Inequalities for Quermassintegrals on k-Convex Domains

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Abstract

In this paper, we study the Aleksandrov-Fenchel inequalities for quermassintegrals on a class of non-convex domains. Our proof uses optimal transport maps as a tool to relate curvature quantities of different orders defined on the boundary of the domain.

1 Introduction

In this paper, we study the classical Aleksandrov-Fenchel inequalities for quermassintegrals on convex domains and extend these inequalities to a class of non-convex domains on the Euclidean space. We obtain a family of geometric inequalities, each relating some nonlinear curvature quantities of different order on the boundary of the domain.

Let Ω in $\mathbb{R}^{n+1}$ be a bounded convex set. We denote the m dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ by $\mathcal{H}^m$. Consider the set

$$\Omega + tB := \{x + ty | x \in \Omega, y \in B\}$$

for $t > 0$, the volume of which, by a theorem of Minkowski [25], is an $n+1$ degree polynomial in t, whose expansion is given by

$$\mathrm{Vol}(\Omega + tB) = \mathcal{H}^{n+1}(\Omega + tB) = \sum_{m=0}^{n+1} C_{n+1}^m W_m(\Omega) t^m.$$

Where $W_m(\Omega)$ for $m = 0, ..., n+1$ are coefficients determined by the set Ω, and $C_{n+1}^m = \frac{(n+1)!}{m!(n+1-m)!}$. The m-th quermassintegral V_m is defined as a multiple of the coefficient $W_{n+1-m}(\Omega)$.

$$V_m(\Omega) := \frac{\omega_m}{\omega_{n+1}} W_{n+1-m}(\Omega). \tag{1}$$

Clearly, for arbitrary domain Ω, $V_{n+1}(\Omega) = \mathcal{H}^{n+1}(\Omega)$.

If Ω has smooth boundary (denoted by M), the quermassintegrals can also be represented as the integrals of invariants of the second fundamental form: Let L_{ij} be the second fundamental form

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on M, and let $\sigma_k(L)$ with $k = 0, ..., n$ be the k-th elementary symmetric function of the eigenvalues of L. (Define $\sigma_0(\lambda) = 1$.) Then

$$V_{n+1-m}(\Omega) := \frac{(n+1-m)!(m-1)!}{(n+1)!} \frac{\omega_{n+1-m}}{\omega_{n+1}} \int_M \sigma_{m-1}(L) d\mu_M, \tag{2}$$

for $m = 1, ..., n+1$. From the above definition, one can see that $V_0(\Omega) = 1$, and $V_n(\Omega) = \frac{\omega_n}{(n+1)\omega_{n+1}} \mathcal{H}^n(\partial\Omega)$, where $\mathcal{H}^n(\partial\Omega)$ is the area of the boundary $\partial\Omega$. From this definition, as a consequence of the Aleksandrov-Fenchel inequalities [1], [2], one obtains the following family of inequalities: if Ω is a convex domain in $\mathbb{R}^{n+1}$ with smooth boundary, then for $0 \le l \le n$,

$$\left(\frac{V_{l+1}(\Omega)}{V_{l+1}(B)}\right)^{\frac{1}{l+1}} \le \left(\frac{V_l(\Omega)}{V_l(B)}\right)^{\frac{1}{l}}, \tag{3}$$

(3) is equivalent to

$$\left(\int_M \sigma_{m-1}(L) d\mu_M\right)^{\frac{1}{n-m+1}} \le C \left(\int_M \sigma_m(L) d\mu_M\right)^{\frac{1}{n-m}}, \tag{4}$$

for $m = n - l$, $1 \le m \le n$. And here $C = C(k, n)$ denotes the constant which is obtained when M is the n-sphere and the inequality becomes an equality. When $m = 0$, (3) is the well-known isoperimetric inequality

$$\mathcal{H}^{n+1}(\Omega)^{\frac{n}{n+1}} \le \frac{\omega_{n+1}^{\frac{1}{n+1}}}{n+1} \mathcal{H}^n(\partial\Omega).$$

The inequalities (3) for convex domains were originally proved using the theory of Minkowski's mixed volume. The original argument in establishing the inequalities in [1], [2] depends strongly on the assumption that the domains dealt with are convex. Since then there have been many different methods to establish these inequalities for convex domains, some without involving the notion of Minkowski's mixed volume (the reader is referred to the book of Hörmander [17] for the subject). In this article, we will study the inequalities for a class of non-convex domains which we will specify below.

The class of domains that we will consider in this paper is the class of k-convex domains defined as follows:

Definition 1.1. *For $\Omega \subset \mathbb{R}^{n+1}$, we say the boundary $M := \partial\Omega$ is k-convex if the second fundamental form $L_{ij}(x) \in \Gamma_k^+$ for all $x \in M$, where Γ_k^+ denotes the Garding's cone*

$$\Gamma_k^+ := \{A \in \mathbb{M}_{n\times n} | \ \sigma_m(A) > 0, \forall\ 1 \le m \le k\}. \tag{5}$$

We remark that with this notation, n-convex is convex in the usual sense, and 1-convex is sometimes referred to as mean convex.

In [15], Guan-Li had applied a fully nonlinear flow to study the inequality (4) for m-convex domains. Namely, one evolves the hypersurface $M := \partial\Omega \subset \mathbb{R}^{n+1}$ along the flow

$$\vec{X}_t = \frac{\sigma_{m-1}}{\sigma_m}(L)\nu, \tag{6}$$

where ν is the unit outer normal of the hypersurface M. The key observation made in [15] is that the ratio

$$\frac{\left(\int_M \sigma_{m-1}(L)d\mu_M\right)^{\frac{1}{n-m+1}}}{\left(\int_M \sigma_m(L)d\mu_M\right)^{\frac{1}{n-m}}} \tag{7}$$

is monotonically increasing along the flow (6). Therefore if the solution of the flow (6) exists for all time $t > 0$ and converges to a round sphere (or up to a rescaling), then one obtains the sharp inequality (4) as a consequence. This type of strategy works for some classes of domains, for example it works for the class of convex domains. In the special case when $m = 1$, (6) is the inverse mean curvature flow, which has been extensively studied in the literature, for example by Evans-Spruck [12], and by Huisken-Ilmanen [20]. We remark that in this special case, under the additional assumption that the domain Ω is outward minimizing, Huisken has proved that the sharp inequality (4) holds. Another class of domains in which this strategy works is when Ω is star-shaped and strictly k-convex. In this case, Gerhardt [14] and Urbas [30] have independently proved that the flow (6) exists for all t and converges to the round sphere. This enables Guan-Li to establish the following result:

Theorem 1.2. *[15] Suppose Ω is a smooth star-shaped domain in $\mathbb{R}^{n+1}$ with k-convex boundary, then the inequality (4) is valid for all $1 \leq m \leq k$; with the equality holds if and only if Ω is a ball.*

We remark in general, without further assumptions on the domain, one anticipates that singularities develop along the flow (6). Hence the flow does not exist for all time.

We would also like to mention that for k-convex domains, a special case of the sharp inequality (3) between V_{n+1} and V_{n-k} was established by Trudinger. (See Section 3 in [29]).

Our main result in this paper is to establish the inequalities of Aleksandrov-Fenchel type at level k for $(k+1)$-convex domains.

Theorem 1.3. *For $k = 2, ..., n-1$, if M is $(k+1)$-convex, then there exists a constant C depending only on n and k, such that for $1 \leq m \leq k$*

$$\left(\int_M \sigma_{m-1}(L)d\mu_M\right)^{\frac{1}{n-m+1}} \leq C\left(\int_M \sigma_m(L)d\mu_M\right)^{\frac{1}{n-m}}.$$

If $k = n$, then the inequality holds when M is n-convex. If $k = 1$, then the inequality holds when M is 1-convex.

Our proof of the above result uses method of optimal transport. The idea to prove geometric inequalities by constructing maps between the domain and the ball was first explored by M. Gromov, (see for example page 47 on [11]). In particular his method was used to prove the classical isoperimetric inequality for domains in $\mathbb{R}^n$. Later in the literature, there are many other geometric inequalities which were established or reproved by exploring properties of maps which are optimal transport maps in special settings. This includes the work of R. McCann [24] on the Brunn-Minkowski inequality, and that of S. Alesker, S. Dar and V. Milman [3] on an Aleksandrov-Fenchel type inequality. In a more recent paper, D. Cordero-Erausquin, B. Nazaret and C. Villani [33] have used the optimal transport map to establish a case of the sharp Sobolev inequalities on $\mathbb{R}^n$. Most recently, P. Castillon [9] gave a reproof of the Michael-Simon inequality on submanifolds of the Euclidean space using methods of optimal transport. In this paper, we will adopt the strategy of the proof of Castillon to a nonlinear setting to prove our main theorem above.

We now recall Michael-Simon inequality:

Theorem 1.4. *[26] Let $i : M^n \to \mathbb{R}^N$ be an isometric immersion ($N > n$). Let U be an open subset of M. For a nonnegative function $u \in C_c^\infty(U)$, there exists a constant C, such that*

$$\left(\int_M u^{\frac{n}{n-1}} d\mu_M\right)^{\frac{n-1}{n}} \leq C \int_M |\vec{H}| \cdot u + |\nabla u| dv_M. \tag{8}$$

In the special case when we take $u \equiv 1$, Michael-Simon inequality gives an inequality between the area of the boundary and the integral of its mean curvature. Thus a natural generalization is to establish inequalities similar to (8) between fully nonlinear curvature quantities $\sigma_{m-1}(L)$ and $\sigma_m(L)$.

Motivated by the same line of ideas, in a subsequent paper, we will establish a family of generalized Michael-Simon inequalities for codimension 1 hypersurfaces M.

Theorem 1.5. *Let $i : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion. Let U be an open subset of M and $u \in C_c^\infty(U)$ be a nonnegative function. For $k = 2, ..., n-1$, if M is $(k+1)$-convex, then there exists a constant C depending only on n and k, such that for $1 \leq m \leq k$*

$$\left(\int_M \sigma_{m-1}(L) u^{\frac{n-m+1}{n-m}} d\mu_M\right)^{\frac{n-m}{n-m+1}} \leq C \int_M (\sigma_m(L)u + \sigma_{m-1}(L)|\nabla u| +, ..., +|\nabla^m u|) d\mu_M.$$

If $k = n$, then the inequality holds when M is n-convex. If $k = 1$, then the inequality holds when M is 1-convex. ($k = 1$ case is a corollary of the Michael-Simon inequality.)

There are three main ingredients in the proof of our main theorem (Theorem 1.3). The first is that we have applied the theory of optimal transport to relate the curvature terms $\sigma_k(L)$ for different k via suitable mass transport equations. The second ingredient is that we have related the quantity of $\sigma_k(L)$ defined on the boundary of the domain via the Gauss-Codazzi equation to the curvature terms of the induced metric defined on the boundary of the domain. The third ingredient is that we have applied the structure equations and Garding's inequality in analyzing the fully nonlinear terms $\sigma_k(L)$.

The organization of this paper is as follows. In Section 2, we will review some basic properties of k-th elementary symmetric function $\sigma_k(\lambda)$. In particular, we highlight those inequalities which are verified by applying Garding's theory of hyperbolic polynomials. In this section, we will also review some well-known facts of optimal transport maps which will be used in the rest of the paper.

In Section 3 of the paper, assuming the main technical proposition (Proposition 3.1), we finish the proof of our main theorem. The proof follows the outline similar to that in the paper by P. Castillon, but to deal with the fully nonlinear quantities of the curvature, we explore the concavity properties of the elementary symmetric functions $\sigma_k(A)$ for matrix A in the Garding's cone. Another difficulty we face is that for non-convex domains, the Hessian of the convex potential of the optimal transport map only exists in general in the Alexandrov sense, which is sufficient for the purpose of studying the Laplacian of the potential function as in the work of Castillon; but it is not clear how to define the notion of σ_k of the Hessian of the potential function in this generalized setting. To overcome this difficulty, we have first applied the regularity results of the optimal maps established earlier by L. Caffarelli ([5], [6], [7]) for convex domains and then applied an approximation argument to finish the proof of the desired inequalities.

We then establish Proposition 3.1 in the remaining sections of the paper. To illustrate the complicated induction steps in the proof, we first present the proof of the proposition for the special case $k = 2$ in Section 4 (where only the size of the optimal map is relevant), and the special case $k = 3$ in Section 5 (where the convexity property of the map plays a crucial role). Finally, in Section 6 we prove Proposition 3.1 for all integers k by a multi-layer inductive argument.

An expository version of this article, where more background of the subject was provided and the main ideas of the proof were outlined, has been published as the lecture notes of the Riemann International School of Mathematics in Verbania, Italy, 2010. ([10])

We remark that in view of the result of Guan-Li ([15]), the most natural assumption in the statement of our theorem should be that the domain is k-convex instead of $k+1$-convex; but at the moment, our proof relies heavily on the extra one level of convexity property of the domain. We also remark that the proof we present here does not yield any sharp constants for the inequalities.

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2 Preliminaries

2.1 Γ_k^+ cone

In this subsection, we will describe some properties of σ_k function and its associated convex cone.

2.1.1 Definitions and Concavity

Definition 2.1. *The k-th elementary symmetric function for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ is*

$$\sigma_k(\lambda) := \sum_{i_1 < ... < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

The elementary symmetric functions are special cases of hyperbolic polynomials introduced by Garding [13], which enjoy the following properties in their associated positive cones.

Definition 2.2.

$$\Gamma_k^+ := \{\lambda \in \mathbb{R}^n | \textit{the connected component of } \sigma_k(\lambda) > 0 \textit{ which contains the identity} = (1, ..., 1)\}$$

is called the positive k-cone.
Equivalently,

$$\Gamma_k^+ = \{\lambda \in \mathbb{R}^n |\ \sigma_1(\lambda) > 0, ..., \sigma_k(\lambda) > 0\}.$$

In particular, Γ_n^+ is the positive cone

$$\{\lambda \in \mathbb{R}^n |\ \lambda_1 > 0, ..., \lambda_n > 0\},$$

and Γ_1^+ is the half space $\{\lambda \in \mathbb{R}^n | \lambda_1 + \cdots + \lambda_n > 0\}$. It is also obvious from Definition 2.2 that Γ_k^+ is an open convex cone and that

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \cdots \subset \Gamma_1^+.$$

Applying Garding's theory of hyperbolic polynomials [13], one concludes that $\sigma_k^{\frac{1}{k}}(\cdot)$ is a concave function in Γ_k^+. Thus

$$\frac{\sigma_k^{\frac{1}{k}}(\lambda) + \sigma_k^{\frac{1}{k}}(\mu)}{2} \leq \sigma_k^{\frac{1}{k}}\left(\frac{\lambda + \mu}{2}\right), \tag{9}$$

for $\lambda, \mu \in \Gamma_k^+$. By the homogeneity of $\sigma_k^{\frac{1}{k}}$, one gets from (9) that for $\lambda, \mu \in \Gamma_k^+$

$$\sigma_k^{\frac{1}{k}}(\lambda) < \sigma_k^{\frac{1}{k}}(\lambda + \mu). \tag{10}$$

Also, $(\frac{\sigma_k(\cdot)}{\sigma_l(\cdot)})^{\frac{1}{k-l}}$ $(k > l)$ is concave in Γ_k^+. Therefore

$$(\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)})^{\frac{1}{k-l}} < (\frac{\sigma_k(\lambda+\mu)}{\sigma_l(\lambda+\mu)})^{\frac{1}{k-l}}, \tag{11}$$

for $\lambda, \mu \in \Gamma_k^+$.

Definition 2.3. *A symmetric matrix A is in $\tilde{\Gamma}_k^+$ cone, if its eigenvalues*

$$\lambda(A) = (\lambda_1(A), ..., \lambda_n(A)) \in \Gamma_k^+.$$

Suppose f is a function on Γ_k^+. $F = f(\lambda(A))$ is the extension of f on $\tilde{\Gamma}_k^+$. Due to a result in [8], f is concave in Γ_k^+ implies F is concave in $\tilde{\Gamma}_k^+$. When there is no confusion, we will denote $\tilde{\Gamma}_k^+$ by Γ_k^+ and $\sigma_k(\lambda(A))$ by $\sigma_k(A)$ for simplicity.

2.1.2 The polarization of σ_k

Notice that $\sigma_n(A) = \det(A)$. An equivalent definition of $\det(A)$ is

$$\det A = \frac{1}{n!}\delta_{j_1,...,j_n}^{i_1,...,i_n} A_{i_1 j_1} \cdots A_{i_n j_n}, \tag{12}$$

where $\delta_{j_1,...,j_n}^{i_1,...,i_n}$ is the generalized Kronecker delta; it is zero if $\{i_1, ..., i_k\} \neq \{j_1, ..., j_k\}$, equals to 1 (or -1) if $(i_1, ..., i_k)$ and $(j_1, ..., j_k)$ differ by an even (or odd) permutation. Inspired by (12), an equivalent way of writing σ_k is that

$$\sigma_k(A) := \frac{1}{k!}\delta_{j_1,...,j_k}^{i_1,...,i_k} A_{i_1 j_1} \cdots A_{i_k j_k}.$$

The Newton transformation tensor is defined as

$$[T_k]_{ij}(A_1, ..., A_k) := \frac{1}{k!}\delta_{j,j_1,...,j_k}^{i,i_1,...,i_k}(A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \tag{13}$$

Definition 2.4. *With the notion of $[T_k]_{ij}$, one may define the polarization of σ_k by*

$$\Sigma_k(A_1, ..., A_k) := A_{1ij} \cdot [T_{k-1}]_{ij}(A_2, ..., A_k) = \frac{1}{(k-1)!}\delta_{j_1,...,j_k}^{i_1,...,i_k}(A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \tag{14}$$

It is called the polarization of σ_k because if we take $A_1 = \cdots = A_k = A$, then $\Sigma_k(A, ..., A)$ is equal to $\sigma_k(A)$ up to a constant. Namely,

$$\sigma_k(A) = \frac{1}{k}\Sigma_k(A, ..., A).$$

Also, from the right hand side of the definition 2.4, we see that Σ_k is symmetric and linear in each component.

Notation 2.5. *When some components are the same, we adopt the notational convention that*

$$\Sigma_k(\overbrace{B,...,B}^{l},C,...,C) := \Sigma_k(\overbrace{B,...,B}^{l},\overbrace{C,...,C}^{k-l}),$$

and

$$[T_k]_{ij}(\overbrace{B,...,B}^{l},C,...,C) := [T_k]_{ij}(\overbrace{B,...,B}^{l},\overbrace{C,...,C}^{k-l}).$$

Also for simplicity, we denote

$$[T_k]_{ij}(A) := [T_k]_{ij}(\overbrace{A,...,A}^{k}).$$

Some relations between the Newton transformation tensor T_k and σ_k are listed below. For any symmetric matrix A, if we denote the trace by Tr, then

$$\sigma_k(A) = \frac{1}{n-k} Tr([T_k]_{ij})(A), \tag{15}$$

and

$$\sigma_{k+1}(A) = \frac{1}{k+1} Tr([T_k]_{im}(A) \cdot A_{mj}). \tag{16}$$

On the other hand, one can write $[T_k]_{ij}$ in terms of σ_k by the formula

$$[T_{k-1}]_{ij}(A) = \frac{\partial \sigma_k(A)}{\partial A_{ij}},$$

and

$$[T_k]_{ij}(A) = \sigma_k(A)\delta_{ij} - [T_{k-1}]_{im}(A)A_{mj}. \tag{17}$$

This last formula implies the following fact which we will repeatedly use later in our proof.

Lemma 2.6. *Suppose B and C are two symmetric matrices, then*

$$\begin{aligned}&[T_{k-1}]_{im}(B,C,...,C)C_{mj}\\ =&\frac{1}{k-1}\Sigma_k(B,C,...,C)\delta_{ij} - \frac{k}{k-1}[T_k]_{ij}(B,C,...,C) - \frac{1}{k-1}[T_{k-1}]_{im}(C,...,C)B_{mj}.\end{aligned} \tag{18}$$

Proof. Since $[T_k]_{ij}$ is multilinear, $[T_k]_{ij}(C+\epsilon B)$ is a degree k polynomial in ϵ, in which

$$\begin{aligned}&\text{the coefficient of the term } \epsilon \text{ in } [T_k]_{ij}(C+\epsilon B,...,C+\epsilon B)\\ =&k \cdot [T_k]_{ij}(B,C,...,C).\end{aligned} \tag{19}$$

Also $[T_k]_{ij}(A) = \sigma_k(A)\delta_{ij} - [T_{k-1}]_{im}(A)A_{mj}$. Thus when we plug in $A = C+\epsilon B$ and expand out the right hand side, we get

$$\begin{aligned}&\text{the coefficient of the term } \epsilon \text{ in } \sigma_k(C+\epsilon B)\delta_{ij} - [T_{k-1}]_{im}(C+\epsilon B)(C+\epsilon B)_{mj}\\ =&\Sigma_k(B,C,...,C)\delta_{ij} - (k-1)[T_{k-1}]_{im}(B,C,...,C)C_{mj} - [T_{k-1}]_{im}(C,...,C)B_{mj}.\end{aligned} \tag{20}$$

Therefore

$$\begin{aligned}&[T_{k-1}]_{im}(B,C,...,C)C_{mj}\\ =&\frac{1}{k-1}\Sigma_k(B,C,...,C)\delta_{ij} - \frac{k}{k-1}[T_k]_{ij}(B,C,...,C) - \frac{1}{k-1}[T_{k-1}]_{im}(C,...,C)B_{mj}.\end{aligned} \tag{21}$$

□

By a similar argument, one has

Lemma 2.7. *Suppose B and C are two symmetric matrices, then*

$$
\begin{aligned}
& [T_{k-1}]_{im}(\overbrace{B,...,B}^{l},C,...,C)C_{mj} \\
= \quad & \frac{C_k^l}{kC_{k-1}^l}\cdot\Sigma_k(\overbrace{B,...,B}^{l},C,...,C)\delta_{ij}-\frac{C_k^l}{C_{k-1}^l}\cdot[T_k]_{ij}(\overbrace{B,...,B}^{l},C,...,C) \\
& -\frac{C_{k-1}^{l-1}}{C_{k-1}^l}\cdot[T_{k-1}]_{im}(\overbrace{B,...,B}^{l-1},C,...,C)B_{mj}.
\end{aligned}
\tag{22}
$$

Proof.

$$
\begin{aligned}
& C_k^l\cdot[T_k]_{ij}(\overbrace{B,...,B}^{l},C,...,C) \\
= \quad & \text{the coefficient of the term } \epsilon^l \text{ in } [T_k]_{ij}(C+\epsilon B,...,C+\epsilon B) \\
= \quad & \text{the coefficient of the term } \epsilon^l \text{ in } \sigma_k(C+\epsilon B)\delta_{ij}-[T_{k-1}]_{im}(C+\epsilon B)(C+\epsilon B)_{mj} \\
= \quad & \frac{C_k^l}{k}\cdot\Sigma_k(\overbrace{B,...,B}^{l},C,...,C)\delta_{ij}-C_{k-1}^l\cdot[T_{k-1}]_{im}(\overbrace{B,...,B}^{l},C,...,C)C_{mj} \\
& -C_{k-1}^{l-1}\cdot[T_{k-1}]_{im}(\overbrace{B,...,B}^{l-1},C,...,C)B_{mj}.
\end{aligned}
\tag{23}
$$

□

2.1.3 Some algebraic inequalities for elements in Γ_k^+ cone

Based on Garding's theory of hyperbolic polynomials [13], we have

(i) if $\lambda\in\Gamma_k^+$, then
$$\frac{\partial\sigma_k(\lambda)}{\partial\lambda_i}>0,\ \text{for } i=1,...,n;$$

(ii) if $A_1,...,A_k\in\Gamma_{k+1}^+$, then $([T_k]_{ij})$ is a positive matrix, i.e.
$$[T_k]_{ij}(A_1,...,A_k)>0;$$

(iii) if $A_1,...,A_k\in\Gamma_k^+$, then
$$\Sigma_k(A_1,...,A_k)>0;$$

(iv) if $A-B\in\Gamma_k^+$ and $A_2,...,A_k\in\Gamma_k^+$, then
$$\Sigma_k(B,A_1...,A_k)<\Sigma_k(A,A_2,...,A_k).$$

Lastly, for nonnegative symmetric matrix A, we have the well-known Newton-MacLaurin inequality: (see e.g. [18])

$$\frac{\sigma_{k+1}(A)\sigma_{k-1}(A)}{\sigma_{k+1}(Id)\sigma_{k-1}(Id)}\leq\frac{\sigma_k^2(A)}{\sigma_k^2(Id)},\tag{24}$$

where Id is the identity matrix.

2.2 Optimal transport map and its regularity

Consider the two Polish spaces D_1 and D_2, with probability measures ω_1 and ω_2 defined on them respectively. Given a cost function $c : D_1 \times D_2 \to \mathbb{R}$. The problem of Monge consists in finding a map $T : D_1 \to D_2$ such that its cost $C(T) := \int_{D_1} c(y_1, T(y_1))d\omega_1$ attains the minimum of the costs among all the maps that push forward ω_1 to ω_2. In general, the problem of Monge may have no solution, however in the special case when D_1 and D_2 are bounded domains defined on the Euclidean space with quadratic cost function, Y. Brenier [4] proved an existence and uniqueness result. More precisely,

Theorem 2.8. *Suppose that D_i (i=1,2) are bounded domains in $\mathbb{R}^n$ with $\mathcal{H}^n(\partial D_i) = 0$ and that the cost function is defined by $c(y_1, y_2) := d(y_1, y_2)^2$, where d is the Euclidean distance. Given two probability measures $\omega_1 := F(y_1)dy_1$, $\omega_2 := G(y_2)dy_2$ defined on D_1, D_2 respectively. Then there exists a unique optimal transport map (solution of the problem of Monge) $T : spt(F) \to spt(G)$. Also T is the gradient of some convex potential function V.*

It is obvious that since the optimal map $T = \nabla V$ pushes forward $F(y_1)dy_1$ to $G(y_2)dy_2$, it satisfies the Monge-Ampère equation in the weak sense.

$$\int_{D_2} \eta(y_2)G(y_2)dy_2 = \int_{D_1} \eta(\nabla V(y_1))F(y_1)dy_1, \tag{25}$$

for any continuous function η.

In general, the potential function V may not be regular, hence it does not satisfy the Monge-Ampère equation $det(D^2_{ij}V(y_1)) = \frac{F(y_1)}{G(\nabla V(y_1))}$ in the classical sense. However, under the additional assumptions on the convexity of D_i, as well as on the smoothness of F and G, Caffarelli has established in his papers [5], [6], [7] the interior and boundary regularity results of such a potential function V. We now state these results of Caffarelli here as we shall apply them later in the proof of our main theorem.

Theorem 2.9. *[6] If D_2 is convex and F, G, $1/F$, $1/G$ are bounded, then V is strictly convex and $C^{1,\beta}$ for some β.*
If F and G are continuous, then $V \in W^{2,p}_{loc}$ for every p.
If F and G are C^{k,α_0}, then $V \in C^{k+2,\alpha}$ for any $0 < \alpha < \alpha_0$.

For the boundary regularity, one needs to assume D_1 to be convex as well:

Theorem 2.10. *[7] If both D_i are C^2 and strictly convex, and F, $G \in C^\alpha$ are bounded away from zero and infinity, then the convex potential function V is $C^{2,\beta}$ up to ∂D_i for some $\beta > 0$. Both β and $\|V\|_{C^{2,\beta}}$ depend only on the maximum and minimum diameter of D_i and the bounds on F, G. Higher regularity of V follows from assumptions on the higher regularity of F and G by the standard elliptic theory.*

From these two theorems, we know that if D_i are smooth and strictly convex, and F, G are both smooth and bounded away from zero and infinity up to the boundary, then the potential function is smooth up to the boundary as well. For more results on the regularity of optimal transport maps between manifolds, we refer the readers to [27], [22], [31], etc.

2.3 Restriction of a convex function to a submanifold

Consider an isometric embedding $i : M^n \to \mathbb{R}^{n+1}$. Let $\vec{n}(x)$ be the inner unit normal at $x \in M$. Let ∇ and D^2 (resp. $\bar{\nabla}$ and $\bar{D}^2$) be the gradient and the Hessian on M (resp. on $\mathbb{R}^{n+1}$); let $\vec{L}(\cdot,\cdot)(x) = L(\cdot,\cdot)(x)\vec{n}(x)$ be the second fundamental form at $x \in M$. Suppose $\bar{V} : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function and $v = \bar{V}|_M$ is its restriction to M. Then the Hessian of v with respect to the metric on M relates to the Hessian of $\bar{V}$ on the ambient space $\mathbb{R}^{n+1}$ in the following way: for all $x \in M$ and all $\xi, \eta \in T_xM$,

$$\begin{aligned} D^2 v(\xi,\eta)(x) =& \bar{D}^2\bar{V}(\xi,\eta)(x) + \langle(\bar{\nabla}\bar{V}), \vec{L}(\xi,\eta)\rangle(x) \\ =& \bar{D}^2\bar{V}(\xi,\eta)(x) + b(x)\cdot L(\xi,\eta)(x), \end{aligned} \tag{26}$$

where $b(x) := \langle(\bar{\nabla}\bar{V}), \vec{n}\rangle(x)$. We remark in general $b(x)$ changes sign on M. Finally we recall the well-known Gauss equation and Codazzi equation that are satisfied by the curvature tensors defined on the embedded submanifold. Denote the curvature tensor of M by R_{ijkl} and the curvature tensor of the ambient space $\mathbb{R}^{n+1}$ by $\bar{R}_{ijkl}$. Then

$$0 = \bar{R}_{ijkl} = R_{ijkl} - L_{ik}L_{jl} + L_{il}L_{jk}, \quad \text{(Gauss equation)} \tag{27}$$

and

$$L_{ij,k} = L_{ik,j}. \quad \text{(Codazzi equation)} \tag{28}$$

3 Proof of the main theorem

Theorem 1.3 (Main Theorem): *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain whose boundary $\partial\Omega$ is an n-dimensional closed hypersurface, denoted by M. Let $L_{ij}(x)$ be the 2nd fundamental form at $x \in M$. Suppose M is $(k+1)$-convex when $2 \le k \le n-1$, i.e. the second fundamental form $L_{ij} \in \Gamma^+_{k+1}$; and suppose M is n-convex when $k = n$. Then for $m \le k$, there exists a constant C depending only on m and n such that*

$$\left(\int_{M^n} \sigma_{m-1}(L)d\mu_M\right)^{\frac{1}{n-(m-1)}} \le C\left(\int_{M^n} \sigma_m(L)d\mu_M\right)^{\frac{1}{n-m}}. \tag{29}$$

The proof of our main theorem hinges on the following proposition (Proposition 3.1), the proof of which is the main technical part of this paper.

Proposition 3.1. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional linear subspace, and p be the orthogonal projection from $\mathbb{R}^{n+1}$ to E. Suppose $V : E \to \mathbb{R}$ is a C^3 convex function that satisfies $|\nabla V| \le 1$. Define its extension to $\mathbb{R}^{n+1}$ by $\bar{V} := V \circ p$, and define the restriction of $\bar{V}$ to the closed hypersurface M by v. Suppose also that M is $(k+1)$-convex if $2 \le k \le n-1$, i.e. the second fundamental form $L_{ij} \in \Gamma^+_{k+1}$. And suppose that M is n-convex if $k = n$. Then for each k and each constant $a > 1$, there exists a constant C, which depends only on k, n and a, such that*

$$\int_M \sigma_k(D^2 v + aL)d\mu_M \le C\int_M \sigma_k(L)d\mu_M. \tag{30}$$

Note that C does not depend on v.

Our proof of Proposition 3.1 uses a multi-layer induction process and is quite complicated. We will first illustrate the idea of the proof of the proposition for the (easy) case $k=2$ in Section 4, where the role of Gauss-Codazzi equation plays a central role; then for the case $k=3$ in Section 5, where in addition, the convexity of the Brenier function in the mass transport equation is crucial in establishing the inequality; finally we will finish the proof for all integers k in Section 6.

In the rest of this section, we will prove our main theorem assuming Proposition 3.1. The first part of our proof uses techniques of optimal transport maps following the same outline as in the work of P. Castillon [9]; we will also apply the concavity properties of σ_k as discussed in Section 2.1.1 of this paper.

Proof of Theorem 1.3. First of all, it is obvious that we only need to prove the inequality for $m=k$ when M is $k+1$-convex, that is we will establish the inequality

$$\left(\int_{M^n} \sigma_{k-1}(L)d\mu_M\right)^{\frac{1}{n-(k-1)}} \leq C\left(\int_{M^n} \sigma_k(L)d\mu_M\right)^{\frac{1}{n-k}}. \tag{31}$$

Let $E\subset\mathbb{R}^{n+1}$ be an n-dimensional linear subspace, $p:\mathbb{R}^{n+1}\to E$ be the orthogonal projection, and J_E be the Jacobian of p. We define

$$f := \frac{\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}}{\int_M \sigma_{k-1}(L)J_E^{\frac{1}{n-k}}d\mu_M}. \tag{32}$$

Note that $\mu := fd\mu_M$ is a probability measure on M. So the pushforward measure $\omega_1 := p_{\#}\mu$ is a probability measure on E. It is absolutely continuous with respect to the Lebesgue measure on E with density $F(y_1)$ given by

$$F(y_1) = \sum_{x\in p^{-1}(y_1)\cap Spt(\mu)} \frac{f(x)}{J_E(x)}. \tag{33}$$

Applying Brenier's theorem, there exists a convex potential V such that ∇V is the solution of Monge problem on E between $(D_1, F(y_1)dy_1)$ and $(D_2, G(y_2)dy_2)$, where $D_1 := Spt(p_{\#}\mu)$; $D_2 := B_E(0,1)$ is the unit ball in E; $F(y_1)$ is defined as above; and $G(y_2)dy_2 := \dfrac{\chi_{B_E(0,1)}}{\omega_n}dy_2$ is the normalized Lebesgue measure on $B_E(0,1)$. Since $\nabla V(Spt(p_{\#}\mu))\subset B_E(0,1)$, we have $|\nabla V|\leq 1$ on D_1.

In general, the convex potential V is only a Lipschitz function. But let us suppose V to be C^3 for a moment to finish the proof of the theorem. Later, we will present an approximation argument to justify this assumption. If V is C^3, then V satisfies the Monge-Ampère equation

$$\omega_n F(y_1) = det(D^2V(y_1))$$

in the classical sense. Define the extension of V by $\bar{V} := V\circ p : \mathbb{R}^{n+1}\to\mathbb{R}$ and its restriction to M by $v(x) := \bar{V}|_M(x) = V\circ p|_M(x)$. Denote the gradient and the Hessian on M by ∇ and D^2 respectively. And denote the gradient and the Hessian on $\mathbb{R}^{n+1}$ by $\bar{\nabla}$ and $\bar{D}^2$ respectively. By (33), for $x\in M$

$$\omega_n\frac{f(x)}{J_E} \leq \omega_n F(p(x)) = det(D^2V(p(x))). \tag{34}$$

By the change of variable formula,

$$det(\bar{D}^2\bar{V}(x)|_{T_xM}) = J_E^2(x)det(D^2V(p(x))).$$

Thus for $x \in M$

$$\omega_n f(x)J_E(x) \leq det(\bar{D}^2\bar{V}(x)|_{T_xM}). \tag{35}$$

Since $\bar{D}^2\bar{V}(x)|_{T_xM}$ is a nonnegative matrix, we take the $n-k+1$-th root on both sides of (35).

$$(\omega_n f(x)J_E(x))^{\frac{1}{n-k+1}} \leq \left(det(\bar{D}^2\bar{V}(x)|_{T_xM})\right)^{\frac{1}{n-k+1}}. \tag{36}$$

To simplify the notation, from now on we will denote $\bar{D}^2\bar{V}(x)|_{T_xM}$ by $\bar{D}^2\bar{V}(x)$.

For each positive constant $a > 1$, multiplying the previous inequality by $\frac{\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V})^{\frac{1}{n-k+1}}}$, we get

$$\begin{aligned}
&(\omega_n f(x)J_E(x))^{\frac{1}{n-k+1}} \cdot \frac{\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V})^{\frac{1}{n-k+1}}} \\
\leq &\left(det(\bar{D}^2\bar{V}(x))\right)^{\frac{1}{n-k+1}} \cdot \frac{\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V})^{\frac{1}{n-k+1}}}.
\end{aligned} \tag{37}$$

Denote the left hand side (resp. right hand side) of this inequality by LHS (resp. RHS). Then

$$RHS = \left(\frac{det(\bar{D}^2\bar{V})}{\sigma_{k-1}(\bar{D}^2\bar{V})}\right)^{\frac{1}{n-k+1}} \sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L). \tag{38}$$

Note that for nonnegative symmetric matrix A, we have the well-known Newton-MacLaurin inequality: (see e.g. [18])

$$\frac{\sigma_{k+1}(A)\sigma_{k-1}(A)}{\sigma_{k+1}(Id)\sigma_{k-1}(Id)} \leq \frac{\sigma_k^2(A)}{\sigma_k^2(Id)}, \tag{39}$$

where Id is the identity matrix. This implies that

$$\frac{\sigma_{k+1}(A)\sigma_k(Id)}{\sigma_k(A)\sigma_{k+1}(Id)} \tag{40}$$

is decreasing in k. Thus

$$\begin{aligned}
\frac{\sigma_n(A)}{\sigma_{k-1}(A)} =&\frac{\sigma_n(A)}{\sigma_{n-1}(A)} \cdots \frac{\sigma_k(A)}{\sigma_{k-1}(A)} \\
\leq &\prod_{i=k}^{n} \frac{\sigma_k(A)\sigma_{k-1}(Id)\sigma_i(Id)}{\sigma_{k-1}(A)\sigma_k(Id)\sigma_{i-1}(Id)} \\
=&C_{n,k}\left(\frac{\sigma_k(A)}{\sigma_{k-1}(A)}\right)^{n-k+1}.
\end{aligned} \tag{41}$$

Therefore

$$\left(\frac{det(\bar{D}^2\bar{V})}{\sigma_{k-1}(\bar{D}^2\bar{V})}\right)^{\frac{1}{n-k+1}} \leq C_{n,k}^{\frac{1}{n-k+1}} \frac{\sigma_k(\bar{D}^2\bar{V})}{\sigma_{k-1}(\bar{D}^2\bar{V})}. \tag{42}$$

Also $(\frac{\sigma_k(A)}{\sigma_j(A)})^{\frac{1}{k-j}}$ is concave in Γ_k^+ for $j<k$. Thus for $L\in\Gamma_k^+$, we have

$$\frac{\sigma_k(\bar{D}^2\bar{V})}{\sigma_{k-1}(\bar{D}^2\bar{V})}\le\frac{\sigma_k(\bar{D}^2\bar{V}+(a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)}. \tag{43}$$

Therefore

$$\begin{aligned}RHS\le&C_{n,k}^{\frac{1}{n-k+1}}\frac{\sigma_k(\bar{D}^2\bar{V}+(a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)}\cdot\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)\\=&C_{n,k}^{\frac{1}{n-k+1}}\sigma_k(\bar{D}^2\bar{V}+(a-1)L).\end{aligned} \tag{44}$$

Note that $D^2v(\xi,\eta)=\bar{D}^2\bar{V}(\xi,\eta)+b(x)\cdot L(\xi,\eta)$ for $\xi,\eta\in T_xM$, where $b(x)=\langle\bar{\nabla}\bar{V}(x),\vec{n}(x)\rangle$.

Since $|\nabla V(x)|\le 1$, we know that $|\bar{\nabla}\bar{V}(x)|\le 1$, and thus $|b(x)|\le 1$. Therefore by Garding's inequality

$$\sigma_k(\bar{D}^2\bar{V}+(a-1)L)=\sigma_k(D^2v+(a-1)L+b(x)L)\le\sigma_k(D^2v+aL).$$

Thus

$$RHS\le C_{n,k}^{\frac{1}{n-k+1}}\sigma_k(D^2v+aL). \tag{45}$$

On the other hand, $\bar{D}^2\bar{V}\in\Gamma_n^+$. Therefore by Garding's inequality, $\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)\ge\sigma_{k-1}((a-1)L)=(a-1)^{k-1}\sigma_{k-1}(L)$. This together with the definition of $f(x)$ in (32) implies that

$$LHS\ge\frac{(a-1)^{(k-1)\cdot(1-\frac{1}{n-k+1})}\omega_n^{\frac{1}{n-k+1}}\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}}{(\int_M\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}d\mu_M)^{\frac{1}{n-k+1}}}. \tag{46}$$

By integrating LHS and RHS in (37) over M, one obtains

$$\begin{aligned}&\frac{(a-1)^{(k-1)\cdot(1-\frac{1}{n-k+1})}\omega_n^{\frac{1}{n-k+1}}\int_M\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}d\mu_M}{(\int_M\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}d\mu_M)^{\frac{1}{n-k+1}}}\\\le&C_{n,k}^{\frac{1}{n-k+1}}\int_M\sigma_k(D^2v+aL)d\mu_M.\end{aligned} \tag{47}$$

Thus

$$\left(\int_M\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}d\mu_M\right)^{1-\frac{1}{n-k+1}}\le(a-1)^{-(k-1)}\omega_n^{\frac{-1}{n-k+1}}C_{n,k}^{\frac{1}{n-k+1}}\int_M\sigma_k(D^2v+aL)d\mu_M. \tag{48}$$

We now apply Proposition 3.1 to V. Then there is a constant C depending only on k, n and a (not on V), such that $\int_M\sigma_k(D^2v+aL)d\mu_M\le C\int_M\sigma_k(L)d\mu_M$. If we apply the above argument to this constant a, then

$$\left(\int_M\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}d\mu_M\right)^{1-\frac{1}{n-k+1}}\le\tilde{C}\int_M\sigma_k(L)d\mu_M, \tag{49}$$

where constant $\tilde{C}$ depends on k, n and a. Fix $a = 2$. Then $\tilde{C}$ depends only on k and n. To get the usual A-F inequality (without the weight function J_E), one can integrate both sides of the above inequality on the Grassmannian $G_{n,n+1}$ of n-planes in $\mathbb{R}^{n+1}$. Since the integration of $\int_{G_{n,n+1}} J_E^{\frac{1}{n-k}} dE$ is invariant in $x \in M$, therefore

$$\left(\int_M \sigma_{k-1}(L) d\mu_M \right)^{\frac{1}{n-k+1}} \leq \tilde{C} \left(\int_M \sigma_k(L) d\mu_M \right)^{\frac{1}{n-k}}, \tag{50}$$

for another constant, still denoted by $\tilde{C}$. As before, $\tilde{C}$ depends only on k and n. This finishes the proof of the theorem under the assumption that V is a C^3 function.

We will now apply Caffarelli's regularity results Theorem 2.10. If the density $F(y_1)$ is bounded away from zero and infinity, and also if D_1 is a strictly convex domain, then by Caffarelli's result, V is a smooth convex potential. We will now describe how to obtain a sequence of smooth maps ∇V_ϵ, such that each transports the measure $F_\epsilon(y_1)dy_1$ to $\frac{\chi_{B_E(0,1)}}{\omega_n} dy_2$ on the unit ball, and we let $F_\epsilon(y_1)dy_1$ approximate to $F(y_1)dy_1$. First of all, there exists a constant $R > 0$, such that D_1 is contained in $B_E(0, R)$, the ball centered at the origin with radius R in E. For $\epsilon > 0$, define the subset $D_1^\epsilon := \{y_1 \in D_1 | \ \epsilon \leq F(y_1) \leq 1/\epsilon\}$. Since $F(y_1)$ is integrable on D_1 and $F(y_1) \geq 0$, we know $D_1^\epsilon \to Spt(F)$, as $\epsilon \to 0$. One can extend $F|_{D_1^\epsilon}$ to $F_\epsilon : B_E(0, R) \to \mathbb{R}$, such that $\frac{\epsilon}{2} \leq F_\epsilon(y_1) \leq \frac{2}{\epsilon}$ on $B_E(0, R)$, and

$$\int_{B_E(0,R)\setminus D_1^\epsilon} F_\epsilon(y_1)dy_1 \leq \epsilon \cdot \omega_n R^n.$$

Such an extension exists because $\epsilon \leq F|_{D_1^\epsilon} \leq \frac{1}{\epsilon}$, and $Vol(B_E(0, R) \setminus D_1^\epsilon) \leq Vol(B_E(0, R)) \leq \omega_n R^n$. Therefore

$$m_\epsilon := \int_{B_E(0,R)} F_\epsilon(y_1)dy_1 = \int_{B_E(0,R)\setminus D_1^\epsilon} F_\epsilon(y_1)dy_1 + \int_{D_1^\epsilon} F_\epsilon(y_1)dy_1 \leq c_0\epsilon + 1, \tag{51}$$

where $c_0 = \omega_n R^n$. Also

$$m_\epsilon \geq \int_{D_1^\epsilon} F_\epsilon(y_1)dy_1 \to 1, \tag{52}$$

as $\epsilon \to 0$. Hence $m_\epsilon \to 1$, as $\epsilon \to 0$. Now for each sufficiently small ϵ, $m_\epsilon > 0$. Thus $\frac{F_\epsilon(y_1)}{m_\epsilon} dy_1$ is a probability measure on $B_E(0, R)$, such that $0 < \frac{\epsilon}{4} < \frac{F_\epsilon(y_1)}{m_\epsilon} \leq \frac{4}{\epsilon}$ on $B_E(0, R)$. As before, Brenier's theorem implies that there exists a convex potential V_ϵ such that ∇V_ϵ is the solution of Monge problem between $(B_E(0, R), \dfrac{F_\epsilon(y_1)}{m_\epsilon} dy_1)$ and $(B_E(0, 1), \dfrac{\chi_{B_E(0,1)}}{\omega_n} dy_2)$. By Theorem 2.10, V_ϵ is a smooth convex potential. Obviously $|\nabla V_\epsilon(y_1)| \leq 1$ for $y_1 \in B_E(0, R)$. Also V_ϵ satisfies the Monge-Ampère equation $\omega_n \dfrac{F_\epsilon(y_1)}{m_\epsilon} = det(D^2 V_\epsilon(y_1))$ in the classical sense. Define the extension of V_ϵ by $\bar{V}_\epsilon := V_\epsilon \circ p : \mathbb{R}^{n+1} \to \mathbb{R}$ and its restriction to M by $v_\epsilon(x) := \bar{V}_\epsilon|_M(x) = V_\epsilon \circ p|_M(x)$. Denote the gradient and the Hessian on M by ∇ and D^2 respectively. And denote the gradient and the Hessian on $\mathbb{R}^{n+1}$ by $\bar{\nabla}$ and $\bar{D}^2$ respectively. Note that on $p^{-1}(D_1^\epsilon)$, $F(y_1) = F_\epsilon(y_1)$. This together with (33) implies that for $x \in p^{-1}(D_1^\epsilon)$

$$\omega_n \frac{f(x)}{m_\epsilon J_E} \leq \omega_n \frac{F(p(x))}{m_\epsilon} = \omega_n \frac{F_\epsilon(p(x))}{m_\epsilon} = det(D^2 V_\epsilon(p(x))). \tag{53}$$

Following the same argument that proves (37) for V, we get for $x \in p^{-1}(D_1^\epsilon)$

$$\begin{aligned}
&\left(\omega_n \frac{f(x)J_E(x)}{m_\epsilon}\right)^{\frac{1}{n-k+1}} \cdot \frac{\sigma_{k-1}(\bar{D}^2\bar{V}_\epsilon + (a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V}_\epsilon)^{\frac{1}{n-k+1}}} \\
\leq &\left(det(\bar{D}^2\bar{V}_\epsilon(x))\right)^{\frac{1}{n-k+1}} \cdot \frac{\sigma_{k-1}(\bar{D}^2\bar{V}_\epsilon + (a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V}_\epsilon)^{\frac{1}{n-k+1}}}.
\end{aligned} \tag{54}$$

Denote the left hand side (resp. right hand side) of this inequality by LHS_ϵ (resp. RHS_ϵ). Then by the same techniques as before

$$RHS_\epsilon \leq C_{n,k}^{\frac{1}{n-k+1}} \sigma_k(D^2 v_\epsilon + aL). \tag{55}$$

And

$$LHS_\epsilon \geq \frac{(a-1)^{(k-1)\cdot(1-\frac{1}{n-k+1})}\omega_n^{\frac{1}{n-k+1}}\sigma_{k-1}(L)J_E^{\frac{1}{n-k}}}{(m_\epsilon \int_M \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M)^{\frac{1}{n-k+1}}}. \tag{56}$$

By integrating LHS_ϵ and RHS_ϵ in (54) over $M \cap p^{-1}(D_1^\epsilon)$, one obtains

$$\begin{aligned}
&\frac{(a-1)^{(k-1)\cdot(1-\frac{1}{n-k+1})}\omega_n^{\frac{1}{n-k+1}} \int_{M\cap p^{-1}(D_1^\epsilon)} \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M}{(m_\epsilon \int_M \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M)^{\frac{1}{n-k+1}}} \\
\leq &C_{n,k}^{\frac{1}{n-k+1}} \int_{M\cap p^{-1}(D_1^\epsilon)} \sigma_k(D^2 v_\epsilon + aL) d\mu_M \\
\leq &C_{n,k}^{\frac{1}{n-k+1}} \int_M \sigma_k(D^2 v_\epsilon + aL) d\mu_M.
\end{aligned} \tag{57}$$

Since V_ϵ is smooth (thus C^3), we may apply the above argument and Proposition 3.1 to obtain for each ϵ, $\int_M \sigma_k(D^2 v_\epsilon + aL) d\mu_M \leq C \int_M \sigma_k(L) d\mu_M$ with the constant C depending only on k, n and a. (Note that C is independent of ϵ.) Fix $a = 2$. Then C depends only on k and n. Thus

$$\frac{\int_{M\cap p^{-1}(D_1^\epsilon)} \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M}{(m_\epsilon \int_M \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M)^{\frac{1}{n-k+1}}} \leq \tilde{C} \int_M \sigma_k(L) d\mu_M, \tag{58}$$

where $\tilde{C}$ depends on k and n, and does not depend on ϵ. Let $\epsilon \to 0$ in this inequality. Since $m_\epsilon \to 1$ and $M \cap p^{-1}(D_1^\epsilon) \to M \cap p^{-1}(Spt(F))$ as $\epsilon \to 0$. By (33), $M \cap Spt(f) \subset M \cap p^{-1}(Spt(F))$. Thus we obtain

$$(\int_M \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M)^{1-\frac{1}{n-k+1}} \leq \tilde{C} \int_M \sigma_k(L) d\mu_M. \tag{59}$$

Equivalently,

$$\int_M \sigma_{k-1}(L)J_E^{\frac{1}{n-k}} d\mu_M \leq (\tilde{C} \int_M \sigma_k(L) d\mu_M)^{\frac{n-k+1}{n-k}}. \tag{60}$$

To get the usual A-F inequality (without the weight function J_E), we can integrate both sides of the above inequality on the Grassmannian $G_{n,n+1}$ of n-planes in $\mathbb{R}^{n+1}$. Since the integration of $\int_{G_{n,n+1}} J_E^{\frac{1}{n-k}} dE$ is invariant in $x \in M$, we have

$$(\int_M \sigma_{k-1}(L) d\mu_M)^{\frac{1}{n-k+1}} \leq \tilde{C}(\int_M \sigma_k(L) d\mu_M)^{\frac{1}{n-k}}, \tag{61}$$

for another constant, still denoted by $\tilde{C}$. As before $\tilde{C}$ depends only on k and n. This finishes the proof of the theorem. □

4 $k = 2$ case of Proposition 3.1

In this section, we are going to prove Proposition 3.1 when $k = 2$. For this special case, only $|\nabla V| \leq 1$ property of the Brenier map is relevant. For simplicity, we choose $a = 2$.

Proof. We first recall that $\frac{1}{2}\Sigma_2(A, A) = \sigma_2(A)$, thus

$$\begin{aligned} \int_M \sigma_2(D^2 v + 2L) d\mu_M &= \int_M \frac{1}{2}\Sigma_2(D^2 v + 2L) d\mu_M \\ &= \int_M \frac{1}{2}[\Sigma_2(D^2 v, D^2 v) + 4\Sigma_2(D^2 v, L) + 4\Sigma_2(L, L)] d\mu_M \\ &= \int_M \sigma_2(D^2 v) + 2\Sigma_2(D^2 v, L) + 4\sigma_2(L) d\mu_M \\ &:= I_{2,2} + 2I_{2,1} + 4I_{2,0}. \end{aligned} \tag{62}$$

By the integration by parts formula,

$$I_{2,2} := \int_M \sigma_2(D^2 v) d\mu_M = \int_M v_{ii} v_{jj} - v_{ij} v_{ij} d\mu_M = \int_M -v_i(v_{jji} - v_{ijj}) d\mu_M. \tag{63}$$

If we apply the curvature equation

$$v_{ijk} - v_{ikj} = R_{mijk} v_m, \tag{64}$$

then

$$I_{2,2} = \int_M v_i Rc_{mi} v_m d\mu_M, \tag{65}$$

where Rc is the Ricci curvature tensor of g on M. By the Gauss equation (27), the Ricci curvature tensor satisfies $Rc_{ik} = L_{jj} L_{ik} - L_{ij} L_{jk}$. If we diagonalize $L_{ij} \sim diag(\lambda_1, ..., \lambda_n)$, then $Rc \sim diag(\lambda_1(H - \lambda_1), ..., \lambda_n(H - \lambda_n))$. Note that

$$\lambda_i(H - \lambda_i) + \frac{\partial \sigma_3(L)}{\partial \lambda_i} = \sigma_2(L) \tag{66}$$

for each $i = 1, ..., n$. Also by our assumption $L \in \Gamma_3^+$, we know that $\frac{\partial \sigma_3(L)}{\partial \lambda_i} > 0$ for each i. Thus $\lambda_i(H - \lambda_i) < \sigma_2(L)$ for each i, i.e. $Rc < \sigma_2(L) \cdot g$. Applying this formula to the inequality (65), we get

$$I_{2,2} \leq \int_M \sigma_2(L)|\nabla v|^2 d\mu_M \leq \int_M \sigma_2(L) d\mu_M, \tag{67}$$

where $|\nabla v| \leq 1$ because $|\bar{\nabla}\bar{V}| \leq 1$. Thus

$$I_{2,2} \leq \int_M \sigma_2(L) d\mu_M. \tag{68}$$

For the term $I_{2,1}$, by definition $\Sigma_2(D^2 v, L) = v_{ii}L_{jj} - v_{ij}L_{ij}$. Thus

$$\begin{aligned} I_{2,1} &:= \int_M \Sigma_2(D^2 v, L) d\mu_M \\ &= \int_M v_{ii}L_{jj} - v_{ij}L_{ij} d\mu_M \\ &= \int_M -v_i L_{jj,i} + v_i L_{ij,j} d\mu_M. \end{aligned} \tag{69}$$

Due to the Codazzi equation (28), $I_{2,1} = 0$.
Finally,

$$I_{2,0} := \int_M \sigma_2(L) d\mu_M. \tag{70}$$

Hence

$$\begin{aligned} \int_M \sigma_2(\bar{D}^2\bar{V}|_{T_xM}) d\mu_M \leq & I_{2,2} + 2I_{2,1} + 4I_{2,0} \\ \leq & 5 \int_M \sigma_2(L) d\mu_M. \end{aligned} \tag{71}$$

This finishes the proof of Proposition 3.1 when $k = 2$. □

5 $k = 3$ case of Proposition 3.1

In this section, we are going to prove Proposition 3.1 when $k = 3$. The convexity property of $\bar{V}$ together with the size estimate $|\bar{\nabla}V| \leq 1$ both play a role in this special case of Proposition 3.1. We still denote $\bar{D}^2\bar{V}|_{T_xM}$ by $\bar{D}^2\bar{V}$ in this section. We will begin by proving the following two lemmas.

Lemma 5.1. *Suppose v and M satisfy the same conditions as in Proposition 3.1. Then*

$$I_{3,1} := \int_M \Sigma_3(D^2 v, L, L) d\mu_M = 0. \tag{72}$$

Proof. The proof of the lemma uses the symmetry of Σ_3 and the Codazzi equation. It proceeds in the following way. By definition of $I_{3,1}$,

$$\begin{aligned} I_{3,1} := \int_M \Sigma_3(D^2 v, L, L) d\mu_M &= \int_M \frac{1}{2!} v_{ij} \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j_1} L_{i_2j_2} d\mu_M \\ &= \int_M \frac{-1}{2!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} \left(L_{i_1j_1,j} L_{i_2j_2} + L_{i_1j_1} L_{i_2j_2,j}\right) d\mu_M. \end{aligned} \tag{73}$$

Since $\delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j_1,j} L_{i_2j_2} = \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j_1} L_{i_2j_2,j}$, we have

$$I_{3,1} = \int_M -v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j_1,j} L_{i_2j_2} d\mu_M. \tag{74}$$

Also, it is not hard to see that $\delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j_1,j} L_{i_2j_2} = \delta^{i,i_1,i_2}_{j_1,j,j_2} L_{i_1j,j_1} L_{i_2j_2}$, because j and j_1 are dummy variables. Also, $\delta^{i,i_1,i_2}_{j_1,j,j_2} = -\delta^{i,i_1,i_2}_{j,j_1,j_2}$. Therefore

$$\begin{aligned} \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j_1,j} L_{i_2j_2} &= -\, \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1j,j_1} L_{i_2j_2} \\ &= \frac{1}{2} \delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{i_1j_1,j} - L_{i_1j,j_1}) L_{i_2j_2}, \end{aligned} \tag{75}$$

which implies that

$$I_{3,1} = \int_M -\frac{1}{2} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{i_1j_1,j} - L_{i_1j,j_1}) L_{i_2j_2} d\mu_M = 0, \tag{76}$$

by the Codazzi equation (28). Thus the lemma holds. □

Lemma 5.2. *Suppose v and M satisfy the same conditions as in Proposition 3.1. Then*

$$I_{3,2} := \int_M \Sigma_3(D^2v, D^2v, L) d\mu_M \le \int_M \sigma_3(L) d\mu_M. \tag{77}$$

Proof. We perform the integration by parts to get

$$\begin{aligned} I_{3,2} :=& \int_M \Sigma_3(D^2v, D^2v, L) d\mu_M \\ =& \int_M \frac{1}{2!} v_{ij} \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1} L_{i_2j_2} d\mu_M \\ =& \int_M \frac{-1}{2!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} \left(v_{i_1j_1j} L_{i_2j_2} + v_{i_1j_1} L_{i_2j_2,j} \right) d\mu_M := A + B. \end{aligned} \tag{78}$$

By the same argument as in (75) and the curvature equation (64),

$$\begin{aligned} A :=& \int_M \frac{-1}{2!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1j} L_{i_2j_2} d\mu_M \\ =& \int_M \frac{-1}{4} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1j_1j} - v_{i_1jj_1}) L_{i_2j_2} d\mu_M \\ =& \int_M \frac{1}{4} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} R_{mi_1jj_1} v_m L_{i_2j_2} d\mu_M. \end{aligned} \tag{79}$$

Using the Gauss equation (27) in (79), we get

$$\begin{aligned} A =& \int_M \frac{1}{4} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{mj} L_{i_1j_1} - L_{mj_1} L_{i_1j}) L_{i_2j_2} d\mu_M \\ =& \int_M \frac{1}{2} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj} L_{i_1j_1} L_{i_2j_2} d\mu_M \\ =& \int_M [T_2]_{ij}(L, L) L_{mj} v_i v_m d\mu_M. \end{aligned} \tag{80}$$

Now, we use the formula (17) for $k = 3$, i.e.

$$[T_2]_{ij}(L,L)L_{mj} = \sigma_3(L)\delta_{im} - [T_3]_{im}(L), \tag{81}$$

and note that when $M \in \Gamma_4^+$, $[T_3]_{im}(L,L,L) \geq 0$. Thus

$$\begin{aligned} A =& \int_M \sigma_3(L)|\nabla v|^2 - [T_3]_{im}(L,L,L)v_i v_m d\mu_M \\ \leq& \int_M \sigma_3(L) d\mu_M. \end{aligned} \tag{82}$$

Also,

$$\begin{aligned} B :=& \int_M \frac{-1}{2!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1} L_{i_2 j_2, j} d\mu_M \\ =& \int_M \frac{-1}{4} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1} \left(L_{i_2 j_2, j} - L_{i_2 j, j_2}\right) d\mu_M = 0, \end{aligned} \tag{83}$$

by the Codazzi equation (28). In conclusion, (82) and (83) imply that

$$I_{3,2} = A + B \leq \int_M \sigma_3(L) d\mu_M. \tag{84}$$

This completes the proof of (77). $\square$

We now prove Proposition 3.1 for $k = 3$. Again, for simplicity, we only demonstrate the proof for $a = 2$.

Proof. By the polarization formula of σ_k,

$$\begin{aligned} \int_M \sigma_3(D^2 v + 2L) d\mu_M =& \int_M \frac{1}{3}\Sigma_3(D^2 v + 2L, D^2 v + 2L, D^2 v + 2L) d\mu_M \\ =& \int_M \frac{1}{3}[\Sigma_3(D^2 v, D^2 v, D^2 v) + 6\Sigma_3(D^2 v, D^2 v, L) + 12\Sigma_3(D^2 v, L, L) \\ &+ 8\Sigma_3(L,L,L)] d\mu_M \\ =& \int_M \sigma_3(D^2 v) + 2\Sigma_3(D^2 v, D^2 v, L) + 4\Sigma_3(D^2 v, L, L) + 8\sigma_3(L) d\mu_M \\ :=& I_{3,3} + 2I_{3,2} + 4I_{3,1} + 8I_{3,0}. \end{aligned} \tag{85}$$

Note that

$$I_{3,0} := \int_M \sigma_3(L) d\mu_M, \tag{86}$$

and by Lemma 5.1 and Lemma 5.2,

$$I_{3,1} = 0. \quad I_{3,2} \leq \int_M \sigma_3(L) d\mu_M.$$

Now we are going to show

$$I_{3,3} \leq \int_M \sigma_3(L) d\mu_M.$$

First of all,

$$
\begin{aligned}
I_{3,3} :=& \int_M \sigma_3(D^2 v) d\mu_M \\
=& \int_M \frac{1}{3!} v_{ij} \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1} v_{i_2 j_2} d\mu_M \\
=& \int_M \frac{-1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} \left(v_{i_1 j_1 j} v_{i_2 j_2} + v_{i_1 j_1} v_{i_2 j_2 j} \right) d\mu_M .
\end{aligned}
\tag{87}
$$

For the same reason as we present in the proof of (72),

$$
\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1 j} v_{i_2 j_2} = \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1} v_{i_2 j_2 j}.
$$

Thus

$$
I_{3,3} = \int_M \frac{-2}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1 j} v_{i_2 j_2} d\mu_M . \tag{88}
$$

Also

$$
\begin{aligned}
\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1 j} v_{i_2 j_2} =& - \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j j_1} v_{i_2 j_2} \\
=& \frac{1}{2} \delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1 j_1 j} - v_{i_1 j j_1}) v_{i_2 j_2}.
\end{aligned}
\tag{89}
$$

This together with the curvature equation (64) gives

$$
\begin{aligned}
I_{3,3} =& \int_M \frac{-1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1 j_1 j} - v_{i_1 j j_1}) v_{i_2 j_2} d\mu_M \\
=& \int_M \frac{1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} R_{m i_1 j j_1} v_m v_{i_2 j_2} d\mu_M .
\end{aligned}
\tag{90}
$$

By the Gauss equation (27),

$$
\begin{aligned}
I_{3,3} =& \int_M \frac{1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{mj} L_{i_1 j_1} - L_{m j_1} L_{i_1 j}) v_m v_{i_2 j_2} d\mu_M \\
=& \int_M \frac{2}{3!} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj} L_{i_1 j_1} v_{i_2 j_2} d\mu_M .
\end{aligned}
\tag{91}
$$

Note that by (13)

$$
[T_2]_{ij}(D^2 v, L) = \frac{1}{2!} \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1 j_1} v_{i_2 j_2}. \tag{92}
$$

Thus

$$
\begin{aligned}
I_{3,3} =& \int_M \frac{2}{3!} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj} L_{i_1 j_1} v_{i_2 j_2} d\mu_M \\
=& \int_M \frac{4}{3!} v_i v_m [T_2]_{ij}(D^2 v, L) L_{mj} d\mu_M .
\end{aligned}
\tag{93}
$$

If we apply Lemma 2.6 to $k = 3$, $B = D^2 v$ and $C = L$, then

$$
[T_2]_{ij}(D^2 v, L) L_{mj} = \frac{1}{2} \Sigma_3(D^2 v, L, L) \delta_{im} - \frac{3}{2} [T_3]_{im}(D^2 v, L, L) - \frac{1}{2} [T_2]_{ij}(L, L) v_{mj}. \tag{94}
$$

We can plug it into (93) to get

$$
\begin{aligned}
I_{3,3} =& \int_M \frac{4}{3!} v_i v_m \left(\frac{1}{2}\Sigma_3(D^2v, L, L)\delta_{im} - \frac{3}{2}[T_3]_{im}(D^2v, L, L) - \frac{1}{2}[T_2]_{ij}(L, L)v_{mj} \right) d\mu_M \\
:=& \frac{1}{3} I_{3,1}^{|\nabla v|^2} + J_{3,1}^{(-1)} + \frac{1}{3} K_{3,0}^{(-1)}.
\end{aligned}
\tag{95}
$$

To estimate $I_{3,1}^{|\nabla v|^2}$, we will use $|\nabla v|$, $|b(x)| \leq 1$. We will also use the fact that $\Sigma_3(\bar{D}^2\bar{V}, L, L) \geq 0$ because $\bar{D}^2\bar{V} \geq 0$ and $L \in \Gamma_3^+$. Therefore if we replace D^2v by $\bar{D}^2\bar{V} + b(x)L$ in $I_{3,1}^{|\nabla v|^2}$, then

$$
\begin{aligned}
I_{3,1}^{|\nabla v|^2} :=& \int_M |\nabla v|^2 \Sigma_3(D^2v, L, L) d\mu_M \\
=& \int_M |\nabla v|^2 \Sigma_3(\bar{D}^2\bar{V} + b(x)L, L, L) d\mu_M \\
\leq& \int_M \Sigma_3(\bar{D}^2\bar{V}, L, L) + \Sigma_3(L, L, L) d\mu_M \\
=& \int_M \Sigma_3(D^2v - b(x)L, L, L) + \Sigma_3(L, L, L) d\mu_M \\
\leq& \int_M \Sigma_3(D^2v, L, L) + 2\Sigma_3(L, L, L) d\mu_M \\
=& \int_M \Sigma_3(D^2v, L, L) + 6\sigma_3(L) d\mu_M.
\end{aligned}
\tag{96}
$$

By Lemma 5.1,

$$
\int_M \Sigma_3(D^2v, L, L) d\mu_M = 0.
$$

So

$$
I_{3,1}^{|\nabla v|^2} \leq 6 \int_M \sigma_3(L) d\mu_M. \tag{97}
$$

To analyze the term $J_{3,1}^{(-1)}$, we use $D^2v = \bar{D}^2\bar{V} + b(x)L$ to get

$$
\begin{aligned}
J_{3,1}^{(-1)} :=& \int_M -v_i v_m [T_3]_{im}(D^2v, L, L) d\mu_M \\
=& \int_M -v_i v_m [T_3]_{im}(\bar{D}^2\bar{V}, L, L) - v_i v_m [T_3]_{im}(L, L, L) b(x) d\mu_M.
\end{aligned}
\tag{98}
$$

Again $\bar{D}^2\bar{V}$ is positive definite and $L \in \Gamma_4^+$. Thus $[T_3]_{im}(\bar{D}^2\bar{V}, L, L) \geq 0$ and $[T_3]_{im}(L, L, L) \geq 0$. Also, $|\nabla v| \leq 1$. Therefore

$$
\begin{aligned}
J_{3,1}^{(-1)} \leq& \int_M Tr([T_3]_{ij})(L, L, L) d\mu_M \\
=& \int_M (n-3)\sigma_3(L) d\mu_M.
\end{aligned}
\tag{99}
$$

For the last term $\frac{1}{3}K_{3,0}^{(-1)}$,

$$\begin{aligned}\frac{1}{3}K_{3,0}^{(-1)} :=& -\frac{1}{3}\int_M v_i v_m [T_2]_{ij}(L,L) v_{mj} d\mu_M \\ =& -\frac{1}{3}\int_M v_i [T_2]_{ij}(L,L)\frac{1}{2}(|\nabla v|^2)_j d\mu_M \\ =& \frac{1}{6}\int_M v_{ij}[T_2]_{ij}(L,L)|\nabla v|^2 d\mu_M + \frac{1}{6}\int_M v_i([T_2]_{ij}(L,L))_j |\nabla v|^2 d\mu_M.\end{aligned} \tag{100}$$

Since $([T_2]_{ij}(L,L))_j = 0$, which is shown in the proof of Lemma 5.1,

$$\begin{aligned}\frac{1}{3}K_{3,0}^{(-1)} =& \frac{1}{6}\int_M v_{ij}[T_2]_{ij}(L,L)|\nabla v|^2 d\mu_M \\ =& \frac{1}{6}\int_M (\bar{D}^2\bar{V}_{ij} + b(x)L_{ij})[T_2]_{ij}(L,L)|\nabla v|^2 d\mu_M.\end{aligned} \tag{101}$$

Now, we can use $|\nabla v| \le 1$, $|b(x)| \le 1$, as well as $\bar{D}^2\bar{V}_{ij} \ge 0$ and $L_{ij} \in \Gamma_4^+$ to obtain that

$$\frac{1}{3}K_{3,0}^{(-1)} \le \frac{1}{6}\int_M \bar{D}^2\bar{V}_{ij}[T_2]_{ij}(L,L) d\mu_M + \frac{1}{6}\int_M [T_2]_{ij}(L,L)L_{ij} d\mu_M. \tag{102}$$

The second term in the above expression is equal to $\frac{1}{2}\int_M \sigma_3(L)d\mu_M$; and the first term in the above expression can be estimated by using $\bar{D}^2\bar{V}_{ij} = v_{ij} - b(x)L_{ij}$. More precisely,

$$\frac{1}{6}\int_M \bar{D}^2\bar{V}_{ij}[T_2]_{ij}(L,L)d\mu_M = \frac{1}{6}\int_M \Sigma_3(L,L,D^2v) - b(x)\Sigma_3(L,L,L)d\mu_M.$$

By Lemma 5.1, $\int_M \Sigma_3(L,L,D^2v)d\mu_M = 0$. Thus

$$\frac{1}{6}\int_M \bar{D}^2\bar{V}_{ij}[T_2]_{ij}(L,L)d\mu_M = \frac{1}{6}\int_M -b(x)\Sigma_3(L,L,L)d\mu_M \le \frac{1}{2}\int_M \sigma_3(L)d\mu_M.$$

Thus $\frac{1}{3}K_{3,0}^{(-1)} \le C\displaystyle\int_M \sigma_3(L)d\mu_M$.

In conclusion $I_{3,3} = \frac{1}{3}I_{3,1}^{|\nabla v|^2} + J_{3,1}^{(-1)} + \frac{1}{3}K_{3,0}^{(-1)} \le C\int_M \sigma_3(L)d\mu_M$. This finishes the estimate of $I_{3,3}$. And thus

$$\int_M \sigma_3(D^2v + 2L)d\mu_M = I_{3,3} + 2I_{3,2} + 4I_{3,1} + 8I_{3,0} \le C\int_M \sigma_3(L)d\mu_M. \tag{103}$$

□

6 General k case of Proposition 3.1

In this section, we are going to prove the following inequality for all integers k.

Proposition 3.1 *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional linear subspace, and p be the orthogonal projection from $\mathbb{R}^{n+1}$ to E. Suppose $V : E \to \mathbb{R}$ is a C^3 convex potential function with $|\nabla V| \le 1$. Define the extension of V to $\mathbb{R}^{n+1}$ by $\bar{V} := V \circ p$. Define the restriction of $\bar{V}$ to the closed* *hypersurface M by $v := \bar{V}|_M$. Denote the Hessian of v by D^2v or v_{ij}. The covariant derivative is with respect to the metric g of M. Suppose also that M is $(k+1)$-convex if $2 \le k \le n-1$, i.e. the second fundamental form $L_{ij} \in \Gamma^+_{k+1}$. And suppose that M is n-convex if $k = n$. Then for each k and each constant $a > 1$, there exists a constant C, which depends only on k, n and a, such that*

$$\int_M \sigma_k(D^2v + aL)d\mu_M \le C \int_M \sigma_k(L)d\mu_M. \tag{104}$$

Remark 6.1. *Note that if $k = 1$, it is obvious that the inequality is true since $\int_M \Delta v dv_m = 0$. If $k = n$, then Γ^+_{k+1} is not well defined; but one can follow the same argument as below to prove that if $L_{ij} \in \Gamma^+_n$ (i.e. Ω is convex), then $\int_M \sigma_n(D^2v + aL)d\mu_M \le C \int_M \sigma_n(L)d\mu_M$. The only difference in the argument is that $[T_n]_{ij}(A) = 0$ for any A. In the following, we will prove the proposition for the cases $k = 2, ..., n-1$.*

In Section 4 and 5, we have shown

$$I_{k,m} := \int_M \Sigma_k(\overbrace{D^2v, ..., D^2v}^{m}, L, ..., L)d\mu_M \le C \int_M \sigma_k(L)d\mu_M, \tag{105}$$

for all $m \le k$ where $k = 2, 3$. We now prove (105) for all $m \le k$ where $k = 2, ..., n-1$. This will imply Proposition 3.1 for general k. Thus we reduce the problem to prove the following Proposition 6.2.

Proposition 6.2. *With the assumptions of Proposition 3.1, for each $m \le k$, where $k = 2, ..., n-1$, there exists a constant C depending only on k and n, such that*

$$I_{k,m} \le C \int_M \sigma_k(L)d\mu_M. \tag{106}$$

The rest of this section will be devoted to the proof of Proposition 6.2 by an inductive argument. As we will see in the argument below, the estimate of $I_{k,m}$ consists of the estimates of three types of terms–I-type, J-type, K-type. We will handle each type individually using inductive arguments.

Proof. We need two initial inequalities to start the inductive argument since in each induction step the index jumps down by 2. First of all, when $m = 1$ the statement is valid. In fact,

$$I_{k,1} := \int_M \Sigma_k(D^2v, L, ..., L)d\mu_M = 0. \tag{107}$$

The proof is the same as that of Lemma 5.1. Thus we omit it here. For $m = 2$,

$$\begin{aligned} I_{k,2} := \int_M \Sigma_k(D^2v, D^2v, L, ..., L)d\mu_M &= \int_M v_{ij}[T_{k-1}]_{ij}(D^2v, L, ..., L)d\mu_M \\ &= \int_M -v_j([T_{k-1}]_{ij}(D^2v, L, ..., L))_i d\mu_M \\ &= \int_M -v_j \delta^{i,i_1,...,i_{k-1}}_{j,j_1,...,j_{k-1}} v_{i_1j_1i}L_{i_2j_2} \cdots L_{i_{k-1}j_{k-1}}d\mu_M. \end{aligned} \tag{108}$$

Here all the terms involving the covariant derivative of L disappear because if we exchange the positions of the dummy indices i and i_2, then

$$\begin{aligned}\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} v_{i_1j_1} L_{i_2j_2,i}\cdots L_{i_{k-1}j_{k-1}} =&\delta^{i_2,i_1,i,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} v_{i_1j_1} L_{ij_2,i_2}\cdots L_{i_{k-1}j_{k-1}} \\ =&-\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} v_{i_1j_1} L_{ij_2,i_2}\cdots L_{i_{k-1}j_{k-1}},\end{aligned} \tag{109}$$

and thus

$$\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} v_{i_1j_1} L_{i_2j_2,i}\cdots L_{i_{k-1}j_{k-1}} =\frac{1}{2}\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} v_{i_1j_1} (L_{i_2j_2,i}-L_{ij_2,i_2})L_{i_3j_3}\cdots L_{i_{k-1}j_{k-1}}. \tag{110}$$

By the Codazzi equation (28), this is equal to 0.
We continue the computation of (108) by an argument similar to that of (110).

$$\begin{aligned}I_{k,2}=&\int_M -v_j([T_{k-1}]_{ij}(D^2v,L,\dots,L))_i d\mu_M \\ =&\int_M -v_j\delta^{i,i_1,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} v_{i_1j_1i}L_{i_2j_2}\cdots L_{i_{k-1}j_{k-1}}d\mu_M \\ =&\frac{1}{2}\int_M -v_j\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} (v_{i_1j_1i}-v_{ij_1i_1})L_{i_2j_2}\cdots L_{i_{k-1}j_{k-1}}d\mu_M.\end{aligned} \tag{111}$$

By the curvature equation (64), it follows that

$$I_{k,2}=-\frac{1}{2}\int_M v_j\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} R_{mj_1i_1i}v_m L_{i_2j_2}\cdots L_{i_{k-1}j_{k-1}}d\mu_M. \tag{112}$$

Again we can apply the Gauss equation (27),

$$I_{k,2}=\frac{1}{2}\int_M \delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} (L_{mi}L_{i_1j_1}-L_{mi_1}L_{ij_1})L_{i_2j_2}\cdots L_{i_{k-1}j_{k-1}}v_jv_m d\mu_M. \tag{113}$$

If we change the positions of the dummy indices i and i_1, and use the fact that $\delta^{i_2,i_1,i,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} = -\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}}$, then

$$\begin{aligned}I_{k,2}=&\frac{1}{2}\int_M \delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} L_{mi}L_{i_1j_1}L_{i_2j_2}\cdots L_{i_{k-1}j_{k-1}}v_jv_m \\ &\qquad+\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,j_2,\dots,j_{k-1}} L_{mi}L_{i_1j_1}L_{i_2j_2}\cdots L_{i_{k-1}j_{k-1}}v_jv_m d\mu_M \\ =&\int_M [T_{k-1}]_{ij}(L,\dots,L)L_{mi}v_jv_m d\mu_M.\end{aligned} \tag{114}$$

We remark that in (111)-(114), we have proved that

$$([T_{k-1}]_{ij}(D^2v,L,\dots,L))_i=-[T_{k-1}]_{ij}(L,\dots,L)L_{mi}v_m. \tag{115}$$

This formula will be used later in this section as well. Since

$$[T_{k-1}]_{ij}(L,\dots,L)L_{mi}=\sigma_k(L)\delta_{mj}-[T_k]_{mj}(L,\dots,L),$$

we have

$$\begin{aligned}I_{k,2} =&\int_M [T_{k-1}]_{ij}(L,...,L)L_{mi}v_jv_m d\mu_M\\=&\int_M \sigma_k(L)|\nabla v|^2 d\mu_M - \int_M [T_k]_{mj}(L,...,L)v_jv_m d\mu_M.\end{aligned}\tag{116}$$

Note that $|\nabla v| \le 1$, so

$$\int_M \sigma_k(L)|\nabla v|^2 d\mu_M \le \int_M \sigma_k(L) d\mu_M.$$

Also, due to the fact that $L \in \Gamma^+_{k+1}$, $[T_k]_{mj}(L,...,L) \ge 0$. Thus

$$-\int_M [T_k]_{mj}(L,...,L)v_jv_m d\mu_M \le 0.\tag{117}$$

Therefore

$$I_{k,2} \le C\int_M \sigma_k(L) d\mu_M.\tag{118}$$

This finishes the proof of inequality (106) for $m = 2$. Notice the assumption $L \in \Gamma^+_{k+1}$ has been used in the estimate of $I_{k,2}$. In the following inductive argument, we will see $L \in \Gamma^+_{k+1}$ is an essential assumption to estimate $I_{k,m}$ for $m \le k$.
To begin the inductive argument, we suppose for $m = 1,...,i_0-1$ where $i_0 \ge 3$, the inequality (106) hold for some constant C depending only on k and n. We will call this assumption the inductive assumption from now on. With this assumption, we want to prove the statement holds for $m = i_0$. Namely, there exists a C, such that

$$I_{k,i_0} \le C\int_M \sigma_k(L) d\mu_M.\tag{119}$$

We remark that in the following, constants denoted by C may have different values from line to line. But all of them depend only on k and n.

To prove the statement for $m = i_0$, we begin by simplifying I_{k,i_0}.

$$\begin{aligned}I_{k,i_0} :=&\int_M \Sigma_k(\overbrace{D^2v,...,D^2v}^{i_0},L,...,L)d\mu_M\\=&\int_M v_{ij}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-1},L,...,L)d\mu_M\\=&\int_M -v_j([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-1},L,...,L))_i d\mu_M\\=&-(i_0-1)\int_M v_j\delta^{i,i_1,...,i_{k-1}}_{j,j_1,...,j_{k-1}}v_{i_1j_1i}v_{i_2j_2}\cdots v_{i_{i_0-1}j_{i_0-1}}L_{i_{i_0}j_{i_0}}\cdots L_{i_{k-1}j_{k-1}}d\mu_M,\end{aligned}\tag{120}$$

where all terms involving the covariant derivative of L disappear for exactly the same reason as stated in (110). Also, similar to (111)-(114), we get

$$\begin{aligned}
&\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} v_{i_1 j_1 i} v_{i_2 j_2} \cdots v_{i_{i_0-1} j_{i_0-1}} L_{i_{i_0} j_{i_0}} \cdots L_{i_{k-1} j_{k-1}} \\
=&\frac{1}{2}\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} (v_{i_1 j_1 i} - v_{i j_1 i_1}) v_{i_2 j_2} \cdots v_{i_{i_0-1} j_{i_0-1}} L_{i_{i_0} j_{i_0}} \cdots L_{i_{k-1} j_{k-1}} \\
=&\frac{1}{2}\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} R_{m j_1 i_1 i} v_m v_{i_2 j_2} \cdots v_{i_{i_0-1} j_{i_0-1}} L_{i_{i_0} j_{i_0}} \cdots L_{i_{k-1} j_{k-1}} \\
=&\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} L_{m i_1} L_{i j_1} v_m v_{i_2 j_2} \cdots v_{i_{i_0-1} j_{i_0-1}} L_{i_{i_0} j_{i_0}} \cdots L_{i_{k-1} j_{k-1}} \\
=&-\delta^{i,i_1,i_2,\dots,i_{k-1}}_{j,j_1,\dots,j_{k-1}} L_{m i} L_{i_1 j_1} v_m v_{i_2 j_2} \cdots v_{i_{i_0-1} j_{i_0-1}} L_{i_{i_0} j_{i_0}} \cdots L_{i_{k-1} j_{k-1}} \\
=&-[T_{k-1}]_{ij}(\overbrace{D^2v,\dots,D^2v}^{i_0-2}, L,\dots,L) L_{mi} v_m.
\end{aligned} \tag{121}$$

Here we remark that in (120)-(121), we have proved that

$$([T_{k-1}]_{ij}(\overbrace{D^2v,\dots,D^2v}^{i_0-1}, L,\dots,L))_i = -(i_0-1)[T_{k-1}]_{ij}(\overbrace{D^2v,\dots,D^2v}^{i_0-2}, L,\dots,L) L_{mi} v_m. \tag{122}$$

Such a formula will be used later in this section as well. Thus

$$I_{k,i_0} = (i_0-1)\int_M [T_{k-1}]_{ij}(\overbrace{D^2v,\dots,D^2v}^{i_0-2}, L,\dots,L) L_{mi} v_j v_m d\mu_M. \tag{123}$$

If we apply Lemma 2.7 to (123) with $l = i_0 - 2$, $B = D^2 v$ and $C = L$, then we get

$$\begin{aligned}
I_{k,i_0} =&(i_0-1)\frac{C_k^{i_0-2}}{kC_{k-1}^{i_0-2}}\int_M \Sigma_k(\overbrace{D^2v,\dots,D^2v}^{i_0-2}, L,\dots,L)|\nabla v|^2 d\mu_M \\
&-(i_0-1)\frac{C_k^{i_0-2}}{C_{k-1}^{i_0-2}}\int_M [T_k]_{mj}(\overbrace{D^2v,\dots,D^2v}^{i_0-2}, L,\dots,L) v_j v_m d\mu_M \\
&-(i_0-1)\frac{C_{k-1}^{i_0-3}}{C_{k-1}^{i_0-2}}\int_M [T_{k-1}]_{ij}(\overbrace{D^2v,\dots,D^2v}^{i_0-3}, L,\dots,L) v_{mi} v_j v_m d\mu_M.
\end{aligned} \tag{124}$$

Define

$$I_{k,l}^{(u)} := \int_M \Sigma_k(\overbrace{D^2v,\dots,D^2v}^{l}, L,\dots,L) u(x) d\mu_M, \tag{125}$$

$$J_{k,l}^{(u)} := \int_M [T_k]_{mj}(\overbrace{D^2v,\dots,D^2v}^{l}, L,\dots,L) v_j v_m u(x) d\mu_M, \tag{126}$$

and

$$K_{k,l}^{(u)} := \int_M [T_{k-1}]_{ij}(\overbrace{D^2v,\dots,D^2v}^{l}, L,\dots,L) v_{mi} v_j v_m u(x) d\mu_M. \tag{127}$$

Then by (124),

$$I_{k,i_0} = (i_0-1)\frac{C_k^{i_0-2}}{kC_{k-1}^{i_0-2}} \cdot I_{k,i_0-2}^{(|\nabla v|^2)} + (i_0-1)\frac{C_k^{i_0-2}}{C_{k-1}^{i_0-2}} \cdot J_{k,i_0-2}^{(-1)} + (i_0-1)\frac{C_{k-1}^{i_0-3}}{C_{k-1}^{i_0-2}} \cdot K_{k,i_0-3}^{(-1)}. \tag{128}$$

In the following we will call any term that takes the form $I_{k,l}^{(u)}$, $J_{k,l}^{(u)}$, $K_{k,l}^{(u)}$ the I-type term, the J-type term, and the K-type term respectively. In the special case when $u=1$, we will denote $I_{k,l}^{(1)}$, $J_{k,l}^{(1)}$, $K_{k,l}^{(1)}$ by $I_{k,l}$, $J_{k,l}$, $K_{k,l}$ for simplicity.

In order to prove (106) for I_{k,i_0}, we need to estimate $I_{k,i_0-2}^{(|\nabla v|^2)}$, $J_{k,i_0-2}^{(-1)}$, $K_{k,i_0-3}^{(-1)}$ individually.

Claim 1: *There exists a constant C depending only on k and n, such that*

$$I_{k,i_0-2}^{(|\nabla v|^2)} \leq C\int_M \sigma_k(L)dv_M. \tag{129}$$

Proof. To estimate $I_{k,i_0-2}^{(|\nabla v|^2)}$, we need the following lemma, which we will prove at the end of this section.

Lemma 6.3. *For any bounded function $u(x)$, let us denote $\max_{x\in M}|u(x)|$ by U. Then for any $l \geq 0$ there exist positive constants $C_0, ..., C_l$ depending on U, k, n, such that*

$$I_{k,l}^{(u)} \leq \sum_{s=0}^{l} C_s I_{k,s}. \tag{130}$$

Also, one can choose $C_l = U$.

We now proceed our argument assuming Lemma 6.3 holds, and apply it to $u(x) = |\nabla v|^2$, $U := \max_{x\in M} u(x) = 1$ and $l = i_0 - 2$. Then

$$I_{k,i_0-2}^{(|\nabla v|^2)} \leq I_{k,i_0-2} + \sum_{s=0}^{i_0-3} C_s I_{k,s}. \tag{131}$$

As one can see, on the right hand side of the above formula, every term is of the form $I_{k,j}$ with $0 \leq j \leq i_0-2$. Therefore by our inductive assumption,

$$I_{k,i_0-2}^{(|\nabla v|^2)} \leq C\int_M \sigma_k(L)dv_M, \tag{132}$$

for some constant C. This finishes the proof of Claim 1.

□

Remark 6.4. *It is obvious that by a similar argument, for any $l \leq i_0 - 1$ and any bounded function $u(x)$ with $U := \max_{x\in M} u(x)$,*

$$I_{k,l}^{(u)} \leq C\int_M \sigma_k(L)dv_M, \tag{133}$$

for some constant C depending only on U, k, n. Formula (133) will be referred to as the I-type estimate. Later it will be used in Claim 3 to estimate $K_{k,i_0-3}^{(-1)}$.

Claim 2:

$$J^{(-1)}_{k,i_0-2} \le C \int_M \sigma_k(L) dv_M, \tag{134}$$

for some constant C. Instead of estimating $J^{(-1)}_{k,i_0-2}$, we want to analyze the more general term $J^{(u)}_{k,i_0-2}$ for any bounded function u on M with bounds depending only on k and n. Recall that

$$J^{(u)}_{k,i_0-2} := \int_M [T_k]_{mj}(\overbrace{D^2v, ..., D^2v}^{i_0-2}, L, ..., L) v_j v_m u(x) d\mu_M. \tag{135}$$

Define $U := \max_{x \in M} |u(x)|$.

Proof of Claim 2. To estimate $J^{(u)}_{k,i_0-2}$, we write

$$\begin{aligned}
J^{(u)}_{k,i_0-2} &:= \int_M [T_k]_{mj}(\overbrace{D^2v, ..., D^2v}^{i_0-2}, L, ..., L) u(x) v_j v_m d\mu_M \\
&= \int_M \Sigma_{k+1}(\overbrace{D^2v, ..., D^2v}^{i_0-2}, L, ..., L, dv \otimes dv) u(x) d\mu_M \\
&= \int_M \Sigma_{k+1}(\overbrace{\bar{D}^2\bar{V} + b(x)L, ..., \bar{D}^2\bar{V} + b(x)L}^{i_0-2}, L, ..., L, dv \otimes dv) u(x) d\mu_M \\
&= \int_M \sum_{j=0}^{i_0-2} C^j_{i_0-2} (b(x))^{i_0-2-j} \Sigma_{k+1}(\overbrace{\bar{D}^2\bar{V}, ..., \bar{D}^2\bar{V}}^{j}, L, ..., L, dv \otimes dv) u(x) d\mu_M.
\end{aligned} \tag{136}$$

Since $L \in \Gamma^+_{k+1}$ and $\bar{D}^2\bar{V} \ge 0$,

$$\Sigma_{k+1}(\overbrace{\bar{D}^2\bar{V}, ..., \bar{D}^2\bar{V}}^{j}, L, ..., L, dv \otimes dv) \ge 0.$$

Also, $|b(x)| \leq 1$ and $|\nabla v| \leq 1$. Thus it follows that

$$
\begin{aligned}
J^{(u)}_{k,i_0-2} &\leq \sum_{j=0}^{i_0-2} U \cdot C^j_{i_0-2} \int_M \Sigma_{k+1}(\overbrace{\bar{D}^2\bar{V}, ..., \bar{D}^2\bar{V}}^{j}, L, ..., L, \delta_{ij}) d\mu_M \\
&= \sum_{j=0}^{i_0-2} U \cdot C^j_{i_0-2} \int_M Tr([T_k]_{ij})(\overbrace{\bar{D}^2\bar{V}, ..., \bar{D}^2\bar{V}}^{j}, L, ..., L) d\mu_M \\
&= \sum_{j=0}^{i_0-2} \frac{n-k}{k} \cdot U \cdot C^j_{i_0-2} \int_M \Sigma_k(\overbrace{\bar{D}^2\bar{V}, ..., \bar{D}^2\bar{V}}^{j}, L, ..., L) d\mu_M \qquad (137) \\
&= \sum_{j=0}^{i_0-2} \frac{n-k}{k} \cdot U \cdot C^j_{i_0-2} \int_M \Sigma_k(\overbrace{D^2v - b(x)L, ..., D^2v - b(x)L}^{j}, L, ..., L) d\mu_M \\
&= \sum_{j=0}^{i_0-2} \int_M u^{(II)}_j(x) \Sigma_k(\overbrace{D^2v, ..., D^2v}^{j}, L, ..., L) d\mu_M.
\end{aligned}
$$

Again $u^{(II)}_j(x)$ $(j = 0, ..., i_0 - 2)$ are some bounded functions which we can estimate in terms of U, k and n. It follows from Lemma 6.3 that there exists nonnegative constants, still denoted by C_s, $s = 0, ..., i_0 - 2$, such that

$$
\sum_{j=0}^{i_0-2} \int_M u^{(II)}_j(x) \Sigma_k(\overbrace{D^2v, ..., D^2v}^{j}, L, ..., L) d\mu_M \leq \sum_{s=0}^{i_0-2} C_s I_{k,s}. \qquad (138)
$$

Thus by (137) and (138)

$$
J^{(u)}_{k,i_0-2} \leq \sum_{s=0}^{i_0-2} C_s I_{k,s}. \qquad (139)
$$

Again, every term on the right hand side is of the form $I_{k,s}$ with $s \leq i_0 - 2$. Thus by our inductive assumption,

$$
J^{(u)}_{k,i_0-2} \leq C \int_M \sigma_k(L) d\mu_M. \qquad (140)
$$

This finishes the estimate of $J^{(u)}_{k,i_0-2}$. It is obvious that $J^{(-1)}_{k,i_0-2}$ is a special case of $J^{(u)}_{k,i_0-2}$ when $u(x) \equiv -1$. Thus (140) holds for $J^{(-1)}_{k,i_0-2}$ as well. This concludes the proof of Claim 2.

□

Remark 6.5. *It is obvious that by a similar argument, for any $l \leq i_0 - 1$ and any bounded function $u(x)$ with $U := \max_{x \in M} u(x)$,*

$$
J^{(u)}_{k,l} \leq C \int_M \sigma_k(L) dv_M, \qquad (141)
$$

for some constant C depending only on U, k, n. Formula (141) will be referred to as J-type estimate. Later it will be used together with Remark 6.4 to estimate $K^{(-1)}_{k,i_0-3}$.

Claim 3:

$$K^{(-1)}_{k,i_0-3} \le C \int_M \sigma_k(L) dv_M, \tag{142}$$

for some constant C.

Proof of Claim 3. If $i_0 = 3$, then it is easy to see that

$$\begin{aligned} K^{(-1)}_{k,i_0-3} &:= -\int_M [T_{k-1}]_{ij}(L,...,L) v_{mi} v_j v_m d\mu_M \\ &= -\int_M [T_{k-1}]_{ij}(L,...,L) \frac{1}{2} v_j (|\nabla v|^2)_i d\mu_M \\ &= \int_M ([T_{k-1}]_{ij}(L,...,L))_i \frac{1}{2} v_j |\nabla v|^2 d\mu_M + \int_M [T_{k-1}]_{ij}(L,...,L) \frac{1}{2} v_{ij} |\nabla v|^2 d\mu_M. \end{aligned} \tag{143}$$

Notice that

$$([T_{k-1}]_{ij}(L,...,L))_i = 0, \tag{144}$$

by the same reason as in the proof of Lemma 5.1.
The second term

$$\int_M [T_{k-1}]_{ij}(L,...,L) \frac{1}{2} v_{ij} |\nabla v|^2 d\mu_M = \frac{1}{2} \int_M \Sigma_k(D^2 v, L, ..., L) |\nabla v|^2 d\mu_M = \frac{1}{2} I^{(|\nabla v|^2)}_{k,1}, \tag{145}$$

by the definition of $I^{(u)}_{k,l}$ in (125). Thus by the I-type estimate (133) in Remark 6.4, $\frac{1}{2} I^{(|\nabla v|^2)}_{k,1} \le C \int_M \sigma_k(L) d\mu_M$ for some constant C depending only on k and n. Therefore $K^{(-1)}_{k,i_0-3} \le C \int_M \sigma_k(L) d\mu_M$.

If $i_0 = 4$, then

$$\begin{aligned} & K^{(-1)}_{k,i_0-3} \\ :=& -\int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) v_{mi} v_j v_m d\mu_M \\ =& -\int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) \frac{1}{2} v_j (|\nabla v|^2)_i d\mu_M \\ =& \int_M ([T_{k-1}]_{ij}(D^2 v, L,...,L))_i \frac{1}{2} v_j |\nabla v|^2 d\mu_M + \int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) \frac{1}{2} v_{ij} |\nabla v|^2 d\mu_M. \end{aligned} \tag{146}$$

The second term in the last line of (146)

$$\int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) \frac{1}{2} v_{ij} |\nabla v|^2 d\mu_M = \frac{1}{2} \int_M \Sigma_k(D^2 v, D^2 v, L, ..., L) |\nabla v|^2 d\mu_M = \frac{1}{2} I^{(|\nabla v|^2)}_{k,2}, \tag{147}$$

by the definition of $I^{(u)}_{k,l}$ in (125). Thus by the I-type estimate (133) in Remark 6.4, $\frac{1}{2} I^{(|\nabla v|^2)}_{k,2} \le C \int_M \sigma_k(L) d\mu_M$, for some constant C. Now we only need to estimate the first term

$$\int_M ([T_{k-1}]_{ij}(D^2 v, L,...,L))_i \frac{1}{2} v_j |\nabla v|^2 d\mu_M$$

in the last line of (146).
To estimate $\int_M ([T_{k-1}]_{ij}(D^2v,L,...,L))_i \frac{1}{2} v_j |\nabla v|^2 d\mu_M$, notice that

$$([T_{k-1}]_{ij}(D^2v,L,...,L))_i = -[T_{k-1}]_{ij}(L,...,L)L_{il}v_l$$

by the argument of (111)-(114). Thus

$$\int_M ([T_{k-1}]_{ij}(D^2v,L,...,L))_i \frac{1}{2} v_j |\nabla v|^2 d\mu_M = -\frac{1}{2}\int_M [T_{k-1}]_{ij}(L,...,L)L_{il}v_l v_j |\nabla v|^2 d\mu_M. \tag{148}$$

By (17), and the definition of $I^{(u)}_{k,l}$, $J^{(u)}_{k,l}$ in (125), (126)

$$\begin{aligned}
&-\frac{1}{2}\int_M [T_{k-1}]_{ij}(L,...,L)L_{il}v_l v_j |\nabla v|^2 d\mu_M \\
=&\int_M \{-C_1\Sigma_k(L,...,L)\delta_{jl} + C_2[T_k]_{jl}(L,...,L)\}\, v_l v_j |\nabla v|^2 d\mu_M \\
=&\int_M -C_1\Sigma_k(L,...,L)|\nabla v|^4 + C_2[T_k]_{jl}(L,...,L)v_l v_j |\nabla v|^2 d\mu_M \\
=&-C_1 I^{(|\nabla v|^4)}_{k,0} + C_2 J^{(|\nabla v|^2)}_{k,0},
\end{aligned} \tag{149}$$

where C_1, C_2 are positive constants depending only on k and n. Notice $|\nabla v| \leq 1$; thus by (133) in Remark 6.4 and (141) in Remark 6.5, the I-type term $-C_1 I^{(|\nabla v|^4)}_{k,0}$ and J-type term $C_2 J^{(|\nabla v|^2)}_{k,0}$ are both bounded by $C\int_M \sigma_k(L)d\mu_M$ for some constant C. Thus in (149)

$$-\frac{1}{2}\int_M [T_{k-1}]_{ij}(L,...,L)L_{il}v_l v_j |\nabla v|^2 d\mu_M \leq C\int_M \sigma_k(L)d\mu_M.$$

Plugging it back to (148), we get $\int_M ([T_{k-1}]_{ij}(D^2v,L,...,L))_i v_j |\nabla v|^2 d\mu_M \leq C\int_M \sigma_k(L)d\mu_M$. This completes the estimate of the first term in the last line of (146). Hence $K^{(-1)}_{k,1} \leq C\int_M \sigma_k(L)d\mu_M$.

We now begin to prove $K^{(-1)}_{k,i_0-3} \leq C\int_M \sigma_k(L)d\mu_M$ for $i_0 \geq 5$. It will be shown shortly that the estimate of $K^{(-1)}_{k,i_0-3}$, by induction, reduces to one of the two cases depending on whether i_0 is an odd or even integer.

First of all,

$$\begin{aligned}
K^{(-1)}_{k,i_0-3} :=& -\int_M [T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L)v_{mi}v_j v_m d\mu_M \\
=& -\int_M [T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L)\frac{1}{2}v_j(|\nabla v|^2)_i d\mu_M \\
=& \int_M ([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L))_i \frac{1}{2}v_j |\nabla v|^2 d\mu_M \\
&+ \int_M [T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L)\frac{1}{2}v_{ij}|\nabla v|^2 d\mu_M \\
=& \int_M ([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L))_i \frac{1}{2}v_j |\nabla v|^2 d\mu_M + \frac{1}{2}I^{(|\nabla v|^2)}_{k,i_0-2},
\end{aligned} \tag{150}$$

by integration by parts and the definition of $I^{(u)}_{k,l}$ in (125). The I-type estimate (133) in Remark 6.4 implies that $\frac{1}{2}I^{(|\nabla v|^2)}_{k,i_0-2} \leq C\int_M \sigma_k(L)d\mu_M$; thus we only need to estimate the first term in the last line of (150), namely $\int_M([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L))_i\frac{1}{2}v_j|\nabla v|^2d\mu_M$. By a similar argument presented in (120)-(121),

$$([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L))_i = -(i_0-3)[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-4},L,...,L)L_{mi}v_m.$$

Thus

$$\begin{aligned}&\int_M([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L))_i\frac{1}{2}v_j|\nabla v|^2d\mu_M\\ =&\int_M -\frac{i_0-3}{2}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-4},L,...,L)L_{mi}v_mv_j|\nabla v|^2d\mu_M.\end{aligned}\tag{151}$$

By Lemma 2.7,

$$\begin{aligned}&[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{l+1},L,...,L)L_{mi}\\ =&\frac{C_k^{l+1}}{kC_{k-1}^{l+1}}\cdot\Sigma_k(\overbrace{D^2v,...,D^2v}^{l+1},L,...,L)\delta_{mj}-\frac{C_k^{l+1}}{C_{k-1}^{l+1}}\cdot[T_k]_{mj}(\overbrace{D^2v,...,D^2v}^{l+1},L,...,L)\\ &-\frac{C_{k-1}^{l}}{C_{k-1}^{l+1}}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{l},L,...,L)v_{mi}\end{aligned}\tag{152}$$

Let $l=i_0-5$, and plug it in (151).

$$\begin{aligned}&\int_M([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-3},L,...,L))_i\frac{1}{2}v_j|\nabla v|^2d\mu_M\\ =&\int_M -\frac{i_0-3}{2}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-4},L,...,L)L_{mi}v_mv_j|\nabla v|^2d\mu_M.\\ =&\int_M -C_1\Sigma_k(\overbrace{D^2v,...,D^2v}^{i_0-4},L,...,L)|\nabla v|^4+C_2[T_k]_{mj}(\overbrace{D^2v,...,D^2v}^{i_0-4},L,...,L)v_mv_j|\nabla v|^2\\ &+C_3[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-5},L,...,L)v_{mi}v_mv_j|\nabla v|^2d\mu_M,\\ =&-C_1I^{(|\nabla v|^4)}_{k,i_0-4}+C_2J^{(|\nabla v|^2)}_{k,i_0-4}+C_3K^{(|\nabla v|^2)}_{k,i_0-5}.\end{aligned}\tag{153}$$

where C_1, C_2, C_3 are constants depending only on k and n. They might have different values from the ones in (149). From now on, the values of C_1, C_2, C_3 may vary from line to line; but they all denote positive constants depending only on k and n. To conclude what we do in (150)-(153), we get for $i_0 \geq 5$

$$K^{(-1)}_{k,i_0-3} = \frac{1}{2} I^{(|\nabla v|^2)}_{k,i_0-2} - C_1 I^{(|\nabla v|^4)}_{k,i_0-4} + C_2 J^{(|\nabla v|^2)}_{k,i_0-4} + C_3 K^{(|\nabla v|^2)}_{k,i_0-5}. \tag{154}$$

By the I-type estimate (133) and the J-type estimate (141) as well as $|\nabla v| \leq 1$, we know

$$\frac{1}{2} I^{(|\nabla v|^2)}_{k,i_0-2} \leq C \int_M \sigma_k(L) d\mu_M; \qquad -C_1 I^{(|\nabla v|^4)}_{k,i_0-4} \leq C \int_M \sigma_k(L) d\mu_M;$$

and

$$C_2 J^{(|\nabla v|^2)}_{k,i_0-4} \leq C \int_M \sigma_k(L) d\mu_M.$$

Therefore

$$K^{(-1)}_{k,i_0-3} \leq C \int_M \sigma_k(L) d\mu_M + C_3 K^{(|\nabla v|^2)}_{k,i_0-5}. \tag{155}$$

Recall that we call any term that takes the form of $K^{(u)}_{k,l}$ the K-type term. In (155), the index l in the K-type term drops from $i_0 - 3$ to $i_0 - 5$. Thus (155) is an inductive inequality of the K-type term, which can be used to decrease the index l. In each step, the index l drops by 2. The induction will stop when either $l = 0$ or $l = 1$. Notice the function u in the K-type term changes from -1 to $|\nabla v|^2$; and in the following induction steps, it will change to $-|\nabla v|^4$, $|\nabla v|^6$ and so on. Since all of them are bounded functions (with bounds 1), this change won't affect the inductive procedure. To see this, we demonstrate one more step of the induction in the following.

Let us assume $i_0 \geq 7$. Otherwise, $i_0 - 5$ is either equal to 0 or 1, so the inductive argument stops. ($i_0 - 5$ is not possible to be a negative integer, since we assumed $i_0 \geq 5$ previously.) With this assumption,

$$\begin{aligned} K^{(|\nabla v|^2)}_{k,i_0-5} &:= \int_M [T_{k-1}]_{ij}(\overbrace{D^2 v, ..., D^2 v}^{i_0-5}, L, ..., L) v_{mi} v_m v_j |\nabla v|^2 d\mu_M \\ &= \int_M [T_{k-1}]_{ij}(\overbrace{D^2 v, ..., D^2 v}^{i_0-5}, L, ..., L) \frac{1}{4} v_j (|\nabla v|^4)_i d\mu_M. \end{aligned} \tag{156}$$

The term on the last line is in the same form as the term on the 2nd line of (150), thus we can start a similar argument using integration by parts.

$$\begin{aligned} K^{(|\nabla v|^2)}_{k,i_0-5} &= \int_M [T_{k-1}]_{ij}(\overbrace{D^2 v, ..., D^2 v}^{i_0-5}, L, ..., L) \frac{1}{4} v_j (|\nabla v|^4)_i d\mu_M \\ &= -\int_M ([T_{k-1}]_{ij}(\overbrace{D^2 v, ..., D^2 v}^{i_0-5}, L, ..., L))_i \frac{1}{4} v_j |\nabla v|^4 d\mu_M \\ &\quad - \int_M [T_{k-1}]_{ij}(\overbrace{D^2 v, ..., D^2 v}^{i_0-5}, L, ..., L) \frac{1}{4} v_{ij} |\nabla v|^4 d\mu_M \\ &= -\int_M ([T_{k-1}]_{ij}(\overbrace{D^2 v, ..., D^2 v}^{i_0-5}, L, ..., L))_i \frac{1}{4} v_j |\nabla v|^2 d\mu_M - \frac{1}{4} I^{(|\nabla v|^4)}_{k,i_0-4}. \end{aligned} \tag{157}$$

Since the I-type estimate (133) in Remark 6.4 implies that $-\frac{1}{4}I^{(|\nabla v|^4)}_{k,i_0-4} \le C\int_M \sigma_k(L)d\mu_M$, we only need to estimate the first term in the last line of (157), namely

$$-\int_M ([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-5},L,...,L))_i \frac{1}{4}v_j|\nabla v|^2 d\mu_M.$$

By a similar argument presented in (120)-(121),

$$([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-5},L,...,L))_i = -(i_0-5)[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-6},L,...,L)L_{mi}v_m.$$

Thus

$$\begin{aligned}&-\int_M ([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-5},L,...,L))_i \frac{1}{4}v_j|\nabla v|^4 d\mu_M\\ =&\int_M \frac{i_0-5}{2}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-6},L,...,L)L_{mi}v_mv_j|\nabla v|^4 d\mu_M.\end{aligned} \tag{158}$$

By Lemma 2.7,

$$\begin{aligned}&[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{l+1},L,...,L)L_{mi}\\ =&\frac{C_k^{l+1}}{kC_{k-1}^{l+1}}\cdot\Sigma_k(\overbrace{D^2v,...,D^2v}^{l+1},L,...,L)\delta_{mj} - \frac{C_k^{l+1}}{C_{k-1}^{l+1}}\cdot[T_k]_{mj}(\overbrace{D^2v,...,D^2v}^{l+1},L,...,L)\\ &-\frac{C_{k-1}^{l}}{C_{k-1}^{l+1}}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{l},L,...,L)v_{mi}\end{aligned} \tag{159}$$

Let $l = i_0-7$, and plug it in (158).

$$\begin{aligned}&-\int_M ([T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-5},L,...,L))_i \frac{1}{4}v_j|\nabla v|^2 d\mu_M\\ =&\int_M \frac{i_0-5}{2}[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-6},L,...,L)L_{mi}v_mv_j|\nabla v|^4 d\mu_M.\\ =&\int_M C_1\Sigma_k(\overbrace{D^2v,...,D^2v}^{i_0-6},L,...,L)|\nabla v|^6 - C_2[T_k]_{mj}(\overbrace{D^2v,...,D^2v}^{i_0-6},L,...,L)v_mv_j|\nabla v|^4\\ &-C_3[T_{k-1}]_{ij}(\overbrace{D^2v,...,D^2v}^{i_0-7},L,...,L)v_{mi}v_mv_j|\nabla v|^4 d\mu_M,\\ =&C_1I^{(|\nabla v|^6)}_{k,i_0-6} - C_2J^{(|\nabla v|^4)}_{k,i_0-6} - C_3K^{(|\nabla v|^4)}_{k,i_0-7}.\end{aligned} \tag{160}$$

In this way, we derive

$$K^{(|\nabla v|^2)}_{k,i_0-5} = -\frac{1}{4}I^{(|\nabla v|^4)}_{k,i_0-4} + C_1I^{(|\nabla v|^6)}_{k,i_0-6} - C_2J^{(|\nabla v|^4)}_{k,i_0-6} - C_3K^{(|\nabla v|^4)}_{k,i_0-7}. \tag{161}$$

By the I-type estimate (133) and the J-type estimate (141) as well as $|\nabla v| \leq 1$, we know

$$-\frac{1}{4} I_{k,i_0-4}^{(|\nabla v|^4)} \leq C \int_M \sigma_k(L) d\mu_M; \qquad C_1 I_{k,i_0-6}^{(|\nabla v|^6)} \leq C \int_M \sigma_k(L) d\mu_M;$$

and

$$-C_2 J_{k,i_0-6}^{(|\nabla v|^4)} \leq C \int_M \sigma_k(L) d\mu_M.$$

Therefore

$$K_{k,i_0-5}^{(|\nabla v|^2)} \leq C \int_M \sigma_k(L) d\mu_M + C_3 K_{k,i_0-7}^{(-|\nabla v|^4)}. \tag{162}$$

(155) together with (162) imply that

$$K_{k,i_0-3}^{(-1)} \leq C \int_M \sigma_k(L) d\mu_M + C_3 K_{k,i_0-7}^{(-|\nabla v|^4)}. \tag{163}$$

This finishes the 2nd step of the induction. As we mentioned before, the function u in the K-type term changes from -1 to $|\nabla v|^2$ in the first step of the induction; and it changes to $-|\nabla v|^4$ in the 2nd step of the induction. Since all of them are bounded functions (with bounds 1), this change won't affect the induction step. Therefore, we conclude that in the q-th step of the induction, we will get

$$K_{k,i_0-3}^{(-1)} \leq C \int_M \sigma_k(L) d\mu_M + C_3 K_{k,i_0-2q-3}^{((-1)^{q+1} \cdot |\nabla v|^{2q})}. \tag{164}$$

The induction stops when $q = \frac{[i_0-3]}{2}$, where $[\cdot]$ denotes the integer part of a number. If i_0 is odd, then when the induction stops we get

$$K_{k,i_0-3}^{(-1)} \leq C \int_M \sigma_k(L) d\mu_M + C_3 K_{k,0}^{((-1)^{\frac{i_0-1}{2}} \cdot |\nabla v|^{i_0-3})}. \tag{165}$$

If i_0 is even, then when the induction stops we get

$$K_{k,i_0-3}^{(-1)} \leq C \int_M \sigma_k(L) d\mu_M + C_3 K_{k,1}^{((-1)^{\frac{i_0-2}{2}} \cdot |\nabla v|^{i_0-4})}. \tag{166}$$

We will prove in the following:

$$K_{k,0}^{(\pm|\nabla v|^{i_0-3})} \leq C \int_M \sigma_k(L) d\mu_M, \quad \text{when } i_0 \text{ is odd;} \tag{167}$$

and

$$K_{k,1}^{(\pm|\nabla v|^{i_0-4})} \leq C \int_M \sigma_k(L) d\mu_M, \quad \text{when } i_0 \text{ is even.} \tag{168}$$

The proofs of these two inequalities are similar to the the arguments of $K_{k,0}^{-1}$, $K_{k,1}^{(-1)}$ respectively, which we have shown at the beginning of the proof of Claim 3.

To prove (167) when i_0 is odd, we first write

$$\begin{aligned} K_{k,0}^{(\pm|\nabla v|^{i_0-3})} &:= \pm \int_M [T_{k-1}]_{ij}(L,...,L) v_{mi} v_j v_m |\nabla v|^{i_0-3} d\mu_M \\ &= \pm \int_M [T_{k-1}]_{ij}(L,...,L) \frac{1}{i_0-1} v_j (|\nabla v|^{i_0-1})_i d\mu_M \\ &= \mp \int_M ([T_{k-1}]_{ij}(L,...,L))_i \frac{1}{i_0-1} v_j |\nabla v|^{i_0-1} d\mu_M \\ &\quad \mp \int_M [T_{k-1}]_{ij}(L,...,L) \frac{1}{i_0-1} v_{ij} |\nabla v|^{i_0-1} d\mu_M. \end{aligned} \tag{169}$$

Notice that

$$([T_{k-1}]_{ij}(L,...,L))_i = 0, \tag{170}$$

by the same reason as in the proof of Lemma 5.1.
So we only need to estimate the term

$$\begin{aligned} &\mp \int_M [T_{k-1}]_{ij}(L,...,L) \frac{1}{i_0-1} v_{ij} |\nabla v|^{i_0-1} d\mu_M \\ =& \mp \frac{1}{i_0-1} \int_M \Sigma_k(D^2 v, L,...,L) |\nabla v|^{i_0-1} d\mu_M = \mp \frac{1}{i_0-1} I_{k,1}^{(|\nabla v|^{i_0-1})}, \end{aligned} \tag{171}$$

by the definition of $I_{k,l}^{(u)}$ in (125). Now by the I-type estimate (133) in Remark 6.4, $\mp \frac{1}{i_0-1} I_{k,1}^{(|\nabla v|^{i_0-1})} \leq C \int_M \sigma_k(L) d\mu_M$ for some constant C depending only on k and n. Therefore $K_{k,0}^{(\pm|\nabla v|^{i_0-3})} \leq C \int_M \sigma_k(L) d\mu_M$.

To prove (168) when i_0 is even, we write

$$\begin{aligned} &K_{k,1}^{(\pm|\nabla v|^{i_0-4})} \\ :=& \pm \int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) v_{mi} v_j v_m |\nabla v|^{i_0-4} d\mu_M \\ =& \pm \int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) \frac{1}{i_0-2} v_j (|\nabla v|^{i_0-2})_i d\mu_M \\ =& \mp \int_M ([T_{k-1}]_{ij}(D^2 v, L,...,L))_i \frac{1}{i_0-2} v_j |\nabla v|^{i_0-2} d\mu_M \\ &\mp \int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) \frac{1}{i_0-2} v_{ij} |\nabla v|^{i_0-2} d\mu_M. \end{aligned} \tag{172}$$

Notice

$$\begin{aligned} &\mp \int_M [T_{k-1}]_{ij}(D^2 v, L,...,L) \frac{1}{i_0-2} v_{ij} |\nabla v|^{i_0-2} d\mu_M \\ =& \mp \frac{1}{i_0-2} \int_M \Sigma_k(D^2 v, D^2 v, L,...,L) |\nabla v|^{i_0-2} d\mu_M = \mp \frac{1}{i_0-2} I_{k,2}^{(|\nabla v|^{i_0-2})}, \end{aligned} \tag{173}$$

by the definition of $I_{k,l}^{(u)}$ in (125). Thus by the I-type estimate (133) in Remark 6.4, $\mp \frac{1}{i_0-2} I_{k,2}^{(|\nabla v|^{i_0-2})} \leq C \int_M \sigma_k(L) d\mu_M$. Now we only need to estimate the term $\mp \int_M ([T_{k-1}]_{ij}(D^2 v, L,...,L))_i \frac{1}{i_0-2} v_j |\nabla v|^{i_0-2} d\mu_M$

in (172).
To estimate $\mp \int_M ([T_{k-1}]_{ij}(D^2 v, L, ..., L))_i \frac{1}{i_0-2} v_j |\nabla v|^{i_0-2} d\mu_M$, we use

$$([T_{k-1}]_{ij}(D^2 v, L, ..., L))_i = -[T_{k-1}]_{ij}(L, ..., L) L_{il} v_l,$$

which is proved in (111)-(114). Thus

$$\begin{aligned} &\mp \int_M ([T_{k-1}]_{ij}(D^2 v, L, ..., L))_i \frac{1}{i_0 - 2} v_j |\nabla v|^{i_0-2} d\mu_M \\ =& \pm \frac{1}{i_0 - 2} \int_M [T_{k-1}]_{ij}(L, ..., L) L_{il} v_l v_j |\nabla v|^{i_0-2} d\mu_M. \end{aligned} \tag{174}$$

By (17), and the definition of $I_{k,l}^{(u)}$, $J_{k,l}^{(u)}$ in (125), (126)

$$\begin{aligned} &\pm \frac{1}{i_0 - 2} \int_M [T_{k-1}]_{ij}(L, ..., L) L_{il} v_l v_j |\nabla v|^{i_0-2} d\mu_M \\ =& \int_M \{\pm C_1 \Sigma_k(L, ..., L)\delta_{jl} \mp C_2 [T_k]_{jl}(L, ..., L)\} v_l v_j |\nabla v|^{i_0-2} d\mu_M \\ =& \int_M \pm C_1 \Sigma_k(L, ..., L)|\nabla v|^{i_0} \mp C_2 [T_k]_{jl}(L, ..., L) v_l v_j |\nabla v|^{i_0-2} d\mu_M \\ =& \pm C_1 I_{k,0}^{(|\nabla v|^{i_0})} \mp C_2 J_{k,0}^{(|\nabla v|^{i_0-2})}, \end{aligned} \tag{175}$$

where C_1, C_2 are positive constants depending only on k and n. Notice $|\nabla v| \leq 1$; thus by (133) in Remark 6.4 and (141) in Remark 6.5, the I-type term $\pm C_1 I_{k,0}^{(|\nabla v|^{i_0})}$ and the J-type term $\mp C_2 J_{k,0}^{(|\nabla v|^{i_0-2})}$ are both bounded by $C \int_M \sigma_k(L) d\mu_M$ for some constant C. Thus in (175)

$$\pm \frac{1}{i_0 - 2} \int_M [T_{k-1}]_{ij}(L, ..., L) L_{il} v_l v_j |\nabla v|^{i_0-2} d\mu_M \leq C \int_M \sigma_k(L) d\mu_M.$$

Plugging it back to (174), we get $\mp \int_M ([T_{k-1}]_{ij}(D^2 v, L, ..., L))_i \frac{1}{i_0-2} v_j |\nabla v|^{i_0-2} d\mu_M \leq C \int_M \sigma_k(L) d\mu_M$. This completes the estimate of the term $\mp \int_M ([T_{k-1}]_{ij}(D^2 v, L, ..., L))_i \frac{1}{i_0-2} v_j |\nabla v|^{i_0-2} d\mu_M$ in (172). Hence $K_{k,1}^{(\pm|\nabla v|^{i_0-4})} \leq C \int_M \sigma_k(L) d\mu_M$.

With inequalities (167) and (168), one can conclude that $K_{k,i_0-3}^{(-1)} \leq C \int_M \sigma_k(L) d\mu_M$. Thus it finishes of the proof of Claim 3. □

By Claim 1, 2, 3,

$$I_{k,i_0} = I_{k,i_0-2}^{(|\nabla v|^2)} + J_{k,i_0-2}^{(-1)} + K_{k,i_0-3}^{(-1)} \leq C \int_M \sigma_k(L) d\mu_M. \tag{176}$$

This finishes the inductive argument. Therefore we have proved Proposition 6.2. □

We finish this section by giving the proof of Lemma 6.3.

Proof. An easy inductive argument would lead us to the conclusion. When $l = 0$, the statement is obviously true. Now suppose this statement holds for $l \leq l_0 - 1$ where $l_0 \geq 0$; we would like to prove that it also holds for $l=l_0$. In fact, since $D^2v=\bar{D}^2\bar{V}+b(x)L$ and $\Sigma_k(\overbrace{\bar{D}^2\bar{V},...,\bar{D}^2\bar{V}}^{j},L,...,L)>0$, we have

$$\begin{aligned}
I_{k,l_0}^{(u)} :=& \int_M \Sigma_k(\overbrace{D^2v,...,D^2v}^{l_0},L,...,L)u(x)d\mu_M \\
=& \int_M \Sigma_k(\overbrace{\bar{D}^2\bar{V}+b(x)L,...,\bar{D}^2\bar{V}+b(x)L}^{l_0},L,...,L)u(x)d\mu_M \\
=& \int_M \Sigma_k(\overbrace{\bar{D}^2\bar{V},...,\bar{D}^2\bar{V}}^{l_0},L,...,L)u(x)+\sum_{j=0}^{l_0-1}C_{l_0}^j\Sigma_k(\overbrace{\bar{D}^2\bar{V},...,\bar{D}^2\bar{V}}^{j},L,...,L)b(x)^{l_0-j}u(x)d\mu_M \\
\leq& \int_M U\cdot\Sigma_k(\overbrace{\bar{D}^2\bar{V},...,\bar{D}^2\bar{V}}^{l_0},L,...,L)d\mu_M+\int_M\sum_{j=0}^{l_0-1}U\cdot C_{l_0}^j\Sigma_k(\overbrace{\bar{D}^2\bar{V},...,\bar{D}^2\bar{V}}^{j},L,...,L)d\mu_M \\
=& \int_M U\cdot\Sigma_k(\overbrace{D^2v-b(x)L,...,D^2v-b(x)L}^{l_0},L,...,L)d\mu_M \\
&+\int_M\sum_{j=0}^{l_0-1}U\cdot C_{l_0}^j\Sigma_k(\overbrace{D^2v-b(x)L,...,D^2v-b(x)L}^{j},L,...,L)d\mu_M \\
=& \int_M U\cdot\Sigma_k(\overbrace{D^2v,...,D^2v}^{l_0},L,...,L)d\mu_M+\sum_{j=0}^{l_0-1}\int_M b_j(x)\Sigma_k(\overbrace{D^2v,...,D^2v}^{j},L,...,L)d\mu_M \\
=& U\cdot I_{k,l_0}+\sum_{j=0}^{l_0-1}I_{k,l}^{(b_j)},
\end{aligned} \tag{177}$$

where $b_j(x)$ are bounded functions whose estimates only depend on U, k and n. Now we choose $C_{l_0}=U$. Also notice that every term in $\sum_{j=0}^{l_0-1}I_{k,l}^{(b_j)}$ falls into the case of our inductive assumption. Thus there exist nonnegative constants $C_0,...,C_{l_0-1}$ (together with $C_{l_0}=U$), such that

$$I_{k,l_0}^{(u)}\leq\sum_{s=0}^{l_0}C_sI_{k,s}. \tag{178}$$

This concludes the proof of Lemma 6.3. $\square$

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