# OPTIMAL TRANSPORTATION ON THE HEMISPHERE

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Abstract. In this paper, we study the optimal transportation on the hemisphere, with the cost function $c(x,y)=\frac{1}{2}d^2(x,y)$, where $d$ is the Riemannian distance of the round sphere. The potential function satisfies a Monge-Ampère type equation with natural boundary condition. We obtain the *a priori* oblique estimate without using any uniform convexity of domains, and in particular for two dimensional case, we obtain the boundary $C^2$ estimate. Our proof does not depend on the smoothness of densities, which is new even for standard Monge-Ampère equations and optimal transportation on Euclidean spaces.

## 1. Introduction

Let $\mathbb{S}^n$ be the $n$-dimensional unit sphere equipped with the standard round metric $g$ and geodesic distance $d$. Denote the northern hemisphere to be $\mathbb{S}^n_+ := \mathbb{S}^n \cap \{x_{n+1} \geq 0\}$. Let $c(x,y)=\frac{1}{2}d^2(x,y)$ be the cost function, $f,g$ be two positive densities on $\mathbb{S}^n_+$, bounded from above and below, and satisfy

$$\int_{\mathbb{S}^n_+} f = \int_{\mathbb{S}^n_+} g. \tag{1.1}$$

In this paper, we study the optimal transportation from $(\mathbb{S}^n_+, f)$ to $(\mathbb{S}^n_+, g)$ and obtain the *a priori* oblique and boundary estimates without using uniform $c$-convexity of domain and smoothness of densities, which are, however, key ingredients in the standard cases.

Let's briefly recall that in optimal transportation $(\Omega, f) \to (\Omega^*, g)$, $f,g>0$ satisfying $\int_\Omega f = \int_{\Omega^*} g$, where $\Omega, \Omega^*$ are the initial and target domains, the optimal mapping $T_u$ is determined by the potential function $u$,

$$Du(x) = -D_x c(x, T_u(x)) \tag{1.2}$$

for a.e. $x \in \Omega$, where the cost function $c$ satisfies conditions (A0)–(A1) in Section 2, and the functions $u,v$ are called potential functions as $(u,v)$ is a maximiser of

$$\sup\{I(\phi,\psi) : (\phi,\psi) \in K\},$$

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where

$$I(\phi,\psi)=\int_\Omega f(x)\phi(x)+\int_{\Omega^*} g(y)\psi(y),$$
$$K=\{(\phi,\psi)\in C(\Omega)\times C(\Omega^*)\,:\,\phi(x)+\psi(y)\geq -c(x,y)\}.$$

When $u$ is smooth, it solves a Monge-Ampère type equation

$$\det\left[D^2u+D_x^2c\right]=|\det D_{xy}^2c|\frac{f}{g\circ T_u}\quad\text{in }\Omega,\tag{1.3}$$

with a natural boundary condition

$$T_u(\Omega)=\Omega^*.\tag{1.4}$$

In the Euclidean case, when $\Omega,\Omega^*$ are two bounded domains in $\mathbb{R}^n$, the global regularity of (1.3)–(1.4) is obtained in [13] by assuming that $\Omega,\Omega^*$ are uniformly $c$-convex with respect to each other and the densities $f,g$ are correspondingly smooth. The uniform $c$-convexity of $\Omega$ with respect to $\Omega^*$ means that the image $c_y(\cdot,y)(\Omega)$ is uniformly convex in the usual sense for each $y\in\Omega^*$, while analogously $\Omega^*$ is uniformly $c$-convex with respect to $\Omega$, if the image $c_x(x,\cdot)(\Omega^*)$ is uniformly convex for each $x$ in $\Omega$.

In the special case $c(x,y)=-x\cdot y$, the $c$-convexity is equivalent to the usual convexity, and (1.3) reduces to the standard Monge-Ampère equation with the boundary condition of prescribing the image of gradient mapping,

$$\begin{cases}\det D^2u=h(x,Du) & \text{in }\Omega,\\ Du(\Omega)=\Omega^*.\end{cases}\tag{1.5}$$

The boundary problem (1.5) has been extensively studied by many mathematicians, for example, see [2, 3, 15] and references therein. A crucial assumption in those work is the uniform convexity of domains $\Omega$ and $\Omega^*$.

Note that when $c(x,y)=-x\cdot y$, (1.2) implies that $T_u=Du$ for a convex potential $u$. In our case $c=d^2/2$, where $d$ is the geodesic distance on $(\mathbb{S}^n,g)$. From the result of [12], the optimal mapping can be expressed by the exponential mapping

$$T_u(x)=\exp_x(\nabla_g u(x)),$$

where $\nabla_g$ denotes the gradient with respect to the round metric $g$ on $\mathbb{S}^n$, and $u$ is a $c$-convex potential. For $\Omega,\Omega^*\subset\mathbb{S}^n$, $\Omega$ is uniformly $c$-convex with respect to $\Omega^*$ is equivalent to $\exp_y^{-1}(\Omega)$ is uniformly convex in $\mathbb{R}^n$ for each $y\in\Omega^*$, while analogously $\Omega^*$ is uniformly $c$-convex with respect to $\Omega$ if $\exp_x^{-1}(\Omega^*)$ is uniformly convex for each $x$ in $\Omega$. However, this is not the case when $\Omega=\Omega^*=\mathbb{S}^n_+$, as pick $y_0\in\mathbb{S}^n_+\cap\{x_n=0\}$,

$$\exp_{y_0}^{-1}(\Omega)=\{z\in\mathbb{R}^n\,:\,z\in B_\pi(0)\cap\{z_n\geq 0\}\text{ or }|z|=\pi\},$$

which is even not connected. One can see that for $\Omega=\Omega^*=\mathbb{S}^n_\varepsilon:=\mathbb{S}^n\cap\{x_{n+1}\geq\varepsilon\}$, they are uniformly $c$-convex to each other for any positive constant $\varepsilon>0$. Therefore, the hemisphere $\mathbb{S}^n_+$ is a critical case in the above sense.

From here on, we use $X = (X_1, \cdots, X_n, X_{n+1})$ to represent a point on $\mathbb{S}^n_+$, while $x = (x_1, \cdots, x_n)$ represents a point in $\mathbb{R}^n$. We use the stereographic projection from the south pole to transform $\mathbb{S}^n_+$ into $\Pi(\mathbb{S}^n_+) = B_1(0) \subset \mathbb{R}^n$ by $x = \Pi(X)$ and

$$X = \Pi^{-1}(x) = \left( \frac{2x_1}{1+|x|^2}, \cdots, \frac{2x_n}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right), \tag{1.6}$$

where $x = (x_1, \cdots, x_n) \in B_1(0)$.

Utilising the ambient Euclidean geometry of $\mathbb{R}^{n+1}$, it is an elementary calculation that

$$d(X, Y) = \arccos(X \cdot Y), \tag{1.7}$$

for $X, Y \in \mathbb{S}^n$ and $d$ the geodesic distance. Under the stereographic projection $\Pi$, one has the optimal transportation from $\Omega = B_1(0)$ to $\Omega^* = B_1(0)$ with the cost function

$$\begin{aligned} \bar{c}(x, y) &= c(\Pi^{-1}(x), \Pi^{-1}(y)) \\ &= \frac{1}{2} \left( \arccos \left( \frac{4(x \cdot y)}{(1+|x|^2)(1+|y|^2)} + \frac{(1-|x|^2)(1-|y|^2)}{(1+|x|^2)(1+|y|^2)} \right) \right)^2, \end{aligned} \tag{1.8}$$

for $x, y \in B_1(0)$. Correspondingly, the potential $u$ and the optimal mapping $T$ will become

$$\bar{u}(x) = u \circ \Pi^{-1}(x) \quad \text{and} \quad \bar{T}(x) = \Pi \circ T \circ \Pi^{-1}(x), \quad \text{for } x \in B_1(0). \tag{1.9}$$

The convexity with respect to $\bar{c}$ is inherited from that of $c$, namely $\bar{u}$ is $\bar{c}$-convex if and only if $u$ is $c$-convex; a domain $E \subset B_1(0)$ is $\bar{c}$-convex with respect to $E^* \subset B_1(0)$ if and only if $\Pi^{-1}(E)$ is $c$-convex with respect to $\Pi^{-1}(E^*)$.

As pointed out, due to the lack of convexity, the standard techniques in dealing with (1.5) are not applicable to (1.3)–(1.4) with the cost function given by (1.8). In this paper, we use an elementary and non-trivial method to establish the a priori oblique and boundary estimates. Our main result is the following:

**Theorem 1.1.** *Assume that the density functions $f, g$ satisfies* (1.1) *and there exists a constant $\lambda > 0$ such that $\lambda^{-1} < f, g < \lambda$. Then we have the a priori estimate*

$$\sum_{i,k=1}^{n} -y_i c^{i,k} x_k \geq c_0, \tag{1.10}$$

*for all $x \in \Omega$ and $y = T(x)$, where $c_0 > 0$ is a constant depending only on $\lambda$.*

*Moreover, when $n = 2$, we have the a priori boundary estimate*

$$\sup_{\partial\Omega} |D^2 u| \leq C, \tag{1.11}$$

*for some constant $C > 0$ depending only on $\lambda$.*

The restriction $n = 2$ is unsatisfactory but it is needed due to a technical reason. On the other hand, to derive the boundary estimate (1.11), usually one need a differentiability of the inhomogeneous term, but here we only need the boundedness. This paper is organised as follows: In Section 2, we introduce some preliminary notations and results. In Section 3, we prove the oblique estimate (1.10), where our approach makes no use of uniform convexity nor duality argument. In Section 4, we prove the boundary $C^2$ estimate (1.11) in the two-dimensional case.

## 2. Preliminaries

First, let's recall some basic notions of optimal transportation on a Riemannian manifold $\mathcal{M}$ with a distance-squared cost function $c(x, y) = \frac{1}{2}d^2(x, y)$.

*Definition* 2.1. Let $\mathcal{M}$ be a compact Riemannian manifold and $d(\cdot, \cdot)$ be its Riemannian distance function. The $c$-transform $u^c$ of a function $u : \mathcal{M} \to \mathbb{R}$ is defined for all $x \in \mathcal{M}$ by

$$u^c(x) = \sup_{y \in \mathcal{M}} \left\{ -\frac{d^2(x, y)}{2} - u(y) \right\}. \tag{2.1}$$

The function $u$ is said to be $c$-convex if $(u^c)^c = u$.

For a $c$-convex function $u$, for any point $x_0 \in \mathcal{M}$, by the above definition there exists $y \in \mathcal{M}$ such that

$$u(x) \geq -\frac{d^2(x, y)}{2} - u^c(y),$$

for all $x \in \mathcal{M}$ with equality holds at $x = x_0$. The function $\varphi(\cdot) = -\frac{d^2(\cdot, y)}{2} - u^c(y)$ is called a $c$-support of $u$ at $x \in \mathcal{M}$. A function $u$ is $c$-convex is equivalent to that for any point $x \in \mathcal{M}$ there exists a $c$-support of $u$ at $x$. Naturally, the potential function $u$ in (1.2) of optimal transportation is $c$-convex.

*Definition* 2.2. Let $u$ be a $c$-convex function, the $c$-normal mapping $T_u$ is defined by

$$T_u(x_0) = \{y \in \mathcal{M} \, : \, u(x) \geq \frac{d^2(x_0, y)}{2} - \frac{d^2(x, y)}{2} + u(x_0), \quad \forall x \in \mathcal{M}\}. \tag{2.2}$$

Note that by duality between $u$ and $u^c$, if $y \in T_u(x_0)$, we have $u^c(y) = -\frac{d^2(x_0, y)}{2} - u(x_0)$, and

$$-D_x c(x_0, y) \in \partial u(x_0),$$

where $\partial u$ is the subgradient of $u$. If $u$ is $C^1$ smooth at $x_0$, then $T_u(x_0)$ is single valued, and is exactly the mapping given by (1.2). In general, $T_u(x)$ is single valued for almost all $x \in \mathcal{M}$ as $u$ is $c$-convex and thus twice differentiable almost everywhere by the well-known theorem of Aleksandrov. If $c(x, y) = -x \cdot y$ and $\mathcal{M}$ is Euclidean, then $T_u$ is the normal mapping for convex functions.

We may extend the $c$-normal mapping to boundary points. Let $x_0 \in \partial\mathcal{M}$ be a boundary point, we denote $T_u(x_0) = \{y \in \mathcal{M} \, : \, y = \lim_{k\to\infty} y_k\}$, where $y_k \in T_u(x_k)$ and $\{x_k\}$ is a sequence of interior points of $\mathcal{M}$ such that $x_k \to x_0$.

Let $\mathcal{U}$ be a subset of $\mathcal{M} \times \mathcal{M}$, which for simplicity we assume compact. Denote $\pi_1, \pi_2$ the usual canonical projections. For any $x \in \pi_1(\mathcal{U})$, we denote by $\mathcal{U}_x$ the set $\mathcal{U} \cap \pi_1^{-1}(x)$. Similarly, we can define $\mathcal{U}_y = \mathcal{U} \cap \pi_2^{-1}(y)$, for any $y \in \pi_2(\mathcal{U})$. Let us introduce the following conditions:

(A0) The cost function $c$ belongs to $C^4(\mathcal{U})$.
(A1) For any $(x, y) \in \mathcal{U}$, $(p, q) \in D_x c(\mathcal{U}) \times D_y c(\mathcal{U})$, there exists unique $Y = Y(x, p), X = X(y, q)$, such that $-D_x c(x, Y) = p, -D_y c(X, y) = q$.
(A2) For any $(x, y) \in \mathcal{U}$, $\det D^2_{x,y} c \neq 0$.

We recall the definition of $c$-convexity for domains (see [11]):

*Definition* 2.3. Let $y \in \pi_2(\mathcal{U})$, a subset $\Omega$ of $\pi_1(\mathcal{U}_y)$ is $c$-convex (resp. uniformly $c$-convex) with respect to $y$ if the set $\{-D_y c(x, y), x \in \Omega\}$ is a convex (resp. uniformly convex) set of $T_y\mathcal{M}$. Whenever $\Omega \times \Omega^* \subset \mathcal{U}$, $\Omega$ is $c$-convex with respect to $\Omega^*$ if it is $c$-convex with respect to every $y \in \Omega^*$.

Similarly we can define $c^*$-convexity of domains by exchanging $x$ and $y$. Without arising any confusion, for simplicity we abuse the notation $c$-convexity to also mean $c^*$-convexity by drop the sup-script. When the cost function $c = d^2/2$, we have $D_x c(x, y) = \exp_x^{-1}(y)$. Therefore, $\Omega^* \subset \mathcal{M}$ is $c$-convex (resp. uniformly $c$-convex) with repsect to $x$ is equivalent to $\exp_x^{-1}(\Omega^*)$ is convex (resp. uniformly convex).

However, conditions (A0)–(A2) are not satisfied on $\mathbb{S}^n_+ \times \mathbb{S}^n_+$ due to the singularities on antipodal points. The next lemma shows that for each point $x \in \partial\mathbb{S}^n_+$, its image under the $c$-normal mapping of a $c$-convex function stays uniformly away from its antipodal point $\hat{x}$. Note that the singularity only occurs on the boundary $\partial\mathbb{S}^n_+ = \mathbb{S}^n_+ \cap \{x_{n+1} = 0\}$ and the antipodal point $\hat{x} = -x$ for $x \in \partial\mathbb{S}^n_+$.

**Lemma 2.1.** *Let $T = T_u$ be the $c$-normal mapping of a $c$-convex potential $u$ such that $T_\# f = g$. Assume that the densities $f, g$ have positive lower and upper bounds. Then there exists a constant $\delta > 0$, such that*

$$d(T(x), \hat{x}) \geq \delta, \tag{2.3}$$

*for any $x \in \partial\mathbb{S}^n_+$.*

*Proof.* The proof essentially follows from [4], where the measures and transport maps are defined on the whole sphere $\mathbb{S}^n$ without boundary. We include it here for completeness. Let $x_0 \in \partial\mathbb{S}^n_+$ be a boundary point, and $\hat{x}_0$ be its antipodal point. We claim that: for almost all $x \in \mathbb{S}^n_+$, $x \neq x_0$,

$$d(T(x), \hat{x}_0) \leq 2\pi \frac{d(T(x_0), \hat{x}_0)}{d(x, x_0)}. \tag{2.4}$$

Then denote $D = \{x \in \mathbb{S}^n_+ : d(x, x_0) \geq \pi/2\}$, a subset of $\mathbb{S}^n_+$. From the preceding inequality, we infer that almost all $x \in D$ are sent by $T$ into $B_\varepsilon(\hat{x}_0)$, where

$$\varepsilon = 2\pi \frac{d(T(x_0), \hat{x}_0)}{\pi/2} = 4d(T(x_0), \hat{x}_0). \tag{2.5}$$

By the measure preserving condition, we then have

$$\int_{B_\varepsilon(\hat{x}_0)} g \geq \int_D f,$$

and thus,

$$d^n(T(x_0), \hat{x}_0) \sup g \geq C \inf f,$$

where $C$ is a constant only depending on $n$. Since $x_0$ is arbitrary, we conclude that there is a constant $\delta > 0$ depending on the lower bound of $f$ and upper bound of $g$ such that (2.3) holds.

Therefore, it suffices to prove the claim (2.4). Fix $x_0 \in \partial\mathbb{S}^n_+$ and another point $x \in \mathbb{S}^n_+$, define the function

$$F(p) = \frac{1}{2}d^2(p, x) - \frac{1}{2}d^2(p, x_0),$$

for $p \in \mathbb{S}^n_+$. The function $F$ satisfies that [4]

$$\mathrm{grad}_p F(p) = \exp_p^{-1}(x_0) - \exp_p^{-1}(x), \tag{2.6}$$

where the gradient is defined everywhere except $\hat{x}_0$. Since our manifold is $\mathbb{S}^n_+$, by comparison with the Euclidean case,

$$|\mathrm{grad}_p F(p)| \geq d(x_0, x'). \tag{2.7}$$

Let us consider on $\mathbb{S}^n_+ \setminus \{\hat{x}_0\}$ the normalized steepest descent equation (with arc-length parameter $s$):

$$\dot{p}(s) = -\frac{\mathrm{grad}_p F[p(s)]}{|\mathrm{grad}_p F[p(s)]|}.$$

From (2.7), any solution $p(s)$ satisfies

$$\frac{d}{ds}F[p(s)] = -|\mathrm{grad}_p F[p(s)]| \leq -d(x_0, x).$$

It is easy to see that for fixed $(x_0, x)$, the function $F(p)$ attains its infimum at $p = \hat{x}_0$. Therefore, starting from $p(0) = p_0$, for some $p_0 \neq \hat{x}_0$, the minimum of $p \mapsto F(p)$ is reached by flowing along an integral curve of length $L \geq d(p_0, \hat{x}_0)$. Writing

$$F(p_0) - F(\hat{x}_0) = -\int_0^L \frac{d}{ds}F[p(s)]ds,$$

we then have

$$\begin{aligned} F(p_0) - F(\hat{x}_0) &\geq \int_0^L d(x_0, x) \\ &\geq d(x_0, x)d(p_0, \hat{x}_0). \end{aligned}$$

It implies that for $x \neq x_0$ and for all $p \in \mathbb{S}^n_+$,

$$d(p, \hat{x}_0) \leq \frac{F(p) - F(\hat{x}_0)}{d(x_0, x)}. \tag{2.8}$$

Next, we show that (2.4) follows from (2.8). We know that the mapping $T$ is a.e. $c$-monotone, see for example [1, 4], which implies that for almost all $x_0 \in \partial\mathbb{S}^n_+$ and $x \in \mathbb{S}^n$,

$$\frac{1}{2}d^2(x_0, T(x_0)) + \frac{1}{2}d^2(x, T(x)) \leq \frac{1}{2}d^2(x_0, T(x)) + \frac{1}{2}d^2(x, T(x_0)).$$

From the definition of function $F$, we get

$$F[T(x)] \leq F[T(x_0)].$$

Now, setting $p = T(x)$ in (2.8), we have

$$\begin{aligned} d(\hat{x}_0, T(x)) &\leq \frac{F(T(x)) - F(\hat{x}_0)}{d(x_0, x)} \\ &\leq \frac{F(T(x_0)) - F(\hat{x}_0)}{d(x_0, x)}, \end{aligned}$$

hence, since $p \mapsto F(p)$ is $2\pi$-Lipschitz, we obtain (2.4), namely

$$d(\hat{x}_0, T(x)) \leq 2\pi \frac{d(T(x_0), x_0)}{d(x_0, x)},$$

for almost all $x \in \mathbb{S}^n_+, x \neq x_0$. The proof is finished. □

We now recall the definition of the cost-sectional curvature [9], and introduce an additional condition on the cost function $c$, which is crucial for regularity estimates [11]:

*Definition* 2.4. Assume that the cost function $c$ satisfies (A0)–(A2) in $\mathcal{U} \subset \mathcal{M} \times \mathcal{M}$. For every $(x_0, y_0) \in \mathcal{U}$, define on $T_{x_0}\mathcal{M} \times T_{x_0}\mathcal{M}$ a real-valued map

$$\mathfrak{S}_c(x_0, y_0)(\xi, \eta) = D^4_{p_\eta p_\eta x_\xi x_\xi} \left[(x, p) \to -c(x, \exp_{x_0}(p))\right]\Big|_{x_0, p_0 = -\nabla_x c(x_0, y_0)}. \tag{2.9}$$

When $\xi, \eta$ are unit orthogonal vectors (with respect to the metric $g$ at $x_0$), $\mathfrak{S}_c(x_0, y_0)(\xi, \eta)$ defines the cost-sectional curvature from $x_0$ to $y_0$ in directions $(\xi, \eta)$.

In fact, definition (2.9) is equivalent to the following

$$\mathfrak{S}_c(x_0, y_0)(\xi, \eta) = D^2_{tt} D^2_{ss} \left[(t, s) \to -c(\exp_{x_0}(t\xi), \exp_{x_0}(p_0 + s\eta))\right]\Big|_{t,s=0}. \tag{2.10}$$

Moreover, the definition of $\mathfrak{S}_c(x_0, y_0)(\xi, \eta)$ is intrinsic, only depends on the points $(x_0, y_0) \in \mathcal{U}$ and vectors $(\xi, \eta)$, but not on the choice of local coordinates around $x_0$ or $y_0$, [6, 9]. We are now ready to introduce the condition

(A3) For any $(x, y) \in \mathcal{U}$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$\mathfrak{S}_c(x, y)(\xi, \eta) \geq c_0 |\xi|^2 |\eta|^2, \tag{2.11}$$

where $c_0$ is a positive constant.

It has been verified [10] that the cost function $c = d^2/2$ over $\mathbb{S}^n$ satisfies (A3) for any $x, y$ such that $d(x, y) < \pi$. Then under the assumptions of Lemma 2.1, we have

**Corollary 2.1.** *The cost function $c$ satisfies conditions (A0)–(A3) on the graph of $T_u$, $\mathcal{G}_T := \{(x, T_u(x)) : x \in \mathbb{S}^n_+\}$.*

**Corollary 2.2.** *Let $u$ be a $c$-convex potential on $\mathbb{S}^n_+$. The densities $f$ and $g$ are bounded from above and below. Then there exists a constant $\alpha \in (0, 1)$ such that $u \in C^{1,\alpha}(\overline{\mathbb{S}^n_+})$.*

*Proof.* Using the stereographic projection, it suffices to show that $\bar{u} \in C^{1,\alpha}(\overline{B_1})$ for some constant $\alpha \in (0, 1)$, where $\bar{u}$ is given in (1.9). By Corollary 2.1, the cost function $\bar{c}$ satisfies (A0)–(A3) on the graph $\mathcal{G}_{\bar{T}} = \{(x, \bar{T}(x)) : x \in B_1(0)\}$. The proof then follows from [8] by using a similar argument. The global $C^{1,\alpha}$ regularity was previously obtained by Loeper in [9] for Euclidean domains and in [10] for spheres $\mathbb{S}^n$.

Let $x_0 \in B_1$ be an interior point and $N_r(x_0) := B_r(x_0) \cap \overline{B}_1(0)$ be a small neighborhood of $x_0$. By Lemma 2.1, for each $y_0 \in \bar{T}(x_0)$, $N_r(x_0)$ is $\bar{c}$-convex with respect to $y_0$ when $r > 0$ is sufficicently small. Let $\varphi = \bar{c}(\cdot, y_0) + a_0$ be a $c$-support of $\bar{u}$ at $x_0$, where $a_0$ is a constant. Then we can define the sub-level set

$$S^0_{h,\bar{u}}(x_0) = \{x \in N_r(x_0) : \bar{u}(x) < \varphi(x) + h\}$$

for $h > 0$ small. Since $\bar{c}$ satisfies (A3), $S^0_{h,\bar{u}}(x_0)$ is $\bar{c}$-convex with respect to $y_0$. Thus by the coordinate transform $x \mapsto D_y\bar{c}(x, y_0)$, $S^0_{h,\bar{u}}(x_0)$ becomes a convex set. By applying the normalization argument in [8], we can obtain $\bar{u} \in C^{1,\alpha}(x_0)$.

For the boundary regularity, it can be reduced into the interior case since the argument in [8] allows that the initial density $f$ vanishes. But we need to extend the function $\bar{u}$ to a neighborhood of $B_1(0)$ by setting

$$\bar{u}(x) = \sup\{-c(x, y) - \bar{u}^c(y) : y \in B_1(0)\}$$

where $\bar{u}^c$ is the dual potential function of $\bar{u}$. □

## 3. Obliqueness

In this section we focus on the optimal transportation after the stereographic transformation, which is from $\Omega = B_1(0)$ to $\Omega^* = B_1(0)$ with the cost function given in (1.8). We drop off the bars over the functions $c, u$ for simplicity, so that the potential function $u$ satisfies equation (1.3) with boundary condition (1.4), where the optimal mapping $T_u$ is determined by (1.2). We will prove (1.10) in the following lemma.

Recall that a boundary condition of the form

$$G(x, u, Du) = 0 \quad \text{on } \partial\Omega \tag{3.1}$$

for a second order partial differential equation in a domain $\Omega$ is called *oblique*, (or *degenerate oblique*) if

$$G_p \cdot \nu \geq c_0 > 0, \quad (\text{or } \geq 0) \tag{3.2}$$

for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$, where $c_0$ is a positive constant, $\nu$ denotes the unit outer normal to $\partial\Omega$. Let $\phi(x) = \frac{1}{2}(|x|^2 - 1)$ and $\phi^*(y) = \frac{1}{2}(|y|^2 - 1)$ be smooth defining functions for $\Omega$ and $\Omega^*$, respectively. Then (1.4) can be written as

$$\begin{aligned} \phi^*(T_u) &= 0 \qquad \text{on } \partial\Omega, \\ \phi^*(T_u) &< 0 \qquad \text{near } \partial\Omega. \end{aligned} \tag{3.3}$$

Set $G(x, u, Du) := \phi^* \circ T_u(x, Du)$. The main estimate in this section is the following

**Lemma 3.1.** *Under the assumptions of Theorem 1.1, the boundary condition* (1.4) *satisfies a strict obliqueness estimate* (3.2).

*Proof.* Let $u \in C^2(\overline{\Omega})$ be an elliptic solution of (1.3)–(1.4), and denote $y = T_u(x)$. By differentiation, we have

$$\phi_k^* D_i y_k \tau_i = 0 \quad \text{on } \partial\Omega \tag{3.4}$$

for any unit tangential vector $\tau$ on $\partial\Omega$, and

$$\phi_k^* D_\nu y_k \geq 0 \quad \text{on } \partial\Omega \tag{3.5}$$

where $\nu$ is the outer normal to $\partial\Omega$, whence

$$\phi_i^* D_j y_i = \chi \nu_j \tag{3.6}$$

for some $\chi \geq 0$. Consequently, from (1.2)

$$-\phi_i^* c^{i,k} w_{jk} = \chi \nu_j, \tag{3.7}$$

where $\{c^{i,j}\} = \{c_{i,j}\}^{-1}$ and

$$w_{ij} := u_{ij} + c_{ij}. \tag{3.8}$$

At this point we observe that $\chi > 0$ on $\partial\Omega$ since $|\nabla\phi^*| \neq 0$ on $\partial\Omega$ and $\det DT \neq 0$. Using the ellipticity of (1.3) and letting $\{w^{ij}\}$ denote the inverse matrix of $\{w_{ij}\}$, we then have

$$-\phi_i^* c^{i,k} = \chi w^{jk} \nu_j. \tag{3.9}$$

Now writing $G(x, p) = \phi^* \circ T_u(x, p) = \phi^*(y)$, by differentiating

$$\begin{aligned} G_{p_k} &= \phi_i^* D_{p_k} y_i = -\phi_i^* c^{i,k} \\ &= \chi w^{jk} \nu_j, \end{aligned} \tag{3.10}$$

thus

$$\begin{aligned} G_p \cdot \nu &= \chi w^{ij} \nu_i \nu_j \\ &> 0, \end{aligned} \tag{3.11}$$

on $\partial\Omega$. Next, we obtain a uniform positive lower bound for $G_p \cdot \nu$ as follows. On the boundary $\partial\Omega \times \partial\Omega^*$, the unit outer normal $\nu_i = x_i$ and $\phi_i^* = y_i$, for $i = 1, \cdots, n$. From (3.10), we have

$$G_p \cdot \nu = \sum_{i,j=1}^{n} -y_i c^{i,j}(x,y) x_j. \tag{3.12}$$

We claim that for $(x,y) \in \partial\Omega \times \partial\Omega^*$, for any $1 \le i \le n$

$$\sum_{j=1}^{n} c_{i,j} y_j = -\arccos(x \cdot y) \frac{x_i}{\sqrt{1-(x\cdot y)^2}}. \tag{3.13}$$

Hence, $\sum_{i=1}^{n} c^{i,j} x_j = -\frac{y_i\sqrt{1-(x\cdot y)^2}}{\arccos(x\cdot y)}$ and then

$$G_p \cdot \nu = \frac{|y^2|\sqrt{1-(x\cdot y)^2}}{\arccos(x\cdot y)} = \frac{\sqrt{1-(x\cdot y)^2}}{\arccos(x\cdot y)}. \tag{3.14}$$

By Lemma 2.1, $1 + x \cdot y \ge \varepsilon_0$ for some positive constant $\varepsilon_0$. Therefore, there exists a constant $c_0 > 0$ such that (3.2) holds. The proof of Lemma 3.1 is completed.

Let us now prove the claim (3.13) at $(x,y) \in \partial B_1 \times \partial B_1$ with the cost function $c$ given in (1.8). Denote

$$\theta = \frac{4(x\cdot y)}{(1+|x|^2)(1+|y|^2)} + \frac{(1-|x|^2)(1-|y|^2)}{(1+|x|^2)(1+|y|^2)}, \tag{3.15}$$

the cost function $c(x,y) = \frac{1}{2}\arccos^2\theta$. By differentiating, the first order derivatives are

$$c_i = \frac{\partial c}{\partial x_i} = -\frac{\arccos\theta}{\sqrt{1-\theta^2}} \frac{1}{1+|y|^2} \left( \frac{4y_i}{1+|x|^2} - \frac{8(x\cdot y)x_i}{(1+|x|^2)^2} - \frac{4x_i(1-|y|^2)}{(1+|x|^2)^2} \right) \tag{3.16}$$

for all $i = 1, \cdots, n$.

At $(x,y) \in \partial B_1 \times \partial B_1$, $|x| = |y| = 1$ and the function $\theta = x \cdot y$, thus

$$c_i = \frac{-\arccos(x\cdot y)}{\sqrt{1-(x\cdot y)^2}} \left( y_i - (x\cdot y)x_i \right). \tag{3.17}$$

Therefore, we obtain the relation $x \cdot D_x c = \sum_{i=1}^{n} x_i c_i = 0$. We point this out because it will be used on the boundary estimates in the next section.

By differentiating $\theta$ in (3.15) with respect to $y$ variable, we have

$$\begin{aligned} \frac{\partial\theta}{\partial y_i} &= \frac{1}{1+|x|^2} \left( \frac{4x_i}{1+|y|^2} - \frac{8(x\cdot y)y_i}{(1+|y|^2)^2} - \frac{4y_i(1-|x|^2)}{(1+|y|^2)^2} \right) \\ &= x_i - (x\cdot y)y_i, \end{aligned} \tag{3.18}$$

for all $(x, y) \in \partial B_1 \times \partial B_1$ and $i = 1, \cdots, n$. By a further differentiation of (3.16) with respect to $y$ variable, the mixed second order derivatives are

$$
\begin{aligned}
c_{i,j} \;=\; & \left(\frac{1}{1-\theta^2} - \frac{\theta \arccos\theta}{(1-\theta^2)^{3/2}}\right) \frac{\partial\theta}{\partial y_j} \frac{1}{1+|y|^2} \left(\frac{4y_i}{1+|x|^2} - \frac{8(x\cdot y)x_i}{(1+|x|^2)^2} - \frac{4x_i(1-|y|^2)}{(1+|x|^2)^2}\right) \\
& + \arccos\theta \frac{1}{\sqrt{1-\theta^2}} \frac{2y_j}{(1+|y|^2)^2} \left(\frac{4y_i}{1+|x|^2} - \frac{8(x\cdot y)x_i}{(1+|x|^2)^2} - \frac{4x_i(1-|y|^2)}{(1+|x|^2)^2}\right) \\
& - \arccos\theta \frac{1}{\sqrt{1-\theta^2}} \frac{1}{1+|y|^2} \frac{1}{1+|x|^2} \left(4\delta_{ij} - \frac{8x_ix_j}{1+|x|^2} + \frac{8x_iy_j}{1+|x|^2}\right).
\end{aligned} \tag{3.19}
$$

Combining (3.18) with (3.19) and noting that $\theta = x \cdot y$, $|x| = |y| = 1$, we have

$$
\begin{aligned}
c_{i,j} \;=\; & \left(\frac{1}{1-\theta^2} - \frac{\theta \arccos\theta}{(1-\theta^2)^{3/2}}\right) (x_j - (x\cdot y)y_j)\,(y_i - (x\cdot y)x_i) \\
& + \frac{\arccos\theta}{\sqrt{1-\theta^2}} \left(y_j\,(y_i - (x\cdot y)x_i) - (\delta_{ij} - x_ix_j + x_iy_j)\right).
\end{aligned} \tag{3.20}
$$

Therefore, the sum in (3.13) becomes

$$
\begin{aligned}
\sum_{j=1}^{n} c_{i,j} y_j \;=\; & \left(\frac{1}{1-\theta^2} - \frac{\theta \arccos\theta}{(1-\theta^2)^{3/2}}\right) (x\cdot y - x\cdot y)\,(y_i - (x\cdot y)x_i) \\
& + \frac{\arccos\theta}{\sqrt{1-\theta^2}} \left((y_i - (x\cdot y)x_i) - (y_j - (x\cdot y)x_i + x_i)\right) \\
=\; & -\arccos(x\cdot y) \frac{x_i}{\sqrt{1-(x\cdot y)^2}}.
\end{aligned} \tag{3.21}
$$

The claim (3.13) is proved, and thus (3.17) holds and Lemma 3.1 is proved. □

## 4. Boundary $C^2$ estimate

In this section we prove the boundary $C^2$ estimate (1.11) in the two dimensional case. Recall that $\Omega, \Omega^* = B_1(0)$ and the boundary condition is written as

$$
\phi^*(T_u) = 0 \quad \text{on } \partial B_1(0), \tag{4.1}
$$

where $\phi^*(y) = \frac{1}{2}(|y|^2 - 1)$, and $T_u = T(\cdot, Du)$ is the optimal mapping.

It is convenient to denote the vector field $\beta = (\beta_1, \beta_2)$ where

$$
\beta_k := \frac{\partial \phi^*}{\partial p_k} = \phi^*_i D_{p_k} y_i = -\phi^*_i c^{i,k}. \tag{4.2}
$$

Differentiating along any tangential vector field $\tau$ on $\partial B_1$, we have

$$
0 = \phi^*_k D_i y_k \tau_i = -\phi^*_k c^{k,j} w_{ji} \tau_i = w_{\tau\beta} \quad \text{on } \partial B_1. \tag{4.3}
$$

Let $\nu$ be the unit outer normal of $\partial B_1$, by differentiating

$$
0 \le \phi^*_k D_i y_k \nu_i = -\phi^*_k c^{k,j} w_{ji} \nu_i = w_{\nu\beta} \quad \text{on } \partial B_1. \tag{4.4}
$$

Here and below we use the notation $w_{\xi\eta}$ to denote $w_{ij}\xi_i\eta_j$ even if $\xi$ and $\eta$ are not unit vector fields.

Suppose $w_{\xi\xi}$ takes its maximum over $\partial B_1$ and unit vector $\xi$ at $x_0 \in \partial B_1$. Note that we may write $\xi$ in terms of a tangential component $\tau(\xi)$ and a component in the direction of $\beta$, namely

$$\xi = \tau(\xi) + \frac{\nu \cdot \xi}{\beta \cdot \nu}\beta$$

where

$$\tau(\xi) = \xi - (\nu \cdot \xi)\nu - \frac{\nu \cdot \xi}{\beta \cdot \nu}\beta^T$$

and

$$\beta^T = \beta - (\beta \cdot \nu)\nu.$$

Thanks to the oblique estimate in Lemma 3.1, we have

$$|\tau(\xi)|^2 = 1 - \left(1 - \frac{|\beta^T|^2}{(\beta \cdot \nu)^2}\right)(\nu \cdot \xi)^2 - 2(\nu \cdot \xi)\frac{\beta^T \cdot \xi}{\beta \cdot \nu} \leq C.$$

Thus,

$$\begin{aligned} w_{\xi\xi} &= w_{\tau(\xi)\tau(\xi)} + \frac{2\nu \cdot \xi}{\beta \cdot \nu} w_{\tau\beta} + \frac{(\nu \cdot \xi)^2}{(\beta \cdot \nu)^2} w_{\beta\beta} \\ &\leq |\tau(\xi)|^2 w_{\tau_0\tau_0} + \frac{(\nu \cdot \xi)^2}{(\beta \cdot \nu)^2} w_{\beta\beta}, \end{aligned} \tag{4.5}$$

Namely, it suffices to control $w_{\tau_0\tau_0}$ and $w_{\beta\beta}$.

From (3.17) and (1.2), $x \cdot Du \equiv 0$ on $\partial B_1$. Without loss of generality, we may assume $x_0 = (0, 1)$ and locally $\partial B_1$ can be represented by $x_2 = \rho(x_1) = \sqrt{1 - |x_1|^2}$. By tangential differentiation at $x_0$,

$$\begin{aligned} 0 &= u_1 + x_k u_{k1} + (u_2 + x_k u_{k2})\rho', \\ 0 &= 2u_{11} + x_k u_{k11} + (u_2 + x_k u_{k2})\rho'' \\ &= 2u_{11} + u_{211} + (u_2 + u_{22}). \end{aligned}$$

As $e_1, e_2$ are the unit tangential and outer normal vectors at $x_0$, repsectively, we have

$$u_{\nu 11} \leq O(1 + w_{ii}). \tag{4.6}$$

We now tangentially differentiate the boundary condition $\phi^*(T_u)$ twice in the $e_1$ direction at $x_0$, to obtain

$$\phi^*_{ij}\frac{\partial y_i}{\partial x_1}\frac{\partial y_j}{\partial x_1} + \phi^*_i \frac{\partial^2 y_i}{\partial x_1^2} + \phi^*_i \frac{\partial y_i}{\partial x_2} = 0,$$

and thus

$$\sum_i \left(c^{i,k} w_{k1}\right)^2 - y_i \frac{\partial}{\partial x_1}\left(c^{i,k} w_{k1}\right) - y_i c^{i,k} w_{k2} = 0,$$

which implies

$$
\begin{aligned}
w_{11}^2 &\le y_i\left(c_{1,}^{i,k} w_{k1} + c^{i,k} D_1 w_{k1}\right) + w_{\beta 2} \\
&= y_i c_{1,}^{i,k} w_{k1} + y_i c^{i,k}(u_{k11} + c_{k11,} + c_{k1,p} c^{p,q} w_{q1}) + w_{\beta 2} \\
&= u_{\beta 11} + O(1 + w_{ii}).
\end{aligned} \tag{4.7}
$$

Let us assume that the maximal double-tangential term $w_{\tau\tau}$ occurs at $x_0$ in a tangential direction $e_1$, i.e. $w_{11}(x_0)$. Hence, $D_\tau w_{11}(x_0) = 0$, which gives

$$
u_{\tau 11} \le O(1 + w_{ii}). \tag{4.8}
$$

Therefore, from (4.6) and (4.8)

$$
u_{\beta 11} \le O(1 + w_{ii}),
$$

and by (4.7)

$$
w_{11}^2 \le O(1 + w_{ii}).
$$

From the fact that $\lambda^{-1} < \det w_{ij} < \lambda$ for some constant $\lambda > 0$, we conclude that

$$
w_{11}(x_0) \le C. \tag{4.9}
$$

It remains to bound $w_{\beta\beta}(x_0)$. By contradiction, we may assume $w_{\beta\beta}(x_0)$ is arbitrarily large. Note that we can decompose $\nu (= e_2)$ in terms of

$$
\nu = -\frac{1}{\beta \cdot \nu}\left(\beta - (\beta \cdot \nu)\nu\right) + \frac{1}{\beta \cdot \nu}\beta.
$$

There exists a matrix $A = (a_{ij})$ such that at $x_0$,

$$
\begin{bmatrix} 1 & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \tau \\ \beta \end{bmatrix} = \begin{bmatrix} \tau \\ \nu \end{bmatrix},
$$

where $0 < a_{22} = \frac{1}{\beta \cdot \nu} \le C$ by the obliqueness, and thus $\det A \le C$. From the decomposition,

$$
\begin{aligned}
&w_{11} = w_{\tau\tau}, \quad w_{12} = a_{21} w_{\tau\tau}, \\
&w_{22} = a_{22}^2 w_{\beta\beta} + a_{21}^2 w_{\tau\tau}.
\end{aligned}
$$

Since $|a_{21}|, w_{\tau\tau}$ are bounded, $w_{22}$ will be arbitrarily large if $w_{\beta\beta}$ is (by assumption).

Next we invoke the dual problem: Let $u^*$ denote the $c$-transform of $u$, defined for $y = T_u(x) \in \Omega^*$ by

$$
u^*(y) = -c(x, y) - u(x).
$$

It follows that

$$
\begin{aligned}
Du^*(y) &= -c_y(x, y) \\
&= -c_y(T^*_{u^*}(y), y),
\end{aligned}
$$

where

$$
T^*_{u^*}(y) = (T_u)^{-1}(y),
$$

and the dual equation is

$$\begin{aligned} |\det D_y(T^*_{u^*})| &= g(y)/f(T^*_{u^*}) \quad \text{in } \Omega^*, \\ T^*_{u^*}(\Omega^*) &= \Omega. \end{aligned}$$

Furthermore, by differentiation at $y = T_u(x)$,

$$w^{ij}(x) = w^*_{kl}(y)c^{k,i}c^{l,j}(x,y), \tag{4.10}$$

where $w^*_{kl}(y) = u^*_{y_k y_l}(y) + c_{,kl}(x,y)$ and $(w^{ij})$ is the inverse of $(w_{ij})$. By a similar analysis as for (4.9), we have

$$w^*_{\tau^*\tau^*}(y_0) \leq C, \tag{4.11}$$

where $y_0 = T_u(x_0)$ and $\tau^*$ is the tangential direction at $y_0$.

Define

$$\tilde{\tau}_k := \tau^*_i c_{i,k}(x_0, y_0). \tag{4.12}$$

Then by (4.10) and (4.11) we have

$$\begin{aligned} C &\geq w^*_{ij}\tau^*_i\tau^*_j \\ &= \left(w^*_{ij}c^{i,k}c^{j,l}\right)\tilde{\tau}_k\tilde{\tau}_l \\ &= w^{kl}\tilde{\tau}_k\tilde{\tau}_l \\ &= w^{11}\tilde{\tau}_1^2 + 2w^{12}\tilde{\tau}_1\tilde{\tau}_2 + w^{22}\tilde{\tau}_2^2. \end{aligned}$$

It is easy to see that the last two terms are bounded because of (4.3) and (4.9). If we can show $\tilde{\tau}_1^2 \geq \delta_0$ for some constant $\delta_0 > 0$, then we have a contradiction as $w^{11} = w_{22}/\det w_{ij}$ will become arbitrary large (by assumption).

At $(x_0, y_0)$, by the obliqueness estimate (1.10)

$$-c^{2,1}y_1 - c^{2,2}y_2 \geq c_0,$$

where $c_0 > 0$ is constant. This is equivalent to

$$\frac{1}{\det c_{i,j}}\left(c_{2,1}y_1 - c_{1,1}y_2\right) \geq c_0,$$

and

$$(c_{2,1}y_1 - c_{1,1}y_2)^2 \geq c_0^2(\det c_{i,j})^2 =: \delta_0 > 0.$$

At $y_0$, the tangential $\tau^* = (y_2, -y_1)$. From (4.12)

$$\begin{aligned} \tilde{\tau}_1 &= c_{1,1}\tau^*_1 + c_{2,1}\tau^*_2 \\ &= c_{1,1}y_2 - c_{2,1}y_1, \end{aligned}$$

and thus we obtain

$$\tilde{\tau}_1^2 \geq \delta_0 > 0.$$

The above contradiction implies that $w_{\beta\beta}(x) \le C$ for all $x \in \partial B_1$. Therefore, by (4.5) we conclude the estimate (1.11) and Theorem 1.1 is proved.

*Remark* 4.1. By Corollary 2.1, the cost function $c$ satisfies the condition (A3). To obtain the global $C^2$ and higher order estimates, we assume furthermore that the densities $f$ and $g$ are $C^2$ smooth. From [13], we have the estimate

$$\sup_{\Omega} |D^2 u| \le C(1 + \sup_{\partial\Omega} |D^2 u|), \tag{4.13}$$

and combining with (1.11) we obtain the global $C^2$ estimate.

Once the second derivatives are bounded, equations (1.3)–(1.4) are uniformly elliptic. This combined with the obliqueness estimate (1.10) yields global $C^{2,\alpha}$ estimates [7]. Moreover, the higher order estimates follow from the theory of linear elliptic equations with oblique boundary conditions [5].

## References


[1] Ambrosio, L. and Pratelli, A., *Existence and stability results in the $L^1$ theory of optimal transportation*, LNM 1813, L.A. Caffarelli and S. Salsa (Eds.), pp. 123–160, 2003.

[2] Caffarelli, L. A., *Boundary regularity of maps with convex potentials II*, Ann. of Math., 144 (1996), 453–496.

[3] Delanoë, P., *Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator*, Ann. Inst. Henri Poincaré, Analyse Non Linéaire, 8 (1991), 443–457.

[4] Delanoë, P. and Loeper, G., Gradient estimates for potentials of invertible gradient-mappings on the sphere, *Calc. Var. PDE.*, 26 (2006), 297–311.

[5] Gilbarg, D. and Trudinger, N., *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, 1983.

[6] Kim, Y.-H. and McCann, R. J., Continuity, curvature, and the general covariance of optimal transportation, *J. Eur. Math. Soc.*, 12 (2010), 1009–1040.

[7] Lieberman, G. and Trudinger, N., Nonlinear oblique boundary value problems for nonlinear elliptic equations, *Trans. Amer. Math. Soc.*, 295 (1986), 509–546.

[8] Liu, J., Hölder regularity of optimal mappings in optimal transportation, *Calc. Var. PDE.*, 34 (2009), 435–451.

[9] Loeper, G., On the regularity of solutions of optimal transportation problems, *Acta Math.* 202 (2009), 241–283.

[10] Loeper, G., Regularity of optimal maps on the sphere: the quadratic cost and the reflector antenna, *Arch. Rat. Mech. Anal.*, 199 (2011), 269–289.

[11] Ma, X. N.; Trudinger, N. S. and Wang, X.-J., Regularity of potential functions of the optimal transportation problem, *Arch. Rat. Mech. Anal.*, 177 (2005), 151–183.

[12] McCann, R. J., Polar factorization of maps on Riemannian manifolds, *Geom. Funct. Anal.* 11 (2001), 589–608.

[13] Trudinger, N. S. and Wang, X.-J., On the second boundary value problem for Monge-Ampère type equations and optimal transportation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, 8 (2009), no. 1, 143–174.

[14] Trudinger, N. S. and Wang, X.-J., On strict convexity and continuous differentiability of potential functions in optimal transportation. *Arch. Rat. Mech. Anal.*, 192 (2009), 403–418.

[15] Urbas, J., *On the second boundary value problem for equations of Monge-Ampère type*, J. Reine Angew. Math. 487 (1997), 115–124.

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