{K}(f-\rho))\right|^2 \tag{3.30}$$

Therefore two solutions will approach each other if the difference of their potentials does. More precisely, if

$$\int_0^\infty \sup_X \left|\nabla_g\mathcal{K}(f-\rho)\right|^2 dt < \infty$$

then the two solutions stay close to each other if they start out close. Note that, in the case of Maier-Saupe potentials the condition is a natural generalization of the condition obtained in the absence of an ambient fluid. In general one has

$$\sup_X \left|\nabla_g\mathcal{K}(f-\rho)\right| \le C\sup_{x\in\mathbb{R}^3}\|f(x,\cdot,t)-\rho(x,\cdot,t)\|_{L^1(M)}$$

4. Smoluchowski - Navier-Stokes Systems

The full system, in which the fluid is influenced by the particles involves Navier-Stokes Equations

$$\begin{cases} \nabla_x\cdot u = 0, \\ \partial_t u + u\cdot\nabla_x u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma \end{cases} \tag{4.1}$$

The tensor $\sigma_{ij}(x,t)$ represents the added stress in the fluid. The added stress tensor is given by an expansion

$$\sigma(x,t) = \sum_{k=1}^{\infty}\sigma^{(k)}(x,t) \tag{4.2}$$

where

$$\sigma_{ij}^{(k)}(x,t) = \int_{M\times\cdots\times M}\gamma_{ij}^{(k)}(m_1,\dots,m_k)f(x,m_1,t)f(x,m_2,t)\cdots f(x,m_k,t)dm_1\cdots dm_k \tag{4.3}$$

Often, only few terms are retained

$$\sigma(x,t) = \sigma^{(1)}(x,t) + \sigma^{(2)}(x,t) \tag{4.4}$$

where

$$\sigma^{(1)}_{ij}(x,t) = \int_M \gamma^{(1)}_{ij}(m) f(x,m,t)\,dm \tag{4.5}$$

and

$$\sigma^{(2)}_{ij}(x,t) = \int\limits_{M\times M} \gamma^{(2)}_{ij}(m,n) f(x,m,t) f(x,n,t)\,dm\,dn. \tag{4.6}$$

The ypical example, for rod-like particles is:

$$\sigma^{(1)}_{ij}(x,t) = \frac{kT}{4\pi}\int\limits_{\mathbb{S}^2}\left(m_i m_j - \frac{\delta_{ij}}{3}\right) f(x,m,t)\,dm \tag{4.7}$$

The function f is a probability density in M for each x,t

$$\int_M f(x,m,t)\,dm = 1 \tag{4.8}$$

for every x, $t \geq 0$. Expansions of this kind for σ are encountered in the polymer literature ([**9**]). In ([**2**]) it was proved that only two structure coefficients in the expansion, $\gamma^{(1)}_{ij}, \gamma^{(2)}_{ij}$ are needed in order to have energetically balanced equations, provided certain constitutive relations are imposed. The energy balance confers stability to certain time-independent solutions of the equations. In this section we are interested only in general existence results, and do not need to use special constitutive relations. We will only use the fact that the coefficients $\gamma^{(k)}_{ij}$ are smooth, time independent, x independent, f independent, and, when infinitely many coefficients are present, we will use a finiteness condition assuming that the series

$$\sum_{k=1}^{\infty} k^3 \|\gamma^{(k)}_{ij}\|_{H^{\rho_k}(M\times\cdots\times M)} \tag{4.9}$$

converges for a sequence $\rho_k > \frac{k+4d+6}{2}$. The particles interact in a mean-field fashion, through potentials that depend linearly and nonlocally on the particle density distribution f ([**25**]).

The evolution of the density f is governed by a Smoluchowski equation

$$\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Gf) = \epsilon\Delta_g f + \kappa\Delta_x f \tag{4.10}$$

The coefficient $\epsilon \geq 0$ is an inverse time scale associated to diffusion among the microscopic particles. The coefficient $\kappa \geq 0$ is a diffusivity constant, associated to the diffusion of particles across physical scales. The tensor G is made of two parts

$$G = \nabla_g U + W, \tag{4.11}$$

The $(0,1)$ tensor field W is obtained from the macroscopic gradient of velocity in a linear smooth fashion, given locally as

$$W(x,m,t) = (W_\alpha(x,m,t))_{\alpha=1,\dots,d} = \left(\sum_{i,j=1}^{n} c^{ij}_{\alpha}(m)\frac{\partial u_i}{\partial x_j}(x,t)\right)_{\alpha=1,\dots,d.} \tag{4.12}$$

The smooth coefficients $c_\alpha^{ij}(m)$ do not depend on the solution, time or x and, like the coefficients $\gamma_{ij}^{(k)}$, they are a constitutive part of the model. The typical example, for rod-like particles is:

$$W(x,m,t) = (\nabla_x u(x,t))m - ((\nabla_x u(x,t))m \cdot m)m. \tag{4.13}$$

The potential U is given by

$$U(x,m,t) = \frac{1}{\tau}\left(\mathcal{K}f\right)(x,m,t) \tag{4.14}$$

where τ is a time scale associated with the microscopic interactions. The nonlocal microscopic interaction potential

$$\left(\mathcal{K}f\right)(x,m,t) = \int_M K(m,q)f(x,q,t)dq \tag{4.15}$$

is given by an integral operator with kernel $K(m,q)$ which is a smooth, time independent, x independent, symmetric function $K : M \times M \to \mathbb{R}$. The forces applied by the particles are obtained after f is integrated along with smooth coefficients $\gamma_{ij}^{(k)}$ on M in order to produce σ. Therefore, only very weak regularity of f with respect to the microscopic variables m is sufficient to control σ. In order to take advantage of this, we consider the $L^2(M)$ selfadjoint pseudodifferential operator

$$R = (-\Delta_g + \mathbf{I})^{-\frac{s}{2}} \tag{4.16}$$

with $s > \frac{d}{2} + 1$. We will use the following properties of R:

$$[R, \nabla_x] = 0, \tag{4.17}$$

$$R\nabla_g : L^1(M) \to L^2(M) \quad \text{is bounded}, \tag{4.18}$$

$$R\nabla_g : L^2(M) \to L^\infty(M) \quad \text{is bounded}, \tag{4.19}$$

$$[\nabla_g c, R^{-1}] : H^s(M) \to L^2(M) \quad \text{is bounded}, \tag{4.20}$$

for any smooth function $c : M \to \mathbb{R}$, and

$$R : L^2(M) \to H^s(M) \quad \text{is bounded}. \tag{4.21}$$

We differentiate (4.10) with respect to x, apply R, multiply by $R\nabla_x f$ and integrate on M. Let us denote by

$$N(x,t)^2 = \int_M \left|R\nabla_x f(x,m,t)\right|^2 dm \tag{4.22}$$

the square of the L^2 norm of $R\nabla_x f$ on M. The following lemma was proved in ([**2**]) for $\kappa = 0$:

Lemma 4.1. *Let $u(x,t)$ be a smooth, divergence-free function and let f solve (4.10). There exists an absolute constant $c > 0$ (depending only on dimensions of space, the coefficients c_α^{ij} and M, but not on u, f, ϵ, κ, τ) so that*

$$\frac{1}{2}\left(\partial_t + u \cdot \nabla_x - \kappa\Delta_x\right) N^2 \le c(|\nabla_x u| + \frac{1}{\tau})N^2 + c|\nabla_x \nabla_x u|N \tag{4.23}$$

holds pointwise in (x,t).

The proof is given below in the Appendix for completeness. It works independently of the dimension n of the variables x.

From (4.2, 4.9) it follows that

$$|\sigma(x,t)| \le c \tag{4.24}$$

holds with a constant that depends only on the coefficients $\gamma_{ij}^{(k)}$. Differentiating (4.3) with respect to x it follows from (4.9) that

$$|\nabla_x \sigma(x,t)| \le cN(x,t) \tag{4.25}$$

holds with a constant c that depends only on the smooth coefficients $\gamma_{ij}^{(k)}$. Indeed,

$$\nabla_x \sigma^{(k)}(x,t) = \sum_{l=1}^{k} I_l(x,t)$$

with

$$I_l = \int_{M\times\cdots\times M} \left(R_l^{-1}\gamma^{(k)}(m_1,\dots,m_k)\right)\left(R_l\nabla_x f(x,m_l,t)\right) dm_l \Pi_{r\neq l} f(x,m_r,t) dm_r$$

where R_l means R acting on the varialble m_l. The inequality (4.25) follows from (4.9) and the positivity and normalization of f, using

$$\sup_{(m_1,\dots,\widehat{m_l},\dots m_k)} \left[\int_{m_l\in M} \left(R_l^{-1}\gamma^{(k)}(m_1\dots m_k)\right)^2 dm_l\right]^{\frac{1}{2}} \le \|\gamma^{(k)}\|_{H^\rho(M\times\cdots\times M)}$$

with $\rho > s + \frac{k-1}{2}$.

The inequalities (4.24) and (4.25) are the only information concerning the relationship between σ and f that we need for regularity results. We used the detailed form (4.2) and the finiteness condition (4.9) to deduce them, but we could just as well require them, instead of (4.2).

The case in which u is given by a steady Stokes equation in $n = 3$, $M = \mathbb{S}^2$ with σ given by a relation (4.5) was studied by Otto and Tzavaras ([**26**]) for the case of $\epsilon > 0,\ \kappa = 0$ and a linear Fokker-Planck equation ($\tau = \infty$). The case $\epsilon \ge 0,\ \kappa = 0$, $\tau \le \infty$ and general M was studied in ([**2**]).

4.1. Global Regularity for coupling to Stokes Equations in 3D. We consider a nonlinear Fokker-Planck system

$$\partial_t f + u\cdot\nabla_x f + \mathrm{div}_g(Wf) + \frac{1}{\tau}\mathrm{div}_g(f\nabla_g(\mathcal{K}f)) = \epsilon\Delta_g f \tag{4.26}$$

where W is given in (4.12) and $\mathcal{K}$ in (4.15). The velocity is related to f via the Stokes equations:

$$-\nu\Delta_x u + \nabla_x p = \mathrm{div}_x\sigma + F, \quad \nabla_x\cdot u = 0. \tag{4.27}$$

The added stresses are given by the relation (4.2, 4.3). We take periodic boundary conditions for the Stokes equations in 3D.

Theorem 4.2. *([**2**]) Let $\epsilon \ge 0$, $\tau \in (0,\infty]$. If the initial distribution $f_0(x,m)$ is smooth ($f_0 \in \mathcal{C}\cap\nabla_x f_0 \in L^q(dx;L^2(M))$ for $q > 3$) , positive and normalized $\int_M f_0(x,m)dm = 1$, then the system has global smooth solutions and*

$$\sup_{t\le T}\|\nabla_x f\|_{L^q(dx;L^2(M))} < \infty.$$

4.2. Heat of Stokes. In this section we discuss the equation (4.10) coupled with

$$\partial_t u - \nu \Delta_x u + \nabla p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0, \tag{4.28}$$

in $n = 3$ with (4.2, 4.9). We consider periodic boundary conditions, for simplicity. The coefficient $\nu > 0$ is the kinematic viscosity. From (4.28) we get

$$\nabla_x u(x,t) = e^{\nu t \Delta} \nabla_x u_0 - \int_0^t e^{\nu(t-s)\Delta} \Delta \mathbb{H} \sigma(s) ds \tag{4.29}$$

with

$$(\mathbb{H}\sigma)_{ij} = R_j \left(\delta_{il} + R_i R_l\right) R_k \sigma_{lk} \tag{4.30}$$

and R_j are Riesz transforms.

We need to control $\int_0^T \|\nabla_x u\|_{L^\infty(dx)} dt$, the stretching factor for gradients of f. We will use the fact that the linear operator

$$h(t) \mapsto \mathcal{T} h = \int_0^t e^{\nu(t-s)\Delta} \Delta \mathbb{H} h(s) ds$$

is bounded in $L^p(dt; L^q(dx))$ for $1 < p, q < \infty$ (see, for example ([**19**])). We start by estimating ∇u from (4.29) using the smoothness of the kernel of the heat equation which results in the bound (see also ([**19**]))

$$\|e^{\nu(t-s)\Delta} \Delta \mathbb{H} \sigma(s)\|_{L^\infty(dx)} \le c(\nu(t-s))^{-1} \|\sigma(s)\|_{L^\infty(dx)},$$

and obtain, in view of (4.24)

$$\|\nabla_x u\|_{L^\infty(dx)} \le C_0 + c \int_0^{t-l} (\nu(t-s))^{-1} ds + \int_{t-l}^t \|e^{\nu(t-s)\Delta} \Lambda \mathbb{H} \Lambda \sigma(s)\|_{L^\infty(dx)} ds.$$

We measure N in $L^p(dt : L^q(dx))$ for $p > \frac{2q}{q-3}$, $q > 3$. Then we get, using (4.25)

$$\|\nabla_x u(\cdot, t)\|_{L^\infty} \le C_0 + c \log\left(\frac{t}{l}\right) + c \int_{t-l}^t (\nu(t-s))^{-\frac{q+3}{2q}} \|N(\cdot, s)\|_{L^q(dx)} ds$$

and thus, by the Hölder inequality in time with p, p^* we obtain that the last term is bounded by

$$l^{\left(\frac{1}{p^*} - \frac{q+3}{2q}\right)} Y_{pq}(t)$$

with

$$Y_{pq}(t) = \left(\int_0^t \|N\|^p_{L^q(dx)} ds\right)^{\frac{1}{p}}.$$

The choice of p was so that the power of l is positive. Then, choosing l in terms of Y_{pq} we get:

$$\|\nabla_x u(\cdot, t)\|_{L^\infty(dx)} \le C_0 + C_T \log_+ \left(Y_{pq}(t)\right) \tag{4.31}$$

for $0 \le t \le T$, with C_0 depending on inital data, C_T depending on T. From (4.29) we get

$$\nabla_x \nabla_x u = e^{\nu t \Delta} \nabla_x \nabla_x u_0 + \int_0^t e^{\nu(t-s)\Delta} \Delta (\mathbb{H} \nabla_x \sigma(s)) ds$$

and therefore, using the boundedness of $\mathcal{T}$ in $L^p(dt; L^q(dx))$, we deduce

$$\|\nabla_x \nabla_x u\|_{L^p((0,t),L^q(\mathbb{T}^2))} \le C_0 + cY_{pq}(t). \tag{4.32}$$

Now we go to (4.23), multiply by N^{q-2}, integrate and multiply by $\|N\|^{p-q}_{L^q(dx)}$ and use Hölder inequalities in both space and time:

$$\frac{d}{pdt}\|N(\cdot,t)\|^p_{L^q(dx)} \le$$
$$\le c\|\nabla_x\nabla_x u(\cdot,t)\|^p_{L^q(dx)} + c\left(\|\nabla_x u(\cdot,t)\|_{L^\infty(dx)} + \tau^{-1} + 1\right)\|N(\cdot,t)\|^p_{L^q(dx)}$$

Integrating in time using (4.31), (4.32), and using the fact that $\log_+(Y_{pq}(t))$ is a non-decreasing function of time, we have

$$\frac{d}{dt}Y_{pq}(t) \le C_0 + c(1 + \tau^{-1} + \log_+ Y_{pq}(t))Y_{pq}(t)$$

which shows that $Y_{pq}(t)$ is bounded by an explicit function of initial data and T, no matter what T. This argument proves:

THEOREM 4.3. *([**6**]) Assume u_0 is divergence-free and belongs to $W^{2,q}(\mathbb{T}^3)$, $q > 3$, assume that f_0 is positive, normalized (4.8) and $f_0 \in L^\infty(dx; \mathcal{C}(M)) \cap \nabla_x f_0 \in L^q(dx; L^2(M))$. Then the solution of (4.10, 4.28, 4.4 4.3) exists for all time and*

$$\|u\|_{L^p[(0,T);W^{2,q}(dx)]} \le C_{pq}(T)$$

for any $p > \frac{2q}{q-3}$, $T > 0$. The constant $C_{pq}(T)$ depends on viscosity $\nu > 0$ and on the size of the initial data but it is bounded uniformly for $\kappa \ge 0$, $\epsilon \ge 0$.

5. Smoluchowski Navier-Stokes Systems in 2D

We consider now the equation (4.10) coupled with two dimensional Navier-Stokes equations

$$\partial_t u + u\cdot\nabla_x u - \nu\Delta_x u + \nabla_x p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0. \tag{5.1}$$

The added stresses are given by the relations (4.4), with (4.3, 4.9). We take periodic boundary conditions in both directions, with equal periods L. Because of the bound (4.24) we obtain the familiar energy inequality in the Navier-Stokes equations

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2(dx)} + \int_0^T \|\nabla_x u\|^2_{L^2(dx)}dt \le C_0 \tag{5.2}$$

where C_0 depends on initial data and the kinematic viscosity $\nu > 0$. We will use $\kappa > 0$.

LEMMA 5.1. *Let $f_0 \in L^\infty(dx; \mathcal{C}(M))$ with $\nabla_x f_0 \in L^2(dx; L^2(M))$, be a positive function obeying (4.8) a.e. Let $u(x,t)$ be a smooth divergence-free function obeying (5.2) and let f obey (4.10). Let*

$$N_0^2(x,t) = \int_M |Rf(x,m,t)|^2\, dm \tag{5.3}$$

and let $N(x,t)$ be defined as above in (4.22). There exists a constant $c > 0$, depending only on the coefficients c_α^{ij}, K and on dimension of space so that

$$\frac{1}{2}\left(\partial_t + u\cdot\nabla_x - \kappa\Delta_x\right)N_0^2 + \kappa N^2 \le c(|\nabla_x u| + \tau^{-1})N_0 \tag{5.4}$$

holds pointwise. Consequently,

$$\int_0^T \|N\|^2_{L^2(dx)}dt \le C_1(T) \tag{5.5}$$

holds with $C_1(T)$ a constant depending only on $\kappa > 0$, the spatial period L, initial data $N_0(x,0)$, τ, T and the constant C_0 in (5.2).

Proof. We apply R to (4.10), multiply by Rf and integrate on M. We obtain

$$\begin{array}{c}\frac{1}{2}\left(\partial_t + u\cdot\nabla_x - \kappa\Delta_x\right)N_0^2(x,t) + \kappa N^2(x,t) = \\ -\epsilon\int_M |\nabla_g Rf|^2\, dm - \int_M (R\mathrm{div}_g(Gf))(Rf)dm.\end{array}$$

The first term in the right-hand side is nonpositive and will be discarded. The second term is

$$\begin{array}{c}-\int_M (R\mathrm{div}_g(Gf))Rf dm = \\ -\left(\frac{\partial u_i(x,t)}{\partial x_j}\right)\int_M (R\mathrm{div}_g(c^{ij}f))(Rf) - \frac{1}{\tau}\int_M (R\mathrm{div}_g(f\nabla_g\mathcal{K}f))(Rf)dm\end{array}$$

Now, using property (4.18), Schwartz inequalities, the normalization (4.8) and the smoothness of the kernel K, we obtain easily (5.4). The inequality (5.5) follows after integration in x and time, using the fact that

$$\sup_t \|N_0\|_{L^2(dx)} \le C$$

follows from the normalization (4.8), the property (4.21), the Sobolev embedding $H^s(M) \subset L^\infty(M)$, and the finite volume of both M and $\mathbb{T}^2$. The proof of the lemma is complete.

In order to continue, we note that (4.25) and the bound (5.5) imply that the forces $\mathrm{div}_x\sigma$ in the Navier-Stokes equations (5.1) are bounded a priori in $L^2(dxdt)$. It is well known (see, for instance [**3**]) that for 2D Navier-Stokes equations with such forces, the solutions are strong, that is

$$\sup_{t\in[0,T]}\|u\|_{H^1(dx)} + \int_0^T \|u\|^2_{H^2(dx)}dt \le C_1(T) \tag{5.6}$$

holds with a constant depending on initial data and viscosity $\nu > 0$. We can explain this fact formally by using the vorticity equation

$$\partial_t\omega + u\cdot\nabla_x\omega - \nu\Delta\omega = \nabla_x^\perp \mathrm{div}_x\sigma. \tag{5.7}$$

We multiply by ω, integrate by parts, use a Schwartz inequality and the bound on the forces to deduce (5.6).

Lemma 5.2. *There exists a constant $C_2(T)$ depending on the initial data, $\kappa > 0$, $\nu > 0$, L, τ and $C_1(T)$ such that*

$$\sup_{t\in[0,T]}\|N\|_{L^2} + \int_0^T \|\Delta_x Rf\|^2_{L^2(dxdm)}dt \le C_2(T) \tag{5.8}$$

holds.

Proof. We apply R to (4.10), multiply by $-\Delta_x Rf$ and integrate $dxdm$. We obtain

$$\frac{d}{2dt}\int |\nabla_x Rf|^2\, dxdm + \kappa\int |\Delta_x Rf|^2\, dxdm \le I + II + III$$

where

$$I = \int (u\cdot\nabla_x Rf)(\Delta_x Rf)dxdm,$$

$$II = \int R\mathrm{div}_g(Wf)(\Delta_x Rf)dxdm$$

and

$$III = \frac{1}{\tau}R\mathrm{div}_g(f\nabla_g(\mathcal{K}f))(\Delta_x Rf)dxdm$$

Clearly

$$|I| \le \|u\|_{L^\infty}\|\nabla_x Rf\|_{L^2(dxdm)}\|\Delta_x Rf\|_{L^2(dxdm)}.$$

Using the fact (4.18) that $R\nabla_g : L^1(dm) \to L^2(dm)$ is bounded, the product structure of W (4.12) and the normalization (4.8) we obtain

$$|II| \le C\|\nabla_x u\|_{L^2}\|\Delta_x Rf\|_{L^2(dxdm)}$$

The last term is bounded using the smoothness of K, (4.8), the property (4.18), and the finite volume of the spatial domain:

$$|III| \le C\tau^{-1}\|\Delta_x Rf\|_{L^2 dxdm}.$$

Because

$$\int_0^T \|u\|_{L^\infty}^2 dt \le c\int_0^T \|u\|_{H^2}^2 dt \tag{5.9}$$

and because of (5.6) the inequality (5.8) follows after applying the Gronwall lemma.

LEMMA 5.3. *There exists a constant $C_3(T)$ depending on $\kappa > 0$, $\nu > 0$, τ, L, $C_2(T)$ and the initial data, such that*

$$\int_0^T \|\Delta_x\sigma\|_{L^2(dx)}^2 dt \le C_3(T) \tag{5.10}$$

holds. Consequently, there exists a constant $C_4(T)$, depending on the $C_3(T)$ above, the initial data and κ, ν, L, τ such that

$$\int_0^T \|\nabla_x u\|_{L^\infty(dx)} \le C_4(T) \tag{5.11}$$

holds.

Proof. We start by the chain rule in (4.3). In order to simplify the notation, we write $\mathbf{m}_k = (m_1, \ldots, m_k)$, $\mathbf{M}_k = M\times\cdots\times M$, $d\mu_l = dm_l\Pi_{r\ne l}f(x, m_r, t)dm_r$ and

$d\mu_{lr} = dm_l dm_r \Pi_{q\neq l,r} f(x, m_q, t)dm_q$. Then we have for $k \geq 2$,

$$\begin{array}{c}\Delta_x \sigma^{(k)}(x,t) = \\ = \sum_{l=1}^{k} \int\limits_{\mathbf{M}_k} \gamma^{(k)}(\mathbf{m}_k)\left(\Delta_x f(x,m_l,t)\right) d\mu_l(\mathbf{m}_k) + \\ \sum_{l\neq r} \int\limits_{\mathbf{M}_k} \gamma^{(k)}(\mathbf{m}_k)(\nabla_x f(x,m_l,t)) \cdot (\nabla_x f(x,m_r,t)) d\mu_{lr}(\mathbf{m}_k) = \\ \sum_{l=1}^{k} \int\limits_{\mathbf{M}_k} \left(R_l^{-1}\gamma^{(k)}(\mathbf{m}_k)\right)\left(R_l \Delta_x f(x,m_l,t)\right) d\mu_l(\mathbf{m}_k) + \\ \sum_{l\neq r} \int\limits_{\mathbf{M}_k} \left(R_l^{-2}R_r^{-2}(\gamma^{(k)}(\mathbf{m}_k)\right)(R_l^2 \nabla_x f(x,m_l,t))(R_r^2 \nabla_x f(x,m_r,t)) d\mu_{lr}(\mathbf{m}_k)\end{array}$$

where, as before R_l means R acting in the variables m_l. Using the fact that

$$\sup_{\mathbf{m}_{k-1}} \left| R_l^{-1}\gamma^{(k)}(\mathbf{m}_k) \right|$$

is an $L^2(dm_l)$ function with norm bounded by $\|\gamma^{(k)}\|_{H^{\rho_k}(\mathbf{M}_k)}$ and the fact that

$$\sup_{\mathbf{m}_{k-2}} \left| R_l^{-2} R_r^{-2} \gamma^{(k)}(\mathbf{m}_k) \right|$$

is an $L^2(M \times M)$ function with norm bounded by the same number

$$\|\gamma^{(k)}\|_{H^{\rho_k}(\mathbf{M}_k)} = C_k$$

we deduce the pointwise inequality

$$\begin{array}{c}\left|\Delta_x \sigma^{(k)}(x,t)\right|^2 \leq \\ \leq ck^2 C_k^2 \int\limits_M |\Delta_x Rf(x,m,t)|^2 \, dm + ck^4 C_k^2 \left(\int\limits_M \left|\nabla_x R^2 f(x,m,t)\right|^2 dm \right)^2 \\ \leq ck^4 C_k^2 \int\limits_M \left[|\Delta_x Rf(x,m,t)|^2 + \left|\nabla_x R^2 f(x,m,t)\right|^4 \right] dm\end{array}$$

Integrating in x we obtain

$$\begin{array}{c}\|\Delta_x \sigma^{(k)}\|_{L^2(dx)}^2 \leq k^4 C_k^2 \int |R\Delta_x f|^2 \, dxdm + \\ + k^4 C_k^2 \int\limits_M \left(\int \left|R^2 \nabla_x f\right|^4 dx \right) dm\end{array}$$

Now we use the well-known 2D inequality :

$$\|h\|_{L^4}^4 \leq C\|h\|_{L^2}^2 \|\nabla_x h\|_{L^2}^2$$

to bound the second term

$$\begin{array}{c}\int\limits_M \left(\int \left|R^2 \nabla_x f\right|^4 dx \right) dm \\ \leq \|R^2 \nabla_x f\|_{L^\infty(dm;L^2)}^2 \int \left|\Delta_x R^2 f\right|^2 dxdm\end{array}$$

Now, the fact that $R : L^2(dm) \to L^\infty(dm)$ is bounded implies the embedding inequality:

$$\|Rg\|_{L^\infty(dm;L^2)} \leq C\|g\|_{L^2(dmdx)}.$$

Indeed, if $\hat{g}(j,m)$ are the x Fourier coefficients of $g(x,m)$,

$$|R\hat{g}(j,m)|^2 \leq C \int\limits_M |\hat{g}(j,n)|^2 \, dn$$

holds m -a.e. in M, for all $j \in \mathbb{Z}$. Summing in j gives the embedding inequality. Therefore the L^4 term is bounded by

$$C\|R\nabla_x f\|^2_{L^2(dxdm)}\|R\Delta_x f\|^2_{L^2(dxdm)}$$

so that,

$$\|\Delta_x \sigma^{(k)}\|^2_{L^2(dx)} \le$$
$$k^4 C_k^2 \left[1 + \|N\|^2_{L^2(dx)}\right] \int |R\Delta_x f|^2 \, dxdm$$

Our assumption (4.9) ($\sum k^3 C_k < \infty$) implies that

$$\sum_{k=1}^{\infty} k^6 C_k^2 < \infty$$

and therefore

$$\|\Delta_x \sigma\|^2_{L^2(dx)} \le$$
$$C \left[1 + \|N\|^2_{L^2(dx)}\right] \int |R\Delta_x f|^2 \, dxdm$$

In view of (5.8) the inequality (5.10) follows by time integration.

The inequality (5.11) follows now from (5.10) and well known results about 2D Navier-Stokes equations with $L^2(dt)(H^1)$ forcing. In fact, multiplying (5.7) by $-\Delta_x \omega$ we deduce that

THEOREM 5.4. *Let $u_0 \in H^2(\mathbb{T}^2)$ be a divergence-free function and let $f_0(x,m)$ be a positive continuous function, normalized (4.8), with $\nabla_x f_0 \in L^2(dxdm)$. Then the solution of the system (4.10, 5.1, 4.4) exists for all time and satisifies*

$$\sup_{t\in[0,T]} \|u\|_{H^2(dx)} + \int_0^T \|u\|^2_{H^3} dt \le C_5(T) \tag{5.12}$$

with $C_5(T)$ a constant depending on the initial data and $\kappa > 0, \nu > 0, L, \tau,$.

6. Appendix: Proof of Lemma 1

The evolution equation of N is

$$\frac{1}{2}\left(\partial_t + u\cdot\nabla_x - \kappa\Delta_x\right) N^2 = -D_1 - D + I + II + III + IV \tag{6.1}$$

with

$$D_1 = \kappa \int_M |\nabla_x \nabla_x R f|^2 \, dm, \tag{6.2}$$

$$D = \epsilon \int_M |\nabla_g R \nabla_x f|^2 \, dm \tag{6.3}$$

$$I = -\frac{\partial u_j}{\partial x_k} \int_M \left(R\frac{\partial f}{\partial x_j}\right)\left(R\frac{\partial f}{\partial x_k}\right) dm \tag{6.4}$$

$$II = -\sum_{\alpha=1}^{2} (\nabla_x \frac{\partial u_i}{\partial x_j}) \int_M (R\mathrm{div}_g(c_\alpha^{ij} f))(\nabla_x R f) dm, \tag{6.5}$$

$$III = -\sum_{\alpha=1}^{2} \frac{\partial u_i}{\partial x_j} \int_M (R\mathrm{div}_g(c_\alpha^{ij}\nabla_x f))(R\nabla_x f)dm, \tag{6.6}$$

and

$$IV = -\frac{1}{\tau}\int_M R\mathrm{div}_g(\nabla_x\{f\nabla_g(\mathcal{K}f)\})R\nabla_x f dm. \tag{6.7}$$

Now we start estimating terms. $D_1 \geq 0$ and $D \geq 0$ will be discarded. Clearly

$$|I| \leq c|\nabla_x u|N^2. \tag{6.8}$$

In order to bound II we use (4.18) to bound

$$\|R\nabla_g(c_\alpha^{ij}f)\|_{L^2(M)} \leq c\|f\|_{L^1(M)} = c$$

so that we have

$$|II| \leq c\,|\nabla_x\nabla_x u|\,N. \tag{6.9}$$

In order to bound III we need to use the commutator carefully. We start by writing

$$R\mathrm{div}_g(c_\alpha^{ij}\nabla_x f) = R\mathrm{div}_g(c_\alpha^{ij}R^{-1}R\nabla_x f) =$$
$$\mathrm{div}_g(c_\alpha^{ij}R\nabla_x f) + \left[R\mathrm{div}_g c_\alpha^{ij}, R^{-1}\right]R\nabla_x f.$$

The second term obeys

$$\|\left[R\mathrm{div}_g c_\alpha^{ij}, R^{-1}\right]R\nabla_x f\|_{L^2(M)} \leq cN$$

because, in view of (4.20) and (4.21) one has that

$$\left[R\mathrm{div}_g c_\alpha^{ij}, R^{-1}\right] : L^2(M) \to L^2(M) \quad \text{is bounded.}$$

The first term needs to be integrated against $R\nabla_x f$ and integration by parts gives

$$\int_M (\mathrm{div}_g(c_\alpha^{ij}R\nabla_x f))R\nabla_x f dm = \frac{1}{2}\int_M (\mathrm{div}_g c_\alpha^{ij})|R\nabla_x f|^2 dm.$$

We obtain thus

$$|III| \leq c\,|\nabla_x u|\,N^2 \tag{6.10}$$

The term IV is split in two, $IV = A + B$

$$A = -\frac{1}{\tau}\int_M R\mathrm{div}_g(\{(\nabla_x f)\nabla_g(\mathcal{K}f)\})R\nabla_x f dm \tag{6.11}$$

and

$$B = -\frac{1}{\tau}\int_M R\mathrm{div}_g(\{f\nabla_g(\mathcal{K}\nabla_x f)\})R\nabla_x f dm. \tag{6.12}$$

The $(0,1)$ tensor $\Phi(x,m,t) = (\nabla_g\mathcal{K}f)(x,m,t)$ is smooth in m for fixed x, t and

$$\|\Phi(x,\cdot,t)\|_{W^{s,\infty}(M)} \leq c_s$$

holds for any s, with c_s depending only on the kernel K. We write the term A

$$\begin{aligned} A &= -\tfrac{1}{\tau}\int_M R\mathrm{div}_g(\{(\nabla_x f)\Phi\})R\nabla_x f dm \\ &= \tfrac{1}{\tau}\int_M R^{-1}(R\nabla_x f)\{\Phi\cdot\nabla_g R^2\nabla_x f)\}dm \\ &= -\tfrac{1}{2\tau}\int_M \mathrm{div}_g\{\Phi\}\,|R\nabla_x f|^2\,dm + \tfrac{1}{\tau}\int_M (R\nabla_x f)\left[R^{-1}, \Phi\nabla_g\right]R(R\nabla_x f)dm \end{aligned}$$

In view of (4.20), (4.21), the operator

$$\left[R^{-1}, \Phi\nabla_g\right] R : L^2(M) \to L^2(M)$$

is bounded with norm bounded by an a priori constant. It follows that

$$|A| \le \frac{c}{\tau} N^2(x,t)$$

holds. The term B is easier to bound, because

$$(\mathcal{K}\nabla_x f)(x,m,t) = \int_M R^{-1}K(m,n)R\nabla_x f(x,n,t)dn$$

and thus

$$\|(\nabla_g \mathcal{K}\nabla_x f)(x,\cdot,t)\|_{L^\infty(M)} \le cN(x,t).$$

Using (4.18) it follows that

$$|B| \le \frac{c}{\tau} N^2(x,t)$$

and consequently

$$|IV| \le \frac{c}{\tau} N^2(x,t). \tag{6.13}$$

Putting together the inequalities (6.8), (6.9), (6.10) and (6.13) we finished the proof of the lemma.

References

[1] J.W. Barrett, C. Schwab, E. Suli, Existence of global weak solutions for some polymeric flow models, Math. Models and Methods in Appl. Science, **15** (2005)

[2] P. Constantin, Nonlinear Fokker-Planck Navier-Stokes Systems, Commun. Math. Sci. **3** (2005), 531-544.

[3] P. Constantin, C. Foias, Navier-Stokes equations, U. Chicago Press, Chicago (1988).

[4] P. Constantin, I. Kevrekidis, E. Titi, Asymptotic states of a Smoluchowski equation, ARMA **174** (2004), 365-384.

[5] P. Constantin, I. Kevrekidis, E. S. Titi, Remarks on a Smoluchowski Equations, Discrete and Continuous Dyn. Syst, **11** (2004) 101-112.

[6] P. Constantin, Ch. Fefferman, E.S. Titi, Global Regularity for Smoluchowski equations coupled to fluids, article in preparation.

[7] P. Constantin, E.S. Titi, J. Vukadinovic, Dissipativity and Gevrey Regularity of a Smoluchowski Equation, Indiana U. Math J., **54** (2005), 949-969.

[8] P. Constantin, J. Vukadinovic, Note on the number of steady states for a 2D Smoluchowski equation, Nonlinearity **18** (2005) 441-443.

[9] M. Doi, S.F. Edwards, The Theory of Polymer Dynamics, Oxford University Press, Oxford 1988.

[10] W. E, T.J. Li, P-W. Zhang, Well-posedness for the dumbell model of polymeric fluids, Commun. Math. Phys. **248** (2004), 409-427.

[11] I. Fatkullin, V. Slastikov, A note on the Onsager model of nematic phase transitions, Commun. Math. Sciences. **3** (2005), 21-.

[12] H. Zhou, H. Wang, M.G. Forest, Q. Wang, A new proof on axisymmetric equilibria of a three-dimensional Smoluchowski equation, Nonlinearity **18** (2005) 2815-2825.

[13] G. Forest, Q. Wang, R. Zhou, The weak shear phase diagram for nematic polymers, Rheologica Acta, **43** (2004), 17-37.

[14] S. Helgason, Differential Geometry, Lie Groups and Symmetric spaces, Academic Press, London 1978.

[15] L. Hormander, The analysis of linear partial differential operators, vol. 3. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[16] B. Jourdain, T. Lelievre, C. Le Bris, Numerical analysis of micro-macro simulations of polymeric flows: a simple case, Math. Models in Appl. Science, **12** (2002), 1205-1243.

[17] B. Jourdain, T. Lelievre, C. Le Bris, Existence of solutions for a micro-macro model of polymeric fluid: the FENE model.
[18] B. Jourdain, C. Le Bris, T. Lelievre, F. Otto, Long time asymptotics of a multisclae model for polymeric fluid flows, preprint, March 2005.
[19] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapmann and Hall/CRC, Research Notes in Mathematics 431, CRC, Boca Raton, 2002.
[20] F. Lin, C. Liu, P. Zhang, On a micro-macro model of polymeric fluids near equilibrium, preprint (2005).
[21] C. Luo, H. Zhang, P-W. Zhang, The structure of the equilibrium solutions of one dimensional Doi equation, Nonlinearity **18** (2005), 379-389.
[22] H. Liu, H. Zhang, P-W Zhang, Axial symmetry and classification of stationary solutions of Doi-Onsager Equation on the sphere with Maier-Saupe potential. Comm. Math. Sci. **3** (2005), 201-218.
[23] T.Li, H. Zhang, P-W. Zhang, Local existence for the dumbell model of polymeric fluids, Commun. PDE 29 (2004), 903-923.
[24] P. Michel, S. Mischler, B. Perthame, General entropy equations for structured population models and scattering, C.R. Acad. Sci. Paris, Ser. I **338** (2004), 697-702.
[25] L. Onsager, The effects of shape on the interaction of colloidal particles, Ann. N.Y. Acad. Sci **51** (1949), 627-659.
[26] F. Otto, A. Tzavaras, Continuity of velocity gradients in suspensions of rod-like molecules, SFB Preprint **147** (2004).
[27] B. Perthame, Talk at Northwestern University, June 2005.
[28] M. Renardy, An existence theorem for model equations resulting from kinetic theories of polymer solutions, SIAM J. Math. Analysis **23** (1991), 313-327.
[29] A. Zarnescu, Thesis, the University of Chicago, in preparation.

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